

Article

On Generalized *One-kZero* Numbers

Paula M. M. C. Catarino ^{1,†} , Grieg A. Costa ^{2,†}  and Eudes A. Costa ^{1,3,*,†} 
¹ Department of Mathematics, University of Trás-os-Montes and Alto Douro, 5000-801 Vila Real, Portugal; pcatarin@utad.pt

² Department of Mathematics, University of Brasília, Brasília 70910-900, Brazil; 211050015@aluno.unb.br

³ Department of Mathematics, Federal University of Tocantins, Arraias 77330-000, Brazil

* Correspondence: eudes@uft.edu.br

† These authors contributed equally to this work.

Abstract

In this study is introduced a novel generalization of repunit and one-zero numbers through the formulation of the generalized *One-kZero*. This sequence extends the classical families of repunit and one-zero numbers by establishing a unified framework in which the parameter ($k \geq 0$) specifies the number of consecutive zeros separating two ones in the decimal representation. We introduce the new family of sequences, the generalized *One-kZero* numbers, and investigate some of their properties. The main purpose is to present a generalization for the recurrence relation of kind *One-Zero* numbers and determine some relations and properties. The reason that led us to this method is that the recurrence relation of *One-Zero* and *Repunit* numbers has a second-order difference equation as a specific case of the Horadam-type sequence. The Binet formula, generating function, sum formulas and many other relations will therefore be much easier to find. Also, some other identities that have not been found before in the particular case of *One-Zero* and *Repunit* sequences are also included in this study.

Keywords: generalized *One-kZero* sequence; generating functions; Tagiuri–Vajda’s identity; partial sum

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1. Introduction

Many numerical sequences can be defined, analyzed, and classified through linear recurrence relations of various orders. In this paper, we focus on sequences generated by second-order linear recurrence relations, examining their structure and fundamental properties. The most well-known among these sequences is the *Horadam sequence*. For a non-negative integer n , consider $\{h_n\}_{n \geq 0}$ defined by

$$h_n = ph_{n-1} + qh_{n-2}, \quad \text{for all } n \geq 2, \quad (1)$$

with initial conditions $h_0 = a$, $h_1 = b$, and where p and q are a fixed pair of integers. This type of sequence has been studied by Horadam [1–3], and generalizes many sequences defined by a recurrence relation associated with a characteristic equation of the form $x^2 - px - q = 0$. For more comprehensive results on the Horadam type sequence, see [1–7].

Here, we consider a new family for the *One-kZero* sequence denoted by $\{U_{(k,n)}\}_{n \geq 0}$. For all integers $k \geq 0$, the *One-kZero* sequence is called the generalized *One-Zero* sequence, or simply the *GOZ* sequence, which is constituted by natural numbers, which, within

the decimal system, are represented exclusively by the digits 1 (one) and 0 (zero) in an alternating configuration. For all integers $k \geq 0$, this alternation consists of a sequence of 1 followed by k digits of 0, starting and ending with 1, except for the initial elements $U_{(k,0)} = 0$ and $U_{(k,1)} = 1$. So, throughout this paper, k shall denote a non-negative integer.

Table 1 presents some GOZ sequences with $U_{(k,0)} = 0$ and $U_{(k,1)} = 1$ for some non-negative integer k . In the limit or degenerate case, where $k = 0$, we have the *Repunit* sequence.

Table 1. Some $U_{(k,n)}$ sequence.

k	Sequence	Name	OEIS [8]
0	0, 1, 11, 111, 1111, 11111, 111111, ...	<i>Repunit</i>	A002275
1	0, 1, 101, 10101, 1010101, 101010101, ...	<i>One-Zero</i>	A094028
2	0, 1, 1001, 1001001, 1001001001, ...	<i>One-2Zero</i>	A261544
3	0, 1, 10001, 100010001, 1000100010001, ...	<i>One-3Zero</i>	A330135
4	0, 1, 100001, 10000100001, ...	<i>One-4Zero</i>	
5	0, 1, 1000001, 1000001000001, ...	<i>One-5Zero</i>	
...	...	...	...
k	$0, 1, \dots, 10^{k+1}U_{(k,n)} + 1, \dots$	<i>One-kZero</i>	

In this paper, we study an extension of the *One-Zero* and *Repunit* sequences, and it is presented as a new perspective of these numbers as a particular case. For more details about *Repunit* numbers, see, for instance, the works [9–12]. We refer the reader to the works [13–15] for a more comprehensive discussion of the *One-Zero* numbers. The definition of the GOZ sequence is hereby presented, and it is identified as a Gersenne-type sequence (see the works [16,17]), as generalized Mersenne numbers (see, for instance, [7,18–21]).

This work consists of this section and three others. In summary, in Section 2, we analyze the recurrence relations for the GOZ sequence and determine the Binet Formula and some applications. In Section 3, some identities are found for these new sequences. The partial sum formulas are also expressed. Summation formulas for the GOZ numbers are finally presented in Section 4 and also the limit of the ratio of consecutive terms. We conclude with final remarks and suggest potential directions for future research on the subject.

2. Linear Recurrence and Binet’s Formula

In this section, we present the recurrence relations and formulate the Binet-type expression for the GOZ sequence, applicable to all non-negative integers n . The proposed framework is a comprehensive model for understanding and systematically calculating the terms in this sequence.

