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Solution of Nonhomogeneous Linear System of Caputo Fractional Differential Equations with Initial Conditions

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Abstract

The solution of a nonhomogeneous linear Caputo fractional differential equation of order $nq, (n-1) < nq < n$ with Caputo fractional initial conditions can be expressed using suitable Mittag–Leffler functions. In order to extend this result to such a nonhomogeneous linear Caputo fractional differential equation of order $nq, (n-1) < nq < n$, that also includes lower order fractional derivative terms, we can reduce such a problem to an n-system of Caputo fractional differential equations of order $q,\ 0 < q < 1$, with corresponding initial conditions. In this work, we use an approximation method to solve the resulting system of Caputo fractional differential equations of order q with initial conditions, using the fundamental matrix solutions involving the matrix Mittag–Leffler functions. Furthermore, we compute the fundamental matrix solution using the standard eigenvalue method. This fundamental matrix solution then allows us to express the component-wise solutions of the system using initial conditions, similar to the scalar case. As a consequence, we obtain solutions to linear nonhomogeneous Caputo fractional differential equations of order $nq, (n-1) < nq < n$, with Caputo fractional initial conditions having lower-order Caputo derivative terms. We illustrate the method with several examples for two and three system, considering cases where the eigenvalues are real and distinct, real and repeated, or complex conjugates.

Keywords: caputo fractional derivative; fractional initial value problem; matrix Mittag–Leffler function

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1. Introduction

The idea of fractional-order derivative dates back to the 17th century. Later, in the 19th century, the foundational mathematics of fractional-order derivatives had been significantly developed. Slowly, it became evident that the fractional derivative of arbitrary order offers a strong framework for modeling complex real-world phenomena across various scientific and engineering disciplines. See [1–8] for more recent applications on fractional differential equations. Fractional differential systems generalize the classical integer-order differential system. Integer-order derivatives are local in nature, whereas fractional derivatives possess a global nature, making them more suitable for modeling dynamic systems. For more on the theory of fractional differential equations, see [9,10]. In fractional derivatives, q can be chosen as a parameter, which enables better alignment with

empirical data. This adaptability has attracted growing interest among researchers, leading to widespread applications and advancements in fractional-order systems. See [11], where the authors fitted the realistic data using the proper value of q as a parameter. This requires the analytical and numerical computation of the solution of the linear Caputo fractional differential equation with initial and boundary conditions.

Using both approximation and Laplace transform methods, we can obtain the solution of the linear nonhomogeneous Caputo fractional differential equation of order nq, where $(n-1) < nq < n$,

$$ {}^{c}D_{a+}^{nq} u(t) = \pm\lambda u(t) + f(t), \quad \lambda \in \mathbb{R}, \tag{1}$$

subject to the fractional initial conditions

$$ {}^{c}D_{a+}^{kq} u(t)\big|_{t=a} = a_k, \quad \text{for } k = 0, 1, 2, \ldots, n-1. \tag{2}$$

In particular, if $f(t) = 0$, then the explicit solution in terms of the Mittag–Leffler function is given by

$$ u(t) = \sum_{k=0}^{n-1} a_k (t-a)^{kq} E_{nq,kq+1}(\pm\lambda (t-a)^{nq}). \tag{3}$$

In particular, when $q = 1$, the solution of the corresponding integer-order differential equation can be obtained as a special case. It has been established in [12] that the operator ${}^{c}D_{a+}^{nq} u(t)$, is sequential of order q, providing an elegant way to obtain the solution. In this work, we address the question of whether the existing results can be extended to linear nonhomogeneous Caputo fractional differential equations that include lower-order fractional derivatives along with fractional initial conditions. The presence of fractional initial conditions is essential for the formulation. Under this formulation, such equations can be reduced to a system of linear Caputo fractional differential equations of order q with corresponding initial conditions. In this work, we provide a methodology for solving linear nonhomogeneous system of Caputo fractional differential equations of order $q, 0 < q < 1$, with initial conditions. We use an approximate method, starting with the initial condition as the first approximation. The final form of the solution is then obtained using the eigenvalue method associated with the constant matrix A. This enables us to find the fundamental matrix solution of the Caputo fractional differential system of order q in terms of appropriate vector Mittag–Leffler functions. The resulting solution of the Caputo fractional differential system with initial conditions yields the integer-order solution of the system with initial conditions of the form:

$$ u' = Au, \quad u(a) = u_0. $$

Then the solution

$$ u = e^{A(t-a)} u_0, $$

can be obtained as a special case.

