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The Existence and Uniqueness of Mild Solutions for Fuzzy Hilfer Fractional Evolution Equations with Non-Local Conditions

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Abstract

In this paper, we investigate a fuzzy Hilfer fractional evolution equation of type $0 < \beta < 1$ and order $1 < \alpha < 2$ subject to nonlocal conditions. Using the infinitesimal generator of a strongly continuous cosine family, we define a mild solution for the proposed system. The existence and uniqueness of such mild solutions are established through Schauder's fixed-point theorem and the Banach contraction principle. An illustrative application to a fuzzy fractional wave equation is presented to demonstrate the effectiveness of the developed approach. The main contribution of this study lies in the unified treatment of fuzzy Hilfer fractional evolution equations under nonlocal conditions, which generalizes and extends several existing results and provides a solid analytical foundation for modeling systems with memory and uncertainty.

Keywords: fuzzy Hilfer fractional derivative; evolution equation; mild solution; Schauder's fixed-point theorem

MSC: 34A08; 34A12; 34G99; 34K99; 34A60



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1. Introduction

In recent times, researchers have increasingly recognized the considerable importance of fuzzy fractional equations as powerful tools for modeling various phenomena characterized by uncertainty and historical dependency. By merging fractional calculus with fuzzy set theory, these equations effectively capture the inherent uncertainty found in a variety of domains, including engineering, physics, biology, economics, and control systems [1,2].

In particular, fractional evolution equations employ fractional derivatives—notably, Caputo derivatives—due to their intrinsic capacity to incorporate memory effects into mathematical formulations. This feature makes these equations suitable for tackling complicated initial and boundary value problems in which the current state relies significantly on past states. Additionally, the integration of fuzzy set theory into these equations enhances their flexibility by allowing parameters, initial conditions, and solutions to be represented as fuzzy numbers, thus more accurately reflecting real-world uncertainties [3,4].

Non-local boundary conditions represent a significant improvement in the theory of fuzzy fractional evolution equations. Unlike classical boundary conditions, which focus exclusively on individual boundary points, non-local conditions include integral or functional relationships that reflect broader interactions within the system. This extension increases the applicability of such equations, providing more realistic frameworks for modeling complex scenarios where boundary interactions are influenced by the overall state of the system [5].

Studying the mild solutions to fuzzy fractional evolution equations subject to non-local boundary conditions is critical to ensuring the accuracy and predictability of these mathematical models. Researchers have employed various fixed-point methods—such as Schauder’s, Krasnoselskii’s, and Darbo’s fixed-point theorems—along with measures of non-compactness to derive robust criteria for the existence and uniqueness of solutions. These theoretical advancements have significantly expanded the practical utility of fuzzy fractional equations, enhancing their effectiveness in solving real-world problems [6].

The study of existence and uniqueness findings for fractional difference equations of the nabla and delta types under fixed point frameworks has seen significant interest in recent advances in discrete fractional calculus. For example, Dimitrov and Jonnalagadda [7] used Banach, Brouwer, and Leray–Schauder fixed-point theorems to prove existence findings for nabla fractional problems with anti-periodic boundary conditions. Dimitrov explored numerous positive solutions for nabla fractional equations under summation boundary conditions in another recent study [8]. Moreover, mixed delta-nabla frameworks have been investigated: Using Banach and Schauder theorems, Reunsumrit and Sitthiwiratham [9] examined delta–nabla difference equations with mixed boundary conditions.

Additionally, Dimitrov and Jonnalagadda [10] made a significant contribution that emphasises the robustness and qualitative behaviour of discrete fractional models by examining the existence, uniqueness, and stability of solutions for nabla fractional difference equations under multipoint summation boundary conditions.

Several notable contributions have significantly advanced the theory and application of fuzzy fractional evolution equations. El Ghazouani et al. [11] discussed the non-local Cauchy problem for fuzzy fractional evolution equations within an arbitrary Banach space for fractional order $\omega \in (1, 2)$, defined as

$$\begin{cases} {}^C_{gH}\mathcal{D}^\omega \mathcal{X}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{F}(t, \mathcal{X}(t), {}^C_{gH}\mathcal{D}^{\omega-1}\mathcal{X}(t)), & t \in J = [0, a], \\ \mathcal{X}(0) = x_0, \\ {}^C_{gH}\mathcal{D}^{\omega-i}\mathcal{X}(0) = x_i, & i = 1, \dots, |\omega|, \end{cases}$$

where $\mathcal{A}$ is a linear operator and $\mathcal{F}$ is a continuous function. They employed Schauder’s fixed-point theorem to derive criteria for the existence and uniqueness of mild fuzzy solutions. Their work provided a clear methodological framework and they offered practical examples to demonstrate the applicability of their theoretical findings.

In a related study, Airou, El Mfadel, and Elomari [12] examined the existence of two distinct types of fuzzy mild solutions for fuzzy fractional evolution equations under Caputo’s generalized Hukuhara (gH) differentiability. Their analysis utilized fuzzy strongly continuous semigroups and a novel version of Krasnoselskii’s fixed-point theorem specifically tailored to fuzzy metric spaces, thus enriching theoretical understanding and expanding potential applications by clearly distinguishing different types of solutions within fuzzy metric spaces.

Moreover, El Ghazouani, Elomari, and Melliani [13] explored the qualitative behaviours and existence of mild solutions to fuzzy boundary value problems with non-local conditions for Caputo-type fractional differential equations of order $1 < \nu < 2$, defined as

$$\begin{cases} \mathcal{D}^\nu \mathcal{Z}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{G}(t, \mathcal{Z}(t)) + \int_0^t p(t-s)f(s, \mathcal{Z}(t))ds, & t \in J = [0, T], \\ \mathcal{Z}(0) + M(\mathcal{Z}) = \mathcal{Z}_0, \\ \mathcal{Z}'(0) + N(\mathcal{Z}) = \mathcal{Z}_1, \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $\mathcal{E}_{v,m}(\mathcal{A}t^v)$ on $\mathcal{T}$ and $\mathcal{G}, f$ are continuous. Their approach combined Darbo's fixed-point theorem with the measure of non-compactness, addressing the generalized Ulam-Hyers stability. The study demonstrated broad applicability and improved upon previous findings, as well as providing illustrative examples to validate the theoretical results.

Real-world systems that exhibit memory, non-local interactions, and uncertainty find a natural representation in fractional models employing the Hilfer derivative. For example, Bulavatsky [14] demonstrated the utility of the Hilfer operator in modeling anomalous diffusion processes, deriving analytical solutions that describe complex transport in heterogeneous media.

Further highlighting its relevance, Ramalakshmi [15] constructed a Θ -Hilfer fractional order model to investigate the transmission dynamics and optimal control measures of the Hepatitis B virus. The study showed how the fractional order parameter affects the efficiency of control measures as well as the spread of infection. In a related domain, Borja-Jaimes et al. [16] provided a comprehensive review on the use of fractional calculus in modeling energy storage systems for electric vehicles, emphasizing the improved predictive accuracy for components like batteries and supercapacitors. These studies collectively affirm the broad relevance of Hilfer-type fractional frameworks across physics, biology, and engineering, thereby motivating the theoretical developments for fuzzy fractional systems with nonlocal conditions presented herein. The analytical complexity of nonlinear fractional differential equations, compounded by the fuzzy framework, often precludes closed-form solutions. This reality bifurcates the research landscape into two complementary paths: the qualitative analysis of solutions establishing their existence, uniqueness, and stability and the quantitative or numerical approximation of these solutions through computational techniques. While the present work is firmly positioned within the former category, providing rigorous theoretical guarantees of existence and uniqueness under suitable conditions, it simultaneously acknowledges the indispensable role of numerical and engineering advances in validating and implementing such theoretical results. In this regard, recent developments in computational and applied fractional calculus such as the adaptive dynamic programming based finite time control of fractional order chaotic systems [17] and the realization of fractional order current mode multifunction filters using the Oustaloup numerical approximation [18]. Motivated by this modeling imperative, we direct our focus to the following fuzzy Hilfer fractional evolution equation incorporating non-local conditions:

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}_{0+}^{\alpha,\beta}\mathcal{U}(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t, \mathcal{U}(t)), & t \in J = [0, a], \\ \mathcal{I}^{2-v}\mathcal{U}(0) = u_0 - \mu(\mathcal{U}), \\ \mathcal{I}^{2-v}\mathcal{U}'(0) = u_1 - \nu(\mathcal{U}), \end{cases} \quad u_i \in \mathbb{E}, i = 0, 1, \quad (1)$$

where ${}^{\mathcal{H}}\mathcal{D}_{0+}^{\alpha,\beta}(\cdot)$ is a fuzzy Hilfer–Caputo fractional derivative of order $1 < \alpha \leq 2$ and type $0 < \beta < 1$, and $v = \alpha + \beta(2 - \alpha)$. Furthermore, $\mathcal{F} : J \times \mathbb{E} \rightarrow \mathbb{E}$ is a continuous function. Let $\mathcal{A}$ be an infinitesimal generator of a strongly continuous cosine family $\{\mathcal{K}(\xi)\}_{\xi \geq 0}$ of uniformly bounded linear operators defined on a fuzzy Banach space $\mathbb{E}$. Also, let $\mathcal{C} = \mathcal{C}([0, a], \mathbb{X})$ be the Banach space of continuous and bounded functions from $[0, a]$ into $\mathbb{E}$ is given as follows, endowed with the topology

$$\|\mathcal{U}\|_{\mathcal{C}} = \sup_{t \in [0, a]} |\mathcal{U}(t)|.$$

In addition, as $\{\mathcal{K}(t)\}_{t \geq 0}$ is cosine family on $\mathbb{X}$, then there exists $\mathcal{M} \geq 1$ where

$$\|\mathcal{K}(t)\| \leq \mathcal{M}. \quad (2)$$

The decision to employ non-local conditions, encapsulated by the functionals $\mu(\mathcal{U})$ and $\nu(\mathcal{U})$, is substantiated by their physical and practical relevance, extending beyond a purely theoretical exercise. Consider the following illustrative scenarios:

- Within continuum mechanics, the initial configuration of a material might be defined by a spatial average across its domain, naturally leading to a non-local specification.
- In the context of infectious disease modeling, the seed number of infections could be derived from an aggregate of cases reported over a preceding period or from interconnected populations.
- For engineered systems, the initial state might be determined by a control input that aggregates historical sensor data.

Such non-local conditions offer a superior descriptive capability for these contexts compared to classical local initial conditions ($\mathcal{U}(0) = u_0$), as they inherently account for distributed influences, cumulative historical data, and feedback mechanisms. Consequently, the resulting models possess enhanced robustness for application-oriented problems. From an analytical perspective, incorporating non-local conditions introduces additional complexity, particularly in proving solution existence and uniqueness, which constitutes a central objective of the present work.

The originality of our contribution is thus highlighted by two key aspects. Firstly, this work represents a pioneering analysis of a fuzzy Hilfer fractional evolution equation in the higher-order range $1 < \alpha \leq 2$, specifically formulated with non-local conditions. The Hilfer operator serves as a generalization that interpolates between the Riemann-Liouville and Caputo derivatives, thus providing a more versatile modeling tool via the parameter β . Secondly, the synthesis of this sophisticated fractional calculus with the formalism of fuzzy set theory and non-local conditions yields a comprehensive framework. This integrated approach is adept at modeling complex systems that concurrently display hereditary properties (through the fractional derivative), epistemic uncertainties (through fuzzy-valued mappings), and global dependencies (through the non-local initial conditions). To our knowledge, this particular confluence of features has not been previously explored in the literature, thereby signifying a substantive extension of the current theoretical landscape.

The impetus for our investigation arises from the demand for advanced mathematical models capable of capturing the intricate dynamics of systems characterized by memory effects, imprecision, and global interactions. Although fuzzy fractional evolution equations subject to local conditions have received considerable attention, numerous real-world processes are more faithfully represented by conditions that incorporate information from the system's state across a temporal continuum or at multiple instances, rather than being confined to a single initial point. This conceptual shift is embodied in the use of non-local conditions.

