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Sharp Unified Smoothness Theory for Cavalieri Estimation via Fourier Decay

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Abstract

Cavalieri estimation is a widely used technique in stereology (applied geometric sampling) for approximating the volume of a solid by sampling cross-sectional areas along a fixed axis. Classical theory shows that, under systematic equidistant sampling (the well-known Cavalieri estimator), the variance decay depends on the smoothness of the area function, which is essentially measured by the number of continuous derivatives. This paper focuses on the natural assumptions under which the theory holds. We first obtain sharp and explicit variance rates: when the Fourier decay is of order $s > 1/2$, the variance of the Cavalieri estimator decays as t^{2s} with a constant independent of t . Building on this, we show that the smoothness condition expressed in terms of the algebraic Fourier decay subsumes both integer- and fractional-order frameworks used to date. Finally, we establish a matching converse showing that, under general assumptions, no broader smoothness framework extends the theory; that is, any algebraic variance decay implies the corresponding Fourier decay.

Keywords: Fourier decay; Cavalieri estimator; systematic sampling; numerical quadrature; computational stereology

MSC: 42A38; 65D30; 41A25; 42A16; 65T50



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1. Introduction

Randomized quadrature formulas for numerical integration are widely used in stereology (applied geometric sampling) to estimate parameters—e.g., the volume of spatial structures. According to Cavalieri’s principle, the volume of a solid $Y \subseteq \mathbb{R}^3$ can be obtained by integrating the areas of its two-dimensional sections along a fixed axis perpendicular to the cuts. Consequently, the classical approach consists of intersecting Y with equidistant orthogonal planes at a distance of $t > 0$ along a fixed axis and measuring the resulting areas (the measurement function). The Cavalieri principle underpins routine volume estimation in stereology across several domains. For example, in neuroscience and pathology, it is used to estimate regional brain volumes (e.g., gray matter). Systematic sampling with a random start is widely adopted in these settings because it is simple to implement and operationally robust. Practitioners can control precision through the section spacing, and costs scale linearly with the number of sections; see, e.g., ref. [1] for further details and additional applications.

Formally, the measurement function is modeled by an integrable real-valued function $f(x)$ with compact support, representing the area of the intersection of Y with the plane

positioned at $x \in \mathbb{R}$ along the axis. Assuming, without loss of generality, that the support of f is $[0, 1]$, the volume of interest is

$$V = \int_0^1 f(x) dx.$$

The (classical) Cavalieri estimator $\hat{V}$ is based on equidistant sampling of f :

$$\hat{V} = t \sum_{x \in X} f(x), \quad (1)$$

where

$$X = t(U + \mathbb{Z}),$$

with $0 < t \leq 1$ representing the sampling step and U denoting a random shift uniformly distributed on $[0, 1]$; see, for example, ([2], Chapter 7).

The Cavalieri estimator in Equation (1) is unbiased, and its variance depends on the smoothness of the measurement function f . Classical stereology uses the notion of $(m, 1)$ piecewise smooth functions: f is $(m - 1)$ times continuously differentiable, and the m -th and $(m + 1)$ -th derivatives exist and are continuous except at finitely many points where finite jumps may occur (an integer-order smoothness notion) [3,4]. Motivated by practical examples from synthetic and real data, this framework was extended towards fractional smoothness based on continuous orders [5,6]. Subsequently, in a series of works treating Newton–Cotes estimators as improvements of the Cavalieri estimator under non-equidistant sampling, weak $(m, 1)$ piecewise smoothness was introduced—allowing $f^{(m+1)}$ to have finitely many (possibly infinite) jumps [7]—and the integer notion was further relaxed within a Sobolev-type function space [8]. However, these two lines of development remain somewhat disjoint: neither the continuous extensions nor the integer relaxations align seamlessly with each other, which hampers a unified theoretical understanding. This motivates searching for smoothness assumptions that both include the above and cannot be improved within the scope where the Cavalieri theory applies.

Therefore, existing smoothness frameworks for Cavalieri estimation have advanced along two largely separate lines. Integer-order piecewise smoothness yields clean rates but can be overly restrictive; fractional extensions capture relevant edge singularities yet do not align transparently with integer relaxations; and neither direction has offered a general necessary condition linking smoothness to variance decay. The key contribution of this paper is to bridge this gap by providing a unified Fourier-decay notion of smoothness that subsumes both lines and yields a necessary condition.

Our idea is to characterize the smoothness of f through the algebraic decay of its Fourier transform, as is standard in other areas of numerical integration such as quasi-Monte Carlo; see, e.g., ([9], Chapter 4). This Fourier-analytic viewpoint aligns with classical Fourier analyses of uniform sampling (via Poisson summation and trapezoidal-rule arguments) [10]. It is also consistent with lattice/digital-net quadrature, where error is governed by the decay of Fourier/Walsh coefficients [11–15].