In the integer-order case, e^{-At} is the inverse of the matrix e^{At}. However, this is not true for the Mittag–Leffler functions, as they do not possess the same structure or properties as the exponential function.

We use an approximate method to obtain a similar solution form which yields the integer-order result as a special case, when $q = 1$. A key advantage of this formulation is that the value of q can be used as a parameter to improve the accuracy of the underlying mathematical model.

We provide several illustrative examples where the eigenvalues of the system matrix A are real and distinct, real and repeated, or form complex conjugate pairs. Our result extends the known solution of scalar Caputo fractional differential equations of order nq, with lower-order fractional derivatives with fractional initial conditions, to the system case.

The organization of this article is as follows. In Section 2, we recall relevant definitions and known results that form the foundation for our main contributions. Section 3 is devoted to developing a method for solving linear nonhomogeneous system of Caputo fractional differential equations with constant coefficients and fractional initial conditions. We present an elegant approach for expressing solutions of the linear nonhomogeneous Caputo fractional differential equations of order nq, where $(n-1) < nq < n$, using appropriate Mittag–Leffler functions. Several examples, including scalar Caputo fractional differential equations of order $2q$ and $3q$ with lower-order terms, are provided to illustrate the applicability and effectiveness of our results.

2. Preliminary Results

In this section, we recall necessary definitions and known results that play a significant role in our main results. In our definitions, Γ represents the Gamma function.

Definition 1. *The Riemann–Liouville (left-sided) fractional derivative of $u(t)$ of order nq, where $(n-1) < nq < n$, is defined by*

$$D^{q}_{a+}u(t) = \frac{1}{\Gamma(n-nq)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-nq-1}u(s)ds,\ t > a. \tag{4}$$

Definition 2. *The Caputo (left-sided) fractional derivative of $u(t)$ of order nq, $(n-1) < nq < n$, is defined by*

$${}^{c}D^{nq}_{a+}u(t) = \frac{1}{\Gamma(n-nq)}\int_{a}^{t}(t-s)^{n-nq-1}u^{(n)}(s)ds,\ t > a, \tag{5}$$

where $u^{(n)}(t) = \frac{d^n}{dt^n}u(t)$.

See [4,7,13] for more details on the Caputo and Riemann–Liouville fractional derivatives.

Definition 3. *The Caputo (left) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:*

$${}^{c}D^{q}_{a+}u(t) = \frac{1}{\Gamma(1-q)}\int_{a}^{t}(t-s)^{-q}u'(s)\mathrm{d}s. \tag{6}$$

In our definition of fractional derivative, we have used the notation ${}^{c}D^{q}_{a+}u(t)$, where the letter c indicates that it is a Caputo fractional derivative. Furthermore, when $q = 1$, it yields the integer-order derivative result as a special case.

The next definition is useful in our basic Caputo fractional differential inequalities.

Definition 4. *If the Caputo derivative of a function $u(t)$ of order $q, 0 < q \leq 1$, exists on an interval $J = [a, T], T > a$, then we say, $u \in C^{q}[J, R)$, where $J = [a, T], T > a$.*

If $u \in C^{1}[a, T]$, then certainly ${}^{c}D^{q}_{a+}u(t)$, exists on $[a, T]$. Note that all C^{1} function on $[a, T]$ are C^{q} functions on $[a, T]$. However, the converse need not be true. For example, the function $f(t) = (t-a)^{\omega}$, for any ω is a C^{q} function, when $q \leq \omega < 1$. However, it is easy to see that $f'(t)$ does not exist at $t = a$.

Next, we define the two-parameter Mittag–Leffler function, which will be useful in solving the system of linear Caputo fractional differential equations.

Definition 5. *The two-parameter Mittag–Leffler function is defined as*

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)}, \tag{7}$$

where $q, r > 0$, and λ is a constant. Furthermore, for $r = q$, (7) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, \quad q > 0. \tag{8}$$

If $r = 1$ in (7), then

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}, \quad q > 0. \tag{9}$$

If $q = r = 1$, in the above definition (7), then we have the usual exponential function $e^{\lambda t}$.

In the above definition (7), λ is a real number and that can be replaced by $\lambda + i\mu$, where λ and μ are real numbers. See [13,14] for more details on the Mittag–Leffler functions.

If λ is replaced by a square matrix A in (7), then we get the corresponding matrix Mittag–Leffler function.