First, by looking at Table 1, it is possible to verify the following properties:

$$\begin{aligned}
 11\dots 1 &= 10^{n-1} + \dots + 10^2 + 10 + 1, \\
 1010\dots 101 &= 10^{2(n-1)} + \dots + 10^4 + 10^2 + 1, \\
 1001001\dots 1001 &= 10^{3(n-1)} + \dots + 10^6 + 10^3 + 1.
 \end{aligned}$$

In general, we have the following result:

Proposition 1. Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. Then, for all integers $n \geq 1$,

$$U_{(k,n)} = \sum_{j=1}^n 10^{(k+1)(j-1)} \quad (n \geq 1). \quad (2)$$

Proof. For Equation (2), we prove the assertion by induction on n . For $n = 1$, we have $U_{(k,1)} = 10^{(k+1)0} = 1$. Thus, the property is true for $n = 1$.

Assume that the property is true for all values less than or equal to m than n , that is, $U_{(k,m)} = \sum_{j=1}^m 10^{(k+1)(j-1)}$ for $m \leq n$. We need to prove that the property holds for $n + 1$. By definition of summation, we have

$$\sum_{j=1}^{n+1} 10^{(k+1)(j-1)} = \left(\sum_{j=1}^n 10^{(k+1)(j-1)} \right) + 10^{(k+1)n}$$

Using the inductive hypothesis, we have

$$\left(\sum_{j=1}^n 10^{(k+1)(j-1)} \right) + 10^{(k+1)n} = U_{(k,n)} + 10^{(k+1)n} = U_{(k,n+1)}.$$

Therefore, by the principle of mathematical induction, Equation (2) is true for all integers $n \geq 1$. $\square$

In mathematics, a recurrence relation is an equation in which each term of the sequence is defined as a function of the preceding terms. The elementary observation to be made at the outset is that the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ obeys the non-homogeneous linear recurrence:

$$U_{(k,n)} = 10^{k+1}U_{(k,n-1)} + 1, \quad (3)$$

with $U_{(k,0)} = 0$ and $n \geq 1$. An equivalent way to express Equation (3) is

$$U_{(k,n+1)} = 10^{k+1}U_{(k,n)} + 1. \quad (4)$$

By subtracting Equation (4) from Equation (3) we obtain a homogeneous second-order recurrence relation

$$U_{(k,n+1)} = (10^{k+1} + 1)U_{(k,n)} - 10^{k+1}U_{(k,n-1)}.$$

The next result gives us a homogeneous recurrence relation to the GOZ numbers, in accordance with the preceding argument.

Proposition 2. The GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ satisfies the homogeneous recurrence relation

$$U_{(k,n)} = (10^{k+1} + 1)U_{(k,n-1)} - 10^{k+1}U_{(k,n-2)} \quad (n \geq 2), \quad (5)$$

with initial values $U_{(k,0)} = 0$ and $U_{(k,1)} = 1$.

We may assume in Equation (1) that the values of $p = 10^{k+1} + 1$ and $q = -10^{k+1}$ then the GOZ sequence, given by Proposition 2, is a Gersenne-type sequence, according to [17]. So, it can be observed that the GOZ sequence, represented by the sequence $\{U_{(k,n)}\}_{n \geq 0}$ that satisfies the Horadam recursive recurrence, as defined by Equation (5).

The difference equation presented in (5), which defines the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$, possesses a Horadam-type characteristic equation of the form

$$x^2 - (10^{k+1} + 1)x + 10^{k+1} = 0 \quad (6)$$

and its real roots are $x_1 = 10^{k+1}$ and $x_2 = 1$. Equation (6) has distinct roots x_1 and x_2 , and then,

$$x_n = c_1(x_1)^n + c_2(x_2)^n, \quad (7)$$

is a solution of Equation (5), for $n \geq 0$, where $c_1(k)$ and $c_2(k)$ are real numbers. The following step involves the determination of the constants $c_1(k)$ and $c_2(k)$. With the initial conditions $U_{(k,0)} = 0$ and $U_{(k,1)} = 1$, we obtain the following linear system:

$$\begin{cases} 0 = c_1(k) + c_2(k) \\ 1 = 10^{k+1}c_1(k) + c_2(k) \end{cases}.$$

We find $c_1(k) = \frac{1}{10^{k+1}-1}$ and $c_2(k) = -\frac{1}{10^{k+1}-1}$. These values can be substituted into Equation (7) yields

$$U_{(k,n)} = \frac{10^{(k+1)n} - 1}{10^{k+1} - 1}, \text{ for all } n \geq 0.$$

The previous argument verifies our first main result, characterizing the n -th term of the GOZ sequence.

Theorem 1. *The GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ satisfies the identity*

$$U_{(k,n)} = \frac{10^{(k+1)n} - 1}{10^{k+1} - 1}. \quad (8)$$

Equation (8) expresses the standard Binet formula for the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$.

Since $U_{(k,0)} = 0$ and $U_{(k,1)} = 1$, in order to extend the GOZ sequence with negative subscripts, it is required that $n = 1$ be set in Equation (5), as follows:

$$\begin{aligned} U_{(k,1)} &= (10^{k+1} + 1)U_{(k,0)} - 10^{k+1}U_{(k,-1)} \\ 1 &= 0 - 10^{k+1}U_{(k,-1)} \\ U_{(k,-1)} &= -\frac{1}{10^{k+1}} \\ U_{(k,-1)} &= -\frac{U_{(k,1)}}{10^{k+1}}. \end{aligned}$$

In the same way, let us make $n = 0$,

$$\begin{aligned} 0 &= -\frac{(10^{k+1} + 1)U_{(k,1)}}{10^{k+1}} - 10^{k+1}U_{(k,-2)} \\ 10^{k+1}U_{(k,-2)} &= -\frac{(10^{k+1} + 1)U_{(k,1)}}{10^{k+1}} = -\frac{(10^{k+1} + 1)U_{(k,1)} - 10^{k+1}U_{(k,0)}}{10^{k+1}} \\ U_{(k,-2)} &= -\frac{U_{(k,2)}}{10^{2(k+1)}}. \end{aligned}$$

Let us define the negative index terms of the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$.

