

## Article

# Completeness Theorems for Impulsive Dirac Operator with Discontinuity

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## Abstract

In this work, the discontinuous Dirac operator with weight is studied. We prove the completeness theorems of the system of eigenfunctions for the discontinuous Dirac operator.

**Keywords:** completeness theorems; impulsive Dirac operator; discontinuous condition

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## 1. Introduction

Boundary value problems with a discontinuity are associated with discontinuous characteristics of materials and have been studied by many authors (see, e.g., [1–4]). Consider the following impulsive Dirac operator:

$$ly := By'(x) + \Omega(x)y(x) = \lambda \rho(x)y(x), \quad x \in I := (0, b) \cup (b, \pi), \quad (1)$$

with boundary conditions  $(\alpha, \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}))$

$$U(y) := y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad (2)$$

$$V(y) := y_1(\pi) \cos \beta + y_2(\pi) \sin \beta = 0, \quad (3)$$

and jump conditions

$$y_1(b+0) = a_1 y_1(b-0), \quad y_2(b+0) = a_1^{-1} y_2(b-0) + a_2 y_1(b-0), \quad (4)$$

where

$$\rho(x) = \begin{cases} 1, & x \in (0, b), \\ \gamma, & x \in (b, \pi), \end{cases} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$  and  $q(x)$  are real-valued functions in  $L^2(0, \pi)$ ,  $\lambda$  is the spectral parameter,  $0 < \gamma < 1$ ,  $a_1 > 0$ , and  $a_2$  is real. Denote problem (1)–(4) by  $L = L(\Omega(x), \rho(x), \alpha, \beta, b, a_1, a_2)$ , and this kind of problem is self-adjoint.



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The completeness theorems of the continuous Dirac systems have been studied by many authors. For example, the qualities of the eigenvalues and eigenfunctions of the discontinuous Dirac system were considered by Huseynov and Latifova (see [5]). They gave the completeness theorem of eigenfunctions.

In [6], Oridoroga and Hassi presented Dirac-type operators with boundary conditions depending on the spectral parameter and proved the completeness theorems given some assumptions, and in [7], they also investigated the completeness and Riesz basis property of systems of eigenfunctions and associated functions for these operators. Later, the same authors (see [8]) extended these results to the system of integro-differential equations with some  $\lambda$ -depending boundary conditions and obtained the completeness with a similar method.

Some authors also considered the completeness theorems in discontinuous case. In [9], Tuna and Eryilmaz studied the completeness for the Dirac operator with conditions different from (2)–(4). Tuna and Kendüzler (see [10]) dealt with the Dirac operator with an eigenparameter in the boundary condition and proved the corresponding completeness.

Regarding the completeness of Dirac operators in the continuous case, several scholars have established conclusive results, demonstrating that the system of eigenfunctions for the continuous Dirac problem is complete in Hilbert space. However, no corresponding conclusions have been reached for the discontinuous case. This motivates us to investigate whether similar completeness properties hold for discontinuous Dirac operators. Consequently, we extend the completeness results of the continuous Dirac operator to the discontinuous case.

This paper generalizes the results in [5] to the discontinuous case and addresses the completeness for the operator discussed in the text in the form of (1)–(4). In Section 1, we present the completeness conclusion of the Dirac operator in the continuous case. In Section 2, the spectral properties, such as eigenvalues and eigenfunctions, are presented, providing a theoretical basis for the completeness conclusion. In Section 3, the completeness theorems and its proof are given.

## 2. Preliminaries

Let

$$\varphi(x, \lambda) = \begin{cases} \varphi_1(x, \lambda) = \begin{pmatrix} \varphi_{11}(x, \lambda) \\ \varphi_{12}(x, \lambda) \end{pmatrix}, & x \in (0, b), \\ \varphi_2(x, \lambda) = \begin{pmatrix} \varphi_{21}(x, \lambda) \\ \varphi_{22}(x, \lambda) \end{pmatrix}, & x \in (b, \pi), \end{cases}$$

