

Article

# Construction Theorem and Application of a Semi-Discrete Hilbert-Type Inequality Involving Partial Sums and Variable Upper Limit Integral Functions

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## Abstract

By using the construction theorem of semi-discrete Hilbert-type inequalities with quasi-homogeneous kernels and real analysis techniques, this paper establishes a semi-discrete Hilbert-type inequality involving partial sums and variable upper limit integral functions, obtains the necessary and sufficient condition for constructing such an inequality, and, under certain conditions, derives the computational expression of the best constant factor. Finally, we discuss the boundedness and operator norm of the corresponding operator using the obtained results.

**Keywords:** semi-discrete Hilbert-type inequality; partial sum; variable upper limit integral function; bounded operator; operator norm

**MSC:** 26D15; 26D10; 47A05



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## 1. Introduction and Preliminary Knowledge

Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\tilde{a} = \{a_m\}$ , and  $\tilde{b} = \{b_n\}$ . Then, we have the following classical discrete Hilbert inequality [1]:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} |a_m|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |b_n|^q \right)^{1/q} = \frac{\pi}{\sin(\pi/p)} \|\tilde{a}\|_p \|\tilde{b}\|_q,$$

where the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. Correspondingly, integral and semi-discrete forms exist:

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dx dy &\leq \frac{\pi}{\sin(\pi/p)} \left( \int_0^{+\infty} |f(x)|^p dx \right)^{1/p} \left( \int_0^{+\infty} |g(y)|^q dy \right)^{1/q} \\ &= \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \end{aligned}$$

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{n+x} dx \leq \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left( \int_0^{+\infty} |f(x)|^q dx \right)^{1/q} = \frac{\pi}{\sin(\pi/p)} \|\tilde{a}\|_p \|f\|_q,$$

where the constant factors are also the best possible. Hilbert-type inequalities, due to their important applications in operator theory, have received extensive attention. For further

research, it is necessary to generalize  $L_p(0, +\infty)$  and  $l_p$  to weighted Lebesgue spaces and weighted Hilbert-type spaces:

$$L_p^\alpha(0, +\infty) = \left\{ f(x) : \|f\|_{p,\alpha} = \left( \int_0^{+\infty} x^\alpha |f(x)|^p dx \right)^{1/p} < +\infty \right\},$$

$$l_p^\alpha = \left\{ \tilde{a} = \{a_n\} : \|\tilde{a}\|_{p,\alpha} = \left( \sum_{n=1}^{\infty} n^\alpha |a_n|^p \right)^{1/p} < +\infty \right\},$$

where  $p > 1$  and  $\alpha > 0$ . In weighted spaces, different authors have introduced many independent parameters and discussed Hilbert-type inequalities with homogeneous kernels, quasi-homogeneous kernels, and several non-homogeneous kernels such as

$$\frac{1}{(x+y)^\lambda}, \frac{1}{m^{\lambda_1} + n^{\lambda_2}}, \frac{\ln(x/y)}{x^\lambda - y^\lambda}, \frac{|x^{\lambda_1} - y^{\lambda_2}|}{\min\{x^{\lambda_1}, y^{\lambda_2}\}}, \frac{(m^{\lambda_1} n^{\lambda_2})^s}{(m^{\lambda_1} + n^{\lambda_2})^r}$$

and the boundedness of their corresponding operators (see [2–8]). In [9] from 2016, the authors first discussed the problem of best matching parameters for Hilbert-type inequalities, solving the problem of parameter rules for Hilbert-type inequalities with best constant factors. In [10] from 2017, the authors further explored the construction conditions and best constant factor estimation for Hilbert-type inequalities. Since then, relevant discussions have continued and have been published in various academic journals (see [11–13]).

In [14] from 2019, the authors studied Hilbert-type inequalities from a new perspective and presented a Hilbert-type inequality involving partial sums. Afterwards, references [15–20] made various improvements and generalizations. For example, under certain parameter restrictions, the authors in [21] obtained an inequality involving partial sums and multiple variable upper limit functions  $F_m(x)$ :

$$\int_0^{+\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)}{(x+n^\alpha)^\lambda} dx \leq \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)} \left( \frac{1}{\alpha} K_{\lambda+m+1}(\lambda_2+1) \right)^{1/p} (K_{\lambda+m+1}(\lambda_1+m))^{1/q}$$

$$\times \left( \int_0^{+\infty} x^{-p(\tilde{\lambda}_1+m-1)-1} F_m^p(x) dx \right)^{1/p} \left( \sum_{n=1}^{\infty} n^{-q[(\tilde{\lambda}_2+1)-1]-1} A_n^q \right)^{1/q}, \quad (1)$$

where  $\tilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$ ,  $\tilde{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$ , and  $\Gamma(\cdot)$  is the Gamma function. It was proven that the constant factor in (1) is the best possible when  $\alpha = 1$  and  $\lambda_1 + \lambda_2 = \lambda$ .