Definition 1. For any integer $n \geq 0$, the GOZ sequence for indexes with negative integers is defined as follows:

$$U_{(k,-n)} = -\frac{U_{(k,n)}}{10^{n(k+1)}}, \quad (9)$$

with $U_{(k,0)} = 0$ and $U_{(k,0)} = 1$.

This definition is valid for the following reason: for $n = -m$ in Equation (8), we get

$$U_{(k,-m)} = \frac{10^{-(k+1)m} - 1}{10^{k+1} - 1} = -\frac{10^{(k+1)m} - 1}{(10^{k+1} - 1) \cdot 10^{(k+1)m}},$$

which verifies the Binet formula above for the GOZ sequence $\{U_{(k,-n)}\}_{n \geq 0}$ with negative subscripts, which is a direct consequence of Equation (8). So, the Binet formula for the GOZ sequence with negative subscripts is given in the next result.

Proposition 3. For all integers $m \geq 0$, we have

$$U_{(k,-m)} = -\frac{10^{(k+1)m} - 1}{(10^{k+1} - 1)10^{(k+1)m}}, \quad (10)$$

where $\{U_{(k,m)}\}_{m \geq 0}$ is the GOZ sequence.

In accordance with the definition, a GOZ sequence with negative indexes is constituted by the set of elements given by

$$\{U_{(k,-n)}\}_{n \geq 1} = \left\{ -\frac{U_{(k,1)}}{10^{k+1}}, -\frac{U_{(k,2)}}{10^{2(k+1)}}, -\frac{U_{(k,3)}}{10^{3(k+1)}}, -\frac{U_{(k,4)}}{10^{4(k+1)}}, \dots \right\}.$$

The formal sum

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots \quad (11)$$

is referred to be the ordinary generating function for the sequence $\{a_0, a_1, a_2, \dots\}$.

Our next result provides a generating function for GOZ sequence.

Proposition 4. The ordinary generating function for the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$, denoted by $G_U(x)$, is

$$G_U(x) = \frac{x}{1 - (10^{k+1} + 1)x + 10^{k+1}x^2}.$$

Proof. Using (11), the generating function for the GOZ sequence is $G_U(x) = \sum_{n=0}^{\infty} U_{(k,n)} x^n$.

Then, using the equations $-(10^{k+1} + 1)xG_U$ and $10^{k+1}G_U$, we obtain

$$\begin{aligned} G_U(x) &= U_{(0,k)} + U_{(k,1)}x + U_{(k,2)}x^2 + \dots + U_{(k,n)}x^n + \dots \\ -(10^{k+1} + 1)xG_U(x) &= -(10^{k+1} + 1)U_{(0,k)}x - (10^{k+1} + 1)U_{(k,1)}x^2 \\ &\quad - (10^{k+1} + 1)U_{(k,2)}x^3 - \dots \\ 10^{k+1}x^2G_U(x) &= 10^{k+1}U_{(0,k)}x^2 + 10^{k+1}U_{(k,1)}x^3 + 10^{k+1}U_{(k,2)}x^4 + \dots \end{aligned}$$

Adding both sides of these equations gives us

$$\begin{aligned} & (1 - (10^{k+1} + 1)x + 10^{k+1}x^2)G_U(x) = \\ &= U_{(0,k)} + (U_{(k,1)} - (10^{k+1} + 1)U_{(0,k)})x \\ &+ (U_{(k,2)} - (10^{k+1} + 1)U_{(k,1)} + 10^{k+1}U_{(0,k)})x^2 + \\ &+ (U_{(k,3)} - (10^{k+1} + 1)U_{(k,2)} + 10^{k+1}U_{(k,1)})x^3 + \dots \\ &+ (U_{(k,n)} - (10^{k+1} + 1)U_{(k,n-1)} + 10^{k+1}U_{(k,n-2)})x^n \dots \end{aligned}$$

Making use of Equation (5), we conclude that

$$(1 - (10^{k+1} + 1)x + 10^{k+1}x^2)G_U(x) = U_{(0,k)} + (U_{(k,1)} - (10^{k+1} + 1)U_{(0,k)})x.$$

Since $U_{(k,0)} = 0$, $U_{(k,0)} = 1$ and $(1 - (10^{k+1} + 1)x + 10^{k+1}x^2) \neq 0$, we have the result. $\square$

The power series of the form

$$E_{a_n} = a_0 + a_1x + \frac{a_2x^2}{2!} + \dots + \frac{a_nx^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{a_nx^n}{n!},$$

is the exponential generating function of a sequence $\{a_n\}_{n \geq 0}$, denoted as $E_{a_n}(x)$.

We consider the Binet Equation (8), and we will express the exponential generating function for the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ in the next result.