Definition 6. *The matrix Mittag–Leffler function $E_{q,r}(At^q)$, is defined by the series*

$$E_{q,r}(At^q) = \sum_{k=0}^{\infty} \frac{(At^q)^k}{\Gamma(kq+r)},$$

where A is a square matrix and $q, r > 0$.

In this work, we will only need the matrix Mittag–Leffler function for the cases when $r = 1$ and $r = q$. See [15] for some work on matrix Mittag–Leffler functions.

Next, we have defined the generalized fractional trigonometric functions of order q, which will be required in our main results.

Definition 7. *The generalized fractional trigonometric functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,1}((\lambda + i\mu)t^q)$, are given by*

$$G\sin_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2i}[E_{q,1}((\lambda + i\mu)t^q) - E_{q,1}((\lambda - i\mu)t^q)], \tag{10}$$

and

$$G\cos_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda + i\mu)t^q) + E_{q,1}((\lambda - i\mu)t^q)], \tag{11}$$

respectively.

We can also define $G\sin_{q,q}((\lambda + i\mu)t^q)$ and $G\cos_{q,q}((\lambda + i\mu)t^q)$ in a similar way.

In above definitions, if $\lambda = 0$, then it will give $\sin_{q,1}(\mu t^q)$ and $\cos_{q,1}(\mu t^q)$ functions. Our definition is consistent with the classical trigonometric functions of the integer-order when $q = 1$.

It is easy to observe that in the definition of

$$G\sin_{q,1}((\lambda + i\mu)t^q) \quad \text{and} \quad G\cos_{q,1}((\lambda + i\mu)t^q),$$

if μ is replaced by $-\mu$, then

$$G\sin_{q,1}((\lambda - i\mu)t^q) = -Gsin_{q,1}((\lambda + i\mu)t^q), \tag{12}$$

and

$$G\cos_{q,1}((\lambda - i\mu)t^q) = G\cos_{q,1}((\lambda + i\mu)t^q). \tag{13}$$

Furthermore, it is easy to see that,

$${}^cD^q{}_{0+}G\sin_{q,1}((\lambda + i\mu)t^q) = \lambda G\sin_{q,1}((\lambda + i\mu)t^q) + \mu G\cos_{q,1}((\lambda + i\mu)t^q), \tag{14}$$

and

$${}^cD^q{}_{0+}G\cos_{q,1}((\lambda + i\mu)t^q) = \lambda G\cos_{q,1}((\lambda + i\mu)t^q) - \mu G\sin_{q,1}((\lambda + i\mu)t^q), \tag{15}$$

respectively.

Consider the scalar Caputo fractional linear initial value problem

$${}^cD^q{}_{a+}u = \lambda u + h(t), \quad u(a) = u_0, \tag{16}$$

with $t \in J$ and $u \in C^q(J)$. Here, $h(t) \in C(J \times \mathbb{R}, \mathbb{R})$, the space of continuous functions from J to $\mathbb{R}$. As seen in [4,7,16], if $u \in C^q(J)$, then the solution of (16) can be written as

$$u(t) = u_0E_{q,1}(\lambda t^q) + \int_a^t (t-s)^{(q-1)}E_{q,q}(\lambda(t-s)^q)h(s)ds. \tag{17}$$

Note that this solution is a C^q solution on the interval $[a, T]$ for any $T > a$. See [17] for the existence and uniqueness of solutions to the Caputo fractional differential equations.

Also, consider the linear nonhomogeneous Caputo fractional differential equation of order nq, where $(n-1) < nq < n$:

$${}^cD^{nq}_{a+}u(t) = \lambda u(t) + f(t), \quad \lambda \in \mathbb{R}, \tag{18}$$

together with the fractional initial conditions:

$${}^cD^{kq}_{a+}u(t)\big|_{t=a} = a_k, \quad \text{for } k = 0, 1, 2, \ldots, n-1. \tag{19}$$

We assume that $f(t) \in C[a, T)$, where $T > a$. The solution to the above equation, along with the given initial conditions, resembles the solution of an integer-order differential equation of order n. However, in this fractional case, the initial conditions involve derivatives of fractional order. Since the Caputo fractional derivative of order nq is a sequential Caputo derivative of order q, the initial conditions are expressed in terms of the Caputo fractional derivatives of order kq, where $k = 0, 1, 2, \ldots, n-1$.