The authors in [14] considered the discrete inequality with the homogeneous kernel  $\frac{1}{(m+n)^\lambda}$ , and the authors in [21] discussed the semi-discrete inequality with the quasi-homogeneous kernel  $\frac{1}{(x+n^\alpha)^\lambda}$ . The parameter relations in them are relatively cumbersome, and the necessity of these parameter relations is not discussed. This paper considers the parameter conditions and best constant factor problem of a semi-discrete Hilbert-type inequality involving partial sums and variable upper limit integral functions with kernel  $\frac{1}{(n^{\lambda_1} + x^{\lambda_2})^\lambda}$ . The parameter condition relationships in the results are simple and clear, facilitating the application of the inequality. Meanwhile, by making full use of the construction theorem of semi-discrete Hilbert-type inequalities with quasi-homogeneous kernels, the proof process is optimized. Finally, its application in operator theory is discussed.

## 2. Notation Conventions and Preliminary Lemmas

To avoid unnecessary repetitions, this paper makes the following conventions. If  $\tilde{a} = \{a_n\}$  and  $f(x)$  is integrable, denote

$$A_n = \sum_{i=1}^n a_i \quad (n = 1, 2, \dots), \quad \tilde{F}(f)(x) = \int_0^x f(t) dt.$$

If  $K(u, v)$  is a binary measurable function, and  $\lambda_1 > 0, \lambda_2 > 0$ , denote

$$W_1(s) = \int_0^{+\infty} K(1, t^{\lambda_2}) t^s dt, \quad W_2(s) = \int_0^{+\infty} K(t^{\lambda_1}, 1) t^s dt.$$

**Lemma 1** ([22]). Let  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \sigma, \alpha, \beta \in \mathbb{R}, \lambda_1, \lambda_2 > 0$ , and let  $K(u, v)$  be a homogeneous function of degree  $\sigma$ , that is,  $K(tu, tv) = t^\sigma K(u, v) (t > 0)$ . Suppose  $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} - \left(\frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p} + \sigma\right) = c$ , and both  $K(t^{\lambda_1}, 1) t^{-\frac{\alpha+1}{p}}$  and  $K(t^{\lambda_1}, 1) t^{-\frac{\alpha+1}{p} + \lambda_1 c}$  are decreasing on  $(0, +\infty)$ , with  $W_1\left(-\frac{\beta+1}{q}\right) < +\infty$ . Then,

(i) There exists a constant  $M > 0$  if and only if  $c \geq 0$  such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} K(n^{\lambda_1}, x^{\lambda_2}) a_n f(x) dx \leq M \|\tilde{a}\|_{p, \alpha} \|f\|_{q, \beta}. \quad (2)$$

(ii) If  $c = 0$ , i.e.,  $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} = \sigma + \frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p}$ , then the best constant factor in (2) is

$$\inf\{M\} = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} W_1\left(-\frac{\beta+1}{q}\right) = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/p} W_2\left(-\frac{\alpha+1}{p}\right).$$

**Lemma 2.** Let  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda, \alpha, \beta \in \mathbb{R}, \lambda_1 > 0, \lambda_2 > 0, \lambda > 0, \frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda = c$ ,  $\alpha > \max\{(\lambda_1 - 1)p - 1, (\lambda_1 - 1)p - 1 + \lambda_1 p c\}$ , and  $-(\lambda + 1)\lambda_2 q - 1 < \beta < -1$ .

(i) There exists a constant  $\bar{M} > 0$  if and only if  $c \geq 0$  such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{n^{\lambda_1-1} x^{\lambda_2-1}}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda+2}} A_n F(x) dx \leq \bar{M} \|\tilde{A}\|_{p, \alpha} \|F\|_{q, \beta}, \quad (3)$$

where  $\tilde{A} = \{A_n\}$ .

(ii) When  $c = 0$ , the best constant factor in (3) is

$$\inf\{\bar{M}\} = \frac{1}{\lambda_1^{1/q} \lambda_2^{1/p} \Gamma(\lambda + 2)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right).$$

**Proof.** Let

$$K(u, v) = \frac{u^{1-\frac{1}{\lambda_1}} v^{1-\frac{1}{\lambda_2}}}{(u+v)^{\lambda+2}}, \quad u > 0, v > 0.$$

Then,  $K(u, v)$  is a non-negative homogeneous function of order  $\sigma = 2 - \frac{1}{\lambda_1} - \frac{1}{\lambda_2} - (\lambda + 2) = -\lambda - \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$ , and

$$K(n^{\lambda_1}, x^{\lambda_2}) = \frac{n^{\lambda_1-1} x^{\lambda_2-1}}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda+2}}.$$

Thus,

$$K(t^{\lambda_1}, 1) t^{-\frac{\alpha+1}{p}} = \frac{1}{(t^{\lambda_1} + 1)^{\lambda+2}} t^{\lambda_1-1-\frac{\alpha+1}{p}},$$

$$K(t^{\lambda_1}, 1)t^{-\frac{\alpha+1}{p}+\lambda_1 c} = \frac{1}{(t^{\lambda_1} + 1)^{\lambda+2}} t^{\lambda_1-1-\frac{\alpha+1}{p}+\lambda_1 c}.$$

Since  $\alpha > (\lambda_1 - 1)p - 1$ , and  $\alpha > (\lambda_1 - 1)p - 1 + \lambda_1 pc$ , we have  $\lambda_1 - 1 - \frac{\alpha+1}{p} < 0$ , and  $\lambda_1 - 1 - \frac{\alpha+1}{p} + \lambda_1 c < 0$ . Also, since  $\lambda_1 > 0, \lambda > 0$ , both  $K(t^{\lambda_1}, 1)t^{-\frac{\alpha+1}{p}}$  and  $K(t^{\lambda_1}, 1)t^{-\frac{\alpha+1}{p}+\lambda_1 c}$  are decreasing on  $(0, +\infty)$ .