The structure of this work is as follows. In Section 2, we recall several preliminary notions and lemmas needed for our analysis. Section 3 is devoted to establishing the concept of a mild solution for (1), where the operator $\mathcal{A}$ is assumed to be the infinitesimal generator of a strongly continuous cosine family $\{\mathcal{K}(t)\}_{t \geq 0}$. In Section 4, we investigate the existence and uniqueness of the mild solution through the application of fixed point techniques. Section 5 provides a numerical example to demonstrate the effectiveness of the obtained theoretical results.

2. Preliminaries

The purpose of this part is to provide some basic concepts and mathematical background on fractional operators of the Hilfer type, cosine family theory, and fuzzy-number-valued functions. The primary findings covered in the parts that follow must be formulated using these ideas.

Definition 1 ([19]). A fuzzynumber is a fuzzy subset $x : \mathbb{R} \rightarrow [0, 1]$ that satisfies the following essential properties:

(a) **Normality:** There exists at least one point $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$.

(b) **Fuzzy convexity:** For all $t_1, t_2 \in \mathbb{R}$ and every $\lambda \in [0, 1]$, the inequality

$$x(\lambda t_1 + (1 - \lambda)t_2) \geq \min\{x(t_1), x(t_2)\}$$

holds.

(c) **Upper semi-continuity:** The membership function x is upper semi-continuous on $\mathbb{R}$.

(d) **Compact support:** The support of x , defined by

$$\text{supp}(x) = \{t \in \mathbb{R} : x(t) > 0\},$$

is a compact subset of $\mathbb{R}$.

The set of all fuzzy numbers defined on the real line is denoted by $\mathfrak{X}$.

We denote the space of all fuzzy numbers on $\mathbb{R}$ by $\mathfrak{X}$.

$$\mathfrak{X} = \{x : \mathbb{R} \rightarrow [0, 1] : x \text{ satisfies a-d above}\}.$$

For all $\alpha \in (0, 1]$, the α -cut of an element $x \in \mathfrak{X}$ is defined by

$$x^\alpha = [\underline{x}(\alpha), \bar{x}(\alpha)].$$

where, $\bar{x}$ denotes the upper bound and $\underline{x}$ denotes the lower bound of the fuzzy number x .

The distance function that measures the separation between two fuzzy numbers $x, y \in \mathfrak{X}$ is expressed as

$$d(x, y) = \sup_{\alpha \in (0, 1]} \max\{|\bar{x}(\alpha) - \bar{y}(\alpha)|, |\underline{x}(\alpha) - \underline{y}(\alpha)|\}.$$

Note that this distance function satisfies the following properties:

(i) The metric is translation invariant, that is,

$$d(x + z, y + z) = d(x, y).$$

(ii) The metric is homogeneous with respect to scalar multiplication, namely,

$$d(\lambda x, \lambda y) = |\lambda| d(x, y), \quad \lambda \in \mathbb{R}.$$

(iii) The metric satisfies the triangle-type inequality,

$$d(x + z, w + z) \leq d(x, w) + d(y, z).$$

Since, $\mathbb{R}_{\mathcal{F}}$ denoted of all fuzzy number on the real line, the operations of addition and scalar multiplication for fuzzy numbers on $\mathbb{R}_{\mathcal{F}}$, where. are defined as follows:

$$[x \oplus y]^{\alpha} = [x]^{\alpha} + [y]^{\alpha} \text{ and } [\lambda \odot x]^{\alpha} = \lambda[x]^{\alpha}, \lambda \in \mathbb{R},$$

where

$$[x]^{\alpha} + [y]^{\alpha} = \{a + b : a \in [x]^{\alpha}, b \in [y]^{\alpha}\}$$

is the Minkowski sum of $[x]^{\alpha}$ and $[y]^{\alpha}$, and

$$\lambda[x]^{\alpha} = \{\lambda a : a \in [x]^{\alpha}\}.$$

Definition 2 ([20]). For $x, y \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara (gH) difference of x and y , denoted by $x \ominus_{gH} y$, is characterized as the element $z \in \mathbb{R}_{\mathcal{F}}$ where

$$x \ominus_{gH} y = z \iff \{(i) \ x = y + z, \text{ or } (ii) \ y = x + (-1)z\}.$$

In terms of α -cuts, we have

$$(x \ominus_{gH} y)^{\alpha} = [\min\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}, \max\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}].$$

where, $\bar{x}$ denotes the upper bound and $\underline{x}$ denotes the lower bound of the fuzzy number x .

Let the set of all triangular fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$ denote by $\mathcal{T}$, then we take $(\mathcal{T}, d_{\infty})$ as a subspace of the metric space $(\mathbb{R}_{\mathcal{F}}, d_{\infty})$, which is a complete metric space.

Furthermore, as demonstrated by Bede and Stefanini [20], if $x, y \in \mathcal{T}$, then the difference $x \ominus_{gH} y$ always occurs in $\mathcal{T}$ and $x \ominus_{gH} y = (-1) \odot (x \ominus_{gH} y)$. Let $\mathcal{E}$ be a subset of $\mathbb{R}_{\mathcal{F}}$, $J \subset \mathbb{R}$, and denote by $\mathcal{C}(J, \mathcal{E})$ the set of all continuous mappings $f : J \rightarrow \mathcal{E}$.

Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathfrak{X}$ be a fuzzy-valued function. The α -cut of f is given by

$$f(t, \alpha) = [\underline{f}(t, \alpha), \bar{f}(t, \alpha)], \text{ for all } t \in [a, b], \alpha \in [0, 1].$$

Definition 3 ([21]). Let $t_0 \in (a, b)$ and define h such that $t_0 + h \in (a, b)$. Then, the generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \rightarrow \mathfrak{X}$ at t_0 , denoted $f'_{gH}(t_0)$ is defined as

$$\lim_{h \rightarrow 0} \left\| \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h} \ominus_{gH} f'_{gH}(t_0) \right\| = 0.$$

where, $\ominus$ denoted the generalized Hukuhara difference, and $\|\cdot\|$ is the Hausdorff norm on fuzzy numbers.

Definition 4 ([3]). Let $f : [a, b] \rightarrow \mathfrak{X}$ and $t_0 \in (a, b)$, with $\underline{f}(t, \alpha)$ and $\bar{f}(t, \alpha)$ both differentiable at t_0 . We say that:

1. at t_0 f is (i, gH) -differentiable if

$$f'_{gH}(t_0) = [\underline{f}'(t_0, \alpha), \bar{f}'(t_0, \alpha)].$$

2. at t_0 f is (ii, gH) -differentiable if

$$f'_{gH}(t_0) = [\bar{f}'(t_0, \alpha), \underline{f}'(t_0, \alpha)].$$

Theorem 1 ([3]). Let $f : L \subset \mathbb{R} \rightarrow \mathfrak{X}$, $g : L \rightarrow \mathbb{R}$, and $t \in L$. Suppose that $g(t)$ is differentiable at t and the fuzzy-valued function $f(t)$ is gH -differentiable at t . Then,

$$(fg)'_{gH}(t) = (f'g)_{gH}(t) + (fg')_{gH}(t).$$

Definition 5 ([22]). Let $f: [a, b] \rightarrow \mathfrak{X}$ and assume that $f'_{gH}(t)$ is gH -differentiable at $t_0 \in (a, b)$, without changing its type of differentiability on (a, b) . Moreover, suppose that both $\underline{f}(t, \alpha)$ and $\overline{f}(t, \alpha)$ are differentiable at t_0 . We say that:

1. at t_0 f is (i, gH) -twice differentiable if

$$f''_{gH}(t_0) = [\underline{f}''(t_0, \alpha), \overline{f}''(t_0, \alpha)].$$

2. at t_0 f is (ii, gH) -twice differentiable if

$$f''_{gH}(t_0) = [\overline{f}''(t_0, \alpha), \underline{f}''(t_0, \alpha)].$$

Definition 6 ([23]). Let $f \in L^{\mathfrak{X}}([a, b])$. The following is the definition of the fuzzy Riemann–Liouville integral of a fuzzy-valued function f

$$\mathcal{RL}\mathcal{I}^q f(t) = \frac{1}{\Gamma(q)} \odot \int_a^t (t-s)^{q-1} \odot f(s) ds, \quad a < s < t, \quad 0 < q < 1.$$

Definition 7. Let $f \in L^{\mathfrak{X}}([a, b])$ be a fuzzy-valued function. Then, we define

$$\mathcal{RL}_{gH}\mathcal{D}^q f(t) = \frac{1}{\Gamma(n-q)} \odot \left(\frac{d}{ds} \right)^n \int_a^t (t-s)^{n-q-1} \odot f(s) ds, \quad n-1 < q < n,$$

and, if $q = n-1$, then

$$\mathcal{RL}_{gH}\mathcal{D}^q f(t) = \left(\frac{d}{ds} \right)^{n-1} f(t).$$

Definition 8 ([24]). Let $f \in \mathcal{C}_{\mathfrak{X}}[a, b]$, and let $\mathcal{C}_{\mathfrak{X}}$ be the space of all continuous fuzzy-valued functions from $[a, b]$ into $\mathfrak{X}$. The fuzzy Hilfer fractional derivative of order α and type β is defined by

$$(\mathcal{H}\mathcal{D}_{0+}^{\alpha, \beta} f)(t) = \mathcal{I}_{0+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} \mathcal{I}_{0+}^{(1-\beta)(n-\alpha)} f(t), \quad t > 0$$

if there is a gH -derivative $x'_{1-\alpha}(t)$, where $0 \leq \beta \leq 1$ and $n-1 < \alpha < n$.

Definition 9 ([25]). Let $0 < \operatorname{Re}(\alpha) < 1$ and $\mathcal{F}(\zeta) \in AC^n[0, b]$ for any $b > 0$. Then, $|\mathcal{F}(\zeta)| \leq Be^{q_0\zeta}$, where $\zeta > b > 0$ and $B, q_0 > 0$, and the finite limits are $\lim_{\zeta \rightarrow 0+} [\mathcal{RL}\mathcal{D}_{0+}^k \mathcal{I}_{0+}^{n-\alpha} \mathcal{F}(\zeta)]$ and $\lim_{\zeta \rightarrow \infty} [\mathcal{RL}\mathcal{D}_{0+}^k \mathcal{I}_{0+}^{n-\alpha} \mathcal{F}(\zeta)]$, where $k = 0, 1, \dots, n-1$. The Laplace transform is defined as follows:

$$\mathcal{L}\{\mathcal{RL}\mathcal{D}_{0+}^{\alpha} \mathcal{F}(\zeta)\}(\lambda) = \lambda^{\alpha} \mathcal{L}\{\mathcal{F}(\zeta)\}(\lambda) - (\mathcal{I}_{0+}^{n-\alpha} \mathcal{F}(\zeta))(0+). \quad (3)$$

Here $\operatorname{Re}(\alpha)$ denotes the real part of α , $AC^n[0, b]$ is the space of n -times absolutely continuous functions on $[0, b]$, and $\mathcal{RL}\mathcal{D}_{0+}^{\alpha}$, $\mathcal{I}_{0+}^{\alpha}$ denote the Riemann–Liouville fractional derivative and integral, respectively.