We first show that the classical theory carries through: if the Fourier decay is of order $s > 1/2$, then the variance of the Cavalieri estimator in Equation (1) decays at a rate of t^{2s} with a constant independent of t . Consequently,

$$\text{var}(\hat{V}) \leq C t^{2s}. \quad (2)$$

We then prove that this Fourier-decay-based notion of smoothness is not only natural as a framework for continuous smoothness but also encompasses all previous notions, thereby providing the largest function space in which the theory holds, at least for the

classical Cavalieri estimator. Furthermore, we establish that, in a broad sense, this Fourier-decay characterization cannot be improved (via a matching converse), which underscores the appropriateness of formulating Cavalieri estimation in these terms.

We note that the matching converse (inverse problem), to our knowledge, is formulated here for the first time in the Cavalieri setting. It has important theoretical implications: First, it rules out broader, ad hoc smoothness notions from yielding faster variance decay for the classical estimator given in Equation (1). Second, it prevents underestimating smoothness: whenever algebraic variance decay is observed, the converse implies the corresponding Fourier decay.

The outline is as follows: In Section 2 we introduce the Fourier-decay function class and relate it to the variance of $\hat{V}$. In Section 3 we review the literature on fractional and relaxed integer smoothness and prove that our notion encompasses these concepts, making it the weakest assumption under which the Cavalieri volume estimation theory works. Section 4 presents an inverse result showing that algebraic Fourier decay cannot be improved in this setting. Finally, Section 5 summarizes our contributions and provides final comments.

2. Fourier-Decay-Type Function Space

We work with the following notation (see, Table 1).

Table 1. Notation used in Section 2 (Fourier-decay framework).

Symbol	Meaning
$V = \int_0^1 f(x) dx$	Target volume.
$t \in (0, 1]$	Section spacing.
$U \sim \text{Unif}[0, 1]$	Random start for systematic sampling.
$X = t(U + \mathbb{Z})$	Sampling grid (section locations).
$\hat{V} = t \sum_{x \in X} f(x)$	Cavalieri estimator (Equation (1)).
$\mathbb{E}[\cdot], \text{var}(\cdot)$	Expectation and variance.
$L^1_{[0,1]}(\mathbb{R})$	Integrable functions vanishing a.e. outside $[0, 1]$.
$\hat{f}(\xi)$	Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$.
$\mathcal{E}^s_{[0,1]}(\mathbb{R})$	Fourier-decay class: $ \hat{f}(\xi) \leq C(1 + \xi ^2)^{-s/2}$, $s > 1/2$.
$f_T(x) = f(x \bmod T)$	T -periodic extension.
$g_T(z)$	T -periodic covariogram: $g_T(z) = \int_0^T f_T(x) f_T(x+z) dx$.
$\hat{f}_T(k)$	k -th Fourier coefficient: $\hat{f}_T(k) = \frac{1}{T} \int_0^T f_T(x) e^{-2\pi i k x/T} dx$.
$\check{f}_T(x) = f_T(-x)$	Reflection.

Let $L^1_{[0,1]}(\mathbb{R})$ denote the space of integrable functions on $\mathbb{R}$ that vanish outside $[0, 1]$ up to sets of measure zero. Let $s > 1/2$ be a smoothness parameter. We consider the Fourier-decay-type function space [11,12,16] as

$$\mathcal{E}^s_{[0,1]}(\mathbb{R}) = \left\{ f \in L^1_{[0,1]}(\mathbb{R}) : \exists C > 0 \text{ such that } |\hat{f}(\xi)| \leq \frac{C}{(1 + |\xi|^2)^{s/2}} \quad \forall \xi \in \mathbb{R} \right\} \quad (3)$$

where $\hat{f}(\xi)$ denotes the Fourier transform of f at $\xi \in \mathbb{R}$, defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx.$$

Let $f \in L^1_{[0,1]}(\mathbb{R})$ and some real number $T \geq 1$. The T -periodic extension of f is defined by

$$f_T(x) = f(x \bmod T), \quad x \in \mathbb{R}.$$

The T -periodic covariogram of f , denoted by g_T , is defined as the following integral function:

$$g_T(z) = \int_0^T f_T(x) f_T(x+z) dx, \quad z \in \mathbb{R}.$$

We summarize some of its basic properties in the following lemma (cf. ([17], Lemma 3.2) for related arguments). We abuse the notation by writing the Fourier coefficients of f_T and any T -periodic function at $k \in \mathbb{Z}$ as

$$\hat{f}_T(k) = \frac{1}{T} \int_0^T f_T(x) e^{-\frac{2\pi i}{T} kx} dx.$$

Lemma 1. Let $f \in L^1_{[0,1]}(\mathbb{R})$, and let g_T be its T -periodic covariogram; then

- (i) $g_T = f_T * \check{f}_T$, where $\check{f}_T(x) = f_T(-x)$ and $*$ denotes the convolution.
- (ii) $\hat{g}_T = T |\hat{f}_T|^2$.
- (iii) $\int_0^T g_T(z) dz = V^2$.
- (iv) If, in addition, $f \in \varepsilon^s_{[0,1]}(\mathbb{R})$, then the Fourier series of g_T converges absolutely on $[0, T]$.

Proof. (i) By definition,

$$g_T(z) = \int_0^T f_T(x) f_T(x+z) dx = \int_0^T f_T(x) \check{f}_T(z-x) dx = (f_T * \check{f}_T)(z),$$

which is a periodic convolution on the circle of length T .

