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About Uniqueness of Steady Ricci Schwarzschild Solitons

Orchidea Maria Lecian

Department of Clinical and Molecular Medicine, Sapienza University of Rome, Viale Regina Elena, 324, 00185 Rome, Italy; orchideamaria.lecian@uniroma1.it

Abstract

In this paper, the uniqueness of steady Schwarzschild gradient Ricci solitons is studied. The role of the weight functions is analyzed. The generalized steady Schwarzschild gradient Ricci solitons are investigated; the implications of the rotational ansatz of Bryant are developed; and the new Generalized Schwarzschildsteady gradient solitons are defined. The aspects of the weight functions of the latter type of solitons are researched as well. The new most-accurate curvature bound of the steady Ricci gradient solitons is provided. The uniqueness of the Schwarzschild solitons is discussed. The Ricci flow is reconciled with the Einstein Field Equations such that the weight functions are utilized to spell out the determinant of the metric tensor, the procedure for which is commented on following the use of the appropriate geometrical objects. The mean curvature is discussed. The configurations of the observer are issued from the geodesics spheres of the solitonic structures.

Keywords: steady Ricci solitons; steady Ricci Schwarzschild solitons; curvature theorems

MSC: 53A99

1. Introduction

It is the aim of the present paper to prove that the Schwarzschild gradient Ricci steady (non-expanding) soliton (with $\lambda = 0$) exists and that it is unique up to the choice of the potential functions f.

In the work of Leandro et al. [1], the steady gradient Ricci solitons are studied. These solitons are proven to be either Ricci flat or the products of the opportune geometrical objects.

In the work of Smoller et al. [2], the Buchdal stability theorem $\frac{9}{8}^{ths} r_S$ is refined.

In the work of Brendle [3], self-similar solutions of the Ricci flow are looked for in the instances of steady Ricci solitons defined as

$$R_{\mu\nu} + \frac{1}{2}\pounds_{\xi} g_{\mu\nu} = 0, \tag{1}$$

with $\pounds\xi$ being the directional derivative with respect to some vector field ξ.

In the work of Bryant [4], the Bryant soliton is defined. Ibidem, the proof is provided that there exists a unique, steady complete soliton with $R > 0$.

From the work of Leandro et al. [1], the flat gradient steady space-dimensional Ricci soliton is isometric to the Bryant soliton. It is the one of the accomplishments of the present work to further extend these concepts.

Examples of solitons are provided, e.g., in the work of Cao [5], in the work of Ivey [6], and in the work of Dancer [7].

In particular, in the work of Cao [5], the Kaehler–Ricci flow is considered in the noncompact case.

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Furthermore, in the work of Ivey [6], a one-parameter family of metrics to which there are corresponding complete non-compact Ricci solitons is provided.

Moreover, in the work of Dancer et al. [7], a metric of the form of multiple warped products is looked for: the soliton potential is written accordingly.

In the works of Chow [8] and in the work of Deng et al. [9], the non-compact cases are investigated; meanwhile, in the work of Fernandez–Lopez et al. the steady gradient Ricci solitons with $\mid R_{ij} \mid^2 \leq \frac{1}{2}R^2$ are researched. In the present paper the constraints arising from general–relativity in $4 = 1 + 3$ spacetime dimensions are focused upon.

Examples of Ricci solitons were summarized in [10]. In the present paper, the indices from the middle of the Latin alphabet refer to 3-dimensional space indices, while the indices from the Greek alphabet account for four-spacetime dimensions indices. We remark that the case of 2-dimensional spaces is not investigated here, as the Killing vectors are trivial.

In the present paper, the work of Leandro et al. [1] will be developed to study the gradient steady Ricci Schwarzschild solitons and the gradient steady Generalized Schwarzschild solitons.

The here-studied solitons are Ricci solitons in the sense that they obey the Ricci flow equations (i.e., in this case, the time–evolution equation newly defined on the Schwarzschild spacetime and on the Generalized Schwarzschild spacetimes).

As results, the work of Brendle [3] and from [11] Proposition 5.2 ibidem are now written for the Schwarzschild soliton (Theorem 22) and for the Generalized Schwarzschild steady gradient soliton (Theorem 23), which constitutes the beginning steps of the new here-presented investigations.

The work of Brendle [3] will be specified to the studied general-relativistic structures; more in detail, the generic vector field ξ, as in Equation (1), will be found. In particular, the vector field in the soliton isometric to the Brendle soliton is found as the 4-velocity vector. The presence of macroscopic general-relativistic matter is defined.

The found solitons are proven to be 'asymptotically' isomorphic to the Bryan soliton.

It is stressed here that the notion of asymtpotical isomorphism follows directly from the Birkhoff Theorem of the Schwarzschild spacetime, according to which one works out that the Schwarzschild spacetime is the only solution of the EFEs which is asymptotically Minkowski flat. Due to the latter implication, it is one of the strategies of the present paper to reconcile the Ricci with the EFEs.

The Schwarzschild soliton and the steady gradient Generalized Schwarzschild soliton are newly characterized according to the properties of the Ricci curvature and to the rotational invariance in Theorems 16, 17, 32–35.

New constraints for the weight function of the Generalized Schwarzschild solitons are found in Theorem 30.

It is a further aim of the present work to qualify the vector field from Equation (1) in $4 = 1 + 3$ general-relativistic scenario as from the 4-velocity 4-vector.

In the work of Cao et al. [12], the $n > 2$ 'complete noncompact locally conformally flat gradient steady solitons' are classified. Ibidem, the 'complete noncompact non-flat conformally flat gradient steady Ricci soliton' is proven to coincide with the Bryant soliton (with respect to scaling).

In [13], the Schwarzschild solitons are looked for in dimensions $n = 2, 3$; the results do not compare because of the non-trivial Killing vector fields.

In [14], the problem of conformal geodesics in General Relativity is envisaged.

The example of a soliton as a warp product is given in Example 1.

The most accurate curvature bound of the steady Ricci gradient soliton is given in Theorem 9.