From  $-(\lambda + 1)\lambda_2 q - 1 < \beta < -1 < \lambda_2 q - 1$ , we have  $1 - \frac{\beta+1}{\lambda_2 q} > 0$  and  $\lambda + 2 - \left(1 - \frac{\beta+1}{\lambda_2 q}\right) > 0$ . Thus,

$$\begin{aligned} W_1\left(-\frac{\beta+1}{q}\right) &= \int_0^{+\infty} K(1, t^{\lambda_2}) t^{-\frac{\beta+1}{q}} dt = \int_0^{+\infty} \frac{t^{\lambda_2-1}}{(1+t^{\lambda_2})^{\lambda+2}} t^{-\frac{\beta+1}{q}} dt \\ &= \frac{1}{\lambda_2} \int_0^{+\infty} \frac{1}{(1+u)^{\lambda+2}} u^{(1-\frac{\beta+1}{\lambda_2 q})-1} du = \frac{1}{\lambda_2} B\left(1 - \frac{\beta+1}{\lambda_2 q}, \lambda + 2 - \left(1 - \frac{\beta+1}{\lambda_2 q}\right)\right) < +\infty, \end{aligned}$$

where  $B(\cdot, \cdot)$  is the beta function. Since  $K(u, v)$  is a homogeneous function of degree  $\sigma = -\lambda - \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$ , the condition  $\frac{\alpha}{\lambda_1 p} + \frac{\beta}{\lambda_2 q} - \left(\frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p} + \sigma\right) = c$  in Lemma 1 is transformed into  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda = c$ .

When  $c = 0$ , we have  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda = 0$ ; hence,  $\lambda + 2 - \left(1 - \frac{\beta+1}{\lambda_2 q}\right) = 1 - \frac{\alpha+1}{\lambda_1 p}$ . Thus,

$$\begin{aligned} \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} W_1\left(-\frac{\beta+1}{q}\right) &= \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} \frac{1}{\lambda_2} B\left(1 - \frac{\beta+1}{\lambda_2 q}, 1 - \frac{\alpha+1}{\lambda_1 p}\right) \\ &= \frac{1}{\lambda_1^{1/q} \lambda_2^{1/p} \Gamma(\lambda+2)} \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right) \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right). \end{aligned}$$

In summary, according to Lemma 1, Lemma 2 holds.  $\square$

**Lemma 3.** Let  $t > 0, 0 < \lambda_1 \leq 1, A_n = \sum_{k=1}^n a_k$  ( $n = 1, 2, \dots$ ), and  $\lim_{n \rightarrow +\infty} e^{-tn^{\lambda_1}} A_n = 0$ . Then,

$$\sum_{n=1}^{\infty} e^{-tn^{\lambda_1}} a_n \leq \lambda_1 t \sum_{n=1}^{\infty} n^{\lambda_1-1} e^{-tn^{\lambda_1}} A_n. \quad (4)$$

**Proof.** By Abel's summation formula and differential mean value theorem, we have

$$\begin{aligned} \sum_{n=1}^N e^{-tn^{\lambda_1}} a_n &= e^{-tN^{\lambda_1}} A_N - \sum_{n=1}^{N-1} A_n \left(e^{-t(n+1)^{\lambda_1}} - e^{-tn^{\lambda_1}}\right) \\ &= e^{-tN^{\lambda_1}} A_N + \lambda_1 t \sum_{n=1}^{N-1} \xi_n^{\lambda_1-1} e^{-t\xi_n^{\lambda_1}} A_n, \end{aligned}$$

where  $n < \xi_n < n + 1$ . Since  $0 < \lambda_1 \leq 1$ , we have  $\xi_n^{\lambda_1-1} e^{-t\xi_n^{\lambda_1}} < n^{\lambda_1-1} e^{-tn^{\lambda_1}}$ , and thus

$$\sum_{n=1}^N e^{-tn^{\lambda_1}} a_n \leq e^{-tN^{\lambda_1}} A_N + \lambda_1 t \sum_{n=1}^{N-1} n^{\lambda_1-1} e^{-tn^{\lambda_1}} A_n.$$

Let  $N \rightarrow +\infty$ , and note that  $\lim_{n \rightarrow +\infty} e^{-tn^{\lambda_1}} A_n = 0$ . Then, (4) is obtained.  $\square$