Using the basis solutions of the homogeneous equation ${}^cD^{nq}_{a+}u = 0$, namely

$$1, \quad \frac{(t-a)^q}{\Gamma(q+1)}, \quad \ldots, \quad \frac{(t-a)^{(n-1)q}}{\Gamma((n-1)q+1)},$$

we can construct the general solution to Equation (18) as a C^{nq} function on the interval $[a, T)$, for any $T > a$.

The solution $u(t)$ is then given by:

$$u(t) = \sum_{k=0}^{n-1} a_k(t-a)^{kq}E_{nq,\,kq+1}(\lambda(t-a)^{nq}) + \int_a^t (t-s)^{nq-1}E_{nq,\,nq}(\lambda(t-s)^{nq})f(s)\,ds. \tag{20}$$

This form of the solution can be obtained using both the approximation method and the Laplace transform method. However, if lower-order Caputo fractional derivative terms are present in Equation (18), neither the approximation method nor the Laplace transform

method will yield a solution for arbitrary n, where n is the integer associated with the Caputo derivative of order nq.

The sequential nature of the Caputo fractional derivative $({}^{c}D_{a+}^{kq})u(t), (k = 2, \ldots, n)$, was recently established in [12]. Using this sequential nature of the Caputo fractional derivative and fractional initial conditions, we can reduce the equation

$$ {}^{c}D_{a+}^{nq}u(t) = \sum_{k=1}^{n-1} b_k \, {}^{c}D_{a+}^{kq}u(t) + \lambda u(t) + f(t), \quad \lambda \in \mathbb{R}, \tag{21}$$

to a system of linear nonhomogeneous Caputo fractional differential equations of order q with initial conditions. This process is very similar to the nth order linear nonhomogeneous integer case with initial conditions. Note that (18) is a special case of (21).

This observation forms the motivation for our main result. However, similar to the classical integer-order case, a system of n Caputo fractional differential equations of order q, where $0 < q < 1$, with initial conditions cannot, in general, be reduced to a scalar nq order Caputo fractional differential equation with fractional initial conditions. Therefore, our main result includes a method for obtaining the solution of Equation (21) as a special case.

3. Main Results

In this section, we develop a method to solve the nonhomogeneous linear system of Caputo fractional differential Equation (constant matrix) with initial conditions. An elegant approach exists for expressing the solution of a linear nonhomogeneous Caputo fractional differential equation of order nq, where $(n-1) < nq < n$, with fractional initial conditions, in terms of the appropriate Mittag–Leffler functions. A key advantage of this solution is that it can be used to solve an n^{th}-order integer differential equation without the need to find the n roots of the characteristic algebraic equation $m^n = \lambda$. However, this method cannot be extended directly to the case of linear Caputo fractional differential equations of order nq, where $(n-1) < nq < n$, with fractional initial conditions that include lower-order fractional derivative terms. It is established in [12] that the Caputo fractional derivative $({}^{c}D_{a+}^{q})u$, can be computed only when the function is in the form $(t-a)^{\Omega}$, where $\Omega \geq q$. In addition, it is established that the Caputo fractional derivative of order nq, where $(n-1) < nq < n$, is sequential of order q. Using this fact, a linear nonhomogeneous Caputo fractional differential equation of order nq, where $(n-1) < nq < n$, with fractional initial conditions, can be reduced to a system of n linear Caputo fractional differential equations of order q with appropriate initial conditions. It is important to note that if the initial conditions are given in terms of integer-order derivatives, this reduction is not possible.

Consider the linear nonhomogeneous system of Caputo fractional differential equations of order q with initial conditions of the form:

$$ {}^{c}D_{a+}^{q}u(t) = Au(t) + f(t), \tag{22}$$

subject to the initial condition:

$$ u(t)\big|_{t=a} = u_0. \tag{23}$$

Here, $u(t)$ is an $n \times 1$ column vector and u_0 is a given $n \times 1$ column vector. The matrix A is an $n \times n$ constant matrix, and $f(t)$ is a given continuous $n \times 1$ column vector function. We assume that $f_i(t) \in C[a, T)$, for some $T > a$, and for $i = 1, 2, \ldots, n$.

The linear nonhomogeneous Caputo fractional differential Equation (21) with fractional initial conditions can be reduced to the Caputo fractional differential system of order q with initial conditions. This will be the special case of the fractional differential system (22) and (23). For more details on solving systems of two and

three Caputo fractional differential equations with initial conditions using the Laplace transform method, see [18,19].