$$\psi(x, \lambda) = \begin{cases} \psi_1(x, \lambda) = \begin{pmatrix} \psi_{11}(x, \lambda) \\ \psi_{12}(x, \lambda) \end{pmatrix}, & x \in (0, b), \\ \psi_2(x, \lambda) = \begin{pmatrix} \psi_{21}(x, \lambda) \\ \psi_{21}(x, \lambda) \end{pmatrix}, & x \in (b, \pi) \end{cases}$$

be the solutions of (1), satisfying the initial conditions

$$\varphi(0, \lambda) = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \end{pmatrix}, \quad \psi(\pi, \lambda) = \begin{pmatrix} \sin \beta \\ -\cos \beta \end{pmatrix} \quad (5)$$

and jump condition (4)). From [11,12], we can calculate that  $\varphi(x, \lambda)$  has the following representation:

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x K_1(x, t) \begin{pmatrix} \sin(\lambda\sigma(x)+\alpha) \\ -\cos(\lambda\sigma(x)+\alpha) \end{pmatrix} dt, \quad (6)$$

Here,  $\varphi_0(x, \lambda) = (\varphi_{01}(x, \lambda), \varphi_{02}(x, \lambda))^T$  satisfies the following forms:

$$\varphi_{01}(x, \lambda) = \begin{cases} \sin(\lambda\sigma(x)+\alpha), & 0 < x < b, \\ a_1^+ \sin(\lambda\sigma(x)+\alpha) + a_1^- \sin(\lambda(\pi-\sigma(x))+\alpha), & b < x < \pi, \end{cases} \quad (7)$$

$$\varphi_{02}(x, \lambda) = \begin{cases} -\cos(\lambda\sigma(x)+\alpha), & 0 < x < b, \\ -a_1^+ \cos(\lambda\sigma(x)+\alpha) + a_1^- \cos(\lambda(\pi-\sigma(x))+\alpha), & b < x < \pi. \end{cases} \quad (8)$$

Similarly, we can compute that the following representation holds for  $\psi(x, \lambda)$ :

$$\psi(x, \lambda) = \psi_0(x, \lambda) + \int_x^\pi K_2(x, t) \begin{pmatrix} \sin(\lambda(\sigma(\pi) - \sigma(x)) + \beta) \\ -\cos(\lambda(\sigma(\pi) - \sigma(x)) + \beta) \end{pmatrix} dt, \quad (9)$$

Here,  $\psi_0(x, \lambda) = (\psi_{01}(x, \lambda), \psi_{02}(x, \lambda))^T$  satisfies the following forms:

$$\psi_{01}(x, \lambda) = \begin{cases} a_2^+ \sin(\lambda(\sigma(\pi) - \sigma(x)) + \beta) + a_2^- \sin(\lambda(\sigma(\pi) + \sigma(x) - 2\sigma(b)) + \beta), & 0 < x < b, \\ \sin(\lambda(\sigma(\pi) - \sigma(x)) + \beta), & b < x < \pi, \end{cases} \quad (10)$$

$$\psi_{02}(x, \lambda) = \begin{cases} -a_2^+ \cos(\lambda(\sigma(\pi) - \sigma(x)) + \beta) + a_2^- \cos(\lambda(\sigma(\pi) + \sigma(x) - 2\sigma(b)) + \beta), & 0 < x < b, \\ -\cos(\lambda(\sigma(\pi) - \sigma(x)) + \beta), & b < x < \pi, \end{cases} \quad (11)$$

where  $\sigma(x) = \int_0^x \sqrt{\rho(t)} dt$ ,  $a_1^\pm = \frac{1}{2}(a_1 \pm \frac{1}{\gamma a_1})$ ,  $a_2^\pm = \frac{1}{2}(\frac{1}{a_1} \pm \gamma a_1)$ , and  $K_n(x, t) = (K_{ijn}(x, t))_{i,j=1,2}$  ( $n = 1, 2$ ) with  $K_{ijn}(x, t)$  are real-valued continuous functions for  $i, j = 1, 2$ .