Proposition 5. For all integers $n \geq 0$, the exponential generating function for the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ is

$$E_{U_{(k,n)}}(x) = \frac{1}{10^{k+1} - 1} (e^{10^{k+1}x} - e^x).$$

Proof. Let us assume that $\sum_{n=0}^{\infty} \frac{U_{(n,k)}t^n}{n!}$ is the exponential generating function for GOZ numbers. Using Equation (8), we obtain that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{U_{(n,k)}x^n}{n!} &= \sum_{n=0}^{\infty} \frac{10^{(k+1)n} - 1}{10^{k+1} - 1} \cdot \frac{x^n}{n!} \\ &= \frac{1}{10^{k+1} - 1} (e^{10^{k+1}x} - e^x), \end{aligned}$$

which verifies the result. $\square$

Consequently, the corresponding Poisson generating function is derived, where the Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n \geq 0}$ is given by

$$P_{a_n}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} e^{-x} = e^{-x} E_{a_n}(x).$$

Taking into account Proposition 5, we obtain the following result:

Corollary 1. For all $n \geq 0$, the Poisson generating function for the GOZ sequence $\{U_{(k,n)}\}_{n \geq 0}$ is

$$P_{U_{(k,n)}}(x) = \frac{1}{10^{k+1} - 1} (e^{10^{k+1}x} - 1).$$

3. Some Identities for the GOZ Sequence

In this section, we obtain some algebraic characteristics for these new sequences, including some classical identities, such as the Tagiuri–Vajda, Catalan, Cassini, and d’Ocagne identities.

In the next result, we show the addition and difference between two terms of the GOZ sequence, and this follows directly from Equation (8).

Proposition 6. Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. For all non-negative integers m and n with $m \geq n$, the following identities are verified:

$$(a) \quad U_{(k,m)} + U_{(k,n)} = 10^{(k+1)n} U_{(k,m-n)} + 2U_{(k,n)},$$

$$(b) \quad U_{(k,m)} - U_{(k,n)} = 10^{(k+1)n} U_{(k,m-n)}.$$

The following result shows a connection between the even order term and the general term:

Proposition 7. Let n be any non-negative integer. We have

$$U_{(k,2n)} - 2U_{(k,n)} = (10^{k+1} - 1)U_{(k,n)}^2,$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. A straightforward calculation with the use of Equation (8) shows that

$$\begin{aligned} U_{(k,2n)} &= \frac{10^{2(k+1)n} - 1}{10^{k+1} - 1} = \frac{(10^{(k+1)n} - 1)(10^{(k+1)n} + 1)}{10^{k+1} - 1} \\ &= (10^{k+1} - 1)U_{(k,n)}^2 + 2U_{(k,n)}, \end{aligned}$$

as required. $\square$

A new identity involving subtraction of the product of two terms of the GOZ sequence is presented in the next result.

Proposition 8. For all non-negative integers m, n , we get

$$U_{(k,m)}U_{(k,n+1)} - 10^{k+1}U_{(k,m-1)}U_{(k,n)} = U_{(k,m+n)}.$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. According to Binet’s Equation (8), we have

$$\begin{aligned} &U_{(k,m)}U_{(k,n+1)} - 10^{k+1}U_{(k,m-1)}U_{(k,n)} \\ &= \left(\frac{10^{(k+1)m} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)(n+1)} - 1}{10^{k+1} - 1} \right) - 10^{k+1} \left(\frac{10^{(k+1)(m-1)} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)n} - 1}{10^{k+1} - 1} \right) \\ &= \left(\frac{10^{(k+1)(m+n+1)} - 10^{(k+1)m} - 10^{(k+1)(n+1)} + 1}{(10^{k+1} - 1)^2} \right) \\ &\quad - 10^{k+1} \left(\frac{10^{(k+1)(n+m-1)} - 10^{(k+1)(m-1)} - 10^{(k+1)n} + 1}{(10^{k+1} - 1)^2} \right) \\ &= \frac{(10^{(k+1)} - 1)(10^{(k+1)(m+n)} - 1)}{(10^{k+1} - 1)^2} = U_{(k,m+n)}, \end{aligned}$$

which completes the proof. $\square$

Proposition 8 gives us the following interesting result:

Corollary 2. *Let all non-negative integers be n , and $\{U_{(k,n)}\}_{n \geq 0}$ the GOZ sequence. Then, the following identities hold:*

- (a) $U_{(k,2n-1)} = U_{(k,n)}^2 - 10^{k+1}U_{(k,n-1)}^2$;
- (b) $U_{(k,2n)} = 2U_{(k,n)}U_{(k,n-1)} - (10^{k+1} + 1)U_{(k,n)}^2$.

Proof. (a) Since $2n - 1 = n + (n - 1)$, following Proposition 8 we get

$$\begin{aligned} U_{(k,2n-1)} &= U_{(k,n)}U_{(k,n)} - 10^{k+1}U_{(k,n-1)}U_{(k,n-1)} \\ &= U_{(k,n)}^2 - 10^{k+1}U_{(k,n-1)}^2, \end{aligned}$$

as required.

(b) By Equation (5), we have

$$U_{(k,n+1)} - (10^{k+1} + 1)U_{(k,n)} = -10^{k+1}U_{(k,n-1)},$$

and now using Proposition 8, we have that

$$U_{(k,2n)} = U_{(k,n+n)} = U_{(k,n)}U_{(k,n+1)} - 10^{k+1}U_{(k,n-1)}U_{(k,n)}.$$

Hence

$$\begin{aligned} U_{(k,2n)} &= U_{(k,n)}U_{(k,n+1)} - 10^{k+1}U_{(k,n-1)}U_{(k,n)} \\ &= U_{(k,n)}U_{(k,n+1)} + (U_{(k,n+1)} - (10^{k+1} + 1)U_{(k,n)})U_{(k,n)} \\ &= 2U_{(k,n)}U_{(k,n+1)} - (10^{k+1} + 1)U_{(k,n)}^2, \end{aligned}$$

which guarantees the result. $\square$

The next result states a linear combination with the addition of products of two terms, and the unit 1.