When A is a constant matrix and $f(t) \in C[a,T)$, for some $T > a$, then the integral representation of the solution of (22) and (23) is given by

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}Au(s)\,ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}If(s)\,ds. \tag{24}$$

Theorem 1. *Let A be a constant matrix and $f(t) \in C[a,T)$, for some $T > a$, and u_0 be a known vector. Using the integral representation (24), the initial approximation $u_1(t)$ is given by*

$$u_1(t) = Iu_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}Au_0(s)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}If(s)ds. \tag{25}$$

The successive approximation $u_m(t)$ for $m = 2,3,4\ldots$ is given by

$$u_m(t) = Iu_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}Au_{m-1}(s)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}If(s)ds. \tag{26}$$

Then, as $m \to \infty$, the solution of the Caputo fractional differential system (22) and (23) is given by

$$u(t) = E_{q,1}(A(t-a)^q)u_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s)ds. \tag{27}$$

Proof. Using (26), we can get,

$$u_2(t) = Iu_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}Au_1(s)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}If(s)ds. \tag{28}$$

Substituting $u_1(t)$ from (25) to $u_2(t)$ and simplifying it, we get

$$u_2(t) = \left[I + A\frac{(t-a)^q}{\Gamma(q+1)} + A^2\frac{(t-a)^{2q}}{\Gamma(2q+1)}\right]u_0 + \int_a^t (t-s)^{q-1}\left[\frac{I}{\Gamma(q)} + A\frac{(t-s)^q}{\Gamma(2q)}\right]f(s)\,ds. \tag{29}$$

Continuing this process, we can compute $u_m(t)$ as

$$u_m(t) = \left[\sum_{k=0}^{m} A^k\frac{(t-a)^{kq}}{\Gamma(kq+1)}\right]u_0 + \int_a^t (t-s)^{q-1}\left[\sum_{k=0}^{m-1} A^k\frac{(t-s)^{kq}}{\Gamma((k+1)q)}\right]f(s)\,ds. \tag{30}$$

Since A is bounded linear operator and $f_i(t) \in C[a,T)$ for some $T > a$ for $i = 1,2,\ldots,n$, taking the limit as $m \to \infty$ in the relation (30), we obtain

$$u(t) = E_{q,1}(A(t-a)^q)u_0 + \int_a^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s)ds. \tag{31}$$

□

Remark 1. *The result yields the solution of the integer-order result as a special case. That is, when $q = 1$, in the homogeneous case, the solution reduces to the classical form*

$$u(t) = e^{A(t-a)}u_0.$$

In the integer-order case, we obtain the solution using the relation $e^{-At} = (e^{At})^{-1}$. However, in the case of the Mittag–Leffler function, such an inverse expression is not readily available, as the Mittag–Leffler function does not possess the exponential inverse property. Using the eigenvalue approach for the constant matrix A, we can compute $E_{q,1}(A(t-a)^q)$, on the same lines.

It is important to note that the fundamental matrix evaluated at $t = a$, may not be the identity matrix. That is,

$$E_{q,1}(A(t-a)^q)\big|_{t=a} = B \neq I.$$

In such a case, the fundamental matrix solution should be modified to ensure identity at the initial time by using

$$E_{q,1}(A(t-a)^q)B^{-1},$$

where B^{-1} is the inverse of the matrix B.

This adjustment is precisely what is implemented in most of the examples presented below. Thus, the main result presented here is a generalization of the scalar nq-order linear Caputo fractional differential equation. Once the modified fundamental matrix $\widehat{\Phi}(t)$ is determined, which is an identity matrix at $t = a$, the solution can be written directly using the initial conditions as in the integer-order case.

Example 1. *Consider the scalar Caputo fractional differential equation*

$${}^cD^{2q}_{0+}u(t) - {}^c D^q_{0+}u(t) - 2u(t) = 0, \quad u(0) = u_{01}, \quad {}^cD^q_{0+}u(t)\Big|_{t=0} = u_{02}.$$

Letting $u(t) = u_1(t)$ and ${}^cD^q_{0+}u_1(t) = u_2(t)$, the equation can be reduced to a system of two linear Caputo fractional differential equations of order q, written as

$$\begin{bmatrix} {}^cD^q_{0+}u_1 \\ {}^cD^q_{0+}u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