Definition 10 ([24]). If $(\mathcal{I}_{0+}^{2-v} \mathcal{U})(t)$ is continuous and $(\mathcal{I}_{0+}^{2-v} \mathcal{U})'(t)$ is absolutely continuous, then

$${}_gH\mathcal{I}_{0+}^v \odot \left(\mathcal{RL}_{gH}\mathcal{D}^v \mathcal{U}(t) \right) = \mathcal{U}(t) - \frac{(\mathcal{I}_{0+}^{2-v} \mathcal{U})(0)}{\Gamma(v-1)} \odot t^{v-2} - \frac{(\mathcal{I}_{0+}^{2-v} \mathcal{U})'(0)}{\Gamma(v)} \odot t^{v-1}, \quad 1 < v < 2.$$

Definition 11 ([24]). Let $f \in \mathfrak{X}$ be a fuzzy-valued function and $t \in [a, b]$. We define the Caputo gH derivative of $f(t)$ as

$${}^c\mathcal{D}_{gH}^q f(t) = \frac{1}{\Gamma(n-q)} \odot \int_a^t (t-s)^{n-q-1} \odot f_{gH}^n(s) ds, \quad n-1 < q < n,$$

and, if $q = n-1$, then

$${}^c_{gH}\mathcal{D}^q f(t) = \left(\frac{d}{ds}\right)^{n-1} f(t).$$

Furthermore, we say that f is (i, gH) -differentiable at t_0 if

$${}_{gH}\mathcal{D}_i^q f(x, t; \alpha) = \left[\mathcal{D}^q \underline{f}(x, t; \alpha), \mathcal{D}^q \bar{f}(x, t; \alpha) \right] \quad \text{for all } q \in (0, 1),$$

and we say that f is (ii, gH) -differentiable at t_0 , if

$${}_{gH}\mathcal{D}_i^q f(x, t; \alpha) = \left[\mathcal{D}^q \bar{f}(x, t; \alpha), \mathcal{D}^q \underline{f}(x, t; \alpha) \right] \quad \text{for all } q \in (0, 1).$$

Definition 12 ([26]). [Fuzzy Laplace Transform] Let $f: [0, +\infty) \rightarrow X \subset \mathbb{R}_F$ be a continuous function such that

$$e^{-st} \odot f(t)$$

is integrable on $[0, +\infty)$. Then, the fuzzy Laplace transform of f , denoted by $\mathcal{L}[f(t)]$, is defined by

$$\mathcal{L}[f(t)](s) := F(s) = \int_0^\infty e^{-st} \odot f(t) dt, \quad s > 0.$$

Additionally, f is considered exponentially bounded of order β if there are constants $M > 0$ and $t_0 > 0$ such that

$$d_\infty^+(f(t), \hat{0}) \leq Me^{\beta t}, \quad \forall t \geq t_0.$$

Lemma 1 ([26]). [Convolution Theorem] If $x: [0, \infty) \rightarrow \mathbb{R}_F$ is piecewise continuous and of exponential order a , and $y: [0, \infty) \rightarrow \mathbb{R}$ is piecewise continuous, then their convolution $(x \star y)$ satisfies

$$\mathcal{L}((x \star y)(t)) = \mathcal{L}[x(t)] \odot \mathcal{L}[y(t)].$$

Definition 13 ([26]). The family of bounded linear operators $\{\mathcal{K}(t)\}_{t \in \mathbb{R}_+}$ maps the Banach space $\mathfrak{X} \rightarrow \mathfrak{X}$ and has just one parameter. Such a family is referred to as a strongly continuous cosine family if and only if

- (i) $\mathcal{K}(0) = I$;
- (ii) $\mathcal{K}(s+t) + \mathcal{K}(s-t) = 2\mathcal{K}(s)\mathcal{K}(t)$ for all $s, t \in \mathbb{R}_+$; and
- (iii) $\mathcal{K}(t)x$ is a continuous on $\mathbb{R}_+$ for any $x \in \mathfrak{X}$.

In particular, the strongly continuous cosine family $\{\mathcal{K}(t)\}_{t \in \mathbb{R}_+}$ (which is connected to the sine family $\{\mathcal{S}(t)\}_{t \in \mathbb{R}_+}$) is defined by

$$\mathcal{K}(t)x = \int_0^t \mathcal{K}(s)x ds, \quad x \in \mathfrak{X}, t \in \mathbb{R}_+.$$

Lemma 2 ([26]). If $\mathcal{A}$ is an infinitesimal generator of a strongly continuous cosine family $\{\mathcal{K}(t)\}_{t \in \mathbb{R}_+}$ on a Banach space $\mathfrak{X}$, then there exist constants $M > 0$ and $\xi \geq 0$ such that

$$\|\mathcal{K}(t)\|_{\mathcal{B}(\mathfrak{X})} \leq Me^{\xi t}, \quad t \in \mathbb{R}_+.$$

Moreover, for every $\lambda > \xi$ and $(\xi^2, \infty) \subset \varrho(\mathcal{A})$ (the resolvent set of $\mathcal{A}$), we obtain

$$\lambda R(\lambda^2; \mathcal{A})x = \int_0^\infty e^{-\lambda t} \mathcal{K}(t)x dt, \quad R(\lambda^2; \mathcal{A})x = \int_0^\infty e^{-\lambda t} \mathcal{R}(t)x dt,$$

for all $x \in \mathfrak{X}$, where the operator $R(\lambda; \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ is the resolvent of $\mathcal{A}$ and $\lambda \in \varrho(\mathcal{A})$.

The operator $\mathcal{A}$ is characterized by

$$\mathcal{A}x = \frac{d^2}{dt^2} \mathcal{K}(0)x, \quad \forall x \in \mathcal{D}(\mathcal{A})$$

where $\mathcal{D}(\mathcal{A}) = \{x \in \mathfrak{X} : \mathcal{K}(t)x \in C^2(\mathbb{R}, \mathfrak{X})\}$. It is evident that the infinitesimal generator $\mathcal{A}$ is a closed, densely defined operator in $\mathfrak{X}$.

Definition 14. The definition of Mainardi's Wright-type function when $t > 0$ is as follows:

$$M_\rho(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n! \Gamma(1 - \rho(n+1))}, \quad \rho \in (0, 1), \quad t \in \mathbb{C},$$

for which we have

$$M_\rho(t) \geq 0 \quad \text{and} \quad \int_0^\infty \theta^\xi M_\rho(\theta) d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\rho\xi)}, \quad \xi > -1.$$

3. Structure of Mild Solution

In this stage, we provide the structure of a mild solution utilizing the cosine family operators and the Laplace transform.

Lemma 3. Problem (1) can be considered in the corresponding integral form, as follows:

$$\mathcal{U}(t) = {}_{gH}\mathcal{I}_{0+}^\alpha [\mathcal{A}\mathcal{U}(t) + \mathcal{F}(t, \mathcal{U}(t))] + \frac{t^{v-2}}{\Gamma(v-1)}(u_0 - \mu(\mathcal{U})) + \frac{t^{v-1}}{\Gamma(v)}(u_1 - \nu(\mathcal{U})). \quad (4)$$

Proof. Taking $0 < t \leq a$, according to Definition 8, we have

$$\begin{aligned} {}_{gH}\mathcal{I}_{0+}^\alpha \left({}_{gH}\mathcal{D}_{0+}^{\alpha, \beta} \mathcal{U} \right)(t) &= {}_{gH}\mathcal{I}_{0+}^\alpha \mathcal{I}_{0+}^{\beta(2-\alpha)} \left(\frac{d^2}{dt^2} \mathcal{I}_{0+}^{(1-\beta)(2-\alpha)} \mathcal{U} \right)(t) \\ &= {}_{gH}\mathcal{I}_{0+}^{\alpha+\beta(2-\alpha)} \left(\mathcal{R}\mathcal{L}\mathcal{D}_{0+}^{\alpha+\beta(2-\alpha)} \mathcal{U} \right)(t) \\ &= {}_{gH}\mathcal{I}_{0+}^v \mathcal{R}\mathcal{L}\mathcal{D}_{0+}^v \mathcal{U}(t). \end{aligned}$$

From Definition 10, we also have

$${}_{gH}\mathcal{I}_{0+}^v \mathcal{R}\mathcal{L}\mathcal{D}_{0+}^v \mathcal{U}(t) = \mathcal{U}(t) - \frac{(\mathcal{I}_{0+}^{2-v} \mathcal{U})(0)}{\Gamma(v-1)} t^{v-2} - \frac{(\mathcal{I}_{0+}^{2-v} \mathcal{U})'(0)}{\Gamma(v)} t^{v-1}. \quad (5)$$

Therefore, applying the operators ${}_{gH}\mathcal{I}_{0+}^v$ to both sides of the equation simultaneously, and utilizing Equations (5) and, leads us to the following result:

$$\mathcal{U}(t) = {}_{gH}\mathcal{I}_{0+}^\alpha [\mathcal{A}\mathcal{U}(t) + \mathcal{F}(t, \mathcal{U}(t))] + \frac{t^{v-2}}{\Gamma(v-1)}(u_0 - \mu(\mathcal{U})) + \frac{t^{v-1}}{\Gamma(v)}(u_1 - \nu(\mathcal{U})).$$

The proof is complete. $\square$

Lemma 4. If $\sigma(t)$ satisfies the integral Equation (4), then

$$\begin{aligned} \mathcal{U}(t) &= {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma} \left(t^{\rho-1} \mathcal{G}_{\rho}(t)(u_0 - \mu(\mathcal{U})) \right) \\ &+ \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_{\rho}(z-t)(u_1 - \nu(\mathcal{U})) dz + \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_{\rho}(t,s) \mathcal{F}(s, \mathcal{U}) ds, t \in (0, a], \\ &\text{such that } \alpha = 2\rho. \text{ Additionally,} \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\rho}(t) &= \int_0^{\infty} \rho \theta \psi_{\rho}(\theta) \mathcal{K}(t^{\rho} \theta) d\theta, \\ \overline{\mathcal{G}}_{\rho}(t,s) &= \int_0^{\infty} \rho \theta \psi_{\rho}(\theta) \mathcal{R}((t-s)^{\rho} \theta) d\theta, \end{aligned}$$

where $\psi_{\rho}(\theta)$ is a probability density function defined according to Definition 14.

Proof. For $t \in [0, a]$, before taking the Laplace transformation of both sides of System (4), if we assume $\operatorname{Re} \lambda > 0$, then we obtain

$$\begin{aligned} \overline{\mathcal{U}}(\lambda) &:= \mathcal{L}\{\mathcal{U}(t)\}(\lambda) = \int_0^{\infty} e^{-\lambda s} \mathcal{U}(s) ds, \\ \overline{\mathcal{F}}(\lambda) &:= \mathcal{L}\{\mathcal{F}(t, \mathcal{U})\}(\lambda) = \int_0^{\infty} e^{-\lambda s} \mathcal{F}(s, \mathcal{U}) ds. \end{aligned}$$

Applying the Laplace transform to both sides of (4), we obtain

$$\begin{aligned} \overline{\mathcal{U}}(\lambda) &= \lambda^{-\alpha} \odot [\mathcal{A} \overline{\mathcal{U}}(\lambda) + \overline{\mathcal{F}}(\lambda)] + \lambda^{1-v} (u_0 - \mu(\mathcal{U})) + \lambda^{-v} (u_1 - \nu(\mathcal{U})), \\ (\lambda^{\alpha} I \ominus \mathcal{A}) \overline{\mathcal{U}}(\lambda) &= \overline{\mathcal{F}}(\lambda) + \lambda^{\alpha-v+1} (u_0 - \mu(\mathcal{U})) + \lambda^{\alpha-v} (u_1 - \nu(\mathcal{U})), \\ \overline{\mathcal{U}}(\lambda) &= (\lambda^{\alpha} I - \mathcal{A})^{-1} [\overline{\mathcal{F}}(\lambda) + \lambda^{\alpha-v+1} (u_0 - \mu(\mathcal{U})) + \lambda^{\alpha-v} (u_1 - \nu(\mathcal{U}))]. \end{aligned}$$

By Lemma 2 and as $\alpha = 2\rho$, we can write

$$\begin{aligned} \overline{\mathcal{U}}(\lambda) &= \int_0^{\infty} e^{-\lambda^{\rho} t} \mathcal{R}(t) \overline{\mathcal{F}}(\lambda) dt + \lambda^{\alpha-v+1} \int_0^{\infty} e^{-\lambda^{\rho} t} \mathcal{K}(t) (u_0 - \mu(\mathcal{U})) dt \\ &+ \lambda^{\alpha-v} \int_0^{\infty} e^{-\lambda^{\rho} t} \mathcal{K}(t) (u_1 - \nu(\mathcal{U})) dt. \end{aligned}$$