The uniqueness of the Schwarzschild soliton is discussed in Theorem 17. The paper is organized as follows:

In Section 2, the new methodologies utilized and the main new results achieved are recapitulated. In Section 3, the introductory material about Ricci solitons is summarized for the present purposes, the Bryant solitons are studied, and the Brendle analysis is recalled. In Section 4, new results about Ricci solitons in $4 = 1 + 3$ spacetime dimensions are introduced, with the scope to study the Schwarzschild solitons and the Generalized Schwarzschild solitons. In Section 5, new theorems about the Schwarzschild solitons are established, with the aim to study the morphology of the pertinent new 4-dimensional spacetime structures: the weight functions are newly written. The case of gradient Schwarzschild solitons is in particular newly observed. In Section 6, new theorems about the Generalized Schwarzschild solitons are worked out, with the purposed to analyze the aspects of the new related $n = 4$-dimensional spacetime structures: the weight functions are spelled out, and the corresponding Ricci scalar is studied. The cases of the Generalized Schwarzschild gradient solitons are newly inspected. In Section 7, the general-relativistic structures corresponding to the Schwarzschild solitons are newly explored, for the metrized space of the Schwarscchild solitons. In Section 8, the anaysis is bourght as fas as the metrized space of the Generalized-Schwarschidl solitons are concerned. In Section 9 the role of the GR observer is defined for the solitonic structure observer (on the geodesics spheres). In Section 10, the role of the general-relativistic macroscopic matter and the phenomenology are provided. In Section 11, the outlook is presented. The Christoffel symbols for the Generalized Schwarzschild metric are summarized in Appendix A. The solutions of the EFEs from the Appendix B have to be looked for after Example 1.

2. Methodology and Results

The present paper is aimed at studying the rotational invariance of the Schwarzschild soliton and that of the Generalized Schwarzschild solitons.

The solitonic structures are issued from the pertinent GR spacetimes, in which macroscopic Relativistic matter is introduced according to the EFEs.

The Ricci flow of the soliton structures are in the present paper studied and reconciled with the EFEs as far as the pertinent GR spacetimes are concerned; the Killing vector field is naturally individuated after the velocity vector field from the GR analysis.

The Einstein spacetime is applied the Birkhoff Theorems, according to which the result is known, that the Schwarzschild spacetime is the only one who is asymptotically Minkowski flat; the properties of the Generalized Schwarzschild spacetimes are studied starting from this property.

The rotational invariance of the Schwarzschild spacetime and that of the Generalized Schwarzschild spacetimes are studied; the geometrical objects are spelled. As one result, it is here newly found that the Schwarzschild soliton and the Generalized Schwarzschild solitons are examples of solitons which admit a GR analogue, which are not a warped product of components of the metric tensor.

The asymptotic flatness of the GR spacetime is utilized to perform the parallelism with the Bryant soliton and with the Brendle soliton.

A further investigation line consists of the study of geodesics spheres in the solitons structures; the relevance of studying the geodesics spheres lies in the necessity to define the initial condition of the solitonic structure: From the initial condition of the solitons, it is possible to determine the configurations of the GR observer.

The validity of the boundary condition is ensured after the rigidity of the Ricci–Bakry–Emery tensor.

The role of the weight functions is spelled out to define both the rotational invariance and the expression of the determinant of the metric.

The definition of the mean curvature, in which the properties of the weight functions are encoded, is applied. The mean curvature is requested to be one with vanishing value, which is here newly proven to be of application after the properties of the weight function and after invoking the Theorema Egregium in GR (as far as the Schwarzschild spacetime and the Generalized Schwarzschild spacetimes are concerned).

The weight functions are then utilized to spell out the determinant of the metric tensor.

The geodesics sphere, on their turn, define their normals; this technical passage allow one to find the new tool to reconcile the EFEs in GR with the Ricci flow of the solitonic structures. As one new result (Theorem 30), new constraints on the weight functions of the Generalized Schwarzschild solitons are newly found.

The geodesics spheres are therefore defined for soliton structures which descend from the GR spacetimes of the Schwarzschild spacetime, of the Generalized Schwarzschild spacetime, of the Schwarzschild–de Sitter spacetime, and of the Schwarzschild–anti-de Sitter spacetime.

3. Introductory Material

3.1. The Bryant Soliton

The Bryant soliton is defined in the work of Bryant [4].

In the work of Bryant [4], the proof is provided that there exists a 1-parameter family of complete expanding solitons with $R > 0$.

Accordingly,

Definition 1. *The gradient Ricci soliton with expansion constant λ is defined as*

$$R_{ij}(g_{kl}) = [\mathcal{H}(g_{kl})]_{ij} f - \lambda g_{ij}. \tag{2}$$

3.2. The Brendle Soliton

It is the purpose of the present Subsection to introduce some further material about solitons in 3-dimensional spaces.

From the work of Leandro et al. [1], the gradient steady Ricci solitons are defined as

Definition 2. *The gradient steady Ricci solitons in 3-dimensional space are defined as*

$$R_{ij} = \mathcal{H}_{ij} f \tag{3}$$

where the Hessian matrix $\mathcal{H}_{ij}$ is calculated with derivatives after the metric tensor, i.e.,

$$\mathcal{H}_{ij} \equiv \left[\mathcal{H}_{g_{lm}}\right]_{ij}. \tag{4}$$

The gradient steady Ricci solitons are ibidem studied as corresponding to translating solutions, and to type II singularities; the singularity analysis of the Ricci flow is studied from [15].

The curvature ansatz of gradient steady Ricci solitons is investigated in [8,9,16,17]. The characteristics pertinent to the present analysis are here studied.

From the work of Leandro et al. [1], the following Theorem is issued:

Theorem 1. *The flat gradient steady n-dimensional-space Ricci soliton is isometric to the Bryant soliton.*

Remark 1 (Remark to Theorem 1)**.** *Indeed, in flat spacetime, the Ricci scalar is vanishing.*

From [18], the complete gradient steady Ricci soliton is investigated as

Definition 3. *The complete gradient steady Ricci solitons are endowed with a Ricci scalar bounded from the weight function f as*

$$R + \mid \nabla f \mid^2 = C_0 \tag{5}$$

with $C_0 > 0$, C_0 being the dimensionfull constant.