Then, the solution is given by

$$u(t) = E_{q,1}(At^q)u_0, \tag{32}$$

where the matrix A is defined as

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

Note that the eigenvalues of the matrix A are $\lambda = 2$ and $\lambda = -1$, which are real and distinct. Then, the solution (32) can be written using the fundamental matrix relative to the matrix A, using the eigenvalue method as in the integer-order case. The explicit solution is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}E_{q,1}(2t^q) + \frac{2}{3}E_{q,1}(-t^q) & \frac{1}{3}E_{q,1}(2t^q) - \frac{1}{3}E_{q,1}(-t^q) \\ \frac{2}{3}E_{q,1}(2t^q) - \frac{2}{3}E_{q,1}(-t^q) & \frac{2}{3}E_{q,1}(2t^q) + \frac{1}{3}E_{q,1}(-t^q) \end{bmatrix}\begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

and the component-wise solution is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3}E_{q,1}(2t^q) + \frac{2}{3}E_{q,1}(-t^q)\right)u_{01} + \left(\frac{1}{3}E_{q,1}(2t^q) - \frac{1}{3}E_{q,1}(-t^q)\right)u_{02} \\ \left(\frac{2}{3}E_{q,1}(2t^q) - \frac{2}{3}E_{q,1}(-t^q)\right)u_{01} + \left(\frac{2}{3}E_{q,1}(2t^q) + \frac{1}{3}E_{q,1}(-t^q)\right)u_{02} \end{bmatrix}.$$

Example 2. *Consider the scalar Caputo fractional differential equation*

$$ {}^cD_{0+}^{2q}u(t) - 2\,{}^cD_{0+}^{q}u(t) + u(t) = 0, \quad u(0) = u_{01}, \quad {}^cD_{0+}^{q}u(t)\Big|_{t=0} = u_{02}. $$

As in the previous example, letting $u(t) = u_1(t)$ and ${}^cD_{0+}^{q}u_1(t) = u_2(t)$, the scalar equation can be reduced to a system of two linear Caputo fractional differential equations of order q, written as

$$ \begin{bmatrix} {}^cD_{0+}^{q}u_1 \\ {}^cD_{0+}^{q}u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. $$

The solution is given by

$$ u(t) = E_{q,1}(At^q)u_0, \tag{33} $$

where the matrix A is

$$ A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad u_0 = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. $$

The solution (33) can be expressed using the fundamental matrix corresponding to A, following the eigenvalue method as in the integer-order case. In this example, the eigenvalues are real and repeated: $\lambda = 1$.

The explicit solution is

$$ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 2E_{q,1}(t^q) - \dfrac{t^q}{q}E_{q,q}(t^q) & \dfrac{t^q}{q}E_{q,q}(t^q) \\ -\dfrac{t^q}{q}E_{q,q}(t^q) & E_{q,1}(t^q) + \dfrac{t^q}{q}E_{q,q}(t^q) \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. $$

Alternatively, the component-wise solution is

$$ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \left(2E_{q,1}(t^q) - \dfrac{t^q}{q}E_{q,q}(t^q)\right)u_{01} + \dfrac{t^q}{q}E_{q,q}(t^q)\,u_{02} \\ -\dfrac{t^q}{q}E_{q,q}(t^q)\,u_{01} + \left(E_{q,1}(t^q) + \dfrac{t^q}{q}E_{q,q}(t^q)\right)u_{02} \end{bmatrix}. $$

Example 3. *Consider the scalar Caputo fractional differential equation*

$$ {}^cD_{0+}^{2q}u(t) - 2\,{}^cD_{0+}^{q}u(t) + 2u(t) = 0, \quad u(0) = u_{01}, \quad {}^cD_{0+}^{q}u(t)\Big|_{t=0} = u_{02}. $$

As in the previous examples, let $u(t) = u_1(t)$ and ${}^cD_{0+}^{q}u_1(t) = u_2(t)$. Then, the scalar equation can be reduced to a system of two linear Caputo fractional differential equations of order q, given by

$$ \begin{bmatrix} {}^cD_{0+}^{q}u_1 \\ {}^cD_{0+}^{q}u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. $$

The solution is given by

$$ u(t) = E_{q,1}(At^q)u_0, \tag{34} $$

where the matrix A is

$$ A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \quad u_0 = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. $$

If $q = 1$, the solution reduces to $u(t) = e^{At}u_0$. Note that the eigenvalues of the matrix A are $\lambda = 1 \pm i$. Then, the solution (34) can be written using the fundamental matrix relative to the matrix A, using the eigenvalue method as in the integer-order case. The explicit solution is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} G\cos_{q,1}((1+i)t^q) & G\cos_{q,1}((1+i)t^q) + G\sin_{q,1}((1+i)t^q) \\ G\cos_{q,1}((1+i)t^q) - G\sin_{q,1}((1+i)t^q) & 2G\cos_{q,1}((1+i)t^q) \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