Define

$$\langle \varphi(x, \lambda), \psi(x, \lambda) \rangle = \begin{cases} \langle \varphi_1(x, \lambda), \psi_1(x, \lambda) \rangle := \varphi_{11}(x, \lambda)\psi_{12}(x, \lambda) - \varphi_{12}(x, \lambda)\psi_{11}(x, \lambda), & 0 < x < b, \\ \langle \varphi_2(x, \lambda), \psi_2(x, \lambda) \rangle := \varphi_{21}(x, \lambda)\psi_{22}(x, \lambda) - \varphi_{22}(x, \lambda)\psi_{21}(x, \lambda), & b < x < \pi. \end{cases}$$

It is easy to verify that  $\langle \varphi(x, \lambda), \psi(x, \lambda) \rangle$  is independent of  $x$  (see [13]), and

$$\langle \varphi(x, \lambda), \psi(x, \lambda) \rangle \Big|_{x=b-0} = \langle \varphi(x, \lambda), \psi(x, \lambda) \rangle \Big|_{x=b+0}. \quad (12)$$

Denote

$$\Delta(\lambda) := \langle \varphi(x, \lambda), \psi(x, \lambda) \rangle = \varphi_1(x, \lambda)\psi_2(x, \lambda) - \varphi_2(x, \lambda)\psi_1(x, \lambda) \quad (13)$$

The function  $\Delta(\lambda)$  is called the characteristic function of  $L$ , which is complete in  $\lambda$ , and it has the zero set  $\{\lambda_n\}_{n \in \mathbb{Z}}$ .

**Lemma 1.** (1) The zero set  $\{\lambda_n\}_{n \in \mathbb{Z}}$  of the characteristic function  $\Delta(\lambda)$  coincides with the eigenvalues of problem L.

(2) The corresponding eigenfunctions  $\varphi(x, \lambda_n)$  and  $\psi(x, \lambda_n)$  satisfy the following equality:

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0, \quad n \in \mathbb{Z}. \quad (14)$$

**Proof.** Following a line of reasoning similar to the proof in [2,4,14], we skip the details here.  $\square$

Next, we denote by  $\alpha_n (n \in \mathbb{Z})$  the normalized constants, which are defined as

$$\alpha_n := \int_0^\pi \|\varphi(x, \lambda_n)\|^2 \rho(x) dx, \quad n \in \mathbb{Z}.$$

**Lemma 2.** The formula holds as follows:

$$\beta_n \alpha_n = -\dot{\Delta}(\lambda_n), \quad (15)$$

where  $\dot{\Delta}(\lambda_n) = \left( \frac{d}{d\lambda} \Delta(\lambda) \right) \Big|_{\lambda=\lambda_n}$ .

**Proof.** Since

$$\begin{aligned} B\psi'(x, \lambda) + \Omega(x)\psi(x, \lambda) &= \lambda\rho(x)\psi(x, \lambda), \\ B\varphi'(x, \lambda_n) + \Omega(x)\varphi(x, \lambda_n) &= \lambda_n\rho(x)\varphi(x, \lambda_n). \end{aligned}$$

It follows from the above formulae that

$$\frac{d}{dx} \langle \varphi(x, \lambda_n), \psi(x, \lambda) \rangle = (\lambda - \lambda_n) \rho(x) \varphi(x, \lambda_n) \psi^T(x, \lambda). \quad (16)$$

The integration of (16) is carried out over the interval  $[0, \pi]$ , and together with (16), we can get

$$\begin{aligned} & \langle \varphi(x, \lambda_n), \psi(x, \lambda) \rangle \Big|_0^{b-0} + \langle \psi(x, \lambda_n), \varphi(x, \lambda) \rangle \Big|_{b+0}^\pi \\ &= (\lambda - \lambda_n) \int_0^\pi \rho(x) \varphi(x, \lambda_n) \psi^T(x, \lambda) dx \\ &= -(\varphi_{21}(\pi, \lambda_n) \cos \beta + \varphi_{21}(\pi, \lambda_n) \sin \beta) - (\psi_{11}(0, \lambda) \cos \alpha + \varphi_{12}(\pi, \lambda_n) \sin \alpha) \\ &= -\Delta(\lambda). \end{aligned}$$