Definition 3 is implemented in [1] from [18] as follows:

Definition 4. *A normalized gradient steady soliton obeys the Ricci curvature bounds*

$$0 \leq R \leq 1. \tag{6}$$

In the work of Brendle [3], the $n = 3$-dimensional-space steady gradient Ricci solitons which are 'asymptotic' (as will be generalized for the present purpose) to the Bryant soliton are shown to be isometric to the Bryant soliton.

The case $\xi = -\nabla \tilde{f}$ from Equation (1) is characterized for the function $\tilde{f}$ under the hypothesis that it is smooth, i.e., as summarized in [19].

Following from the work of [19], further examples of the gradient steady solitons are given in [5,6].

In the work of Brendle [3], it is stated that, $\forall n \geq$ 3-dimensional spaces, $\exists$ n dimensional space steady gradient Ricci soliton which is rotationally symmetric and whose Ricci curvature is hypothesized to be positive.

From the work of Brendle [3] and from the work of Cao et al. [11] Proposition 5.2 ibidem, and from the work of Leandro et al. [1] Theorem 1 ibidem, the following Proposition is recalled:

Proposition 1. *Let $(\mathcal{M}, g, f)$ a complete $n = 3$ gradient steady Ricci soliton; if $R > 0$, and if $lim_{r\to\infty} R(r) = 0$, then there exists the smooth function ψ: $\psi \in (0, 1)$, such that there exists X, a Killing vector field such that*

$$\nabla R + \psi(R)\nabla f = 0. \tag{7}$$

From Definition 5 and from Remark 4, the following Theorem is newly proven:

Theorem 2. *There exists the rotationally invariant Schwarzschild soliton, which is a gradient steady Ricci soliton $\forall \mu = const$, $\forall c$.*

Proof of Theorem 2. For Schwarzschild solitons, one has that $\lambda = 0$ $\forall \mu$. □

Remark 2 (Remark to Theorem 2)**.** *Theorem 2 newly enforces the use of the EFEs in the definitions of solitons.*

The new Theorem is here stated in order to generalize the results from [20].

Theorem 3. *The mean weighed curvature of the Schwarzschild soliton is vanishing.*

Proof of Theorem 3. After the Theorema Egregium, and with

$$\nabla \phi = const \tag{8}$$

with $\phi \equiv lnf$, for the chosen normal–that is, the normal is selected up to choosing a constant slice. □

The importance of Theorem 3 lies in connecting the Ricci scalar with the scaling properties of the metric, as complemented in the Appendix B after the Theorema Egregium. The role of the normals (to the geodesics spheres) is here newly given its geometrical meaning within a soliton structure. In greater detail, the value of the normal is newly related with the properties of the weights, and, in particular, it is here newly outlined that

Theorem 4. *In the case $\nabla\phi \neq 0$, the weights are the tools that not only select the isoperimetrical case, but, more importantly, select the geodesics sphere on which the boundary initial conditions are taken.*

Theorem 4 allows one to select the initial conditions of the solitonic structure.

From [20], one therefore acquires the fact that the rigidity of the Ricci–Bakry–Emery tensor is enabled after the choice of the boundary conditions.

From Example 1, the following Theorem is newly acquired:

Theorem 5. *Given a non-constant, given $(g_{\mu\nu}, f)$, the rotational ansatz holds for $f \to f + a$.*

Proof of Theorem 5. Because $\tilde{\lambda} = \lambda = 0$. □

Remark 3 (Remark to Theorem 5)**.** *Theorem 5 newly establishes the validity of the rotational ansatz for the weights.*

From the work of Bryant [4], the Theorem is now newly put as

Theorem 6. *There exists a unique four-dimensional steady complete (Bryant) soliton with $R > 0$.*

The role of Theorem 6 is the definition of the uniqueness of the steady complete Bryant soliton.

Furthermore, the following Definition is issued:

Definition 5. *Given $(g_{\mu\nu}, f)$, then the soliton $(\mu g_{\mu\nu}, f + c)$ is a gradient Ricci soliton with expansion constant λ/μ.*

The following remark is reported:

Remark 4. *In the case*

$$R_{\mu\nu}[g_{\rho\sigma}] = \mathcal{H}_{\mu\nu}[f] - \lambda g_{\mu\nu} = \mathcal{H}_{\mu\nu}[\tilde{f}] - \tilde{\lambda} g_{\mu\nu} \tag{9}$$

then

$$\mathcal{H}_{\mu\nu}[f] = \alpha g_{\mu\nu}; \tag{10}$$

therefore

$$\alpha = 0 \tag{11}$$

with

$$\lambda = \tilde{\lambda} \tag{12}$$

and

$$f \to f + a \tag{13}$$

The following example is taken:

Example 1. *Let a be non-constant, and let $g_{\mu\nu}$ be a product metric.*

In the work of Cao [5], the Kaehler Ricci flow is studied in Theorems 1 and 2 from ibidem, i.e., as

Theorem 7. $\forall n = 1 \exists$ *complete gradient Ricci soliton with positive sectional curvature and which is rotationally symmetric. It is unique up to rescaling and up to dilation.*

And as

Theorem 8. $\exists$ *the gradient Ricci soliton for the metric from Theorem 7.*

In the work of Ivey [6], the metric g is looked for in the form

$$\sqrt{n}dt = g\mathcal{Y}ds \tag{14}$$

for a first integral

$$\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2 + \mathcal{W}^2 = 1 + \mathcal{G}g^2\mathcal{Y}*^2 \tag{15}$$

with $\mathcal{G}$ being an arbitrary constant; the invariances of the ratio g/f are studied.

In the work of Dancer et al. [7], examples of complete steady gradient Ricci solitons are provoded: the work is noted to generalize the results of Bryant and those of Ivey [6]. Metric of the form

$$ds^2 = dt^2 + \sum_i g_i^2(t)h_i \tag{16}$$

are looked for, i.e., as multiple warped products. The soliton potential is written accordingly.