**Lemma 4.** Let  $t > 0$  and  $\lim_{x \rightarrow +\infty} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) = 0$ . Then,

$$\int_0^{+\infty} e^{-tx^{\lambda_2}} f(x) dx = \lambda_2 t \int_0^{+\infty} x^{\lambda_2-1} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) dx.$$

**Proof.** Since  $\frac{d}{dx} \tilde{F}(f)(x) = f(x)$  and  $\tilde{F}(f)(0) = 0$ , by integration by parts, we have

$$\begin{aligned} \int_0^{+\infty} e^{-tx^{\lambda_2}} f(x) dx &= \int_0^{+\infty} e^{-tx^{\lambda_2}} d\tilde{F}(f)(x) \\ &= \lim_{x \rightarrow +\infty} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) + \lambda_2 t \int_0^{+\infty} x^{\lambda_2-1} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) dx \\ &= \lambda_2 t \int_0^{+\infty} x^{\lambda_2-1} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) dx. \end{aligned}$$

□

### 3. Main Results

**Theorem 1.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda, \alpha, \beta \in \mathbb{R}$ , satisfying  $0 < \lambda_1 \leq 1$ ,  $\lambda_2 > 0$ ,  $\lambda > 0$ , and  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda = c$ , with  $\max\{(\lambda_1 - 1)p - 1, (\lambda_1 - 1)p - 1 + \lambda_1 p c\} < \alpha < -1$ ,  $-(\lambda + 1)\lambda_2 q < \beta < -1$ ,  $\lim_{n \rightarrow +\infty} e^{-tn^{\lambda_1}} A_n = 0$  ( $t > 0$ ), and  $\lim_{x \rightarrow +\infty} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) = 0$  ( $t > 0$ ).

(i) There exists a constant  $M > 0$  if and only if  $c \geq 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda \geq 0$ , such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} dx \leq M \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta}, \quad (5)$$

where  $\tilde{A} = \{A_n\} = \{\sum_{k=1}^n a_k\}$ .

(ii) When  $c = 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda = 0$ , the best constant factor in (5) is

$$M_0 = \lambda_1^{1/p} \lambda_2^{1/q} \frac{1}{\Gamma(\lambda)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right).$$

**Proof.** (i) According to the Gamma function definition, we obtain

$$\frac{1}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1} e^{-t(n^{\lambda_1} + x^{\lambda_2})} dt.$$

Thus, by Lemmas 3 and 4, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} dx &= \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \int_0^{+\infty} a_n f(x) \left( \int_0^{+\infty} t^{\lambda-1} e^{-t(n^{\lambda_1} + x^{\lambda_2})} dt \right) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1} \left( \sum_{n=1}^{\infty} e^{-tn^{\lambda_1}} a_n \right) \left( \int_0^{+\infty} e^{-tx^{\lambda_2}} f(x) dx \right) dt \\ &\leq \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \int_0^{+\infty} t^{(\lambda+2)-1} \left( \sum_{n=1}^{\infty} n^{\lambda_1-1} e^{-tn^{\lambda_1}} A_n \right) \left( \int_0^{+\infty} x^{\lambda_2-1} e^{-tx^{\lambda_2}} \tilde{F}(f)(x) dx \right) dt \\ &= \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \int_0^{+\infty} t^{(\lambda+2)-1} \left( \sum_{n=1}^{\infty} \int_0^{+\infty} n^{\lambda_1-1} x^{\lambda_2-1} e^{-t(n^{\lambda_1} + x^{\lambda_2})} A_n \tilde{F}(f)(x) dx \right) dt \\ &= \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \int_0^{+\infty} n^{\lambda_1-1} x^{\lambda_2-1} A_n \tilde{F}(f)(x) \left( \int_0^{+\infty} t^{(\lambda+2)-1} e^{-t(n^{\lambda_1} + x^{\lambda_2})} dt \right) dx \\ &= \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \Gamma(\lambda+2) \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{n^{\lambda_1-1} x^{\lambda_2-1}}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda+2}} A_n \tilde{F}(f)(x) dx. \end{aligned} \quad (6)$$

By Lemma 2, when  $c \geq 0$ , there exists a constant  $\overline{M} > 0$  such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{n^{\lambda_1-1} x^{\lambda_2-1}}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda+2}} A_n \tilde{F}(f)(x) dx \leq \overline{M} \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta}.$$

Let  $M = \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \Gamma(\lambda + 2) \overline{M}$ , then (5) can be obtained from (6).