Dividing the two sides by  $\lambda - \lambda_n$  and letting  $\lambda \rightarrow \lambda_n$ , we have

$$\int_0^\pi \rho(x) \varphi(x, \lambda_n) \psi^T(x, \lambda_n) dx = -\dot{\Delta}(\lambda_n).$$

Together with the above formula and the definition of  $\alpha_n$ , we arrive at (15).  $\square$

From (7)–(11) and (12)–(13), we can calculate

$$\Delta(\lambda) = \Delta_0(\lambda) + o(\exp |\tau| \sigma(\pi)), \quad (17)$$

where

$$\begin{aligned}\Delta_0(\lambda) &= a_1^+ \sin(\lambda\sigma(\pi) + \alpha - \beta) + a_1^- \sin(\lambda(2\sigma(b) - \sigma(T)) + \alpha + \beta) \\ &\quad + \frac{a_2}{2\gamma} \cos(\lambda\sigma(\pi) + \alpha - \beta) - \frac{a_2}{2\gamma} \cos(\lambda(2\sigma(b) - \sigma(T)) + \alpha + \beta).\end{aligned}\quad (18)$$

Define the sector  $G_\varepsilon := \{\lambda : |\lambda - \lambda_n^0| \geq \varepsilon > 0, n \in \mathbb{Z}\}$ , where  $\lambda_n^0$  are zeros of the function  $\Delta_0(\lambda)$ . Asymptotic formulas (17) and (18) imply

$$|\Delta(\lambda)| \geq C_\varepsilon e^{|\Im \lambda| \sigma(\pi)}, \quad \lambda \in G_\varepsilon, \quad (19)$$

where  $C_\varepsilon$  is a constant.

### 3. Completeness Theorem

Define  $L^2((0, \pi), \mathbb{C}^2, \rho)$  as the Hilbert space of vector-valued functions with two components. This Hilbert space has the following inner product:

$$(f, g) = \int_0^\pi g(x) f^T(x) \rho(x) dx.$$

Next, applying an approach similar to that in [15], we can construct Green's function and represent the solution, which is given below by Green's function.

Let us consider the boundary value problem  $L_f$ , which consists of a non-homogeneous differential equation,

$$ly := By'(x) + \Omega(x)y(x) = \lambda \rho(x)y(x) + f^T(x)\rho(x), \quad x \in I \quad (20)$$

with boundary conditions (2) and jump conditions (4), and then calculate the solution of this problem.

First, consider the solution of problems (1)–(4) with an homogeneous Sturm–Liouville equation, which has the form

$$Y(x, \lambda) = \begin{cases} C_1 \varphi_1(x, \lambda) + S_1 \psi_1(x, \lambda), & 0 < x < b, \\ C_2 \varphi_2(x, \lambda) + S_2 \psi_2(x, \lambda), & b < x < \pi, \end{cases}$$

where  $C_1, C_2, S_1$ , and  $S_2$  are some constants.

Using the method of variation in the constants, we can obtain the solution of  $L_f$  in the form

$$Y(x, \lambda) = \begin{cases} C_1(x, \lambda) \varphi_1(x, \lambda) + S_1(x, \lambda) \psi_1(x, \lambda), & 0 < x < b, \\ C_2(x, \lambda) \varphi_2(x, \lambda) + S_2(x, \lambda) \psi_2(x, \lambda), & b < x < \pi, \end{cases} \quad (21)$$

where functions  $C_1(x, \lambda), S_1(x, \lambda), C_2(x, \lambda)$ , and  $S_2(x, \lambda)$  satisfy the linear system of equations for  $x \in (0, b)$ :

$$\begin{cases} C_1'(x, \lambda) \varphi_1(x, \lambda) + S_1'(x, \lambda) \psi_1(x, \lambda) = 0, \\ C_1'(x, \lambda) \varphi_1'(x, \lambda) + S_1'(x, \lambda) \psi_1'(x, \lambda) = f(x) \rho(x), \end{cases} \quad (22)$$