(ii) The periodic convolution theorem gives

$$\hat{g}_T(k) = \widehat{f_T * \check{f}_T}(k) = T \hat{f}_T(k) \widehat{\check{f}_T}(k).$$

Since $\check{f}_T(x) = f_T(-x)$, we have $\widehat{\check{f}_T}(k) = \overline{\hat{f}_T(k)}$; hence $\hat{g}_T(k) = T |\hat{f}_T(k)|^2$.

(iii) By Fubini's theorem and T -periodicity,

$$\begin{aligned} \int_0^T g_T(z) dz &= \int_0^T \int_0^T f_T(x) f_T(x+z) dx dz = \int_0^T f_T(x) \left(\int_0^T f_T(x+z) dz \right) dx \\ &= \left(\int_0^T f_T(u) du \right) \left(\int_0^T f_T(x) dx \right) = \left(\int_0^T f_T \right)^2 = V^2. \end{aligned}$$

This is because on $[0, T]$, we have $f_T = f$ on $[0, 1]$ and $f_T = 0$ a.e. on $(1, T]$, so $\int_0^T f_T = \int_0^1 f = V$.

(iv) Since $f \in \varepsilon^s_{[0,1]}(\mathbb{R})$ with $s > 1/2$, there exists $C > 0$ such that

$$|\hat{f}(\xi)| \leq \frac{C}{(1 + |\xi|^2)^{s/2}} \quad \text{for all } \xi \in \mathbb{R},$$

where $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$. For the T -periodic extension, one has

$$\hat{f}_T(k) = \frac{1}{T} \int_0^T f_T(x) e^{-\frac{2\pi i}{T} kx} dx = \frac{1}{T} \int_0^1 f(x) e^{-\frac{2\pi i}{T} kx} dx = \frac{1}{T} \hat{f}\left(\frac{k}{T}\right).$$

Hence, for all $k \in \mathbb{Z}$,

$$|\hat{f}_T(k)| \leq \frac{1}{T} \frac{C}{(1 + (k/T)^2)^{s/2}}.$$

According to (ii),

$$|\hat{g}_T(k)| = T |\hat{f}_T(k)|^2 \leq T \left(\frac{C}{T} \right)^2 \frac{1}{(1 + (k/T)^2)^s} = \frac{C^2}{T} \frac{1}{(1 + (k/T)^2)^s}.$$

Since $s > 1/2$, the series $\sum_{k \in \mathbb{Z}} (1 + (k/T)^2)^{-s}$ converges; therefore

$$\sum_{k \in \mathbb{Z}} |\hat{g}_T(k)| \leq \frac{C^2}{T} \sum_{k \in \mathbb{Z}} \frac{1}{(1 + (k/T)^2)^s} < \infty.$$

Thus $\{\hat{g}_T(k)\}_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, and the Fourier series $g_T(z) = \sum_{k \in \mathbb{Z}} \hat{g}_T(k) e^{2\pi i k z/T}$ converges absolutely on $[0, T]$. $\square$

Remark 1. The threshold $s > 1/2$ ensures absolute convergence of the Fourier series of g_T (Lemma 1(iv)) and, equivalently, that $\sum_{k \geq 1} k^{-2s} < \infty$. This justifies the Fourier-series argument and the exchange of sums in Proposition 1, yielding the finite bound in Equation (2). For $s \leq 1/2$, this control may fail.

The result below extends a well-known formula to the setting $f \in \varepsilon_{[0,1]}^s(\mathbb{R})$; see, for instance, ([5], Equation (41)) and references therein.

Proposition 1. Let $f \in \varepsilon_{[0,1]}^s(\mathbb{R})$, and let $\hat{V}$ be defined as in Equation (1). Let $N = \lceil \frac{1}{t} \rceil$, and set $T = Nt$. Then

$$\text{var}(\hat{V}) = 2T \sum_{k=1}^{\infty} \hat{g}_T(kN) = 2T^2 \sum_{k=1}^{\infty} |\hat{f}_T(kN)|^2.$$

Proof. With T as in the statement, we have $t = T/N$, and Equation (1) can be rewritten as

$$\hat{V} = \frac{T}{N} \sum_{i=0}^{N-1} f_T\left(\frac{iT}{N} + U\right),$$

where U is uniform on $[0, T]$. Using Lemma 1(iii),

$$\text{var}(\hat{V}) = \mathbb{E}[\hat{V}^2] - \int_0^T g_T(z) dz.$$

For the first term,

$$\begin{aligned} \mathbb{E}[\hat{V}^2] &= \frac{1}{T} \left(\frac{T}{N} \right)^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_T\left(\frac{(i-j)T}{N}\right) \\ &= \frac{T}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_T\left(\frac{(i-j)T}{N}\right). \end{aligned}$$

Also, by T -periodicity, the summand only depends on $i - j \bmod N$. For each residue $r \in \{0, \dots, N-1\}$ there are exactly N pairs of (i, j) , with $i - j \equiv r \pmod{N}$; hence

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_T\left(\frac{(i-j)T}{N}\right) = N \sum_{r=0}^{N-1} g_T\left(\frac{rT}{N}\right).$$