Let $\theta \in (0, \infty)$. If Mainardi's Wright-type function (defined in Definition 14) refers to $\psi_{\rho}(\theta)$, and

$$\psi_{\rho}(\theta) = \frac{\theta^{\frac{-1}{\rho}-1}}{\rho} \omega_{\rho}(\theta^{\frac{-1}{\rho}}),$$

where $\omega_{\rho}(\theta)$ is the one-sided stable probability defined as follows

$$\omega_{\rho}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\rho n-1} \frac{\Gamma(\rho n+1)}{n!} \sin(\rho n \pi), \quad \theta \in (0, \infty).$$

Therefore, we obtain the following formula:

$$\int_0^{\infty} e^{-\xi \theta} \omega_{\rho}(\theta) d\theta = e^{-\xi^{\rho}}, \text{ for } \rho \in (\frac{1}{2}, 1).$$

Hence, in our situation,

$$\begin{aligned}
 & \int_0^\infty e^{-\lambda t} \mathcal{R}(t) \overline{\mathcal{F}}(\lambda) dt \\
 &= \int_0^\infty e^{-(\lambda t)^\rho} \mathcal{R}(t^\rho) (\rho t^{\rho-1} \overline{\mathcal{F}}(\lambda)) dt \\
 &= \int_0^\infty \int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} \mathcal{R}(t^\rho) e^{-\lambda s} [\mathcal{F}(s, \mathcal{U})] ds dt \\
 &= \int_0^\infty \int_0^\infty \int_0^\infty \rho t^{\rho-1} e^{-\lambda t^\rho} \omega_\rho(\theta) \mathcal{R}(t^\rho) e^{-\lambda s} \mathcal{F}(s, \mathcal{U}) ds dt \\
 &= \int_0^\infty \int_0^\infty \frac{\rho t^{\rho-1}}{\theta^\rho} e^{-\lambda(t+s)} \omega_\rho(\theta) \mathcal{R}\left(\frac{t^\rho}{\theta^\rho}\right) \mathcal{F}(s, \mathcal{U}) d\theta ds dt \\
 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho(t-s)^{\rho-1}}{\theta^\rho} e^{-\lambda t} \odot \omega_\rho(\theta) \mathcal{R}\left(\frac{(t-s)^\rho}{\theta^\rho}\right) \mathcal{F}(s, \mathcal{U}) d\theta ds dt \\
 &= \int_0^\infty e^{-\lambda t} \left\{ \int_0^t \int_0^\infty \rho \theta (t-s)^{\rho-1} \odot \psi_\rho(\theta) \mathcal{R}((t-s)^\rho \theta) \mathcal{F}(s, \mathcal{U}) d\theta ds \right\} dt \\
 &= \int_0^\infty e^{-\lambda t} \left\{ \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_\rho(t-s) \mathcal{F}(s, \mathcal{U}) ds \right\} dt \\
 &= \mathcal{L} \left\{ \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_\rho(t, s) \mathcal{F}(s, \mathcal{U}) ds \right\} (\lambda).
 \end{aligned}$$

As $\alpha - v = -\beta(2 - \alpha) = -\gamma \in (-1, 0)$, and the Laplace inverse transform yields $\mathcal{L}^{-1}(\lambda^{-\gamma}) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}$, we can write

$$\begin{aligned}
 & \lambda^{\alpha-v} \int_0^\infty e^{-\lambda t} \mathcal{K}(t) (u_1 - v(\mathcal{U})) dt = \lambda^{-\gamma} \int_0^\infty e^{-\lambda t} \mathcal{K}(t) (u_1 - v(\mathcal{U})) dt \\
 &= \lambda^{-\gamma} \int_0^\infty \rho t^{\rho-1} e^{-(\lambda t)^\rho} \mathcal{K}(t) (u_1 - v(\mathcal{U})) dt \\
 &= \lambda^{-\gamma} \int_0^\infty \int_0^\infty \rho t^{\rho-1} e^{-\lambda t^\rho} \omega_\rho(\theta) \mathcal{K}(t) (u_1 - v(\mathcal{U})) d\theta dt \\
 &= \lambda^{-\gamma} \int_0^\infty \int_0^\infty \frac{\rho t^{\rho-1}}{\theta^\rho} \rho e^{-\lambda t} \rho \omega_\rho(\theta) \rho \mathcal{K}\left(\left(\frac{t}{\theta}\right)^\rho\right) \rho (u_1 - v(\mathcal{U})) d\theta dt \\
 &= \lambda^{-\gamma} \int_0^\infty e^{-\lambda t} \rho \left\{ \int_0^\infty \rho \theta t^{\rho-1} \rho \left(\rho^{-1} \theta^{\frac{-1}{\rho}-1} \rho \omega_\rho(\theta^{-1}) \right) \rho \mathcal{K}(t^\rho \theta) \rho (u_1 - v(\mathcal{U})) d\theta \right\} dt \\
 &= \lambda^{-\gamma} \int_0^\infty e^{-\lambda t} \left\{ \int_0^\infty \rho \theta t^{\rho-1} \psi_\rho(\theta) \mathcal{K}(t^\rho \theta) (u_1 - v(\mathcal{U})) d\theta \right\} dt \\
 &= \lambda^{-\gamma} \mathcal{L} \left\{ t^{\rho-1} \mathcal{G}_\rho(t) (u_1 - v(\mathcal{U})) \right\} (\lambda) \\
 &= \mathcal{L} \left\{ \frac{t^{\gamma-1}}{\Gamma(\gamma)} \right\} \mathcal{L} \left\{ t^{\rho-1} \mathcal{G}_\rho(t) (u_1 - v(\mathcal{U})) \right\} (\lambda) \\
 &= \mathcal{L} \left\{ \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_\rho(z-t) (u_1 - v(\mathcal{U})) dz \right\} (\lambda).
 \end{aligned}$$

Similarly, as $\alpha - v + 1 = 1 - \beta(2 - \alpha) = 1 - \gamma \in (0, 1)$, according to the Laplace transform of the Riemann–Liouville fractional derivative (3), we get $\lim_{t \rightarrow 0} {}_{gH} \mathcal{I}^\gamma t^{\rho-1} \mathcal{G}_\rho(t) = 0$ and obtain

$$\begin{aligned}
 & \lambda^{\alpha-v+1} \int_0^\infty e^{-\lambda t} \mathcal{K}(t) (u_0 - \mu(\mathcal{U})) dt = \lambda^{1-\gamma} \mathcal{L} \left\{ t^{\rho-1} \mathcal{G}_\rho(t) (u_0 - \mu(\mathcal{U})) \right\} (\lambda) - \lim_{t \rightarrow 0} \mathcal{I}^\gamma t^{\rho-1} \mathcal{G}_\rho(t) \\
 &= \mathcal{L} \left\{ {}_{gH}^{\mathcal{RL}} \mathcal{D}_{0+}^{1-\gamma} \left(t^{\rho-1} \mathcal{G}_\rho(t) (u_0 - \mu(\mathcal{U})) \right) \right\} (\lambda).
 \end{aligned}$$

Finally, we can achieve the required outcome using the inverse Laplace transform, and the proof is concluded. $\square$

Definition 15. A function $\mathcal{U}(t) \in (C[0, a]; \mathfrak{X})$ is considered to be a mild solution of (1) if it satisfies

$$\begin{aligned} \mathcal{U}(t) = & {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma} \left(t^{\rho-1} \mathcal{G}_{\rho}(t)(u_0 - \mu(\mathcal{U})) \right) + \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_{\rho}(z-t)(u_1 - \nu(\mathcal{U})) dz \\ & + \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_{\rho}(t,s) \mathcal{F}(s, \mathcal{U}) ds. \end{aligned} \quad (6)$$

The following auxiliary lemmas are obtained by standard arguments based on the properties of fractional cosine families.

Lemma 5. The following estimates for $\mathcal{G}_{\rho}(t)$ and $\overline{\mathcal{G}}_{\rho}(t)(t,s)$ hold for any fixed $t \geq 0$ and $0 < s < t$:

$$|\mathcal{G}_{\rho}(t)x| \leq \mathcal{M}|x| \quad \text{and} \quad |\overline{\mathcal{G}}_{\rho}(t,s)x| \leq \mathcal{M}_1|x|, \quad \mathcal{M}_1 = \frac{\mathcal{M}a^{\rho}}{\Gamma(2\rho)}.$$

Lemma 6. For any $0 < s < t$ and $t > 0$, the operators $\mathcal{G}_{\rho}(t)$ and $\overline{\mathcal{G}}_{\rho}(t,s)$ are strongly continuous.

Lemma 7. Suppose that $\mathcal{R}(t)$ and $\mathcal{T}(t,s)$ are compact for every $0 < s < t$. Then, for any $0 < s < t$, the operators $\mathcal{G}_{\rho}(t)$ and $\overline{\mathcal{G}}_{\rho}(t,s)$ are compact.

Lemma 8. For each $t > 0$ and $y \in \mathbb{H}$, the following formula is true if (4) holds:

$$\frac{d}{dt}(t^{\rho-1} \mathcal{G}_{\rho}(t)y) = (\rho-1)t^{\rho-1} \mathcal{G}_{\rho}(t)y + t^{2\rho-2} \mathcal{G}_{\rho}^2(t).$$

Furthermore, according to the assumption $(\mathbb{A}\mathbb{S}\mathbb{H}_1)$ and letting $\|\gamma(t, 0, 0)\| = \mathcal{Z}^*$,

$$d_{\infty} \left(\frac{d}{dt}(t^{\rho-1} \mathcal{G}_{\rho}(t)y), \hat{0} \right) \leq \delta d_{\infty}(y, \hat{0}),$$

where

$$\delta = \mathcal{M}(\rho-1)a^{\rho-1} + \frac{2\mathcal{M}a^{3\rho-2}\rho}{\Gamma(2\rho)\Gamma(\rho)}$$

and

$$\mathcal{G}_{\rho}^2(t)y = \int_0^{\infty} \rho^2 \theta^2 \psi_{\rho}(\theta) \mathcal{R}(t^{\rho}\theta)y d\theta.$$

Proof. As $\frac{d}{dt}(\mathcal{K}(t^{\rho}\theta)y) = \rho\theta t^{\rho-1} \mathcal{R}(t^{\rho}\theta)y$ for $t > 0$ and $y \in T$, it can be calculated that

$$\begin{aligned} \frac{d}{dt}(t^{\rho-1} \mathcal{G}_{\rho}(t)y) &= \frac{d}{dt} \int_0^{\infty} \rho\theta \psi_{\rho}(\theta) \mathcal{K}(t^{\rho}\theta)y d\theta \\ &= \int_0^{\infty} \rho\theta \psi_{\rho}(\theta) \frac{d}{dt}(\mathcal{K}(t^{\rho}\theta)y) d\theta \\ &= t^{\rho-1} \int_0^{\infty} \rho\theta \psi_{\rho}(\theta) \mathcal{R}(t^{\rho}\theta)y d\theta. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \frac{d}{dt}(t^{\rho-1} \mathcal{G}_{\rho}(t)y) &= (\rho-1)t^{\rho-1} \mathcal{G}_{\rho}(t)y + t^{2\rho-2} \int_0^{\infty} \rho^2 \theta^2 \psi_{\rho}(\theta) \mathcal{R}(t^{\rho}\theta)y d\theta \\ &= (\rho-1)t^{\rho-1} \mathcal{G}_{\rho}(t)y + t^{2\rho-2} \mathcal{G}_{\rho}^2(t)y. \end{aligned}$$

Additionally, for any $t \in [0, a]$, we get

$$\begin{aligned}
 d_{\infty}\left(\frac{d}{dt}(t^{\rho-1}\mathcal{G}_{\rho}(t)y),\hat{0}\right) &= d_{\infty}\left((\rho-1)t^{\rho-1}\mathcal{G}_{\rho}(t)y+t^{2\rho-2}\int_0^{\infty}\rho^2\theta^2\psi_{\rho}(\theta)\mathcal{R}(t^{\rho}\theta)y,\hat{0}\right) \\
 &\leq d_{\infty}\left((\rho-1)t^{\rho-1}\mathcal{G}_{\rho}(t)y,\hat{0}\right)+d_{\infty}(t^{2\rho-2}\int_0^{\infty}\rho^2\theta^2\psi_{\rho}(\theta)\mathcal{R}(t^{\rho}\theta)y,\hat{0}) \\
 &\leq \mathcal{M}(\rho-1)a^{\rho-1}d_{\infty}(y,\hat{0})+a^{2\rho-2}\rho^2\frac{2\mathcal{M}a^{\rho}}{\rho\Gamma(2\rho)\Gamma(\rho)}d_{\infty}(y,\hat{0}) \\
 &= \left(\mathcal{M}(\rho-1)a^{\rho-1}+\frac{2\mathcal{M}a^{3\rho-2}\rho}{\Gamma(2\rho)\Gamma(\rho)}\right)d_{\infty}(y,\hat{0}):=\delta d_{\infty}(y,\hat{0}).
 \end{aligned}$$

□

Prior to discussing our significant findings, we offer the following hypotheses.