And for  $x \in (b, \pi)$ ,

$$\begin{cases} C_2'(x, \lambda) \varphi_2(x, \lambda) + S_2'(x, \lambda) \psi_2(x, \lambda) = 0, \\ C_2'(x, \lambda) \varphi_2'(x, \lambda) + S_2'(x, \lambda) \psi_2'(x, \lambda) = f(x) \rho(x). \end{cases} \quad (23)$$

Since  $\lambda$  is not an eigenvalue, we have

$$\langle \varphi_1(x, \lambda), \psi_1(x, \lambda) \rangle = \langle \varphi_2(x, \lambda), \psi_2(x, \lambda) \rangle \neq 0.$$

This implies that (22) and (23) have a unique solution, respectively. It follows that for  $x \in (0, b)$ ,

$$C_1'(x, \lambda) = -\frac{1}{\Delta(\lambda)} \psi_1(x, \lambda) f^T(x) \rho(x), \quad S_1'(x, \lambda) = \frac{1}{\Delta(\lambda)} \varphi_1(x, \lambda) f^T(x) \rho(x) \quad (24)$$

and for  $x \in (b, \pi)$

$$C_2'(x, \lambda) = -\frac{1}{\Delta(\lambda)} \psi_2(x, \lambda) f^T(x) \rho(x), \quad S_2'(x, \lambda) = \frac{1}{\Delta(\lambda)} \varphi_2(x, \lambda) f^T(x) \rho(x). \quad (25)$$

From (24) and (25), we can obtain that

$$\begin{aligned} C_1(x, \lambda) &= \frac{1}{\Delta(\lambda)} \int_x^b \psi_1(t, \lambda) f^T(t) \rho(t) dt + A_1, & x \in (0, b), \\ S_1(x, \lambda) &= \frac{1}{\Delta(\lambda)} \int_0^x \varphi_1(t, \lambda) f^T(t) \rho(t) dt + A_2, & x \in (0, b), \\ C_2(x, \lambda) &= \frac{1}{\Delta(\lambda)} \int_x^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + A_3, & x \in (b, \pi), \\ S_2(x, \lambda) &= \frac{1}{\Delta(\lambda)} \int_b^x \varphi_2(t, \lambda) f^T(t) \rho(t) dt + A_4, & x \in (b, \pi), \end{aligned}$$

where  $A_1, A_2, A_3$ , and  $A_4$  are some constants. Substituting the above equalities into (21), we have

$$Y(x, \lambda) = \begin{cases} \frac{\varphi_1(x, \lambda)}{\Delta(\lambda)} \int_x^b \psi_1(t, \lambda) f^T(t) \rho(t) dt + \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} \int_0^x \varphi_1(t, \lambda) f^T(t) \rho(t) dt \\ + A_1 \varphi_1(x, \lambda) + A_2 \psi_1(x, \lambda), & x \in (0, b), \\ \frac{\varphi_2(x, \lambda)}{\Delta(\lambda)} \int_x^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \int_b^x \varphi_2(t, \lambda) f^T(t) \rho(t) dt \\ + A_3 \varphi_2(x, \lambda) + A_4 \psi_2(x, \lambda), & x \in (b, \pi). \end{cases} \quad (26)$$

Let solution  $Y(x, \lambda)$  satisfy boundary conditions  $U(y)$  and  $V(y)$ , respectively. It yields

$$\begin{aligned} U(Y) &= \left( \frac{\varphi_{11}(0, \lambda)}{\Delta(\lambda)} \int_0^b \psi_1(t, \lambda) f^T(t) \rho(t) dt + A_1 \varphi_{11}(0, \lambda) + A_2 \psi_{11}(0, \lambda) \right) \cos \alpha \\ &+ \left( \frac{\varphi_{12}(0, \lambda)}{\Delta(\lambda)} \int_0^b \psi_1(t, \lambda) f^T(t) \rho(t) dt + A_1 \varphi_{12}(0, \lambda) + A_2 \psi_{12}(0, \lambda) \right) \sin \alpha. \end{aligned}$$

Since

$$A_1(\varphi_{11}(0, \lambda) \cos \alpha + \varphi_{12}(0, \lambda) \sin \alpha) = 0$$

and

$$A_2(\psi_{11}(0, \lambda) \cos \alpha + \psi_{12}(0, \lambda) \sin \alpha) = A_2 \Delta(\lambda),$$