Therefore

$$\mathbb{E}[\hat{V}^2] = \frac{T}{N} \sum_{r=0}^{N-1} g_T\left(\frac{rT}{N}\right).$$

According to Lemma 1(iv), expand g_T in the Fourier series to obtain

$$\begin{aligned}\text{var}(\hat{V}) &= T \left(\frac{1}{N} \sum_{r=0}^{N-1} g_T \left(\frac{rT}{N} \right) - \frac{1}{T} \int_0^T g_T(z) dz \right) \\ &= T \left(\frac{1}{N} \sum_{r=0}^{N-1} \sum_{k \in \mathbb{Z}} \hat{g}_T(k) e^{2\pi i k r / N} - \hat{g}_T(0) \right) \\ &= T \left(\sum_{k \in \mathbb{Z}} \hat{g}_T(k) \frac{1}{N} \sum_{r=0}^{N-1} e^{2\pi i k r / N} - \hat{g}_T(0) \right).\end{aligned}$$

Since $\frac{1}{N} \sum_{r=0}^{N-1} e^{2\pi i k r / N} = \mathbf{1}_{\{N|k\}}$, the $k = 0$ term cancels, and we obtain

$$\text{var}(\hat{V}) = T \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{g}_T(kN) = 2T \sum_{k=1}^{\infty} \hat{g}_T(kN).$$

Finally, Lemma 1(ii) yields $\hat{g}_T(kN) = T |\hat{f}_T(kN)|^2$; hence

$$\text{var}(\hat{V}) = 2T^2 \sum_{k=1}^{\infty} |\hat{f}_T(kN)|^2$$

as we claimed. $\square$

Corollary 1. Let $f \in \varepsilon_{[0,1]}^s(\mathbb{R})$ and $0 < t \leq 1$. Then, the Cavalieri estimator $\hat{V}$ exhibits the variance bound given in Equation (2).

Proof. By periodicity,

$$\hat{f}_T(k) = \frac{1}{T} \hat{f} \left(\frac{k}{T} \right), \quad k \in \mathbb{Z}.$$

From Proposition 1, we have

$$\text{var}(\hat{V}) = 2T^2 \sum_{k=1}^{\infty} |\hat{f}_T(kN)|^2 = 2T^2 \sum_{k=1}^{\infty} \left| \frac{1}{T} \hat{f} \left(\frac{kN}{T} \right) \right|^2 = 2 \sum_{k=1}^{\infty} \left| \hat{f} \left(\frac{k}{t} \right) \right|^2,$$

since $T = Nt$ implies $kN/T = k/t$. Using the decay assumption $|\hat{f}(\xi)| \leq C(1 + |\xi|^2)^{-s/2}$,

$$\text{var}(\hat{V}) \leq 2C^2 \sum_{k=1}^{\infty} (1 + (k/t)^2)^{-s} = 2C^2 t^{2s} \sum_{k=1}^{\infty} (t^2 + k^2)^{-s} \leq 2C^2 t^{2s} \sum_{k=1}^{\infty} k^{-2s}.$$

Since $s > 1/2$, the series $\sum_{k=1}^{\infty} k^{-2s}$ converges, and we obtain

$$\text{var}(\hat{V}) \leq C' t^{2s},$$

with $C' = 2C^2 \sum_{k=1}^{\infty} k^{-2s} < \infty$. This proves the claim. $\square$

3. Relation with Previous Smoothness Assumptions

We recall the largest known function space in which the theory of Cavalieri estimation for integer-order smoothness has been developed. This space (originally defined as $W_c^{m,\text{BV}}(\mathbb{R})$ with compactly supported functions, we may, after affine rescaling, restrict the support to $[0, 1]$ without loss of generality), denoted as $W_{[0,1]}^{s,\text{BV}}(\mathbb{R})$, consists of all functions $f \in L_{[0,1]}^1(\mathbb{R})$ that admit m weak derivatives such that the m -th derivative $f^{(m)}$ is of bounded variation, with $f', f'', \dots, f^{(m)} \in L_{[0,1]}^1(\mathbb{R})$ [8].

Proposition 2. For all non-negative integers m , the following inclusion holds:

$$W_{[0,1]}^{m,BV}(\mathbb{R}) \subset \varepsilon_{[0,1]}^{m+1}(\mathbb{R}).$$

Proof. Fix $\xi = 1/t$ with $0 < t \leq 1$ (for $|\xi| < 1$, the bound follows from the L^1 norm of f). Set

$$T = \left\lceil \frac{1}{t} \right\rceil \cdot t.$$

Note that $\hat{f}(\xi) = T\hat{f}_T(k)$. By assumption, f and thus f_T admit m weak derivatives in $L^1_{[0,1]}(\mathbb{R})$. For $k = \lceil \frac{1}{t} \rceil$ we have

$$|\hat{f}(\xi)| = T \left| \frac{1}{(2\pi i \xi)^m} \widehat{f_T^{(m)}}(k) \right| \leq \frac{T}{(2\pi)^m \xi^m} |\widehat{f_T^{(m)}}(k)| \leq \left(1 + \frac{1}{\xi}\right) \frac{1}{(2\pi)^m \xi^m} |\widehat{f_T^{(m)}}(k)|.$$