4. Some New Results About Ricci Solitons in $4 = 1 + 3$ Spacetime Dimensions

It is the aim of the present section to research the some aspects of the Ricci solitons on $4 = 1 + 3$ spacetime dimensions—the generalization being highly non-linear. Indeed, the Killing vectors of the spacetimes here described are non-linear with respect to those described in Section 3.

Definition 6. *The gradient steady Ricci solitons are defined as*

$$R_{\mu\nu} = \mathcal{H}_{\mu\nu}f \tag{17}$$

where the Hessian matrix $\mathcal{H}_{\mu\nu}$ is calculated with derivatives after the metric tensor, i.e.,

$$\mathcal{H}_{\mu\nu} \equiv \left[\mathcal{H}_{g_{\rho\sigma}}\right]_{\mu\nu}. \tag{18}$$

Furthermore, Definition 4 from [1] is now implemented into the GR spacetimes as

Theorem 9. *A normalized Ricci steady gradient soliton obeys the Ricci curvature bound*

$$0 \leq R \leq 1. \tag{19}$$

The specificity of Theorem 9 is the establishment of a normalization of the Ricci scalar of Schwarzschild solitons.

Theorem 1 is now specified for the Ricci soliton in $4 = 1 + 3$ spacetime dimensions as the following new theorem:

Theorem 10. *The* $4 = 1 + 3$*-spacetime dimensional flat gradient steady Ricci soliton is isometric to the correspondingly generalized Bryant soliton.*

Remark 5 (Remark to Theorem 10)**.** *The EFEs are used. The role of the weights is therefore newly outlined as far as the soliton structure is concerned after the use of the EFEs within the construction of the soliton structure.*

From the work of Brendle [3] and from [11] Proposition 5.2 ibidem, the following new theorem holds:

Theorem 11. *Let* $(\mathcal{M}, g.f)$ *be a complete* $n = 1 + 3$ *gradient steady Ricci soliton; the norm of the Ricci scalar is bounded from below as*

$$\mid R \mid \geq 0. \tag{20}$$

Remark 6 (Remark to Theorems 10 and 11)**.** *The role of the Ricci scalar in defining the soliton structure is to select the possible configurations allowed, as discussed, i.e., in Appendix B.*

5. New Theorems About the Gradient Schwarzschild Solitons

The present section is aimed at establishing some new definitions and some new theorems related to the Schwarzschild gradient solitons.

Theorem 12. *The complete gradient steady Schwarzschild soliton are characterized after a weight function*

$$\mid f \mid^2 > C_1, \tag{21}$$

with $C_1 > 0$*,* C_1 *being the dimensionfull constant.*

(a) The case of $\mid f \mid = 0$ *is the case in which no scaling properties are implied.*
(b) The case $\nabla\phi = 0$ *from Theorem 3 corresponds to the choice of a normal for the geodesics sphere, for which the geodesics sphere is chosen as one of fictitious singularity, and to the geodesics sphere that is determined from the former geodesics sphere without the scalings from Appendix B.*
(c) The case of $\nabla\phi = const$ *corresponds to the choice of a normal for the geodesics spheres, according to which the scaling properties of the metric define a soliton structure whose geodesics sphere can be worked out one from the other after scaling of the metric.*

Theorem 12 is aimed at studying the geometric properties of the weight function with respect to the normal of the geodesics spheres.

Remark 7 (Remark to Theorem 12(a–c))**.** *The case* $\mid f \mid = 0$ *is the case in which the isoperimetric invariances reduce to those of the Schwarzschild spacetime only.*

From Definition 5, the following Theorem is newly stated:

Theorem 13. *For Schwarzschild solitons, there exists the rotational-invariant Schwarzschild soliton, which is a gradient steady Ricci soliton* $\forall \mu = const$ $\forall c$*.*

Proof of Theorem 13. In the case of Schwarzschild solitons, $\lambda = 0$. □

The role of Theorem 13 is to introduce the study of the rotational invariance, which is completed with the tools of Appendix B.

The Schwarzschild complete steady gradient solitons are Ricci solitons, as the Ricci flow is described. Specifically,

Theorem 14. *The Schwarzschild complete steady gradient solitons are Ricci complete gradient steady solitons.*

Proof of Theorem 14. The Schwarzschild complete steady gradient solitons obey the flow equation. □

The study of the flow equation of Schwarzschild solitons is here proven for the first time.

Theorem 9 is now verified as far as the gradient steady normalized Schwarzschild solitons are concerned, i.e.,

Proposition 2. *The Schwarzschild normalized gradient solitons have vanishing Ricci curvature.*

Proof. By construction. □

From Proposition 2, the following is proven:

Theorem 15. *The gradient of the weight functions of the gradient steady Schwarzschild soliton is bounded from above as*

$$\mid \nabla f \mid^2 = C_3, \tag{22}$$

with $C_3 > 0$, C_3 *being the dimensionfull constant.*

The relevance of Theorem 15 is to improve Theorem 3.

Theorem 15 is now adapted for the normalized case as

Proposition 3. *The Schwarzschild gradient steady normalized solitons are such that the weight function is bounded as*

$$\mid \nabla f \mid^2 \leq 1. \tag{23}$$

The case $\mid \nabla f \mid^2 = 1$ is therefore to be studied.

From the work of Brendle [3] and from [11] Proposition 5.2 ibidem, the following new theorem holds from Theorem 11:

Theorem 16. *Let* $(\mathcal{M}, g.f)$ *be a complete* $n = 1 + 3$ *gradient steady Schwarzschild soliton; the norm of the Ricci scalar* R_S *is*

$$\mid R_S \mid \geq 0. \tag{24}$$

The importance of Theorem 16 lies in ensuring the vanishing of the Ricci scalar of the Schwarzschild soliton after the geometric properties of the Schwarzschild spacetime in general relativity.