(ASH₁) The function $\mathcal{F} \in \mathcal{C}([0, a] \times \mathcal{T}, \mathcal{T})$ is continuous for almost all $t \in [0, a]$, and the function $\mathcal{F}(\cdot, u): [0, a] \rightarrow \mathcal{T}$ is strongly measurable for all $\mathcal{U} \in \mathcal{C}([0, a], \mathcal{T})$.

(ASH₂) Let $\alpha_1 \in [0, \alpha)$. There exist $\kappa(\cdot), \kappa \in L^{\frac{1}{\alpha_1}}([0, a], \mathbb{R}^+)$, such that for all $\mathcal{U}, \mathcal{V} \in \mathcal{C}([0, a], \mathcal{T})$ we have

$$d_{\infty}(\mathcal{F}(t, \mathcal{U}), \mathcal{F}(t, \mathcal{V})) \leq \kappa(t)d_{\infty}(\mathcal{U}, \mathcal{V}), t \in [0, a].$$

(ASH₃) Let $\alpha_1 \in [0, \alpha)$. There exist $m(\cdot), m \in L^{\frac{1}{\alpha_1}}([0, a], \mathbb{R}^+)$, such that for all $\mathcal{U} \in \mathcal{C}([0, a], \mathcal{T})$ we have

$$d_{\infty}(\mathcal{F}(t, \mathcal{U}), \hat{0}) \leq m(t)d_{\infty}(\mathcal{U}, \hat{0}), t \in [0, a].$$

(ASH₄) $\mu, \nu: \mathcal{T} \rightarrow \overline{\mathcal{DA}}$ is continuous and there exist constants $\mathcal{O}_1, \mathcal{O}_2$ be triangular fuzzy numbers such that

$$d_{\infty}(\mu(\mathcal{U}), \mu(\mathcal{V})) \leq \mathcal{O}_1 d_{\infty}(\mathcal{U}, \mathcal{V}),$$

$$d_{\infty}(\nu(\mathcal{U}), \nu(\mathcal{V})) \leq \mathcal{O}_2 d_{\infty}(\mathcal{U}, \mathcal{V}).$$

Remark 1. Considering the recognized connections between the Riemann–Liouville fractional derivative and the Caputo fractional derivative, for $0 < 1 - \gamma < 1$, we obtain

$${}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}\left(t^{\rho-1}\mathcal{G}_{\rho}(t)(u_0-\mu(\mathcal{U}))\right)={}_H^{\mathcal{C}}\mathcal{D}_{0+}^{1-\gamma}\left(t^{\rho-1}\mathcal{G}_{\rho}(t)(u_0-\mu(\mathcal{U}))\right). \quad (7)$$

4. Existence and Uniqueness Results

The existence and uniqueness of solutions to problem (1) are summarized in this section.

Theorem 2. Assume hypotheses ASH₁ – ASH₄ and

$$\overline{P}=\left(\frac{(\rho-1)\mathcal{O}_1\mathcal{M}a^{\rho-\gamma}}{\Gamma(\gamma)\Gamma(1-\gamma)}+\frac{3\mathcal{M}\rho a^{3\rho-\gamma-1}}{2\Gamma(4\rho)\Gamma(\gamma)\Gamma(1-\gamma)}+\mathcal{M}\mathcal{O}_2a^{\gamma+\rho-1}\mathbb{B}(\gamma,\rho)+\mathcal{M}_1\|\kappa\|_{L^{\frac{1}{\alpha_1}}}a^{\rho}\right)<1.$$

Then, the Cauchy problem (1) has a mild fuzzy solution in the space $\mathcal{C}([0, a], \mathcal{T})$.

Proof. Let us define the set

$$\mathcal{B}_s=\left\{\mathcal{U} \in \mathcal{T}: d_{\infty}(\mathcal{U}, \hat{0}) \leq s, s>0, s \geq \bar{K}+\frac{\mathcal{M}_1\|m\|_{L^{\frac{1}{\alpha_1}}}a^{\rho}}{\rho}\right\} \subset \mathcal{T},$$

where,

$$\bar{K} = \left\{ \delta \left(\frac{a^{1-\gamma}}{\Gamma(1-\gamma)} d_{\infty}(u_0, \hat{0}) + d_{\infty}(\mu(\mathcal{U}), \hat{0}) \right) + \frac{\mathcal{M}a^{\gamma+\rho-1}\mathbb{B}(\gamma, \rho)}{\Gamma(\gamma)} (d_{\infty}(u_1, \hat{0}) + d_{\infty}(\nu(\mathcal{U}), \hat{0})) \right\}.$$

Here, $\mathbb{B}(n, m)$ is a Beta function. Clearly, $\mathcal{B}_s$ is non-empty, convex, bounded, and closed.

Consider the operator $\Phi: \mathcal{C}(J, \mathcal{T}) \rightarrow \mathcal{C}(J, \mathcal{T})$ defined by

$$\begin{aligned} \mathcal{U}(t) &= {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma} \left(t^{\rho-1} \mathcal{G}_{\rho}(t)(u_0 - \mu(\mathcal{U})) \right) \\ &+ \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_{\rho}(z-t)(u_1 - \nu(\mathcal{U})) dz \\ &+ \int_0^t (t-s)^{\rho-1} \bar{\mathcal{G}}_{\rho}(t, s) \mathcal{F}(s, \mathcal{U}) ds. \end{aligned} \quad (8)$$

Due to the continuity of the related functions and the basic characteristics of fractional integrals, the operator Φ introduced in Equation (8) is properly defined. Therefore, proving the existence of a fixed point $\mathcal{U}$ of the operator Φ , which corresponds to a solution of the given equation, is sufficient.

We proceed in two phases to establish that the operator Φ satisfies the hypotheses of the Banach contraction principle, thereby ensuring the existence and uniqueness of a fixed point.

Step 1: $\Phi(\mathcal{B}_s) \subset \mathcal{B}_s$. For $\mathcal{U} \in \mathcal{B}_s$, by $\mathbb{ASH}_2$ and (1) we get

$$\begin{aligned} d_{\infty}(\Phi \mathcal{U}(t), \hat{0}) &\leq d_{\infty} \left({}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma} \left(t^{\rho-1} \mathcal{G}_{\rho}(t)(u_0 - \mu(\mathcal{U})) \right), \hat{0} \right) \\ &+ d_{\infty} \left(\int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_{\rho}(z-t)(u_1 - \nu(\mathcal{U})) dz, \hat{0} \right) \\ &+ d_{\infty} \left(\int_0^t (t-s)^{\rho-1} \bar{\mathcal{G}}_{\rho}(t, s) \mathcal{F}(s, \mathcal{U}) ds, \hat{0} \right) \\ &\leq d_{\infty} \left(\mathcal{I}^{1-\gamma} \frac{d}{dt} \left(t^{\rho-1} \mathcal{G}_{\rho}(t) u_0 - \mu(\mathcal{U}) \right), \hat{0} \right) \\ &+ \mathcal{M} d_{\infty} \left(\int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} (u_1 - \nu(\mathcal{U})) dz, \hat{0} \right) \\ &+ \mathcal{M}_1 d_{\infty} \left(\int_0^t (t-s)^{\rho-1} m(s) ds, \hat{0} \right) \\ &\leq \delta d_{\infty} \left(\frac{1}{\Gamma(1-\gamma)} \int_0^l (l-s)^{-\gamma} (u_0 - \mu(\mathcal{U})) dl, \hat{0} \right) \\ &+ \mathcal{M} d_{\infty} \left(\int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} (u_1 - \nu(\mathcal{U})) dz, \hat{0} \right) \\ &+ \mathcal{M}_1 d_{\infty} \left(\int_0^t (t-s)^{\rho-1} m(s) ds, \hat{0} \right) \\ &\leq \frac{\delta a^{1-\gamma}}{\Gamma(1-\gamma)} d_{\infty}(u_0, \hat{0}) + \delta d_{\infty}(\mu(\mathcal{U}), \hat{0}) \\ &+ \mathcal{M} a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho) d_{\infty}(u_1 - \nu(\mathcal{U}), \hat{0}) \\ &+ \mathcal{M}_1 \|m\|_{L^{\frac{1}{\kappa_1}}} d_{\infty} \left(\int_0^t (t-s)^{\rho-1} ds, \hat{0} \right) \\ &\leq \frac{\delta a^{1-\gamma}}{\Gamma(1-\gamma)} d_{\infty}(u_0, \hat{0}) + \delta d_{\infty}(\mu(\mathcal{U}), \hat{0}) \\ &+ \frac{\mathcal{M} a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho)}{\Gamma(\gamma)} d_{\infty}(u_1, \hat{0}) \\ &+ \frac{\mathcal{M} a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho)}{\Gamma(\gamma)} d_{\infty}(\nu(\mathcal{U}), \hat{0}) + \frac{\mathcal{M}_1 \|m\|_{L^{\frac{1}{\kappa_1}}} a^{\rho}}{\rho} \\ &= \bar{K} + \frac{\mathcal{M}_1 \|m\|_{L^{\frac{1}{\kappa_1}}} a^{\rho}}{\rho} \leq s. \end{aligned}$$

This indicates that $\Phi(\mathcal{B}_s) \subset \mathcal{B}_s$.

Step 2: Φ is a contraction.

From Equations (1) and (14), and by assumptions $\mathbb{ASH}_1$, $\mathbb{ASH}_3$, and $\mathbb{ASH}_4$, we have

$$\begin{aligned} & d_\infty(\Phi \mathcal{U}(t), \Phi \mathcal{V}(t)) \\ & \leq d_\infty\left(\mathcal{RLD}_{0+}^{1-\gamma} t^{\rho-1} \mathcal{G}_\rho(t)(u_0 - \mu(\mathcal{U})), \mathcal{RLD}_{0+}^{1-\gamma} t^{\rho-1} \mathcal{G}_\rho(t)(u_0 - \mu(\mathcal{V}))\right) \\ & + d_\infty\left(\int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_\rho(z-t)(u_1 - \nu(\mathcal{U})) dz, \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_\rho(z-t)(u_1 - \nu(\mathcal{V})) dz\right) \\ & + d_\infty\left(\int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_\rho(t,s) (\mathcal{F}(s, \mathcal{U})) ds, \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_\rho(t,s) (\mathcal{F}(s, \mathcal{V})) ds\right), \\ & \leq \frac{1}{\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} \frac{d}{dt} \left(t^{\rho-1} \mathcal{G}_\rho(t)\right) d_\infty(u_0 - \mu(\mathcal{U}), u_0 - \mu(\mathcal{V})) dl \\ & + \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} \mathcal{G}_\rho(z-t) d_\infty(u_1 - \nu(\mathcal{U}), u_1 - \nu(\mathcal{V})) dz \\ & + \int_0^t (t-s)^{\rho-1} \overline{\mathcal{G}}_\rho(t,s) d_\infty(\mathcal{F}(s, \mathcal{U}), \mathcal{F}(s, \mathcal{V})) ds \\ & \leq \frac{\mathcal{O}_1}{\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} \left((\rho-1)t^{\rho-1} \mathcal{G}_\rho(t) + t^{2\rho-2} \int_0^\infty \rho^2 \theta^2 \psi_\rho(\theta) \mathcal{R}(t^\rho \theta) d\theta\right) d_\infty(\mathcal{U}, \mathcal{V}) dl \\ & + \mathcal{MO}_2 \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t)^{\rho-1} d_\infty(\mathcal{U}, \mathcal{V}) dz + \mathcal{M}_1 \int_0^t (t-s)^{\rho-1} \kappa(s) d_\infty(\mathcal{U}, \mathcal{V}) ds \\ & \leq \frac{(\rho-1)\mathcal{O}_1 \mathcal{M} a^{\rho-1}}{\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} d_\infty(\mathcal{U}, \mathcal{V}) dl \\ & + \frac{a^{2\rho-2} \mathcal{O}_1}{\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} \left(\int_0^\infty \rho^2 \theta^2 \psi_\rho(\theta) \int_0^{t^\rho \theta} \|\mathcal{K}\| ds d\theta\right) d_\infty(\mathcal{U}, \mathcal{V}) dl \\ & + \mathcal{MO}_2 a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho) d_\infty(\mathcal{U}, \mathcal{V}) + \mathcal{M}_1 \|\kappa\|_{L^{\frac{1}{\alpha_1}}} a^\rho d_\infty(\mathcal{U}, \mathcal{V}). \end{aligned}$$