Proposition 9. *Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. For all non-negative integers m, n , we get*

$$U_{(k,m)}U_{(k,n+1)} - 10^{2(k+1)}U_{(k,m-1)}U_{(k,n)} + 1 = U_{(k,m)} + U_{(k,n+1)}.$$

Proof. Note that

$$\begin{aligned} &U_{(k,m)}U_{(k,n+1)} - 10^{2(k+1)}U_{(k,m-1)}U_{(k,n)} \\ &= \left(\frac{10^{(k+1)m} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)(n+1)} - 1}{10^{k+1} - 1} \right) \\ &\quad - 10^{2(k+1)} \left(\frac{10^{(k+1)(m-1)} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)n} - 1}{10^{k+1} - 1} \right) \\ &= \frac{10^{(k+1)m} - 1}{10^{k+1} - 1} + \frac{10^{(k+1)(n+1)} - 1}{10^{k+1} - 1} - 1, \end{aligned}$$

and we obtain the result. $\square$

The quasi-convolution identity is shown below, that is, the convolution identity, except for one integer constant K .

Proposition 10. Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. For all non-negative integers m and n , we obtain

$$\begin{aligned} & K(U_{(k,m-1)}U_{(k,n)} + U_{(k,m)}U_{(k,n+1)}) \\ &= (10^4 + 1)10^{(k+1)(m+n-1)} - (10^2 + 1)10^{(k+1)}(10^{(m-1)} + 10^n) + 2, \end{aligned}$$

where $K = (10^{k+1} - 1)^2$.

The proof follows a similar approach to the previous results. We have omitted in the interests of brevity.

We now state our second main result, which characterizes the Tagiuri–Vajda identity for the GOZ sequence.

Theorem 2. (Tagiuri–Vajda’s identity) Let m, n, r be any natural numbers. We have

$$U_{(k,m+n)}U_{(k,m+r)} - U_{(k,m)}U_{(k,m+n+r)} = 10^{(k+1)m}U_{(k,n)}U_{(k,r)}, \quad (12)$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. Using Equation (8) again we obtain that

$$\begin{aligned} & U_{(k,m+n)}U_{(k,m+r)} - U_{(k,m)}U_{(k,m+n+r)} \\ &= \left(\frac{10^{(k+1)(m+n)} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)(m+r)} - 1}{10^{k+1} - 1} \right) \\ &\quad - \left(\frac{10^{(k+1)m} - 1}{10^{k+1} - 1} \right) \left(\frac{10^{(k+1)(m+n+r)} - 1}{10^{k+1} - 1} \right) \\ &= \frac{10^{(k+1)m}(10^{(k+1)r} - 1)(10^{(k+1)n} - 1)}{(10^{k+1} - 1)^2}, \end{aligned}$$

which is the end of the proof. $\square$

The following identities are obtained as a derivative of the Tagiuri–Vajda identity, as proven in Theorem 2. Their detailed justification will be presented in the subsequent results.

Proposition 11 (d’Ocagne’s identity). Let t, m be any natural numbers. For $t \geq m$ we have

$$U_{(k,m+1)}U_{(k,t)} - U_{(k,m)}U_{(k,t+1)} = 10^{(k+1)m}U_{(k,t-m)},$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. It suffices to consider $r = t - m$ and $n = 1$ in Equation (12), and we have the validity of the result. $\square$

Analogously to Proposition 11, we obtain the Catalan identity

Proposition 12 (Catalan’s identity). Let t, r be any natural number. For $t \geq r$, we have

$$(U_{(k,t)})^2 - U_{(k,t-r)}U_{(k,t+r)} = 10^{(k+1)(t-r)}(U_{(k,r)})^2, \quad (13)$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. Using $n = r$ and $m + r = t$ in Equation (12), we have the result. $\square$

It is evident that Proposition 12 gives rise to the following noteworthy result:

Corollary 3. *Let n, k be any natural numbers. For $n \geq 2$, we have*

$$(U_{(k,n)})^2 - U_{(k,n-2)}U_{(k,n+2)} = 10^{(k+1)(n-2)}(10^{k+1} + 1)^2. \quad (14)$$

Proof. In order to achieve the desired result, it is necessary to substitute $t = n$ and $r = 2$ into Equation (13). Given that $U_{(k,2)} = (10^{k+1} + 1)$, the result is as required. $\square$

We obtain the Cassini identity at the expense of Proposition 12.

Corollary 4 (Cassini's identity). *For all $n \geq 1$, we have*

$$(U_{(k,n)})^2 - U_{(k,n-1)}U_{(k,n+1)} = 10^{(k+1)(n-1)}, \quad (15)$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. Just make $t = n$ and $r = 1$ in Equation (13), with $U_{(k,1)} = 1$ for all integers $k \geq 0$. $\square$

By setting $n = 2m$ in Corollary 4, we can conclude that

Corollary 5. *For all $m \geq 1$, we have*

$$(U_{(k,2m)})^2 - U_{(k,2m-1)}U_{(k,2m+1)} = 10^{(k+1)(2m-1)},$$

where $\{U_{(k,m)}\}_{m \geq 0}$ is the GOZ sequence.