Additionally, since $f_T^{(m)}$ is of bounded variation, it is known that its Fourier coefficients decay as follows ([18], Theorem 4.12):

$$|\widehat{f_T^{(m)}}(k)| \leq \frac{\text{TV}_{[0,T]}(f_T^{(m)})}{2\pi k} \leq \frac{\text{TV}_{[0,T]}(f_T^{(m)})}{2\pi \xi},$$

where $\text{TV}_{[0,T]}(f_T^{(m)})$ denotes the total variation of $f_T^{(m)}$ on $[0, T]$. Thus, we have

$$|\hat{f}(\xi)| \leq \left(1 + \frac{1}{\xi}\right) \frac{\text{TV}_{[0,T]}(f_T^{(m)})}{(2\pi)^m \xi^m} \leq \frac{C \left(\text{TV}_{[0,1]}(f^{(m)}) + |f^{(m)}(0)| + |f^{(m)}(1)| \right)}{(1 + \xi^2)^{\frac{m+1}{2}}}$$

for some $C > 0$. Finally, since $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$, the result follows. $\square$

Also, a smoothness concept that allows for fractional-continuous smoothness was introduced by García-Fiñana and Cruz-Orive [5,6]. Below, we provide the definition of this concept on $[0, 1]$, including the technical condition on the $(m + 1)$ -th derivative, as detailed by [5], Equation (76) with $p = 0$.

Definition 1. Let $q \geq 0$, and set $m = \lceil q \rceil$ and $\alpha = m - q \in [0, 1)$. A function $f \in L^1_{[0,1]}(\mathbb{R})$ is said to be q -smooth if f has continuous derivatives up to order $m - 1$, and there exist finitely many points $a_1, \dots, a_n \in [0, 1]$ such that $f^{(m)}$ has a continuous first derivative on each connected component of $[0, 1] \setminus \{a_1, \dots, a_n\}$. Moreover, for each i there exists $0 < \delta \leq \frac{1}{2}$ and exponents $\alpha_i^\pm \in [0, 1)$ such that as $x \rightarrow a_i^\pm$,

$$f^{(m)}(x) = \begin{cases} d_i^- (a_i - x)^{-\alpha_i^-} + c_{i,0}^- + c_{i,1}^-(x - a_i) + R_i^-(x), & x \in [a_i - \delta, a_i), \\ d_i^+ (x - a_i)^{-\alpha_i^+} + c_{i,0}^+ + c_{i,1}^+(x - a_i) + R_i^+(x), & x \in (a_i, a_i + \delta], \end{cases}$$

where $d_i^\pm \in \mathbb{R}$, $c_{i,0}^\pm, c_{i,1}^\pm \in \mathbb{R}$ and the remainders $R_i^\pm$ have a continuous first derivative on the corresponding half-interval and satisfy $R_i^\pm(x) = o(|x - a_i|)$ as $x \rightarrow a_i^\pm$. Set $\alpha = \max_i \{\alpha_i^-, \alpha_i^+\}$; then by construction $q = m - \alpha$.

We demonstrate that the space of functions defined in Equation (3) includes that notion. The following result will be used in the proof.

Lemma 2. Let $\alpha \in (0, 1)$ and consider $f(x) = x^{-\alpha}$ for $x \in (0, 1]$. Then there exists $C > 0$ such that for all $\zeta \in \mathbb{R} \setminus \{0\}$,

$$\left| \int_0^1 x^{-\alpha} e^{-2\pi i \zeta x} dx \right| \leq C |\zeta|^{\alpha-1}.$$

Proof. We include the proof for completeness; cf. ([19], Example 3.6) for related arguments. If $|\zeta| \leq 1$, then $\left| \int_0^1 x^{-\alpha} e^{-2\pi i \zeta x} dx \right| \leq \int_0^1 x^{-\alpha} dx = \frac{1}{1-\alpha} \leq \frac{1}{1-\alpha} |\zeta|^{\alpha-1}$. Hence assume $|\zeta| > 1$ and, without loss of generality, $\zeta > 0$, since

$$\int_0^1 x^{-\alpha} e^{-2\pi i (-\zeta)x} dx = \overline{\int_0^1 x^{-\alpha} e^{-2\pi i \zeta x} dx}.$$

We now bound the cosine and sine terms separately:

$$\int_0^1 x^{-\alpha} \cos(2\pi \zeta x) dx \quad \text{and} \quad \int_0^1 x^{-\alpha} \sin(2\pi \zeta x) dx.$$

For the first one, by integration by parts,

$$\int_0^1 x^{-\alpha} \cos(2\pi \zeta x) dx = \frac{\sin(2\pi \zeta)}{2\pi \zeta} + \frac{\alpha}{2\pi \zeta} \int_0^1 x^{-\alpha-1} \sin(2\pi \zeta x) dx,$$

and the boundary term at $x = 0$ vanishes since $\lim_{x \rightarrow 0^+} x^{-\alpha} \sin(2\pi \zeta x) / (2\pi \zeta) = 0$ for $\alpha \in (0, 1)$. With $t = \zeta x$,

$$\int_0^1 x^{-\alpha-1} \sin(2\pi \zeta x) dx = \zeta^{\alpha} \int_0^{\zeta} t^{-\alpha-1} \sin(2\pi t) dt.$$