Then, we have

$$U(Y) = A_2 \Delta(\lambda). \quad (27)$$

Similarly,

$$V(Y) = \left( \frac{\psi_{21}(\pi, \lambda)}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + A_3 \varphi_{21}(\pi, \lambda) + A_4 \psi_{21}(\pi, \lambda) \right) \cos \beta \\ + \left( \frac{\psi_{22}(\pi, \lambda)}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + A_3 \varphi_{22}(\pi, \lambda) + A_4 \psi_{22}(\pi, \lambda) \right) \sin \beta.$$

Since

$$A_4(\psi_{21}(\pi, \lambda) \cos \beta + \psi_{22}(\pi, \lambda) \sin \beta) = 0$$

and

$$A_3(\varphi_{21}(0, \lambda) \cos \beta + \varphi_{22}(0, \lambda) \sin \beta) = A_3 \Delta(\lambda),$$

we can obtain

$$V(Y) = A_3 \Delta(\lambda). \quad (28)$$

It follows from (27) and (28) that

$$A_2 = A_3 = 0. \quad (29)$$

Next, we shall calculate constants  $A_1$  and  $A_4$ . Considering jump conditions (4), we have

$$\frac{\varphi_{21}(b+0, \lambda)}{\Delta(\lambda)} \int_{b+0}^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + A_3 \varphi_{21}(b+0, \lambda) + A_4 \psi_{21}(b+0, \lambda) \\ = a_1 \left( \frac{\psi_{11}(b-0, \lambda)}{\Delta(\lambda)} \int_0^{b-0} \varphi_1(t, \lambda) f^T(t) \rho(t) dt + A_1 \varphi_{11}(b-0, \lambda) + A_2 \psi_{11}(b-0, \lambda) \right)$$

and

$$\frac{\varphi_{22}(b+0, \lambda)}{\Delta(\lambda)} \int_{b+0}^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + A_3 \varphi_{22}(b+0, \lambda) + A_4 \psi_{22}(b+0, \lambda) \\ = a_1^{-1} \left( \frac{\psi_{12}(b-0, \lambda)}{\Delta(\lambda)} \int_0^{b-0} \varphi_1(t, \lambda) f^T(t) \rho(t) dt + A_1 \varphi_{12}(b-0, \lambda) + A_2 \psi_{12}(b-0, \lambda) \right) \\ + a_2 \left( \frac{\psi_{11}(b-0, \lambda)}{\Delta(\lambda)} \int_0^{b-0} \varphi_1(t, \lambda) f^T(t) \rho(t) dt + A_1 \varphi_{11}(b-0, \lambda) + A_2 \psi_{11}(b-0, \lambda) \right).$$

Together with the above formulae with (26), we can get

$$\left\{ \begin{array}{l} -A_1 a_1 \varphi_{11}(b-0, \lambda) + A_4 \psi_{21}(b+0, \lambda) \\ = \frac{a_1 \psi_{11}(b-0, \lambda)}{\Delta(\lambda)} \int_0^b \varphi_1(t, \lambda) f^T(t) \rho(t) dt - \frac{\varphi_{21}(b-0, \lambda)}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt, \\ -A_1 (a_2 \varphi_{11}(b-0, \lambda) + a_1^{-1} \varphi_{12}(b-0, \lambda)) + A_4 \psi_{22}(b+0, \lambda) \\ = \frac{\varphi_{22}(b+0, \lambda)}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt + \frac{a_1^{-1} \psi_{12}(b-0, \lambda)}{\Delta(\lambda)} \int_0^b \varphi_1(t, \lambda) f^T(t) \rho(t) dt \\ + \frac{a_2 \psi_{11}(b-0, \lambda)}{\Delta(\lambda)} \int_0^b \varphi_1(t, \lambda) f^T(t) \rho(t) dt, \end{array} \right. \quad (30)$$