Conversely, suppose (5) holds. If  $c < 0$ , take a sufficiently small  $\varepsilon > 0$ , and let

$$a_n = n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1}, \quad n = 1, 2, \dots$$

$$f(x) = \begin{cases} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1}, & x \geq 1, \\ 0, & 0 < x < 1. \end{cases}$$

Since  $\alpha > p(\lambda_1 - 1) > -p - 1$ , we have  $-\frac{\alpha+1}{p} - 1 < 0$ . And since  $\varepsilon > 0$  is sufficiently small,  $-\frac{\alpha+1+\lambda_1\varepsilon}{p} - 1 < 0$ . Also, from  $\alpha < -1$ , and  $\varepsilon > 0$  being sufficiently small, we get  $-\frac{\alpha+1+\lambda_1\varepsilon}{p} > 0$ . Hence,

$$A_n = \sum_{k=1}^n k^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} < \int_0^n t^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} dt = \frac{-p}{\alpha + 1 + \lambda_1\varepsilon} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}}. \quad (7)$$

Since  $\beta < -1$  and  $\varepsilon > 0$  is sufficiently small, we have  $-\frac{\beta+1+\lambda_2\varepsilon}{q} > 0$ . Thus, when  $0 < x < 1$ , we obtain  $\tilde{F}(f)(x) = 0$ . When  $x \geq 1$ , we have

$$\tilde{F}(f)(x) = \int_1^x t^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1} dt = \frac{-q}{\beta + 1 + \lambda_2\varepsilon} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}}. \quad (8)$$

From (7) and (8), we have

$$\begin{aligned} \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta} &= \left( \sum_{n=1}^{\infty} n^{\alpha} |A_n|^p \right)^{1/p} \left( \int_0^{+\infty} x^{\beta} |\tilde{F}(f)(x)|^q dx \right)^{1/q} \\ &\leq \frac{pq}{(\alpha + 1 + \lambda_1\varepsilon)(\beta + 1 + \lambda_2\varepsilon)} \left( \sum_{n=1}^{\infty} n^{-1-\lambda_1\varepsilon} \right)^{1/p} \left( \int_1^{+\infty} x^{-1-\lambda_2\varepsilon} dx \right)^{1/q} \\ &\leq \frac{pq}{(\alpha + 1 + \lambda_1\varepsilon)(\beta + 1 + \lambda_2\varepsilon)} \left( 1 + \int_1^{+\infty} t^{-1-\lambda_1\varepsilon} dt \right)^{1/p} \left( \int_1^{+\infty} x^{-1-\lambda_2\varepsilon} dx \right)^{1/q} \\ &= \frac{pq}{\lambda_1^{1/p} \lambda_2^{1/q} \varepsilon (\alpha + 1 + \lambda_1\varepsilon)(\beta + 1 + \lambda_2\varepsilon)} (1 + \lambda_1\varepsilon)^{1/p}, \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} dx &= \sum_{n=1}^{\infty} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} \left( \int_1^{+\infty} \frac{1}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1} dx \right) \\ &= \sum_{n=1}^{\infty} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1-\lambda\lambda_1} \left( \int_1^{+\infty} \frac{1}{(1 + n^{-\lambda_1} x^{\lambda_2})^{\lambda}} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1} dx \right) \\ &= \frac{1}{\lambda_2} \sum_{n=1}^{\infty} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1-\lambda\lambda_1-\frac{\lambda_1}{\lambda_2} \left( \frac{\beta+1+\lambda_2\varepsilon}{q} + 1 \right) + \frac{\lambda_1}{\lambda_2}} \int_{n^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2\varepsilon}{\lambda_2 q}-1} dt \\ &= \frac{1}{\lambda_2} \sum_{n=1}^{\infty} n^{-1-\lambda_1(c+\varepsilon)} \int_{n^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2\varepsilon}{\lambda_2 q}-1} dt \\ &\geq \frac{1}{\lambda_2} \sum_{n=1}^{\infty} n^{-1-\lambda_1(c+\varepsilon)} \int_1^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2\varepsilon}{\lambda_2 q}-1} dt. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \frac{1}{\lambda_2} \sum_{n=1}^{\infty} n^{-1-\lambda_1(c+\varepsilon)} \int_1^{+\infty} \frac{1}{(1+t)^\lambda} t^{-\frac{\beta+1+\lambda_2\varepsilon}{\lambda_2 q}-1} dt \\ & \leq M \frac{pq}{\lambda_1^{1/p} \lambda_2^{1/q} \varepsilon (\alpha+1+\lambda_1\varepsilon)(\beta+1+\lambda_2\varepsilon)} (1+\lambda_1\varepsilon)^{1/p} < +\infty. \end{aligned} \quad (9)$$

Since  $c < 0$  and  $\varepsilon > 0$  is sufficiently small,  $1 + \lambda_1(c + \varepsilon) < 1$ , we have  $\sum_{n=1}^{\infty} n^{-1-\lambda_1(c+\varepsilon)} = +\infty$ , which contradicts (9). Hence,  $c < 0$  does not hold; in other words,  $c \geq 0$ .