Fix $0 < \varepsilon < 1$. Since $|\sin(2\pi t)| \leq 2\pi |t|$ on $(0, \varepsilon]$ and $|\sin(\cdot)| \leq 1$ on $[\varepsilon, \infty)$,

$$\left| \int_0^{\zeta} t^{-\alpha-1} \sin(2\pi t) dt \right| \leq 2\pi \int_0^{\varepsilon} t^{-\alpha} dt + \int_{\varepsilon}^{\infty} t^{-\alpha-1} dt = C < \infty,$$

independently of ζ . Thus,

$$\left| \int_0^1 x^{-\alpha} \cos(2\pi \zeta x) dx \right| \leq \frac{1}{2\pi |\zeta|} + \frac{\alpha}{2\pi} C |\zeta|^{\alpha-1} \leq \left(\frac{1}{2\pi} + \frac{\alpha}{2\pi} C \right) |\zeta|^{\alpha-1}.$$

For the sine integral, the change $t = \zeta x$ yields

$$\begin{aligned} \int_0^1 x^{-\alpha} \sin(2\pi \zeta x) dx &= \zeta^{\alpha-1} \int_0^{\zeta} t^{-\alpha} \sin(2\pi t) dt \\ &= \zeta^{\alpha-1} \left(\int_0^{\infty} t^{-\alpha} \sin(2\pi t) dt - \int_{\zeta}^{\infty} t^{-\alpha} \sin(2\pi t) dt \right). \end{aligned}$$

The first integral corresponds to the Mellin transform of $\sin(2\pi t)$ evaluated at $\alpha + 1$ and has a known finite value for $\alpha \in (0, 1)$. Since $t^{-\alpha} \sin(2\pi t)$ is integrable on $[1, \infty)$, the tail integral $T(\zeta) = \int_{\zeta}^{\infty} t^{-\alpha} \sin(2\pi t) dt$ is continuous on $[1, \infty)$ and $T(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$; hence $\sup_{\zeta \geq 1} |T(\zeta)| < \infty$. Therefore

$$\left| \int_0^1 x^{-\alpha} \sin(2\pi \zeta x) dx \right| \leq C' |\zeta|^{\alpha-1},$$

finishing the proof. $\square$

Proposition 3. Let $q > 0$. Any q -smooth function belongs to $\varepsilon_{[0,1]}^{q+1}(\mathbb{R})$.

Proof. Let $\xi = 1/t$ with $0 < t \leq 1$ (the case $|\xi| < 1$ follows from the bounding $\hat{f}(\xi)$ in terms of the L^1 norm of f). Set

$$T = \left\lceil \frac{1}{t} \right\rceil t, \quad m = \lceil q \rceil, \quad \alpha = m - q \in [0, 1).$$

By the definition of q -smoothness, f has continuous derivatives up to order $m - 1$, and $f^{(m)}$ is piecewise continuously differentiable on $[0, 1]$, with only finitely many singular points $a_1, \dots, a_n$ at which $f^{(m)}$ has the stated two-sided expansions. In particular, we have $f^{(m)} \in L^1_{[0,1]}(\mathbb{R})$, and for $k = \lceil \frac{1}{t} \rceil$,

$$|\hat{f}_T(k)| = \left| \frac{1}{(2\pi i \xi)^m} \widehat{f_T^{(m)}}(k) \right| \leq \frac{1}{(2\pi)^m |\xi|^m} |\widehat{f_T^{(m)}}(k)|.$$

Moreover,

$$\widehat{f_T^{(s)}}(k) = \frac{1}{T} \int_0^T f^{(s)}(x) e^{-2\pi i \xi x} dx = \frac{1}{T} \int_0^1 f^{(s)}(x) e^{-2\pi i \xi x} dx,$$

since $f_T = f$ on $[0, 1]$ and $f_T = 0$ a.e. on $(1, T]$. Fix $\delta > 0$ as in the definition (shrinking it if needed so that $a_i \pm \delta \in [0, 1]$), and set $a_0 = 0$ and $a_{n+1} = 1$ for notational convenience.

Now, splitting the integral above over $[0, 1]$ into smooth subintervals that avoid the discontinuities, that is, $[a_i + \delta, a_{i+1} - \delta]$, we have

$$\begin{aligned} \int_{a_i + \delta}^{a_{i+1} - \delta} f^{(m)}(x) e^{-2\pi i \xi x} dx &= \left[\frac{f^{(m)}(x) e^{-2\pi i \xi x}}{-2\pi i \xi} \right]_{a_i + \delta}^{a_{i+1} - \delta} \\ &\quad + \frac{1}{2\pi i \xi} \int_{a_i + \delta}^{a_{i+1} - \delta} f^{(m+1)}(x) e^{-2\pi i \xi x} dx = O(\xi^{-1}), \end{aligned}$$

since $f^{(m)}$ is continuously differentiable on such intervals. Here, the boundary terms are bounded because $f^{(m)}$ is continuous up to the endpoints of each smooth piece.