From the work of Brendle [3] and from [19], Theorem 16 is completed as

Theorem 17. *The complete* $n = 1 + 3$ *gradient steady Schwarzschild soliton is unique.*

Proof of Theorem 15. The solutions of the EFEs are unique in the case of the Schwarzschild spacetime; therefore, the Ricci flow of the complete gradient steady Schwarzschild soliton is constrained from the EFEs. The vanishing of the Ricci scalar of the Schwarzschild spacetime renders straightforward that the definition of the needed constants is well-posed to ensure uniqueness. □

Theorem 17 is therefore of momentous importance in the selection of the solitonic structures, which are the $n = 1 + 3$ complete gradient steady Schwarzschild solitons.

From the work of Bryant [4], from Definition 1, the following is obtained:

Theorem 18. *There exists the Schwarzschild gradient steady Ricci soliton (non-expanding) with* $\lambda = 0$*; the choice of the weight function is arbitrary.*

Proof of Theorem 16. The determinant is calculated

$$[\mathcal{H}(g_{\rho\sigma})]_{\mu\nu} f = 0 \tag{25}$$

$\forall f$. □

It is the aim of the present section to develop the tools to study geodesics spheres, on which the configuration of the observer is chosen in GR.

The geodesics sphere from the gradient Schwarzschild solitons is defined after the properties of the weight function, which is related in [20] to the normal of the geodesics sphere.

The properties of the weight function postulated in Theorems 3 and 15 are worked out within the present framework independently of the choice of the normal of the geodesics spheres.

Indeed, it is one result of the present section to demonstrate that the properties of the geodesics sphere are those of the Schwarzschild spacetimes in GR. Starting from this momentous new result, the definition of the properties of the weight functions is also studied according to Appendix B.

6. New Theorems About the Generalized Schwarzschild Gradient Solitons

The aim of the present section is to further investigate the features of the Generalized Schwarzschild gradient solitons. The newly proven theorems open the way to further investigation.

Theorem 19. *The curvature of the Generalized Schwarzschild complete gradient steady soliton is bounded as*

$$R + \mid \nabla f \mid^2 = C_2, \tag{26}$$

with $C_2 > 0$*,* C_2 *being the dimensionfull constant.*

The value of Theorem 19 is to further refine the properties of the normal of the geodesics sphere within a Schwarzschild solitonic structure.

Theorem 20. *The Generalized Schwarzschild complete steady gradient solitons are Ricci complete gradient steady solitons.*

Proof of Theorem 20. The Generalized Schwarzschild complete steady gradient solitons obey the flow equation. □

The significance of Theorem 20 is the comparison of the geodesics spheres of Schwarzschild solitons to those from the Generalized Schwarzschild solitons.

From Proposition 2, the following theorem is obtained:

Theorem 21. *The Schwarzschild complete gradient steady solitons are uniformly bounded as*

$$R + \mid \nabla f \mid^2 = C_4 \tag{27}$$

with $C_4 > 0$*,* C_4 *being the dimensionfull constant.*

The value of Theorem 21 consists in connecting the geometric properties of the normal of the geodesics spheres to those of the Ricci scalar.

From Theorem 11, the following theorem is obtained:

Theorem 22. *Let $(\mathcal{M}, g.f)$ be a complete $n = 1 + 3$ gradient steady Generalized Schwarzschild soliton; the norm of the Ricci scalar is bounded from below as*

$$R > 0. \tag{28}$$

The importance of Theorem 22 lies in setting the properties of the Ricci scalar of the Generalized Schwarzschild solitons for the first time.

Remark 8. *In this case, the uniqueness of the steady complete Generalized Schwarzschild soliton has to be studied after Theorem 1 and after Theorem 6.*

From the work of Brendle [3] and from [11] Proposition 5.2 ibidem, from Theorem 11, and from Theorem 23, the new Theorem is proven.

Theorem 23. *The $n = 1 + 3$-spacetime-dimensional steady gradient Generalized Schwarzschild soliton is unique; it is endowed with the Ricci curvature R_{GS} as*

$$R_{GS} > 0. \tag{29}$$

The role of Theorem 23 is the definition of the uniqueness of the Generalized Schwarzschild soliton for the first time.

The present section is aimed at researching Generalized Schwarzschild gradient solitons.

The Generalized Schwarzschild gradient complete steady solitons are taken into account.

As a new result, the Ricci curvature of the Generalized Schwarzschild solitons is defined, the definition of which has to be applied to the geometrical objects.

The mechanisms envisaged to further study the Brendle structures are thereby provided.

7. Schwarzschild Soliton: Metrized Space

The present section is aimed at defining the metrization of the space of the Schwarzschild soliton. All of the theoretical results here developed are worked as descending from the metric properties of the resulting space.

Theorem 24. *In the metrized space, the weight function f is vanishing.*

Theorem 24 states for the first time that the comparison of all the properties requested for the Schwarzschild soliton, whose limits in general relativity are well-posed, implies that the weight function must be a vanishing one, which will be discussed below.

Proof of Theorem 22. The metrized space of the Schwarzschild gradient steady normalized solitons is a space of convex functions. □

From Proposition 1, the following theorem is obtained:

Theorem 25. *Let $(\mathcal{M}, g, f)$ be a Schwarzschild complete gradient steady $n = 1 + 3$ soliton. There exists Φ, a smooth function, with $c_S \leq 1$ $\Phi \in (c_S, 1)$ such that ψ_S is bounded as*

$$\psi_S < a_S r^2 + b_S r + c_S \tag{30}$$

and such that there exists the Killing vector field vedi(s) such that

$$\psi_S \nabla f = 0 \tag{31}$$

being X the vector field on the Schwarzschild soliton.

The importance of Theorem 25 lies in setting the properties of the Killing vector field.

Theorem 26. *In Theorem 25, the function c_S is now set as $c_S = C_S = 1$.*

The significance of Theorem 26 is the use of the value of the $\nabla f = 0$ for the normalization of the studied bound of ϕ_S.