Given the definition of solutions  $\varphi_i(x, \lambda)$  and  $\psi_i(x, \lambda)$  ( $i = 1, 2$ ), and with  $\lambda$  not being an eigenvalue, the following determinant holds for (30):

$$\begin{aligned} & \begin{vmatrix} -a_1\varphi_{11}(b-0, \lambda) & \psi_{21}(b+0, \lambda) \\ -(a_2\varphi_{11}(b-0, \lambda) + a_1^{-1}\varphi_{12}(b-0, \lambda)) & \psi_{22}(b+0, \lambda) \end{vmatrix} \\ &= \begin{vmatrix} -\varphi_{21}(b+0, \lambda) & \psi_{21}(b+0, \lambda) \\ -\varphi_{22}(b+0, \lambda) & \psi_{22}(b+0, \lambda) \end{vmatrix} \\ &= \Delta(\lambda) \neq 0 \end{aligned}$$

Since this determinant is not equal to zero, the solution of (30) is unique. By solving system (30), we have

$$A_1 = \frac{1}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt, \quad (31)$$

$$A_4 = \frac{1}{\Delta(\lambda)} \int_0^b \varphi_1(t, \lambda) f^T(t) \rho(t) dt. \quad (32)$$

Substituting  $A_i$  ( $i = \overline{1, 4}$ ) into (26), we get

$$Y(x, \lambda) = \begin{cases} \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} \int_0^x \varphi_1(t, \lambda) f^T(t) \rho(t) dt + \frac{\varphi_1(x, \lambda)}{\Delta(\lambda)} \int_x^b \psi_1(t, \lambda) f^T(t) \rho(t) dt \\ + \frac{\varphi_1(x, \lambda)}{\Delta(\lambda)} \int_b^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt, & x \in (0, b), \\ \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \int_b^x \varphi_2(t, \lambda) f^T(t) \rho(t) dt + \frac{\varphi_2(x, \lambda)}{\Delta(\lambda)} \int_x^\pi \psi_2(t, \lambda) f^T(t) \rho(t) dt \\ + \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \int_0^b \varphi_1(t, \lambda) f^T(t) \rho(t) dt, & x \in (b, \pi). \end{cases} \quad (33)$$

Solution (33) can be rewritten as

$$Y(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} \int_0^x \varphi(t, \lambda) f^T(t) \rho(t) dt + \frac{\varphi(x, \lambda)}{\Delta(\lambda)} \int_x^\pi \psi(t, \lambda) f^T(t) \rho(t) dt. \quad (34)$$

Denote Green's function as follows:

$$G(x, t, \lambda) = \begin{cases} \frac{1}{\Delta(\lambda)} \psi(x, \lambda) \varphi(t, \lambda), & 0 \leq t \leq x \leq \pi, x \neq b, y \neq b, \\ \frac{1}{\Delta(\lambda)} \varphi(x, \lambda) \psi(t, \lambda), & 0 \leq x \leq t \leq \pi, x \neq b, y \neq b. \end{cases}$$

We can represent (34) with Green's function as the follows:

$$Y(x, \lambda) = \int_0^\pi G(x, t, \lambda) f^T(t) \rho(t) dt. \quad (35)$$

Finally, the auxiliary lemma below is needed. Since its proof is very similar to the ones in (Ref. [15], Lemma 1.3.1) and (Ref. [2], P. 108), we do not give further details.

**Lemma 3.** For every vector-valued function  $f(x) \in L^2((0, \pi), \mathbb{C}^2, \rho)$ , the following equality is valid:

$$\begin{aligned} & \lim_{|\lambda| \rightarrow \infty} \max_{0 \leq x < b} e^{-|\Im \lambda| \sigma(\pi)} \left| \int_0^x \varphi_1(t, \lambda) f^T(t) \rho(t) dt \right| \\ &= \lim_{|\lambda| \rightarrow \infty} \max_{0 \leq x < b} e^{-|\Im \lambda| (\sigma(\pi) - \sigma(x))} \left| \int_x^b \psi_1(t, \lambda) f^T(t) \rho(t) dt \right| \\ &= \lim_{|\lambda| \rightarrow \infty} e^{-|\Im \lambda| \sigma(\pi)} \left| \int_0^b \varphi_1(t, \lambda_n) f^T(t) \rho(t) dt \right| = 0, \quad x \in (0, b) \end{aligned}$$