Alternatively, the solution can be expressed in the fundamental matrix form as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} G\cos_{q,1}((1+i)t^q) - G\sin_{q,1}((1+i)t^q) & G\sin_{q,1}((1+i)t^q) \\ -2G\sin_{q,1}((1+i)t^q) & G\cos_{q,1}((1+i)t^q) + G\sin_{q,1}((1+i)t^q) \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

Finally, the component-wise solution is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \left[G\cos_{q,1}((1+i)t^q) - G\sin_{q,1}((1+i)t^q)\right]u_{01} + \left[G\sin_{q,1}((1+i)t^q)\right]u_{02} \\ \left[-2G\sin_{q,1}((1+i)t^q)\right]u_{01} + \left[G\cos_{q,1}((1+i)t^q) + G\sin_{q,1}((1+i)t^q)\right]u_{02} \end{bmatrix}.$$

Example 4. *Consider the scalar fractional differential equation*

$${}^cD^{3q}_{0+}u(t) - 6\,{}^cD^{2q}_{0+}u(t) + 11\,{}^cD^{q}_{0+}u(t) - 6u(t) = 0,$$

subject to the initial conditions

$$u(0) = u_{01}, \quad {}^cD^{q}_{0+}u(t)\Big|_{t=0} = u_{02}, \quad {}^cD^{2q}_{0+}u(t)\Big|_{t=0} = u_{03}.$$

Letting $u(t) = u_1(t)$, ${}^cD^{q}_{0+}u_1(t) = u_2(t)$, and ${}^cD^{q}_{0+}u_2(t) = u_3(t)$, the scalar equation can be reduced to a system of three linear Caputo fractional differential equations of order q, which can be written as

$$\begin{bmatrix} {}^cD^{q}_{0+}u_1 \\ {}^cD^{q}_{0+}u_2 \\ {}^cD^{q}_{0+}u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} u_{01} \\ u_{02} \\ u_{03} \end{bmatrix}.$$

It is straightforward to verify that the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix},$$

are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. Using these eigenvalues, one can compute the matrix $E_{q,1}(At^q)$. The explicit solution is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3E_{q,1}(t^q) - 3E_{q,1}(2t^q) + E_{q,1}(3t^q) & -\frac{5}{2}E_{q,1}(t^q) + 4E_{q,1}(2t^q) - \frac{3}{2}E_{q,1}(3t^q) & \frac{1}{2}E_{q,1}(t^q) - E_{q,1}(2t^q) + \frac{1}{2}E_{q,1}(3t^q) \\ 3E_{q,1}(t^q) - 6E_{q,1}(2t^q) + 3E_{q,1}(3t^q) & -\frac{5}{2}E_{q,1}(t^q) + 8E_{q,1}(2t^q) - \frac{9}{2}E_{q,1}(3t^q) & \frac{1}{2}E_{q,1}(t^q) - 2E_{q,1}(2t^q) + \frac{3}{2}E_{q,1}(3t^q) \\ 3E_{q,1}(t^q) - 12E_{q,1}(2t^q) + 9E_{q,1}(3t^q) & -\frac{5}{2}E_{q,1}(t^q) + 16E_{q,1}(2t^q) - \frac{27}{2}E_{q,1}(3t^q) & \frac{1}{2}E_{q,1}(t^q) - 4E_{q,1}(2t^q) + \frac{9}{2}E_{q,1}(3t^q) \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{02} \\ u_{03} \end{bmatrix}.$$

Example 5. *Consider the scalar system of fractional nonhomogeneous differential equations:*

$$\begin{aligned} {}^cD^{q}_{0+}u_1(t) &= 4u_1(t) + 2u_2(t) + 1, \\ {}^cD^{q}_{0+}u_2(t) &= 3u_1(t) - u_2(t) + 1. \end{aligned} \tag{35}$$

with initial conditions

$$u_1(0) = u_{01}, \quad u_2(0) = u_{02}.$$

To solve this nonhomogeneous system, we first solve its homogeneous part. The coefficient matrix A of the homogeneous part of system (35) is given by:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 5$. Using these eigenvalues, the corresponding fundamental matrix $\Phi(t)$ is:

$$\Phi(t) = \begin{bmatrix} E_{q,1}(-2t^q) & 2E_{q,1}(5t^q) \\ -3E_{q,1}(-2t^q) & E_{q,1}(5t^q) \end{bmatrix}.$$

It is easy to observe that $\Phi(0) = B \neq I$, where B is the matrix obtained by evaluating $\Phi(t)$ at $t = 0$.