From Proposition 1 and from Theorem 25, the following Theorem is described:

Theorem 27. *The smooth function ψ_S is such that*

$$\psi \nabla f = 0, \tag{32}$$

i.e., smooth function ψ_S is bounded as

$$\psi(r) < a_S r^2 + b_S r + c_S. \tag{33}$$

The urgency of Theorem 27 is found in the definition of the bounds of $\psi(r)$.

The properties are compatible with those descending from the metrization of the spacetime in GR.

As from the present analysis, one finds that the weight function in the metric space is a vanishing one.

This result would be a priori not influencing the study of the rotational invariance of the Schwarzschild spacetime and that of the Generalized Schwarzschild spacetimes. This understanding notwithstanding, the use of the weight functions in the spelling of the determinant of the metric tensor opens up new, more detailed research guidelines which will be followed up on in further work. The goal achieved in the present section is the description of the role of geodesics spheres in the studied soliton structures, which will be used in isoperimetric (in-)equalities.

8. Generalized Schwarzschild Solitons: Metrized Space

The aim of the present section is the study of the properties of the Generalized Schwarzschild solitons, which are prepared from the corresponding metrized space.

Theorem 28. *In the metrized space,*

$$0 < f \leq 1. \tag{34}$$

Proof of Theorem 26. The metrized space of the Generalized Schwarzschild gradient steady normalized solitons is a space of convex functions. □

The aim of Theorem 28 is the definition of the bound of the weight function of the Generalized Schwarzschild solitons.

From Proposition 1, the following theorem is established:

Theorem 29. *Let $(\mathcal{M}, g, f)$ be a Generalized Schwarzschild complete gradient steady $n = 1 + 3$ soliton; there exists Φ_{GS}, a smooth function, $\Phi \in (c_{GS}, 1)$, with $c_{GS} \leq 1$, such that ψ_{GS} is bounded as*

$$\psi_{GS} < a_{GS} r^2 + b_{GS} r + c_{GS} \tag{35}$$

and such that there exists Killing vector fields such that

$$\psi_{GS} \nabla f = 0 \tag{36}$$

X being the vector field of the Schwarzschild soliton.

The significance of Theorem 29 is the study of the Killing vector field of Generalized Schwarzschild solitons.

From Theorem 29, the following Theorems are worked out:

Theorem 30. *From Theorem 29, the function f is determined such that*

$$\nabla f \geq \frac{1}{r} \nabla R. \tag{37}$$

The momentous role of Theorem 30 is to define a lower bound of $\nabla f$from the Ricci scalar.

From Theorem 30 it is determined that the weight function in the Generalized Schwarzschild steady gradient solitons obeys the following constraints:

Theorem 31. *For the weight function f_{GS}, the Generalized Schwarzschild steady gradient solitons is constrained as*

$$\frac{1}{r} \nabla R \leq f_{GS} \leq 1. \tag{38}$$

Proof of Theorem 31. After the metrization of the spacetime. □

The significance of Theorem 31 is the upgrading of the constraints of ∇R for the Generalized Schwarzschild solitons; the role of the weight function has to studied after the Theorema Egregium.

From the work of Brendle [3] and from the work of Cao [19], the following new Theorems are proven:

Theorem 32. *The Generalized Schwarzschild steady gradient soliton is Ricci; it obeys the time-evolution equation of the gradient flow.*

It is here remarked that the Ricci flow and the EFEs are reconciled after Theorem 32 for solitons for the first time.

It is remarked that

Theorem 33. *The Generalized Schwarzschild steady gradient soliton admits rotational symmetry.*

It is here stressed that a solitonic structure with rotational symmetry which is not a warped product is here found for the first time.

Theorem 34. *The Generalized Schwarzschild steady gradient soliton is isomorphic to the $4 = 1 + 3$-spacetime-dimensional Bryant gradient steady soliton.*

Proof of Theorem 34. From Definition 1. □

It is important to recall that the finding of a new Bryant steady gradient soliton is achieved after Theorem 34.

Furthermore,

Theorem 35. *The Generalized Schwarzschild steady gradient solitons are endowed with a Ricci scalar whose expression is highly non-trivial.*

The pertinent geometrical objects with their scaling properties are listed in Appendix B.

The aim of the present section is to study the properties of the Generalized Schwarzschild solitons which arise from the metrication of the spacetimes, from which the solitons structures are built.

As a new result, new constraints on the weight functions are established.

The Schwarzschild–de Sitter Solitons and the Schwarzschild–Anti-de Sitter Soliton

It is the aim of the present subsection to extend the found results to the Schwarzschild–de Sitter spacetimes and to the Schwarzschild–anti-de Sitter spacetimes, from which the solitons structures are here newly built.

Theorem 36. *The Schwarzschild–de Sitter solitons with* $g_{00} = 1 - \frac{r_S}{r} + C_3 r^2$ *are defined with Ricci scalar* R_{S-dD} *as*

$$R_{S-dD} = -12C_3 \tag{39}$$

with C_3 *being the dimensionfull constant,* $C_3 > 0$.

In Theorem 36, the Schwarzschild–de Sitter soliton is newly derived.

Theorem 37. *The Schwarzschild–anti-de Sitter solitons with* $g_{00} = 1 - \frac{r_S}{r} - C_3 r^2$ *are defined with Ricci scalar* R_{S-a-dD} *as*

$$R_{S-a-dD} = 12C_3 \tag{40}$$

with C_3 *being the dimensionfull constant,* $C_3 < 0$.

The Schwarzschild–anti-de Sitter soliton is newly proven in Theorem 37.

In [12], the result indicating the non-existence of non-Einstein compact steady Ricci solitons is proven. Here, we prove that:

Theorem 38. *There do not exist non-Einstein compact steady Ricci Schwarzschild solitons.*

Theorem 39. *There do not exist non-Einstein compact steady Ricci Schwarzschild–de Sitter solitons.*

Theorem 40. *There do not exist non-Einstein compact steady Ricci Schwarzschild–anti-de Sitter solitons.*

Theorem 38 is relevant in assuring that the found solitonic structures are Einstein, i.e., according to the present findings, their Ricci flow and their EFEs are reconciled.