and

$$\begin{aligned} & \lim_{|\lambda| \rightarrow \infty} \max_{b < x \leq \pi} e^{-|\Im \lambda| \sigma(\pi)} \left| \int_b^x \varphi_2(t, \lambda_n) f^T(t) \rho(t) dt \right| \\ &= \lim_{|\lambda| \rightarrow \infty} \max_{b < x \leq \pi} e^{-|\Im \lambda| (\sigma(\pi) - \sigma(x))} \left| \int_x^\pi \psi_2(t, \lambda_n) f^T(t) \rho(t) dt \right| \\ &= \lim_{|\lambda| \rightarrow \infty} e^{-|\Im \lambda| \sigma(\pi)} \left| \int_b^\pi \psi_2(t, \lambda_n) f^T(t) \rho(t) dt \right| = 0, \quad x \in (b, \pi). \end{aligned}$$

**Theorem 1.** (Completeness theorem) The system of eigenfunctions  $\{\varphi(x, \lambda_n)\}_{n \in \mathbb{Z}}$  of problems (1)–(4) is complete in the space  $L^2((0, \pi), \mathbb{C}^2, \rho)$ .

**Proof.** From (33), we can get

$$\Re_{\lambda=\lambda_n} Y(x, \lambda) = \begin{cases} \frac{\psi_1(x, \lambda_n)}{\Delta(\lambda_n)} \int_0^x \varphi_1(t, \lambda_n) f^T(t) \rho(t) dt + \frac{\varphi_1(x, \lambda_n)}{\Delta(\lambda_n)} \int_x^b \psi_1(t, \lambda_n) f^T(t) \rho(t) dt \\ + \frac{\varphi_1(x, \lambda_n)}{\Delta(\lambda_n)} \int_b^\pi \psi_2(t, \lambda_n) f^T(t) \rho(t) dt, & x \in (0, b), \\ \frac{\psi_2(x, \lambda_n)}{\Delta(\lambda_n)} \int_b^x \varphi_2(t, \lambda_n) f^T(t) \rho(t) dt + \frac{\varphi_2(x, \lambda_n)}{\Delta(\lambda_n)} \int_x^\pi \psi_2(t, \lambda_n) f^T(t) \rho(t) dt \\ + \frac{\psi_2(x, \lambda_n)}{\Delta(\lambda_n)} \int_0^b \varphi_1(t, \lambda_n) f^T(t) \rho(t) dt, & x \in (b, \pi). \end{cases}$$

Using (2) of Lemmas 1 and 2, we have

$$\begin{aligned} \Re_{\lambda=\lambda_n} Y(x, \lambda) &= \frac{\beta_n}{\Delta(\lambda_n)} \varphi(x, \lambda_n) \int_0^\pi \varphi(t, \lambda_n) f^T(t) \rho(t) dt \\ &= -\frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^\pi \varphi(t, \lambda_n) f^T(t) \rho(t) dt, \quad x \in I. \end{aligned} \quad (36)$$

Let  $f(x) \in L^2((0, \pi), \mathbb{C}^2, \rho)$  be such that

$$(f(x), \varphi(x, \lambda_n)) = \int_0^\pi \varphi(t, \lambda_n) f^T(t) \rho(t) dt = 0, \quad n \in \mathbb{Z}.$$

In view of (36), this yields  $\Re_{\lambda=\lambda_n} Y(x, \lambda) = 0$ . It also implies that for every fixed value of  $x \in I$ , function  $Y(x, \lambda)$  is an entire function with respect to  $\lambda$ . By virtue of Lemma 3.1, we have

$$\lim_{\substack{|\lambda| \rightarrow \infty \\ \lambda \in G_\delta}} \max_{x \in I} |Y(x, \lambda)| = 0.$$

This yields  $Y(x, \lambda) \equiv 0$ . Together with (35), it follows that  $f(x) = 0$  a.e. on  $(0, \pi)$ . This completes the proof.  $\square$

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