Near each discontinuity a_i , it suffices to analyze the most singular case with order α . By Lemma 2,

$$\int_{a_i}^{a_i + \delta} (x - a_i)^{-\alpha} e^{-2\pi i \xi x} dx = e^{-2\pi i \xi a_i} \int_0^\delta t^{-\alpha} e^{-2\pi i \xi t} dt = O(\xi^{\alpha-1}),$$

and the same bound holds on $(a_i - \delta, a_i]$. The regular parts in the local expansion contribute lower orders: the constant term yields $O(\xi^{-1})$, and the linear term yields $O(\xi^{-2})$; the remainder $R_i^\pm$, which is continuously differentiable on the corresponding half-interval and satisfies $R_i^\pm(x) = o(|x - a_i|)$, also contributes $O(\xi^{-1})$ by integration by parts. Since $\alpha \in [0, 1)$, we have $\xi^{-1} \leq \xi^{\alpha-1}$ for $\xi \geq 1$, so all these contributions are dominated by $O(\xi^{\alpha-1})$. Summing over all pieces, there exists $C > 0$ such that

$$|\widehat{f_T^{(m)}}(k)| \leq C \xi^{\alpha-1}.$$

Therefore,

$$|\hat{f}_T(k)| \leq \frac{C}{(2\pi)^m \xi^{m+1-\alpha}} = \frac{C}{(2\pi)^m \xi^{q+1}}.$$

To sum up, since $\hat{f}(\xi) = T \hat{f}_T(k)$ with $k/T = \xi$, the same arguments as in Proposition 2 show that $|\hat{f}(\xi)| \leq C'(1 + |\xi|^2)^{-(q+1)/2}$, i.e., $f \in \varepsilon_{[0,1]}^{q+1}(\mathbb{R})$. This completes the proof. $\square$

We note that by taking the smoothness indices from Propositions 2 and 3, namely $s = m + 1$ and $s = q + 1$, and inserting them into the variance bound in Equation (2), we recover the classical rates $O(t^{2m+2})$ ([8], Theorem 1) and $O(t^{2q+2})$ ([5], Proposition 5.1), with constants independent of t . This confirms that the Fourier-decay formulation captures both the integer and fractional smoothness settings within a single framework.

4. Inverse Problem

The following proposition may be interpreted as the inverse problem of the previous results on the variance decay of the Cavalieri estimator, in particular Proposition 1. By the inverse problem, we mean the converse direction: from an algebraic variance decay assumption $\text{var}(\hat{V}) \leq C t^{2\alpha}$ for all $t \in (0, 1]$, deduce the necessary Fourier-decay rate $|\hat{f}(\xi)| \leq C'(1 + |\xi|^2)^{-\alpha/2}$ (cf. Proposition 4). The motivation comes from the example

$$f(x) = x \log x \cdot \mathbf{1}_{[0,1]}(x),$$

where $\mathbf{1}_{[0,1]}(x)$ denotes the indicator of $[0, 1]$ (equal to 1 if $x \in [0, 1]$ and 0 otherwise), which is weakly $(0, 1)$ -piecewise smooth but neither $(0, 1)$ -piecewise smooth nor 0-smooth [7]. Under these notions of smoothness, the strongest conclusion available about the variance decay is of order 2. However, it can be shown that for every $\epsilon > 0$, we have $f \in \epsilon_{[0,1]}^{2-\epsilon}(\mathbb{R})$, which, by Equation (2), yields

$$\text{var}(\hat{V}) \leq C t^{4-2\epsilon}, \quad (4)$$

i.e., a variance decay arbitrarily close to order 4.

Numerical check: To illustrate this behavior, we estimated the empirical variance of the Cavalieri estimator $\hat{V}$ for $f(x) = x \log x \cdot \mathbf{1}_{[0,1]}(x)$ using mean numbers of slices $1/t$ from 2 to 50 (i.e., t decreasing from 0.5 to 0.02). For each t , we computed the empirical variance using 1000 Monte Carlo runs (independent random shifts) and then fitted, on a log–log scale, a least-squares line of variance versus $1/t$ using the largest 10% of the $1/t$ grid (smallest t) to target the asymptotic regime. The fitted slope is -3.66 , consistent with the theoretical rate in Equation (4).

Proposition 4 (Inverse problem). *Let $f \in \epsilon_{[0,1]}^s(\mathbb{R})$, and let $\hat{V}$ be defined as in Equation (1). If for all $0 < t \leq 1$ there exists a constant $C > 0$ such that*

$$\text{var}(\hat{V}) \leq C t^{2\alpha}, \quad (5)$$

then $f \in \epsilon_{[0,1]}^\alpha(\mathbb{R})$.