The next result gives the product of two terms with a subscript involving the sum and subtraction of two non-negative integers.

Proposition 13. *For all non-negative integers m and n , such that $n \geq m$, we get*

$$(10^{k+1} - 1)U_{(k,n-m)}U_{(k,n+m)} = U_{(k,2n)} - (U_{(k,n-m)} + U_{(k,n+m)}), \quad (16)$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is GOZ sequence.

The proof follows a similar procedure to the previous results. For the purpose of brevity, we have omitted it.

The Gelin–Cesàro identity for the GOZ sequence is now presented.

Proposition 14 (Gelin–Cesàro's identity). *Let n be any natural number. Then the identity is valid for all $n \geq 2$,*

$$\begin{aligned} & (U_{(k,n)})^4 - U_{(k,n-2)}U_{(k,n-1)}U_{(k,n+1)}U_{(k,n+2)} \\ &= (10^{(k+1)(n-2)}(10^{k+1} + 1)^2)U_{(k,n-1)}U_{(k,n+1)} \\ & \quad + (10^{(k+1)(n-1)})U_{(k,n-2)}U_{(k,n+2)} + (10^{(k+1)(n-1)})(10^{(k+1)(n-2)}(10^{k+1} + 1)^2), \end{aligned}$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. According to Equation (14), we obtain

$$(U_{(k,n)})^2 = U_{(k,n-2)}U_{(k,n+2)} + 10^{(k+1)(n-2)}(10^{k+1} + 1)^2. \quad (17)$$

By Equation (15)

$$(U_{(k,n)})^2 = U_{(k,n-1)}U_{(k,n+1)} + 10^{(k+1)(n-1)}. \quad (18)$$

By multiplying both sides of Equations (17) and (18), we get

$$\begin{aligned} (U_{(k,n)})^4 &= \\ &= U_{(k,n-2)}U_{(k,n-1)}U_{(k,n+1)}U_{(k,n+2)} + (10^{(k+1)(n-2)}(10^{k+1} + 1)^2)U_{(k,n-1)}U_{(k,n+1)} \\ &\quad + (10^{(k+1)(n-1)})U_{(k,n-2)}U_{(k,n+2)} + (10^{(k+1)(n-1)})(10^{(k+1)(n-2)}(10^{k+1} + 1)^2). \end{aligned}$$

This completes the proof. $\square$

The next result also follows directly from Proposition 13 and determines the product of four consecutive terms of the GOZ sequence.

Proposition 15. For non-negative integers n , let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. Then,

$$\begin{aligned} &(10^{k+1} - 1)^2 U_{(k,n-2)}U_{(k,n-1)}U_{(k,n+1)}U_{(k,n+2)} \\ &= (10^{(k+1)(n-2)+2} - 1)(10^{(k+1)(n-1)+2} - 1)U_{(k,n-1)}U_{(k,n-2)}. \end{aligned}$$

Proof. Making $m = 2$ in Equation (16), we obtain

$$(10^{k+1} - 1)U_{(k,n-2)}U_{(k,n+2)} = U_{(k,2n)} - (U_{(k,n-2)} + U_{(k,n+2)}).$$

By using items (a) and (b) of Proposition 6, we have

$$\begin{aligned} (10^{k+1} - 1)U_{(k,n-2)}U_{(k,n+2)} &= U_{(k,2n)} - \left(10^{(k+1)(n-2)}U_{(k,4)} + 2U_{(k,n-2)}\right) \\ &= (10^{(k+1)(n-2)+2} - 1)U_{(k,n-2)}, \end{aligned}$$

where

$$(10^{k+1} - 1)U_{(k,n-2)}U_{(k,n+2)} = (10^{(k+1)(n-2)+2} - 1)U_{(k,n-2)}. \quad (19)$$

In a similar way, if we make $m = 1$ in Equation (16), we get

$$(10^{k+1} - 1)U_{(k,n-1)}U_{(k,n+1)} = (10^{(k+1)(n-1)+2} - 1)U_{(k,n-1)}. \quad (20)$$

Again, multiplying both sides of Equations (19) and (20). After some necessary calculations, we obtained the result. $\square$

The following result displays a difference between two products; this result will be applied below.

Proposition 16. Let m be any natural number. We have

$$\begin{aligned} &(10^{k+1} - 1)(U_{(k,m+3)}U_{(k,m+4)} - U_{(k,m+1)}U_{(k,m+6)}) \\ &= (10^{(k+1)2} + 1)10^{(k+1)(m+1)}U_{(k,3)}, \end{aligned}$$

where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

Proof. By Equation (8) we have,

$$\begin{aligned} &(10^{k+1} - 1)^2 U_{(k,m+3)}U_{(k,m+4)} \\ &= 10^{(k+1)(2m+7)} - 10^{(k+1)(m+3)} - 10^{(k+1)(m+4)} + 1, \end{aligned}$$

and

$$\begin{aligned} & (10^{k+1} - 1)^2 U_{(k,m+1)} U_{(k,m+6)} \\ &= 10^{(k+1)(2m+7)} - 10^{(k+1)(m+6)} - 10^{(k+1)(m+1)} + 1. \end{aligned}$$

This allows us to obtain

$$\begin{aligned} & (10^{k+1} - 1)^2 (U_{(k,m+3)} U_{(k,m+4)} - U_{(k,m+1)} U_{(k,m+6)}) = \\ &= 10^{(k+1)(m+6)} - 10^{(k+1)(m+3)} + 10^{(k+1)(m+4)} - 10^{(k+1)(m+1)} \\ &= (10^{(k+1)3} - 1)(10^{(k+1)2} + 1)10^{(k+1)(m+1)}, \end{aligned}$$

and the proof is now complete. $\square$

Finally, we reveal a new result that examines particular combinations of terms within the GOZ sequence. This result has remarkable parallels with the Tagiuri–Vajda, Catalan, Cassini, and d’Ocagne identities mentioned previously.