Now, by multiplying $\Phi(t)$ by B^{-1}, we obtain a modified fundamental matrix $\widehat{\Phi}(t)$ such that $\widehat{\Phi}(0) = I$. The modified fundamental matrix is given by:

$$\widehat{\Phi}(t) = \begin{bmatrix} \frac{1}{7}E_{q,1}(-2t^q) + \frac{6}{7}E_{q,1}(5t^q) & \frac{-2}{7}E_{q,1}(-2t^q) + \frac{2}{7}E_{q,1}(5t^q) \\ \frac{-3}{7}E_{q,1}(-2t^q) + \frac{3}{7}E_{q,1}(5t^q) & \frac{6}{7}E_{q,1}(-2t^q) + \frac{1}{7}E_{q,1}(5t^q) \end{bmatrix}.$$

The solution to the homogeneous part of system (35) can be written immediately as

$$u_h(t) = \widehat{\Phi}(t)u_0.$$

For convenience, we denote $\widehat{\Phi}(t)$ by $\widehat{\Phi_{q,1}}(t)$.

The solution of the nonhomogeneous system (35) is given by

$$u(t) = \widehat{\Phi_{q,1}}(t)u_0 + \int_0^t (t-s)^{q-1}\widehat{\Phi_{q,q}}[(t-s)^q]f(s)ds, \tag{36}$$

where

$$\widehat{\Phi_{q,q}}(t-s)^q = \begin{bmatrix} \frac{1}{7}E_{q,q}(-2(t-s)^q) + \frac{6}{7}E_{q,q}(5(t-s)^q) & \frac{-2}{7}E_{q,q}(-2(t-s)^q) + \frac{2}{7}E_{q,q}(5(t-s)^q) \\ \frac{-3}{7}E_{q,q}(-2(t-s)^q) + \frac{3}{7}E_{q,q}(5(t-s)^q) & \frac{6}{7}E_{q,q}(-2(t-s)^q) + \frac{1}{7}E_{q,q}(5(t-s)^q) \end{bmatrix},$$

and $f(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The computation of the integral term (36) can be obtained analytically using the relation from [7]

$$\int_0^t \tau^{q-1}E_{q,q}(\lambda\tau^q)\,d\tau = \frac{1}{\lambda}(E_{q,1}(\lambda t^q) - 1).$$

Integrating term by term and component wise on (36), the solution of the nonhomogenous system (35) becomes

$$u(t) = \widehat{\Phi_{q,1}}(t)u_0 + \begin{bmatrix} -\frac{1}{7}\cdot\frac{-1}{2}\left[E_{q,1}(-2t^q) - 1\right] + \frac{8}{7}\cdot\frac{1}{5}\left[E_{q,1}(5t^q) - 1\right] \\ \frac{3}{7}\cdot\frac{-1}{2}\left[E_{q,1}(-2t^q) - 1\right] + \frac{4}{7}\cdot\frac{1}{5}\left[E_{q,1}(5t^q) - 1\right] \end{bmatrix}. \tag{37}$$

4. Conclusions

We have presented a method for solving linear nonhomogeneous Caputo fractional differential equations of order nq where, $n-1 < nq < n$, with fractional initial conditions having lower-order fractional derivatives in the equation. Note that such a Caputo

fractional differential equation can be reduced to Caputo fractional differential system of order $q, 0 < q < 1$, with the initial condition. In this work, we have obtained the analytical solution of the Caputo fractional differential system of the form

$$^{c}D^{q}_{a+}u(t) = Au(t) + f(t), \quad u(t)|_{t=a} = u_0,$$

where A is an $n \times n$ constant matrix and f is a continuous function on $[0, T]$ for $T > 0$ and u_0 is a given known vector. Using the iterative approximation method, the homogeneous solution of the system for $q < 1$ can be written as:

$$u_h(t) = E_{q,1}(A(t-a)^q)B^{-1}u_0,$$

where $E_{q,1}(A(t-a)^q)\big|_{t=a} = B \neq I$.

This result is valuable for applications, since the value of q can be used as a parameter to enhance the linear system models. In our future works, we will use our result to study the stability of the Lotka–Volterra model, as well as broader biological and physical systems.

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