(ii) When  $c = 0$ , by (6) and Lemma 2, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} dx \leq \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \Gamma(\lambda+2) \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{n^{\lambda_1-1} x^{\lambda_2-1}}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} A_n \tilde{F}(f)(x) dx \\ & \leq \frac{\lambda_1 \lambda_2}{\Gamma(\lambda)} \Gamma(\lambda+2) \frac{1}{\lambda_1^{1/q} \lambda_2^{1/p} \Gamma(\lambda+2)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right) \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta} \\ & = \frac{\lambda_1^{1/p} \lambda_2^{1/q}}{\Gamma(\lambda)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right) \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta} = M_0 \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta}. \end{aligned}$$

If  $M_0$  is not the best constant factor in (5), then there exists a constant  $M_1 < M_0$  such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n \tilde{f}(x)}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} dx \leq M_1 \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta}. \quad (10)$$

Take a sufficiently large  $N > 0$  and a sufficiently small  $\varepsilon > 0$ , and let

$$a_n = \begin{cases} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1}, & n = N, N+1, \dots \\ 0, & n = 1, 2, \dots, N-1 \end{cases}, \quad \tilde{f}(x) = \begin{cases} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1}, & x \geq 1, \\ 0, & 0 < x < 1. \end{cases}$$

Thus, when  $n = 1, 2, \dots, N-1$ , we have  $A_n = 0$ . When  $n = N, N+1, \dots$ , we obtain

$$\begin{aligned} A_n &= \sum_{k=N}^n k^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} < \int_{N-1}^n t^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} dt \\ &\leq \int_0^n t^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}-1} dt = \frac{-p}{\alpha+1+\lambda_1\varepsilon} n^{-\frac{\alpha+1+\lambda_1\varepsilon}{p}}. \end{aligned}$$

When  $0 < x < 1$ ,  $\tilde{F}(f)(x) = 0$ ; when  $x \geq 1$ , we have

$$\tilde{F}(f)(x) = \int_1^x t^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1} dt < \int_0^x t^{-\frac{\beta+1+\lambda_2\varepsilon}{q}-1} dt = \frac{-q}{\beta+1+\lambda_2\varepsilon} x^{-\frac{\beta+1+\lambda_2\varepsilon}{q}}.$$

Thus, we have

$$\begin{aligned} & \|\tilde{A}\|_{p,\alpha} \|\tilde{F}(f)\|_{q,\beta} = \left( \sum_{n=N}^{\infty} n^\alpha |A_n|^p \right)^{1/p} \left( \int_0^{+\infty} x^\beta |\tilde{F}(f)(x)|^q dx \right)^{1/q} \\ & \leq \frac{pq}{(\alpha+1+\lambda_1\varepsilon)(\beta+1+\lambda_2\varepsilon)} \left( \sum_{n=N}^{\infty} n^{-1-\lambda_1\varepsilon} \right)^{1/p} \left( \int_1^{+\infty} x^{-1-\lambda_2\varepsilon} dx \right)^{1/q} \\ & \leq \frac{pq}{(\alpha+1+\lambda_1\varepsilon)(\beta+1+\lambda_2\varepsilon)} \left( \int_{N-1}^{+\infty} t^{-1-\lambda_1\varepsilon} dt \right)^{1/p} \left( \int_1^{+\infty} x^{-1-\lambda_2\varepsilon} dx \right)^{1/q} \\ & = \frac{pq}{(\alpha+1+\lambda_1\varepsilon)(\beta+1+\lambda_2\varepsilon)} \cdot \frac{1}{\varepsilon \lambda_1^{1/p} \lambda_2^{1/q}} (N-1)^{-\frac{\lambda_1\varepsilon}{p}}. \end{aligned} \quad (11)$$

Similarly, we have

$$\begin{aligned}
 & \sum_{n=N}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} dx = \sum_{n=N}^{\infty} n^{-\frac{\alpha+1+\lambda_1 \varepsilon}{p}-1} \left( \int_1^{+\infty} \frac{1}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} x^{-\frac{\beta+1+\lambda_2 \varepsilon}{q}} dx \right) \\
 &= \frac{1}{\lambda_2} \sum_{n=N}^{\infty} n^{-1-\lambda_1 \varepsilon} \int_{n^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2 \varepsilon}{\lambda_2 q}-1} dt \\
 &\geq \frac{1}{\lambda_2} \int_N^{+\infty} t^{-1-\lambda_1 \varepsilon} dt \int_{N^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2 \varepsilon}{\lambda_2 q}-1} dt \\
 &= \frac{1}{\lambda_1 \lambda_2 \varepsilon} N^{-\lambda_1 \varepsilon} \int_{N^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2 \varepsilon}{\lambda_2 q}-1} dt. \tag{12}
 \end{aligned}$$

From (10)–(12), we obtain

$$\begin{aligned}
 & \frac{1}{\lambda_1 \lambda_2 \varepsilon} N^{-\lambda_1 \varepsilon} \int_{N^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2 \varepsilon}{\lambda_2 q}-1} dt \\
 &\leq M_1 \frac{pq}{(\alpha+1+\lambda_1 \varepsilon)(\beta+1+\lambda_2 \varepsilon)} \frac{1}{\varepsilon \lambda_1^{1/p} \lambda_2^{1/q}} (N-1)^{-\frac{\lambda_1 \varepsilon}{p}}.
 \end{aligned}$$

Thus,

$$\frac{(\alpha+1+\lambda_1 \varepsilon)(\beta+1+\lambda_2 \varepsilon)}{\lambda_1^{1/q} \lambda_2^{1/p} pq} N^{-\lambda_1 \varepsilon} \int_{N^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1+\lambda_2 \varepsilon}{\lambda_2 q}-1} dt \leq M_1 (N-1)^{-\frac{\lambda_1 \varepsilon}{p}}.$$