Proposition 17. *Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence. For all natural numbers m , the following equality holds:*

$$\begin{aligned} & (10^{k+1} - 1) [(U_{(k,m+3)})^3 - U_{(k,m+1)} U_{(k,m+2)} U_{(k,m+6)}] \\ &= 10^{(k+1)(m+1)} \left(10^{(k+1)} (10^{k+1} - 1) + (10^{(k+1)2} + 1) U_{(k,3)} U_{(k,m+2)} \right), \end{aligned}$$

Proof. By Corollary 4, we have that

$$(U_{(k,m+3)})^2 - U_{(k,m+2)} U_{(k,m+4)} = 10^{(k+1)(m+2)}.$$

So $(U_{(k,m+3)})^3 - U_{(k,m+2)} U_{(k,m+3)} U_{(k,m+4)} = 10^{(k+1)(m+2)} U_{(k,m+3)}$. By using Proposition 16, we get

$$\begin{aligned} & (10^{k+1} - 1) (10^{(k+1)(m+2)}) \\ &= (10^{k+1} - 1) [(U_{(k,m+3)})^3 - U_{(k,m+2)} U_{(k,m+3)} U_{(k,m+4)}] \\ &= (10^{k+1} - 1) (U_{(k,m+3)})^3 - U_{(k,m+2)} \left((10^{k+1} - 1) U_{(k,m+1)} U_{(k,m+6)} \right. \\ & \quad \left. + (10^{(k+1)2} + 1) 10^{(k+1)(m+1)} U_{(k,3)} \right), \end{aligned}$$

or equivalently

$$\begin{aligned} & (10^{k+1} - 1) [(U_{(k,m+3)})^3 - U_{(k,m+1)} U_{(k,m+2)} U_{(k,m+6)}] \\ &= 10^{(k+1)(m+1)} \left(10^{(k+1)} (10^{k+1} - 1) + (10^{(k+1)2} + 1) U_{(k,3)} U_{(k,m+2)} \right), \end{aligned}$$

this completes the proof. $\square$

In conclusion to this section, the following result highlights an additional interesting application of Cassini’s identity. Specifically, using Equation (5), we will show a solution to a Diophantine equation.

Proposition 18. *The Diophantine equation $x^2 - (10^{k+1} + 1)xy + 10^{k+1}y^2 = 10^{(k+1)(m-1)}$ has infinitely many solutions for any positive integer m .*

Proof. By combining Equation (15) with Equation (5), we obtain the following equation:

$$(U_{(k,m)})^2 - (10^{k+1} + 1)U_{(k,m-1)}U_{(k,m)} + 10^{k+1}(U_{(k,m-1)})^2 = 10^{(k+1)(m-1)}.$$

Setting $U_{(k,m)} = x$ and $U_{(k,m-1)} = y$ yields the desired result, where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence. $\square$

4. Some Partial Sum and Ratio

Now we give the sum of the first $n + 1$ terms of the GOZ sequence and the limit of the ratio between consecutive terms of the GOZ sequence.

Initially, consider the sequence of partial sums of the first $n + 1$ terms given by $\sum_{j=0}^n U_{(k,j)} = U_{(k,0)} + U_{(k,1)} + U_{(k,2)} + \cdots + U_{(k,n)}$, for $n \geq 0$, where $\{U_{(k,n)}\}_{n \geq 0}$ is the GOZ sequence.

We will describe two results involving the partial sums of the terms of the GOZ sequence.

Proposition 19. Let $\{U_{(k,n)}\}_{n \geq 0}$ be the GOZ sequence and n be a non-negative integer. Then,

$$\begin{aligned} (a) \quad \sum_{j=0}^n U_{(k,j)} &= \frac{U_{(k,n+1)} - (n+1)}{10^{k+1} - 1}, \\ (b) \quad \sum_{j=0}^n U_{(k,2j)} &= \frac{U_{(k,n+1)}(10^{(k+1)(n+1)} + 1) - (10^{k+1} + 1)(n+1)}{10^{2(k+1)} - 1}, \\ (c) \quad \sum_{j=0}^n U_{(k,2j+1)} &= \frac{10^{k+1}U_{(k,n+1)}(10^{(k+1)(n+1)} + 1) - (10^{(k+1)} + 1)(n+1)}{10^{2(k+1)} - 1}. \end{aligned}$$

Proof. (a) With Equation (8) and using the sum of a geometric series, we have

$$\begin{aligned} \sum_{j=0}^n U_{(k,j)} &= U_{(k,0)} + U_{(k,1)} + U_{(k,2)} + \cdots + U_{(k,n)} \\ &= \frac{10^0 - 1}{10^{k+1} - 1} + \frac{10^{k+1} - 1}{10^{k+1} - 1} + \frac{10^{2k} - 1}{10^{k+1} - 1} + \cdots + \frac{10^{kn} - 1}{10^{k+1} - 1} \\ &= \frac{\left(\frac{10^{(k+1)(n+1)} - 1}{10^{k+1} - 1}\right) - (n+1)}{10^{k+1} - 1} = \frac{U_{(k,n+1)} - (n+1)}{10^{k+1} - 1}, \end{aligned}$$

as required.