By (14), we can write

$$\int_0^\infty \rho^2 \theta^2 \psi_\rho(\theta) \int_0^{t^\rho \theta} \|\mathcal{K}\| ds d\theta = \mathcal{M} \int_0^\infty \rho^2 \theta^2 \psi_\rho(\theta) t^\rho \theta d\theta \leq \mathcal{M} \rho^2 a^\rho \int_0^\infty \theta^3 \psi_\rho(\theta) d\theta = \frac{3\mathcal{M} \rho^2 a^\rho}{2\rho \Gamma(4\rho)}.$$

Then, the inequality becomes

$$\begin{aligned} & d_\infty(\Phi \mathcal{U}(t), \Phi \mathcal{V}(t)) \leq \\ & \frac{(\rho-1)\mathcal{O}_1 \mathcal{M} a^{\rho-1}}{\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} d_\infty(\mathcal{U}, \mathcal{V}) dl + \frac{3\mathcal{M} \rho a^{3\rho-2}}{2\Gamma(4\rho)\Gamma(\gamma)} \int_0^l (l-s)^{-\gamma} d_\infty(\mathcal{U}, \mathcal{V}) dl \\ & + \mathcal{MO}_2 a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho) d_\infty(\mathcal{U}, \mathcal{V}) + \mathcal{M}_1 \|\kappa\|_{L^{\frac{1}{\alpha_1}}} a^\rho d_\infty(\mathcal{U}, \mathcal{V}) \\ & \leq \left(\frac{(\rho-1)\mathcal{O}_1 \mathcal{M} a^{\rho-\gamma}}{\Gamma(\gamma)\Gamma(1-\gamma)} + \frac{3\mathcal{M} \rho a^{3\rho-\gamma-1}}{2\Gamma(4\rho)\Gamma(\gamma)\Gamma(1-\gamma)} + \mathcal{MO}_2 a^{\gamma+\rho-1} \mathbb{B}(\gamma, \rho) + \mathcal{M}_1 \|\kappa\|_{L^{\frac{1}{\alpha_1}}} a^\rho\right) d_\infty(\mathcal{U}, \mathcal{V}) \\ & := \overline{P} d_\infty(\mathcal{U}, \mathcal{V}). \end{aligned}$$

Thus, $\overline{P} < 1$ is relevant, which ensures that Φ has a unique fixed point and (6) is the unique fuzzy mild solution to problem (1) on $[0, a]$. Therefore, the proof is complete.

□

Our final result in this section is concerned with proving the existence of a solution to problem (1) by applying Schauder's fixed-point theorem.

Theorem 3. Assume that $\mathbb{ASH}_1 - \mathbb{ASH}_4$ hold. Then, the operator $\Phi : C(J, \mathcal{T}) \rightarrow C(J, \mathcal{T})$ defined by equation (8) has at least one fixed point. Thus, the fuzzy fractional Cauchy problem (1) admits at least one mild solution.

Proof. Step 1: $\Phi(\mathcal{B}_s) \subset \mathcal{B}_s$.

As in Step 1 of the proof of Theorem 4.1, Φ indeed maps $\mathcal{B}_s$ into itself.

Step 2: Φ is continuous. Let $\mathcal{U}_n \rightarrow \mathcal{U}$ in $C([0, a], \mathcal{T})$. For any $t \in [0, a]$, consider

$$\begin{aligned} (\Phi \mathcal{U}_n)(t) &= {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t^{\rho-1}\mathcal{G}_\rho(t)(u_0 - \mu(\mathcal{U}_n))) \\ &\quad + \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)}(z-t)^{\rho-1}\mathcal{G}_\rho(z-t)(u_1 - \nu(\mathcal{U}_n))dz \\ &\quad + \int_0^t (t-s)^{\rho-1}\mathcal{G}_\rho(t,s)\mathcal{F}(s, \mathcal{U}_n(s))ds. \end{aligned}$$

By the continuity of the mappings μ , ν , and $\mathcal{F}(\cdot)$, it holds that

$$\mu(\mathcal{U}_n) \rightarrow \mu(\mathcal{U}), \quad \nu(\mathcal{U}_n) \rightarrow \nu(\mathcal{U}), \quad \mathcal{F}(s, \mathcal{U}_n(s)) \rightarrow \mathcal{F}(s, \mathcal{U}(s)) \text{ uniformly in } s.$$

$$\begin{aligned} &\lim_{n \rightarrow \infty} d_\infty((\Phi \mathcal{U}_n)(t), (\Phi \mathcal{U})(t)) \\ &\leq \lim_{n \rightarrow \infty} d_\infty({}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t^{\rho-1}\mathcal{G}_\rho(t)(u_0 - \mu(\mathcal{U}_n))), {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t^{\rho-1}\mathcal{G}_\rho(t)(u_0 - \mu(\mathcal{U}))) \\ &\quad + \lim_{n \rightarrow \infty} d_\infty\left(\int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)}(z-t)^{\rho-1}\mathcal{G}_\rho(z-t)(u_1 - \nu(\mathcal{U}_n))dz, \int_0^t \frac{z^{\gamma-1}}{\Gamma(\gamma)}(z-t)^{\rho-1}\mathcal{G}_\rho(z-t)(u_1 - \nu(\mathcal{U}))dz\right) \\ &\quad + \lim_{n \rightarrow \infty} d_\infty\left(\int_0^t (t-s)^{\rho-1}\mathcal{G}_\rho(t,s)\mathcal{F}(s, \mathcal{U}_n(s))ds, \int_0^t (t-s)^{\rho-1}\mathcal{G}_\rho(t,s)\mathcal{F}(s, \mathcal{U}(s))ds\right) \rightarrow 0, \end{aligned}$$

when $n \rightarrow \infty$. Therefore, the operator Φ is continuous on $C([0, a], \mathcal{T})$.

Step 3: Φ maps bounded sets into equicontinuous sets. Let $\mathcal{B}_s \subset C([0, a], \mathcal{T})$ be a bounded set. For any $\mathcal{U} \in \mathcal{B}_s$ and for $0 \leq t_1 \leq t_2 \leq a$,

$$\begin{aligned} &d_\infty((\Phi \mathcal{U})(t_2), (\Phi \mathcal{U})(t_1)) \\ &\leq d_\infty({}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t_2^{\rho-1}\mathcal{G}_\rho(t_2)(u_0 - \mu(\mathcal{U}))), {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t_1^{\rho-1}\mathcal{G}_\rho(t_1)(u_0 - \mu(\mathcal{U}))) \\ &\quad + d_\infty\left(\int_0^{t_2} \frac{z^{\gamma-1}}{\Gamma(\gamma)}(z-t_2)^{\rho-1}\mathcal{G}_\rho(z-t_2)(u_1 - \nu(\mathcal{U}))dz, \int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)}(z-t_1)^{\rho-1}\mathcal{G}_\rho(z-t_1)(u_1 - \nu(\mathcal{U}))dz\right) \\ &\quad + d_\infty\left(\int_0^{t_2} (t_2-s)^{\rho-1}\mathcal{G}_\rho(t_2,s)\mathcal{F}(s, \mathcal{U}(s))ds, \int_0^{t_1} (t_1-s)^{\rho-1}\mathcal{G}_\rho(t_1,s)\mathcal{F}(s, \mathcal{U}(s))ds\right). \end{aligned}$$

To evaluate the first term,

$$\begin{aligned} &d_\infty({}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t_2^{\rho-1}\mathcal{G}_\rho(t_2)(u_0 - \mu(\mathcal{U}))), {}^{\mathcal{RL}}\mathcal{D}_{0+}^{1-\gamma}(t_1^{\rho-1}\mathcal{G}_\rho(t_1)(u_0 - \mu(\mathcal{U}))) \\ &\leq \frac{1}{\Gamma(\gamma)}d_\infty\left(\int_0^{t_1} (l-s)^{-\gamma}\left((\rho-1)t_2^{\rho-1}\mathcal{G}_\rho(t_2) - (\rho-1)t_1^{\rho-1}\mathcal{G}_\rho(t_1)\right)(u_0 - \mu(\mathcal{U}))dl, \hat{0}\right) \\ &\quad + \frac{1}{\Gamma(\gamma)}d_\infty\left(\int_0^{t_1} (l-s)^{-\gamma}\left(t_2^{2\rho-2}\mathcal{G}_\rho^2(t_2) - t_1^{2\rho-2}\mathcal{G}_\rho^2(t_1)\right)(u_0 - \mu(\mathcal{U}))dl, \hat{0}\right) \\ &\quad + \frac{1}{\Gamma(\gamma)}d_\infty\left(\int_{t_1}^{t_2} (l-s)^{-\gamma}\left((\rho-1)t_2^{\rho-1}\mathcal{G}_\rho(t_2)\right)(u_0 - \mu(\mathcal{U}))dl, \hat{0}\right) \\ &\quad + \frac{1}{\Gamma(\gamma)}d_\infty\left(\int_{t_1}^{t_2} (l-s)^{-\gamma}\left(t_2^{2\rho-2}\mathcal{G}_\rho^2(t_2)\right)(u_0 - \mu(\mathcal{U}))dl, \hat{0}\right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\gamma)} d_{\infty} \left(\int_0^{t_1} (l-s)^{-\gamma} \left((\rho-1)t_2^{\rho-1} \mathcal{G}_{\rho}(t_2) - (\rho-1)t_2^{\rho-1} \mathcal{G}_{\rho}(t_1) \right) (u_0 - \mu(\mathcal{U})) dl, \hat{0} \right) \\
&+ \frac{1}{\Gamma(\gamma)} d_{\infty} \left(\int_0^{t_1} (l-s)^{-\gamma} \left((\rho-1)t_2^{\rho-1} \mathcal{G}_{\rho}(t_1) + (\rho-1)t_1^{\rho-1} \mathcal{G}_{\rho}(t_1) \right) (u_0 - \mu(\mathcal{U})) dl, \hat{0} \right) \\
&+ \frac{1}{\Gamma(\gamma)} d_{\infty} \left(\int_0^{t_1} (l-s)^{-\gamma} \left(t_2^{2\rho-2} \mathcal{G}_{\rho}^2(t_2) - t_2^{2\rho-2} \mathcal{G}_{\rho}^2(t_1) \right) (u_0 - \mu(\mathcal{U})) dl, \hat{0} \right) \\
&+ \frac{1}{\Gamma(\gamma)} d_{\infty} \left(\int_0^{t_1} (l-s)^{-\gamma} \left(t_2^{2\rho-2} \mathcal{G}_{\rho}^2(t_1) - t_1^{2\rho-2} \mathcal{G}_{\rho}^2(t_1) \right) (u_0 - \mu(\mathcal{U})) dl, \hat{0} \right) \\
&\leq \frac{(\rho-1)C_1}{\Gamma(\gamma)} \int_0^{t_1} (l-s)^{-\gamma} t_2^{\rho-1} d_{\infty}(\mathcal{G}_{\rho}(t_2), \mathcal{G}_{\rho}(t_1)) dl \\
&+ \left(\frac{(\rho-1)C_1 \mathcal{M}}{\Gamma(\gamma)} + \frac{C_1}{\Gamma(\gamma)} \frac{3\mathcal{M}\rho^2 a^{\rho}}{2\rho\Gamma(4\rho)} \right) \int_0^{t_1} (l-s)^{-\gamma} (t_2^{\rho-1} - t_1^{\rho-1}) dl \\
&+ \frac{C_1}{\Gamma(\gamma)} \int_0^{t_1} (l-s)^{-\gamma} t_2^{2\rho-2} d_{\infty}(\mathcal{G}_{\rho}^2(t_2), \mathcal{G}_{\rho}^2(t_1)) dl \\
&+ \left(\frac{\mathcal{M}C_1}{\Gamma(\gamma)} + \frac{C_1}{\Gamma(\gamma)} \frac{3\mathcal{M}\rho^2 a^{\rho}}{2\rho\Gamma(4\rho)} \right) \int_{t_1}^{t_2} (l-s)^{-\gamma} t_2^{\rho-1} dl \\
&\leq \frac{(\rho-1)C_1}{\Gamma(\gamma)} \int_0^{t_1} (l-s)^{-\gamma} t_2^{\rho-1} d_{\infty}(\mathcal{G}_{\rho}(t_2) - \mathcal{G}_{\rho}(t_1), \hat{0}) dl \\
&+ \left(\frac{(\rho-1)C_1 \mathcal{M}}{\Gamma(\gamma)} + \frac{C_1}{\Gamma(\gamma)} \frac{3\mathcal{M}\rho^2 a^{\rho}}{2\rho\Gamma(4\rho)} \right) \frac{(t_2^{\rho-1} - t_1^{\rho-1})}{\Gamma(1-\gamma)} \left\{ (t_1-s)^{1-\gamma} - (-s)^{1-\gamma} \right\} \\
&+ \frac{C_1}{\Gamma(\gamma)} \int_0^{t_1} (l-s)^{-\gamma} t_2^{2\rho-2} d_{\infty}(\mathcal{G}_{\rho}^2(t_2), \mathcal{G}_{\rho}^2(t_1)) dl \\
&+ \left(\frac{\mathcal{M}C_1}{\Gamma(\gamma)} + \frac{C_1}{\Gamma(\gamma)} \frac{3\mathcal{M}\rho^2 a^{\rho}}{2\rho\Gamma(4\rho)} \right) \frac{(t_2-s)^{1-\gamma} - (t_1-s)^{1-\gamma}}{\Gamma(1-\gamma)} t_2^{\rho-1} := \mathfrak{I}_1.
\end{aligned}$$