Let  $\varepsilon \rightarrow 0^+$ , we get

$$\frac{(\alpha+1)(\beta+1)}{\lambda_1^{1/q} \lambda_2^{1/p} pq} \int_{N^{-\lambda_1}}^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1}{\lambda_2 q}-1} dt \leq M_1.$$

Further, letting  $N \rightarrow +\infty$ , we obtain

$$\frac{(\alpha+1)(\beta+1)}{\lambda_1^{1/q} \lambda_2^{1/p} pq} \int_0^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{-\frac{\beta+1}{\lambda_2 q}-1} dt \leq M_1.$$

By the properties of the Beta function and Gamma function, when  $\alpha < -1$  and  $\beta < -1$ ,  $-\frac{\alpha+1}{\lambda_1 p} > 0$  and  $-\frac{\beta+1}{\lambda_2 q} > 0$ ; thus,

$$\begin{aligned}
 M_1 &\geq \frac{(\alpha+1)(\beta+1)}{\lambda_1^{1/q} \lambda_2^{1/p} pq} B\left(-\frac{\beta+1}{\lambda_2 q}, \lambda + \frac{\beta+1}{\lambda_2 q}\right) = \frac{(\alpha+1)(\beta+1)}{\lambda_1^{1/q} \lambda_2^{1/p} pq} B\left(-\frac{\alpha+1}{\lambda_1 p}, -\frac{\beta+1}{\lambda_2 q}\right) \\
 &= \frac{1}{\lambda_1^{1/q} \lambda_2^{1/p} \Gamma(\lambda)} \frac{(\alpha+1)(\beta+1)}{pq} \Gamma\left(-\frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(-\frac{\beta+1}{\lambda_2 q}\right) \\
 &= \lambda_1^{1/p} \lambda_2^{1/q} \frac{1}{\Gamma(\lambda)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(1 - \frac{\beta+1}{\lambda_2 q}\right) = M_0.
 \end{aligned}$$

This contradicts  $M_1 < M_0$ ; hence,  $M_0$  is the best constant factor in (5).  $\square$

Similar to the proof of Theorem 1, we can also obtain the following theorem:

**Theorem 2.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda, \alpha, \beta \in \mathbb{R}$ ,  $0 < \lambda_1 \leq 1$ ,  $\lambda_2 > 0$ ,  $\lambda > 0$ ,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} = c$ ,  $\max\{(\lambda_1 - 1)p - 1, (\lambda_1 - 1)p - 1 + \lambda_1 pc\} < \alpha < -1$ ,  $q - 1 - (\lambda + 1)\lambda_2 q < \beta < q - 1$ , and  $\lim_{n \rightarrow +\infty} e^{-tn^{\lambda_1}} A_n = 0$  ( $t > 0$ ).

(i) There exists a constant  $M > 0$  if and only if  $c \geq 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} \geq 0$ , such that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}} dx \leq M \|\tilde{A}\|_{p,\alpha} \|f\|_{q,\beta}, \quad (13)$$

where  $\tilde{A} = \{A_n\} = \{\sum_{k=1}^n a_k\}$  ( $a_k \geq 0$ ).

(ii) When  $c = 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} = 0$ , the best constant factor in (13) is

$$M_0 = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/p} \frac{1}{\Gamma(\lambda)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(\frac{1}{\lambda_2} - \frac{\beta+1}{\lambda_2 q}\right).$$

## 4. Applications

Let  $p > 1$ , and define

$$S_{\lambda_1} l_p^\alpha = \left\{ \tilde{a} = \{a_n\} (a_n \geq 0) : \|\tilde{A}\|_{p,\alpha} = \left(\sum_{n=1}^{\infty} n^\alpha |A_n|^p\right)^{1/p} < +\infty, \lim_{n \rightarrow +\infty} e^{-tn^{\lambda_1}} A_n = 0 (t > 0) \right\}.$$

Below, we discuss the boundedness of the discrete operator from  $S_{\lambda_1} l_p^\alpha$  to  $L_p^\gamma(0, +\infty)$  and the problem of operator norms.

**Theorem 3.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  ( $p > 1$ ),  $\alpha, \beta \in \mathbb{R}$ ,  $0 < \lambda_1 \leq 1$ ,  $\lambda_2 > 0$ ,  $\lambda > 0$ ,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} = c$ ,  $\max\{(\lambda_1 - 1)p - 1, (\lambda_1 - 1)p - 1 + \lambda_1 p c\} < \alpha < -1$ ,  $q - 1 - (\lambda + 1)\lambda_2 q < \beta < q - 1$ , and define the operator  $T$  as

$$T(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}}, \quad \tilde{a} = \{a_n\} \in S_{\lambda_1} l_p^\alpha.$$