Theorems 39 and 40 allow one to ensure that the EFEs are reconciled with the Ricci flow in Schwarzschild–de Sitter solitons and in Schwarzschild–anti-de Sitter solitons.

9. About the Time Evolution of the Observers

It is our aim now to investigate the implications of the well-posedness of the initial conditions for the selection of the configuration of the observer. The rigidity of the Ricci–Bakry–Emery curvature tensor ensures that the configuration of the observer is unique.

This is achieved after studying the features of the geodesic sphere on which the observer is located.

Theorem 41. *The complete noncompact locally conforming flat gradient steady Ricci Schwarzschild soliton is unique.*

Proof. After the Geometrical Objects in Appendix B. □

Theorem 41 ensures that the geodesic spheres are individuated after the normal in Schwarzschild solitons, in Schwarzschild–de Sitter solitons, and in Schwarzschild–anti-de Sitter solitons.

According to the Buchdahl theorems [2,21] about the initial condition, the observer on the 4-geodesics sphere evolves with the 4-geodesics sphere.

10. The Role of General-Relativistic Matter

In [2], the (origin of the gravitational) collapse of a star is worked out independently of the equations of the state of the matter.

The ratio is defined

$$\zeta \equiv \frac{\rho}{\bar{\rho}} \tag{41}$$

with

$$\bar{\rho} = \frac{3}{4}\pi \frac{M(r)}{r^3}. \tag{42}$$

This way, the equations of state of the general-relativistic macroscopic matter are implied after the mass accretion law [22].

The collapse is observed here in general relativity, from [2], to happen after the maximum redshift z factor is achieved.

The application is considered, and the infinite-redshift surfaces are studied.

A further application is considered in which, in this case, from [22], the time derivative of the mass $\dot{M}$ is not constant, i.e., the radius of the fictitious singularity is discussed accordingly.

It is here recalled that matter fields are introduced as macroscopic relativistic matter (dust, fluid, and gas), i.e., as solutions of the EFEs in GR.

The present section is aimed at studying the configurations of the observers in GR when they take place in the solitons structures.

In particular, the reconciliation of the EFEs with the Ricci flow allows one to spell out the geodesics spheres on which the GR observers are aligned. The solidity of the newly-obtained results is enforced after the confirmation of the application of the Schoen-Yau Theorem [23].

11. Outlook

In the work of Chan [16], the work of Munteanu et al. [24] is reappraised; in particular, the n-space-dimensional steady Ricci solitons with non-vanishing Ricci scalar are studied, and curvature estimates are provided from those proposed from Theorem 1 from [24]. Specifically, the solitons that are weight function bounded from above after a constant are treated. In the present work, Theorems 1 and 2 from [16] are completed after new Theorem 31; Equation (6) from [16] is improved as well. It is the aim of further work to study the curvatures estimations.

The relevance of the initial data is now studied.

In detail, from Theorem 2 from [16], $(\mathcal{M}^n, g, f)$ the $n \geq 2$-dimensional complete gradient steady soliton non-vanishing Ricci scalar is considered, with weight function f re-

quested to be bounded from above after a constant, i.e., $f \leq C_7$, C_7 being the dimensionfull constant. Furthermore, the Riemann tensor is requested to be bounded as

$$lim_{r\to\infty} r \mid R_{\mu\nu\rho\sigma} \mid \leq \frac{1}{5}. \tag{43}$$

As a result, there exists $C_8 > 0$, C_8 being the dimensionfull constant, such that, from the initial condition $\vec{x}_0$, the following estimates hold:

$$\mid R_{\mu\nu\rho\sigma} \mid \leq C_8 e^{-\sqrt{|\vec{X}-\vec{X}_0|^2}}. \tag{44}$$

From Theorem 4 ibidem, the behavior of its Ricci scalar is set as

$$lim_{r\to\infty} R = 0 \tag{45}$$

with $r \equiv \sqrt{\mid \vec{X} - \vec{X}_0 \mid^2}$ to obtain the estimate of Equation (43).

In the present context, the following new theorem holds

Theorem 42. *For a Generalized Schwarzschild soliton, the estimate holds*

$$\mid R_{\mu\nu\rho\sigma} \mid \leq C_9, \tag{46}$$

C_9 being the dimensionfull constant.

The significance of Theorem 42 issues from the fact that is is here newly stated that the Riemann tensor of a Generalized Schwarzschild soliton is found to be constant.

It is here commented that all the isoperimetric inequalities which will descend from the metrization of the space of the Generalized Schwarzschild soliton are newly defined after Theorem 42.

Furthermore, Equation (6) from [16] is improved as well.

In the work of Chow et al. [8], non-compact steady gradient Ricci solitons are studied.

In the work of Deng et al. [9], non-compact Ricci solitons are researched. It is the aim of further work to also explore these cases.

Moreover, in the work of Fernandez–Lopez, steady gradient Ricci solitons with $\mid R_{\mu\nu} \mid^2 \leq R/2$ are investigated; it is here stressed that the diverse constraints arising from GR will be addressed in a further work.

The properties defined in [25] do not apply to Schwarzschild solitons; therefore, the coincidence of Bryant Schwarszchild solitons with Brendle Schwarzschild solitons is not only and improvement of the understanding of the rotational symmetries of the Schwarzschild spacetime, but also it opens the investigation of the properties of the tangent cones. Differently, the spacetime structure of the Generalized-Schwarzschild solitons allow for the consideration of implementing curvature estimates.

Notation

In the present paper, the adjective 'dimensionless' refers to objects whose dimensional analysis reveals that they are pure numbers; on the contrary, the adjective 'dimensionfull' is related to objects whose dimensional analysis is measurable in units.