As for the second term,

$$\begin{aligned}
&d_{\infty} \left(\int_0^{t_2} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_2)^{\rho-1} \mathcal{G}_{\rho}(z-t_2) (u_1 - v(\mathcal{U})) dz, \int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_1)^{\rho-1} \mathcal{G}_{\rho}(z-t_1) (u_1 - v(\mathcal{U})) dz \right) \\
&\leq d_{\infty} \left(\int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)} \left((z-t_2)^{\rho-1} \mathcal{G}_{\rho}(z-t_2) - (z-t_1)^{\rho-1} \mathcal{G}_{\rho}(z-t_1) \right) (u_1 - v(\mathcal{U})) dz, \hat{0} \right) \\
&+ d_{\infty} \left(\int_{t_1}^{t_2} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_2)^{\rho-1} \mathcal{G}_{\rho}(z-t_2) (u_1 - v(\mathcal{U})) dz, \hat{0} \right) \\
&\leq C_2 \int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_2)^{\rho-1} d_{\infty}(\mathcal{G}_{\rho}(z-t_2), \mathcal{G}_{\rho}(z-t_1)) dz \\
&+ C_2 \mathcal{M} \int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)} \left((z-t_2)^{\rho-1} - (z-t_1)^{\rho-1} \right) dz \\
&+ C_2 \mathcal{M} \int_{t_1}^{t_2} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_2)^{\rho-1} dz, \\
&\leq C_2 \mathbb{B}(\gamma, \rho) d_{\infty}(\mathcal{G}_{\rho}(z-t_2), \mathcal{G}_{\rho}(z-t_1)) + C_2 \mathcal{M} \int_0^{t_1} \frac{z^{\gamma-1}}{\Gamma(\gamma)} \left((z-t_1)^{\rho-1} - (z-t_2)^{\rho-1} \right) dz \\
&+ C_2 \mathcal{M} \int_{t_1}^{t_2} \frac{z^{\gamma-1}}{\Gamma(\gamma)} (z-t_2)^{\rho-1} dz := \mathfrak{I}_2.
\end{aligned}$$

Additionally,

$$\begin{aligned}
& d_{\infty} \left(\int_0^{t_2} (t_2 - s)^{\rho-1} \mathcal{G}_{\rho}(t_2, s) \mathcal{F}(s, \mathcal{U}(s)) ds, \int_0^{t_1} (t_1 - s)^{\rho-1} \mathcal{G}_{\rho}(t_1, s) \mathcal{F}(s, \mathcal{U}(s)) ds, \right) \\
& \leq \|m\|_{L^{\frac{1}{\alpha_1}}} d_{\infty} \left(\int_0^{t_1} ((t_2 - s)^{\rho-1} \mathcal{G}_{\rho}(t_2, s) - (t_1 - s)^{\rho-1} \mathcal{G}_{\rho}(t_1, s)) ds, \hat{0} \right) + \|m\|_{L^{\frac{1}{\alpha_1}}} \mathcal{M} d_{\infty} \left(\int_{t_1}^{t_2} (t_2 - s)^{\rho-1} ds, \hat{0} \right) \\
& \leq \|m\|_{L^{\frac{1}{\alpha_1}}} \mathcal{M} d_{\infty} \left(\int_0^{t_1} ((t_2 - s)^{\rho-1} - (t_1 - s)^{\rho-1}) ds, \hat{0} \right) + \|m\|_{L^{\frac{1}{\alpha_1}}} \int_0^{t_1} (t_2 - s)^{\rho-1} d_{\infty}(\mathcal{G}_{\rho}(t_2, s), \mathcal{G}_{\rho}(t_1, s)) ds \\
& + \|m\|_{L^{\frac{1}{\alpha_1}}} \mathcal{M} d_{\infty} \left(\int_{t_1}^{t_2} (t_2 - s)^{\rho-1} ds, \hat{0} \right) := \mathfrak{J}_3,
\end{aligned}$$

where $C_1 = \sup_{\mathcal{U} \in \mathcal{B}} d_{\infty}(u_0 - \mu(\mathcal{U}), 0)$ and $C_2 = \sup_{\mathcal{U} \in \mathcal{B}} d_{\infty}(u_1 - \nu(\mathcal{U}), 0)$. Furthermore, by the continuity of $\mathcal{G}_{\rho}(t, s)$ and $\mathcal{G}_{\rho}^2(t)$, we can show that $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3 \rightarrow 0$ as $t_1 \rightarrow t_2$. Thus, Φ is uniformly equicontinuous over $\mathcal{U} \in \mathcal{B}_s$. Combined with its uniform boundedness, the Arzelà–Ascoli theorem implies that Φ is compact.

□

The use of both Banach and Schauder fixed point theorems in this study serves distinct purposes and offers complementary insights. The Banach contraction principle guarantees a **unique** mild solution under the condition $\bar{P} < 1$, which is explicitly computable from system parameters. This is particularly advantageous when precise control over the solution’s uniqueness is required, such as in control theory or well-posed dynamical systems. However, its applicability is limited to contraction mappings, which may not hold for highly nonlinear or non-Lipschitz nonlinearities $\mathcal{F}$.

In contrast, Schauder’s fixed-point theorem ensures existence without requiring contractivity, making it suitable for systems where nonlinearities are only continuous or bounded. This is especially useful in fuzzy settings where the metric structure may not readily yield contraction constants. The trade-off, however, is that uniqueness is not guaranteed, and additional compactness conditions must be verified. In the context of fuzzy Hilfer fractional systems, the combination of both theorems provides a robust framework: Banach’s theorem ensures uniqueness under stricter conditions, while Schauder’s theorem broadens the class of admissible nonlinearities.

5. Applied Examples

Consider the fuzzy Hilfer-type fractional evolution equation

$${}^{\mathcal{H}}_{gH}\mathcal{D}_{0+}^{\alpha, \beta} \mathcal{U}(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t, \mathcal{U}(t)), \quad t \in [0, a]$$

with fuzzy initial conditions

$$\mathcal{I}^{2-v} \mathcal{U}(0) = u_0 - \mu(\mathcal{U}), \quad \mathcal{I}^{2-v} \mathcal{U}'(0) = u_1 - \nu(\mathcal{U}),$$

where $\alpha = 1.8$, $\beta = 0.5$, $\rho = 0.9$, $\gamma = 0.6$, and $v = \alpha + \beta(2 - \alpha) = 1.9$. Let the final time be $a = 1.5$. Furthermore, the operator $\mathcal{A}$ generates a cosine family $\mathcal{K}(t) = \cos(t)I$ with bound $\mathcal{M} = 1$. To verify the hypotheses, let $\mathcal{F} : [0, a] \times \mathcal{T} \rightarrow \mathcal{T}$ be defined by

$$\mathcal{F}(t, \mathcal{U}) := \sin(t) \mathcal{U}^2(t),$$

where $\mathcal{U} \in \mathcal{T} = \mathcal{C}([0, a], \mathbb{E})$. To verify the Lipschitz-type condition in assumption $\mathbb{ASH}_2$, we estimate

$$|\mathcal{F}(t, \mathcal{U}) - \mathcal{F}(t, \mathcal{V})| = |\sin(t)| |\mathcal{U}^2(t) - \mathcal{V}^2(t)| = |\sin(t)| |\mathcal{U}(t) - \mathcal{V}(t)| |\mathcal{U}(t) + \mathcal{V}(t)|.$$

Taking the supremum with respect to the fuzzy metric d_∞ , where, d_∞ denotes the Hausdorff-type supremum metric on the α -cuts of fuzzy sets, that is

$$d_\infty(\mathcal{U}, \mathcal{V}) = \sup_{\alpha \in [0,1]} \max \{ |\underline{\mathcal{U}}(\alpha) - \underline{\mathcal{V}}(\alpha)|, |\overline{\mathcal{U}}(\alpha) - \overline{\mathcal{V}}(\alpha)| \},$$

We obtain

$$d_\infty(\mathcal{F}(t, \mathcal{U}), \mathcal{F}(t, \mathcal{V})) \leq (\|\mathcal{U}\|_\infty + \|\mathcal{V}\|_\infty) d_\infty(\mathcal{U}, \mathcal{V}).$$

Hence, the modulus of continuity is given by

$$\kappa(t) := \|\mathcal{U}\|_\infty + \|\mathcal{V}\|_\infty,$$

which is constant with respect to t and belongs to $L^{1/\alpha_1}([0, a])$ for any finite a and $0 < \alpha_1 < \alpha$. Therefore, $\mathcal{F}(t, \mathcal{U})$ satisfies assumption ASH_2 .

Verification of Hypotheses:

- $\mathcal{F}(t, \mathcal{U})$ is continuous and strongly measurable $\Rightarrow \text{ASH}_1$ holds.
- $\mathcal{F}(t, \mathcal{U})$ is Lipschitz with respect to $\mathcal{U}$, with $\|\kappa\| = 0.15 \in L^{1/\alpha_1}$ for $\alpha_1 = 0.5 \Rightarrow \text{ASH}_2$ and ASH_3 hold.
- We define two non-linear mappings $\mu, \nu : \mathcal{T} \rightarrow \overline{\mathcal{DA}}$, where $\mathcal{T} = \mathcal{C}([0, a], \mathbb{E})$, as follows:

$$\mu(\mathcal{U})(t) := \sin(t) \cdot \mathcal{U}^2(t), \quad \nu(\mathcal{U})(t) := \frac{\mathcal{U}(t)}{1 + \|\mathcal{U}(t)\|}.$$

These mappings are continuous and fuzzy-valued for all $\mathcal{U} \in \mathcal{T}$. We now compute their respective moduli of continuity with respect to the fuzzy metric d_∞ .