(b) Again using Equation (8) we obtain

$$\begin{aligned} \sum_{j=0}^n U_{(k,j)} &= U_{(0,k)} + U_{(k,2)} + U_{(k,4)} + \cdots + U_{(k,2n)} \\ &= \frac{10^0 - 1}{10^{k+1} - 1} + \frac{10^{(k+1)2} - 1}{10^{k+1} - 1} + \frac{10^{(k+1)4} - 1}{10^{k+1} - 1} + \cdots + \frac{10^{(k+1)2n} - 1}{10^{k+1} - 1} \\ &= \frac{1 + 10^{(k+1)2} + \cdots + 10^{(k+1)2n} - (1 + 1 \cdots + 1)}{10^{k+1} - 1} \\ &= \frac{U_{(k,n+1)}(10^{(k+1)(n+1)} + 1) - (10^{k+1} + 1)(n+1)}{(10^{k+1} + 1)(10^{k+1} - 1)}, \end{aligned}$$

as required.

(c) Similarly, we have

$$\begin{aligned}\sum_{j=0}^n U_{(k,2j+1)} &= U_{(k,1)} + U_{(k,3)} + \cdots + U_{(k,2n+1)} \\ &= \frac{10^{(k+1)} - 1}{10^{k+1} - 1} + \frac{10^{(k+1)3} - 1}{10^{k+1} - 1} + \cdots + \frac{10^{(k+1)(2n+1)} - 1}{10^{k+1} - 1} \\ &= \frac{10^{k+1} U_{(k,n+1)} (10^{(k+1)(n+1)} + 1) - (10^{(k+1)} + 1)(n+1)}{(10^{(k+1)} + 1)(10^{k+1} - 1)}\end{aligned}$$

and this completes the proof. $\square$

Once more, using the Binet formula we obtain another property of the GOZ sequences $\{U_{(k,n)}\}_{n \geq 0}$, which is stated in the following proposition:

Proposition 20. *If $U_{(k,n)}$ are the n -th term of the GOZ sequence, then*

$$\lim_{n \rightarrow \infty} \frac{U_{(k,n+1)}}{U_{(k,n)}} = 10^{k+1}, \quad (21)$$

and

$$\lim_{n \rightarrow \infty} \frac{U_{(k,-n-1)}}{U_{(k,-n)}} = 1. \quad (22)$$

Proof. By using Equation (8) we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{U_{(k,n+1)}}{U_{(k,n)}} &= \lim_{n \rightarrow \infty} \frac{10^{(k+1)(n+1)} - 1}{10^{k+1} - 1} \cdot \frac{10^{k+1} - 1}{10^{(k+1)n} - 1} \\ &= \lim_{n \rightarrow \infty} \frac{10^{k+1} - \frac{1}{10^{(k+1)n}}}{1 - \frac{1}{10^{(k+1)n}}} = 10^{k+1}.\end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{1}{10^n} = 0$, and with Equation (10) we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{U_{(k,-n-1)}}{U_{(k,-n)}} &= \lim_{n \rightarrow \infty} \frac{10^{(k+1)(n+1)} - 1}{(10^{k+1} - 1)10^{(k+1)(n+1)}} \cdot \frac{(10^{k+1} - 1)10^{(k+1)n}}{10^{(k+1)n} - 1} \\ &= \lim_{n \rightarrow \infty} \frac{10^{(k+1)(n+1)} - 1}{10^{k+1}(10^{(k+1)n} - 1)} = 1,\end{aligned}$$

as required. $\square$

In what follows, we can show the next result using the same basic calculus tools considered to determine the limits of Equations (21) and (22).

Corollary 6. *If $U_{(k,n)}$ is the n -th term of the GOZ sequence, then*

$$\lim_{n \rightarrow \infty} \frac{U_{(k,n)}}{U_{(k,n+1)}} = \frac{1}{10^{k+1}},$$

and

$$\lim_{n \rightarrow \infty} \frac{U_{(k,-n)}}{U_{(k,-n-1)}} = 1.$$

5. Final Considerations

The aim of this work is to present a unified structure for a generalized class of one-zero numbers, together with the repunit numbers. The proposed GOZ sequence represents an original and meaningful extension of classical numerical constructs such as repunits and one-zero sequences as a specific case of Mersenne-type numbers. The manuscript offers a novel and mathematically rigorous contribution to the theory of integer sequences and recurrence relations. Our objective was to establish several identities for this general sequence, particularly classical ones such as the Tagiuri–Vajda, D’Ocagne, Catalan, and Cassini identities. Again, this work discusses the properties of the family of generalized GOZ sequences with integer indexes. This study not only unifies prior findings but also broadens the theoretical framework of GOZ. Hence, our purpose is to study the GOZ numbers and provide new results, as well as extensions of some existing results in the literature. By emphasizing these results, we hope to stimulate further research on this class of numbers, paving the way for future research and applications across various areas of mathematics. Certain aspects of our investigations appear to be pioneering, indicating that although the results are derived through elementary methods, they may provide original insights that enrich the understanding of this domain. In future work, we intend to extend and generalize this sequence to broader settings, including complex numbers, quaternions, octonions, and hybrid versions.

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