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Appendix A. Geometrical Objects of the Generalized Schwarzschild Metric

For the Generalized Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_S}{r} + \Psi(r)\right)dt^2 - \frac{1}{\left(1 - \frac{r_S}{r} + \Psi(r)\right)}dr^2 - r^2 d\theta^2 - r^2(sin\theta)^2 d\phi^2 \tag{A1}$$

the non-vanishing components of the Riemann tensor $R_{\mu\nu\rho\sigma}$ are calculated as

$$R_{trtr} = -\frac{1}{2}\frac{1}{r^3}\left(r^3\frac{d^2\Psi}{dr^2} - 2r_S\right) = -R_{trrt}, \tag{A2a}$$

$$R_{t\theta t\theta} = -\frac{1}{r^2}(r\Psi + r - r_S)\left(r^2\frac{d^2\Psi}{dr^2} + r_S\right), \tag{A2b}$$

$$R_{r\phi t\phi} = (sin\theta)^2 R_{t\theta t\theta}, \tag{A2c}$$

$$R_{\phi r\phi r} = \frac{1}{2}\frac{1}{r\Psi + r - r_S}\left(r^2\frac{d\Psi}{dr} + r_S\right), \tag{A2d}$$

$$R_{\theta\phi\theta\phi} = -r^2(-r\Psi + r_S) \tag{A2e}$$

and the Ricci scalar is written as

$$R = -\frac{1}{r^2}\left[2\Psi + 4r\frac{d\Phi}{dr} + \frac{d^2}{dr^2}\Psi\right]. \tag{A3}$$

Appendix B. Geometrical Objects of the Scalings

Appendix B.1. The Schwarzschild Metric

For the Schwarzschild metric, the local conformal scaling is described as

$$ds^2 = e^{-2f(r)}\left(1 - \frac{r_S}{r}\right) - \frac{e^{-2f(r)}}{\left(1 - \frac{r_S}{r}\right)} - r^2 e^{-2f(r)} dtheta^2 - r^2(sin\theta)^2 d\phi^2. \tag{A4}$$

The geometrical objects to be discussed are

$$R_{00} = -\frac{(r - r_S)}{r^2}[(r - 2r_s)P + 2(r - rS)N - 2L], \tag{A5a}$$

$$R_{rr} = -\frac{1}{(r - r_S)}[-3(r - r_S)P + 2L], \tag{A5b}$$

$$R_{\theta\theta} = -r(r - r_S)P + 2r(r - r_S)N + (-4r + r_S)L, \tag{A5c}$$

$$R_{\phi\phi} = (sin\theta)^2 R_{\theta\theta}, \tag{A5d}$$

with

$$\frac{d^2 f}{dr^2} \equiv P, \tag{A6a}$$

$$\left(\frac{df}{dr}\right)^2 \equiv N, \tag{A6b}$$

$$\frac{df}{df} \equiv L \tag{A6c}$$

and

$$R = \frac{6}{r^2 e^{-2f}}\left[r(r - r_S)\frac{d^2 f}{dr^2} - r(r - r_S)\left(\frac{df}{dr}\right)^2 + (2r - r_S)\frac{df}{dr}\right]. \tag{A7}$$

The Ricci scalar Equation (A7) is discussed as

$$R = 0 \tag{A8}$$

for the condition on the Schwarzschild radius

$$r_S = r\frac{2r\mu\frac{d^2\mu}{dr^2} - r\left(\frac{d\mu}{dr}\right)^2 + 4\mu\frac{d\mu}{dr}}{2r\mu\frac{d\mu}{dr^2} - r\left(\frac{d\mu}{dr}\right)^2 + 2\frac{d\mu}{dr}} \tag{A9}$$

which can be restated as a condition on the first derivative of $\mu(r)$ as

$$I =\equiv \frac{df}{dr}, \tag{A10a}$$

$$Q \equiv \frac{d^2\mu}{dr^2}, \tag{A10b}$$

$$Y \equiv \left(\frac{df}{dr}\right)^2 \tag{A10c}$$

as

$$I = \frac{r}{2(r-r_S)\mu}[-2\mu(r-r_S)Q + (r-r_S)Y]. \tag{A11}$$

Alternatively, the condition(s) on the first derivative of $\mu(r)$

$$Y = \frac{2\mu}{(r-r_S)}[r(r-r_S)P + (2r-r_S)I] \tag{A12}$$

or the condition on the second derivative of $\mu(r)$ as

$$Q = -\frac{1}{2r(r-r_S)\mu}[-r(r-r_S)Y + 2(2r-r_S)\mu I]. \tag{A13}$$

Appendix B.2. Conformal Scalings of the Generalized Schwarzschild Metric

The scalings of the Generalized Schwarzschild metric are written as

$$ds^2 = e^{-2f(r)}\left(1 - \frac{r_S}{r}\Phi(r)\right)dt^2 - e^{-2f(r)}\frac{1}{\left(1-\frac{r_S}{r}\Phi(r)\right)} - r^2e^{-2f(r)}d\theta^2 - r^2e^{-2f(r)}(sin\theta)^2d\phi^2; \tag{A14}$$

the Ricci scalar to be discussed is written here

$$R = -\frac{1}{r^2e^{-2f(r)}}\left[6r^2N\Phi - 6r^2P\Phi + 6r^2N - 6r_srN - 6r^2L\frac{d\Phi}{dr} - 12rL\Phi - 6r^2P + 6r_SrP + r^2\frac{d^2\Phi}{dr^2} - 12rL + 6r_SL + 4r\frac{d\Phi}{dr} + 2\Phi\right]. \tag{A15}$$

The general spherically symmetric stationary metric endowed with the Schwarzschild-solid-angle element is written as

$$ds^2 = e^{-2f(r)}Fdt^2 - \frac{e^{-2f(r)}}{F} - r^2d\theta^2 - r^2(sin\theta)^2d\psi^2; \tag{A16}$$

the Ricci scalar is written as

$$R = -\frac{1}{r^2e^{-2f(r)}}\left[6rFP - 6r^2L\frac{dF}{dr} - 6r^2FP - 12rLF + r^2\frac{d^2F}{dr^2} + 4r\frac{dF}{dr + 2F - 2}\right]. \tag{A17}$$

It is here commented that the EFEs are solved in the case of macroscopic general-relativistic matter.

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