Lipschitz Continuity of μ : For $\mu(\mathcal{U})(t) = \sin(t) \cdot \mathcal{U}^2(t)$, we estimate

$$|\mu(\mathcal{U})(t) - \mu(\mathcal{V})(t)| = |\sin(t)| \cdot |\mathcal{U}^2(t) - \mathcal{V}^2(t)| \leq |\sin(t)| \cdot |\mathcal{U}(t) - \mathcal{V}(t)| |\mathcal{U}(t) + \mathcal{V}(t)|.$$

Hence,

$$d_\infty(\mu(\mathcal{U}), \mu(\mathcal{V})) \leq 2\|\mathcal{U}\|_\infty d_\infty(\mathcal{U}, \mathcal{V}),$$

which implies that μ is Lipschitz with modulus $\mathcal{O}_1 = 2\|\mathcal{U}\|_\infty$.

Lipschitz Continuity of ν : For $\nu(\mathcal{U})(t) = \frac{\mathcal{U}(t)}{1 + \|\mathcal{U}(t)\|}$, we consider the inequality

$$\left| \frac{\mathcal{U}}{1 + \|\mathcal{U}\|} - \frac{\mathcal{V}}{1 + \|\mathcal{V}\|} \right| \leq \frac{|\mathcal{U} - \mathcal{V}|}{1 + \min\{\|\mathcal{U}\|, \|\mathcal{V}\|\}}.$$

Thus,

$$d_\infty(\nu(\mathcal{U}), \nu(\mathcal{V})) \leq \frac{1}{1 + \min\{\|\mathcal{U}\|_\infty, \|\mathcal{V}\|_\infty\}} d_\infty(\mathcal{U}, \mathcal{V}),$$

which shows that ν is Lipschitz with modulus $\mathcal{O}_2 = \frac{1}{1 + \min\{\|\mathcal{U}\|_\infty, \|\mathcal{V}\|_\infty\}}$.

Computation of the Existence Constant $\bar{P}$: Considering the estimates

$$\Gamma(0.6) \approx 1.032, \quad \Gamma(0.4) \approx 2.218, \quad \Gamma(3.6) \approx 5.697, \quad \mathbb{B}(0.6, 0.9) \approx 1.17,$$

we compute

$$\begin{aligned}\text{term}_1 &= \frac{(\rho - 1)\mathcal{O}_1 a^{\rho - \gamma}}{\Gamma(\gamma)\Gamma(1 - \gamma)} \approx -0.014, \\ \text{term}_2 &= \frac{3\rho a^{3\rho - \gamma - 1}}{2\Gamma(4\rho)\Gamma(\gamma)\Gamma(1 - \gamma)} \approx 0.0469, \\ \text{term}_3 &= \mathcal{O}_2 a^{\gamma + \rho - 1} \mathbb{B}(\gamma, \rho) \approx 0.0656, \\ \text{term}_4 &= \|\kappa\| a^\rho = 0.15 \cdot 1.5^{0.9} \approx 0.2236.\end{aligned}$$

$$\therefore \bar{P} \approx -0.014 + 0.0469 + 0.0656 + 0.2236 = 0.3221 < 1.$$

Discussion of the results: As $\bar{P} < 1$ in Figure 1, all of the assumptions ASH_1 – ASH_4 are met; therefore, the problem admits a unique mild fuzzy solution according to the existence and uniqueness theorem.

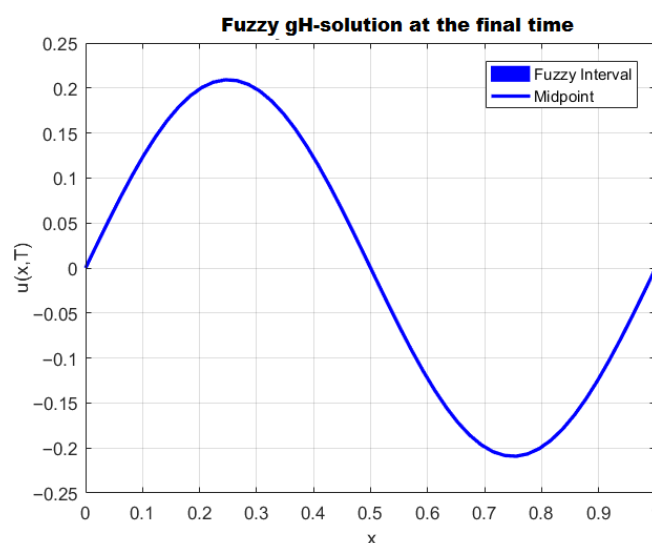


Figure 1. The fuzzy mild solution $\bar{P} = 0.3221$.

The development of the fuzzy solution interval is demonstrated through numerical simulations using $\alpha = 1.5$, $\beta = 0.5$, and $\delta = 0.01$. These simulations demonstrate how uncertainty propagates as a result of fractional dynamics and non-linearity.

The numerical distribution of the fuzzy gH -solution $\mathcal{U}(x, T)$ at the final time T is shown in Figure 2. The answer calculated using the generalized Hukuhara framework is shown as a solid line, representing the fuzzy solution's midpoint, while the shaded fuzzy interval shows the fuzzy function's lower and upper limits. There is noticeable oscillatory behavior in the picture, which symmetrically declines towards the borders and is especially focused near the mid-domain. While the breadth of the fuzzy interval rises dramatically in certain locations—particularly near $x \approx 0.2$ and $x \approx 0.8$ —the midpoint solution stays regular, suggesting increasing uncertainty and unpredictability in the fuzzy state.

The numerical results in Figure 3 demonstrate that the fuzzy solution $\mathcal{U}(t)$ for the Hilfer-type fractional differential equation with order $\alpha = 1.8$ displays dynamic behavior under two distinct settings of the type parameter; namely, $\beta = 0$ and $\beta = 1$. The solution—represented as a fuzzy interval bounded by its lower and upper routes—was computed using the integral form with cosine family operators and a convolution kernel defined by a probability density function. For $\beta = 0$, which corresponds to the Riemann–Liouville-type Hilfer derivative, the solution develops more quickly and has a wider range in its fuzzy limits, indicating higher sensitivity to the initial conditions and greater propagation of uncertainty. This is to be expected, as the Riemann–Liouville derivative gives greater weight to the system's prior conduct. In contrast, in the situation where $\beta = 1$, the Caputo-

type behavior leads to smoother dynamics and a shorter fuzzy range. This implies that the influence of the initial data deteriorates more gradually and that the solution behaves more consistently over time. Moreover, the apparent oscillations in the solution are caused by the cosine family $\mathcal{K}(t) = \cos(\omega t)$, which regulates the structure of the underlying evolution system. These oscillations are superimposed on top of a generally increasing trend due to the non-linearity in the source term $\mathcal{F}(t, \mathcal{U})$ and the interaction with the fractional kernel.

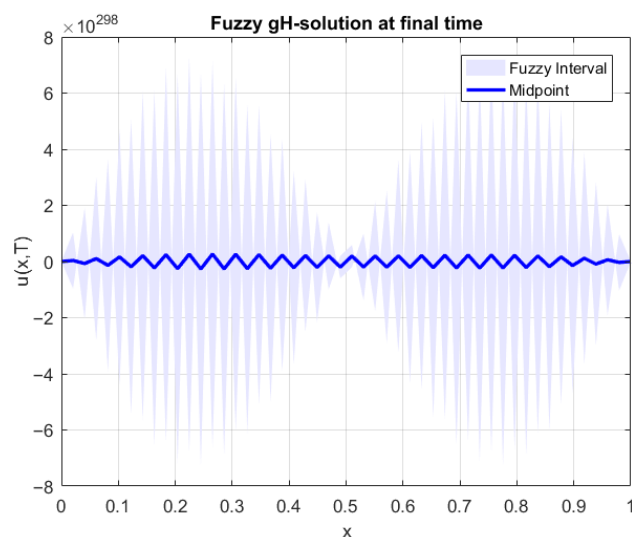


Figure 2. Fuzzy gH -Hilfer fractional derivative numerical solution.

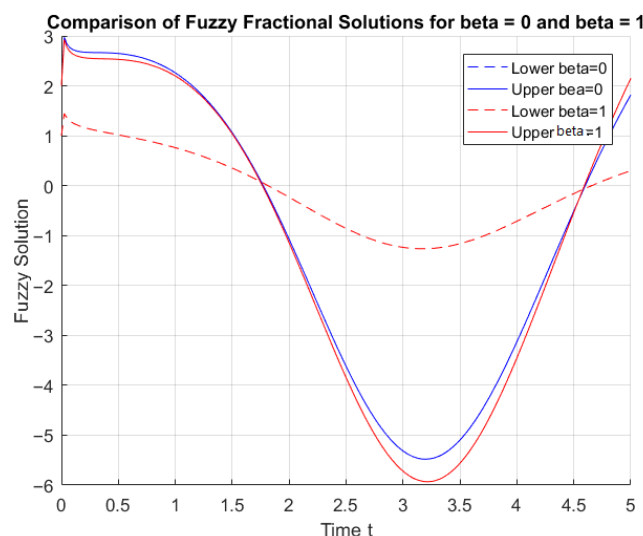


Figure 3. Comparison of fuzzy fractional solutions for $\beta = 0$ and $\beta = 1$.

Overall, the comparison reveals that the qualitative behavior of the fuzzy fractional solution is significantly influenced by the type parameter β . The choice between $\beta = 0$ and $\beta = 1$ when modelling uncertain dynamical systems in real-world scenarios should be based on the required sensitivity and memory properties.

6. Conclusions

In this article, we considered a Hilfer fractional derivative and demonstrated the existence and uniqueness of mild solutions to fractional fuzzy problems of order 1–2. To date, this problem has not been the subject of any scientific investigation. The Banach contraction theorem was used to demonstrate the existence and uniqueness of the mild

solution, which can be verified if certain predetermined requirements (ASH_1 – ASH_4) are satisfied. Furthermore, under the same assumptions, we also used Schauder’s fixed-point theorem to demonstrate the existence of a mild solution. Finally, we applied our findings to create a graphic representation of a fuzzy fractional wave equation through a numerical example.

The theoretical results established in this study may find applications in modeling real-world systems that exhibit both memory and uncertainty. Diffusion processes in porous or heterogeneous media, fuzzy control systems where actuator dynamics contain hereditary effects, and viscoelastic materials with fuzzy constitutive parameters are examples of potential fields. In addition, the suggested fuzzy Hilfer framework might be used to analyse energy storage models in electrical and electrochemical systems, population dynamics, and epidemic propagation under unclear data. These uses demonstrate the adaptability of the current method and its capacity to operate as a link between real-world modelling problems and abstract fractional theory.

Further research might look at a number of interesting avenues. One possible extension is to explore the controllability and Hyers-Ulam stability of fuzzy Hilfer-Katugampola fractional systems using nonlinear Leray-Schauder alternative approaches and measurements of noncompactness. Additionally, it would be beneficial to study impulsive and stochastic versions of fuzzy Hilfer models and to create numerical algorithms that mimic memory and uncertainty in real-world systems.

A significant direction for future work involves extending the present results to more complex settings. This includes generalizing the problem to multi-dimensional fuzzy Banach spaces and considering time-dependent operators $\mathcal{A}(t)$. In multi-dimensional settings, the fuzzy metric structure and the analysis of α -cuts become more complex, requiring careful extension of the current framework. For time-dependent operators, the strongly continuous cosine family $\{\mathcal{K}(t)\}$ would need to be replaced by a two-parameter evolution system $\{\mathcal{K}(t, s)\}$, complicating the representation of mild solutions and requiring techniques from evolution semigroup theory. Key mathematical challenges in such generalizations include establishing uniform bounds for time-dependent generators, handling non-autonomous nonlinearities, and ensuring the compactness of solution operators in the absence of time-translation invariance. Successfully overcoming these challenges would significantly broaden the applicability of the results to non-autonomous and spatially complex physical and engineering systems. Overall, these developments would strengthen fuzzy fractional analysis’s usefulness in physics, engineering, and biological systems.

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