(i)  $T$  is a bounded operator from  $S_{\lambda_1} l_p^\alpha$  to  $L_p^{\beta(p-1)}(0, +\infty)$  if and only if  $c \geq 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} \geq 0$ . Then, there exists a constant  $M > 0$  such that

$$\|T(\tilde{a})\|_{p,\beta(p-1)} \leq M \|\tilde{A}\|_{p,\alpha}.$$

(ii) When  $c = 0$ , i.e.,  $\frac{\alpha+1}{\lambda_1 p} + \frac{\beta+1}{\lambda_2 q} + \lambda - \frac{1}{\lambda_2} = 0$ , the norm of operator  $T$  is

$$\|T\| = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/p} \frac{1}{\Gamma(\lambda)} \Gamma\left(1 - \frac{\alpha+1}{\lambda_1 p}\right) \Gamma\left(\frac{1}{\lambda_2} - \frac{\beta+1}{\lambda_2 q}\right). \quad (14)$$

**Proof.** We first prove that when  $a_n \geq 0$ , (14) is equivalent to (13).

If (13) holds, let

$$f(x) = x^{\beta(p-1)} |T(\tilde{a})(x)|^{p-1} = x^{\beta(p-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{(n^{\lambda_1} + x^{\lambda_2})^{\lambda}}\right)^{p-1}.$$

Then,

$$\begin{aligned}\|T(\tilde{a})\|_{p,\beta(p-1)}^p &= \int_0^{+\infty} x^{\beta(p-1)} |T(\tilde{a})(x)|^p dx \\ &= \int_0^{+\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} dx = \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} dx \\ &\leq M \|\tilde{A}\|_{p,\alpha} \|f\|_{q,\beta} \\ &= M \|\tilde{A}\|_{p,\alpha} \left( \int_0^{+\infty} x^\beta (x^{\beta(1-p)} |T(\tilde{a})(x)|^{p-1})^q dx \right)^{1/q} \\ &= M \|\tilde{A}\|_{p,\alpha} \left( \int_0^{+\infty} x^{\beta(1-p)} |T(\tilde{a})(x)|^p dx \right)^{(p-1)/p} \\ &= M \|\tilde{A}\|_{p,\alpha} \|T(\tilde{a})\|_{p,\beta(1-p)}^{p-1},\end{aligned}$$

from which (14) can be derived.

Conversely, if (14) holds, by Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \int_0^{+\infty} \frac{a_n f(x)}{(n^{\lambda_1} + x^{\lambda_2})^\lambda} dx &= \int_0^{+\infty} f(x) T(\tilde{a})(x) dx \\ &= \int_0^{+\infty} \left( x^{-\frac{\beta}{q}} T(\tilde{a})(x) \right) \left( x^{\frac{\beta}{q}} f(x) \right) dx \\ &\leq \left( \int_0^{+\infty} x^{-\frac{\beta p}{q}} |T(\tilde{a})(x)|^p dx \right)^{1/p} \left( \int_0^{+\infty} x^\beta |f(x)|^q dx \right)^{1/q} \\ &= \left( \int_0^{+\infty} x^{\beta(p-1)} |T(\tilde{a})(x)|^p dx \right)^{1/p} \|f\|_{q,\beta} \\ &= \|T(\tilde{a})\|_{p,\beta(p-1)} \|f\|_{q,\beta} \leq M \|\tilde{A}\|_{p,\alpha} \|f\|_{q,\beta},\end{aligned}$$

Thus, (13) holds. Since (14) is equivalent to (13), Theorem 3 holds by Theorem 2.  $\square$

In Theorem 3, taking  $\lambda_1 = \lambda_2 = \frac{1}{2}$  and  $\lambda = 2$ , we can obtain

**Corollary 1.** Let  $\frac{1}{p} + \frac{1}{q} = 1$  ( $p > 1$ ),  $\alpha, \beta \in \mathbb{R}$ ,  $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 0$ ,  $-\frac{p}{2} - 1 < \alpha < -1$ , and  $-\frac{q}{2} - 1 < \beta < q - 1$ . Then, the operator  $T$ ,

$$T(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{(\sqrt{n} + \sqrt{x})^2}$$

is a bounded operator from  $S_{\frac{1}{2}} l_p^\alpha$  to  $L_p^{\beta(p-1)}(0, +\infty)$ , and the operator norm of  $T$  is

$$\|T\| = \Gamma\left(1 - 2\frac{\alpha+1}{p}\right) \Gamma\left(2\left(1 - \frac{\beta+1}{q}\right)\right).$$

## 5. Conclusions

In this paper, we discuss the more general semi-discrete Hilbert-type inequality involving the partial sum  $A_n$  and the variable upper limit integral function  $\tilde{F}(f)(x)$ , where the kernel is the quasi-homogeneous kernel  $\frac{1}{(n^{\lambda_1} + x^{\lambda_2})^\lambda}$ . We obtain the necessary and sufficient conditions for such inequalities to hold and the expressions of the best constant factors. Due to the full use of the construction theorem of quasi-homogeneous semi-discrete Hilbert-type inequalities, the proof process is optimized; because the parameter structure of the obtained results is simple and clear, our results are more convenient to apply. Finally, we also discuss

the application of the results in operator theory, highlighting the application value of the obtained results. In the future, we will also explore high-dimensional cases.

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