Proof. Fix $\xi = 1/t$ with $t \leq 1$ (for $|\xi| < 1$, the bound follows from the L^1 norm of f). Let $N = \lceil \frac{1}{t} \rceil$, and set $T = Nt$. Since $f \in \epsilon_{[0,1]}^s(\mathbb{R})$ for some smoothness $s > 1/2$, by Proposition 1, it follows that

$$\text{var}(\hat{V}) = 2T \sum_{k=1}^{\infty} \hat{g}_T(kN) = 2T^2 \sum_{k=1}^{\infty} |\hat{f}_T(kN)|^2.$$

From Equation (5) we obtain

$$2T^2 |\hat{f}_T(N)|^2 \leq \text{var}(\hat{V}) \leq C t^{2\alpha}.$$

Therefore,

$$|\hat{f}(\xi)| = T |\hat{f}_T(N)| \leq \frac{C'}{(1 + \xi^2)^{\alpha/2}}.$$

Finally, since $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$, the claim follows. $\square$

5. Conclusions

The smoothness of the measurement function underlying sampling-based volume estimators is a key factor for accuracy. Various notions of smoothness have been introduced independently for systematic and irregular sampling schemes. This work has shown that a Fourier-based notion of smoothness applies to the Cavalieri estimator and subsumes these approaches, providing a natural and continuous framework. Furthermore, we show that this characterization is essentially optimal (inverse problem), as it cannot be improved in general, particularly when variance decay is assessed in an algebraic-rate sense, which is often sufficient in practice. Table 2 summarizes assumptions and guarantees and situates our contribution.

Table 2. Smoothness assumptions for the Cavalieri estimator: summary of guarantees and relationships.

Framework	Assumption (Sketch)	Variance Rate	Contains Others?	Converse (Necessity)
Classical integer $((m, 1)$ -piecewise smooth) [3]	m derivatives continuous; finitely many jumps in higher derivatives	$O(t^{2m+2})$	No	No
Weak integer (BV-Sobolev-type space) [8]	m integrable weak derivatives and $f^{(m)}$ of bounded variation; allows certain jump singularities	$O(t^{2m+2})$	Partially, classical integer	No
Fractional q -smooth [5]	Fractional regularity via local exponents around finitely many points	$O(t^{2q+2})$	Partially, classical integer	No
This paper: Fourier decay	$ \hat{f}(\xi) \leq C(1 + \xi ^2)^{-s/2}$ with $s > 1/2$	$O(t^{2s})$	Yes	Yes

Additionally, we show that posing the inverse problem is practically relevant: earlier smoothness frameworks can be overly pessimistic and therefore overestimate the variance (equivalently, they underestimate smoothness). In Figure 1, the log–log regression of $\widehat{\text{var}}(\hat{V})$ (empirical variance) versus $1/t$ (the mean number of slices) exhibits a slope close to four, in contrast with the order-2 prediction of those frameworks.

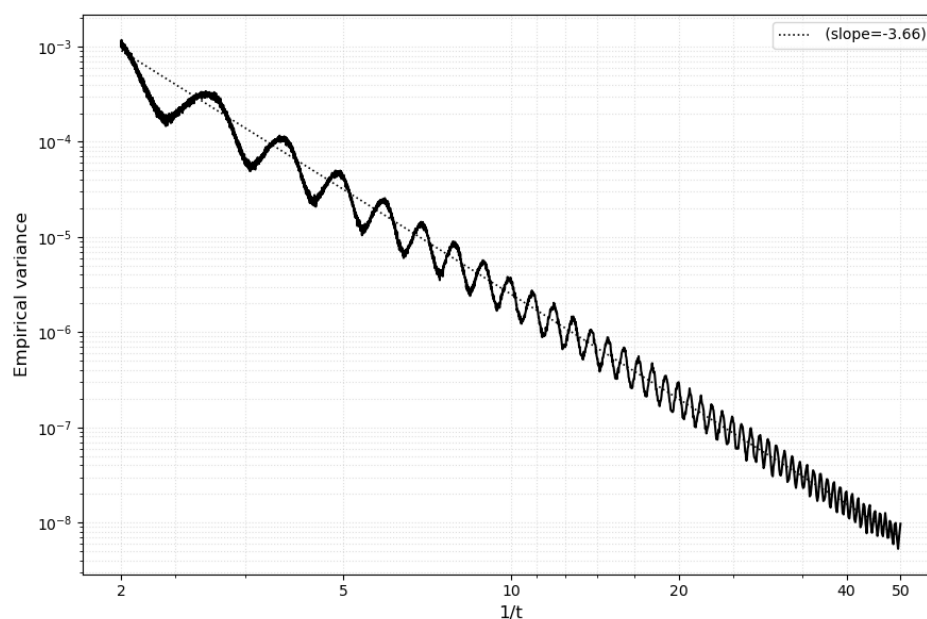


Figure 1. (y-axis) Empirical variance of the Cavalieri estimator for $f(x) = x \log x \cdot \mathbf{1}_{[0,1]}(x)$ versus (x-axis) the mean number of slices $1/t$ (log–log scale). The dashed line is the least-squares fit computed on the smallest 10% of t values; its slope is -3.66 (see the legend).

As future work, we aim to extend the bound in Equation (2) to Cavalieri-type estimators driven by more general spatial point processes (beyond equidistant designs), including recent Newton–Cotes-based schemes; this would further support Fourier decay as a natural framework across a wider family of sampling methods.

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