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Bi-Univalent Function Classes Defined by Imaginary Error Function and Bernoulli Polynomials

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Abstract

In recent years, special functions have played a significant role in the investigation of different subclasses within the class of bi-univalent functions. In this work, we present and investigate two new subclasses of bi-univalent functions defined in $\mathfrak{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, characterized by Bernoulli polynomials associated with imaginary error functions. For functions belonging to these subclasses, we establish bounds for their initial coefficients. For these classes, we also tackle the Fekete–Szegö problem. Several new results are also obtained as special cases by specifying certain parameter values in the general findings.

Keywords: Bernoulli polynomials; regular functions; imaginary error function; bi-univalent functions

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1. Preliminaries

Special polynomials play a pivotal role across various disciplines, including combinatorics, computer science, number theory, numerical analysis, physics, and engineering. Polynomials such as Bernoulli, Chebyshev, Faber, Fibonacci, Gegenbauer, Horadam, Lucas–Lehmer, Pell–Lucas, and their extensions have proven to be fundamental tools in both theoretical and applied contexts. Among these, recent research has shown a growing interest in a particular class, "Bernoulli polynomials", due to their rich mathematical structure and wide-ranging applications. The ability of this family of polynomials to capture complex behavior within a finite set of terms makes them an intriguing basis set for function approximation, especially when working with fractional derivatives, where the derivative is taken to a non-integer power. Fractional calculus offers a strong foundation for precisely simulating systems with memory effects or anomalous diffusion by extending the ideas of differentiation and integration to non-integer orders. In this context, Bernoulli polynomials have found novel applications in the numerical solution of fractional-order differential equations. For instance, a novel approximation technique based on orthonormal Bernoulli

polynomials was created in [1] to resolve fractional Lane–Emden type equations. Additionally, Loh and Phang [2] employed Bernoulli polynomials to numerically solve Fredholm fractional integro-differential equations involving right-sided Caputo derivatives with multi-fractional orders.

The Bernoulli polynomials $B_j(x)$, where $x \in \mathbb{R}$, and with j being a non-negative integer, are commonly defined via the generating function (see, e.g., [3]):

$$\mathfrak{B}(x,\varsigma) = \frac{\varsigma e^{x\varsigma}}{e^{\varsigma}-1} = B_0(x) + B_1(x)\varsigma + B_2(x)\frac{\varsigma^2}{2!} + \cdots .|\varsigma| < 2|\pi. \tag{1}$$

The $B_j(x)$ can be conveniently computed using the following recursion formula:

$$\sum_{n=0}^{j-1} \binom{j}{n} B_n(x) = jx^{j-1}, \quad j = 2,3,4,\cdots,$$

with $B_0(x) = 1$.

The first few $B_j(x)$ are given by

$$B_1(x) = \frac{2x-1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, \cdots . \tag{2}$$

The study of the geometric characteristics and behavior of analytic functions is the focus of Geometric Function Theory (GFT), a foundational field within complex analysis. In recent years, this subfield has attracted substantial scholarly interest, owing to its profound theoretical advancements and its pivotal role in various applications.

Let $\mathfrak{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ denote the open unit disc. The class $\mathcal{A}$ comprises all functions ϕ that are analytic and normalized in $\mathfrak{U}$. Each element of $\mathcal{A}$ admits a representation of the form

$$\phi(\varsigma) = \varsigma + d_2\varsigma^2 + d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma \in \mathfrak{U}, \tag{3}$$

and let $\mathcal{S} = \{\phi \in \mathcal{A} : \phi \text{ is univalent in } \mathfrak{U}\}$.

In [4], Bieberbach conjectured that for every function $\phi \in \mathcal{S}$, the coefficients satisfy $|d_j| \leq j$ for all $j \geq 2$. To address this conjecture, many new subcategories of $\mathcal{S}$ were introduced, and various related results were established. After years of extensive research, Luis de Branges ultimately proved the Bieberbach conjecture for all $j \geq 2$ in [5].

Another important problem in GFT is the Fekete–Szegö Functional (FSF), defined as $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, studied for functions $\phi \in \mathcal{S}$ (see [6]). The problem has been extensively studied for functions in various subclasses of $\mathcal{S}$. One notable subclass of $\mathcal{S}$ is the class of bi-univalent functions, denoted by Σ. The concept of Σ was introduced by Levin in [7]. A function ϕ belongs to this class if both ϕ and $\phi^{-1} = \psi$ are elements of $\mathcal{S}$. According to the Koebe theorem (see [8]), each function $\phi \in \mathcal{S}$ of the form (3) has an inverse given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{4}$$

satisfying $\varsigma = \psi(\phi(\varsigma)), \varsigma \in \mathfrak{U}$ and $w = \phi(\psi(w)), |w| < r_0(\phi), r_0(\phi) \geq 1/4, w \in \mathfrak{U}$.

The class Σ is nonempty, as it includes functions such as $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$, and $-\log(1-\varsigma)$. However, functions like $\varsigma - \frac{\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and $\frac{\varsigma}{(1-\varsigma)^2}$ belong to the class $\mathcal{S}$ but are not members of Σ. For a concise overview and further discussion on the properties of the class Σ, the reader is referred to [9–12]. The renewed interest and significant surge in research on the class Σ were initiated by Srivastava and collaborators [13]. Since then, various subclasses of Σ have attracted considerable attention from researchers; see, for example, [14–17] and the references cited therein.

The error function is a fundamental special function with critical applications across numerous scientific disciplines, including probability theory, statistics, and various branches of engineering. Mathematicians have therefore focused a great deal of attention on it. Several significant related topics were reported for the error function; see [18–20] for examples. The error function

$$erf(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \sum_{\eta=0}^{\infty} \frac{(-1)^{\eta} \varsigma^{2\eta+1}}{(2\eta+1)\eta!}, \varsigma \in \mathbb{C}. \tag{5}$$

and its approximations are commonly used to predict events that hold with probability, high or low. Using Equation (5), Ramachandran et al. [21] investigated a normalized form of the error function, defined by

$$nerf(\varsigma) = \frac{\sqrt{\pi}\varsigma}{2} erf(\sqrt{\varsigma}) = \varsigma + \sum_{\eta=2}^{\infty} \frac{(-1)^{\eta-1} \varsigma^{\eta}}{(2\eta-1)(\eta-1)!}, \varsigma \in \mathbb{C}.$$

By expressing the integrand e^{-y^2} as a series and integrating term by term, one can derive a power series representation of the imaginary error function, denoted by ierf, where "i" represents the imaginary unit. This allows for the inclusion of oscillatory components in solutions, which can be crucial when modeling wave-like phenomena. As explained in ([22,23]), the Maclaurin's series can be obtained as indicated below.

$$ierf(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \sum_{\eta=0}^{\infty} \frac{\varsigma^{2\eta+1}}{(2\eta+1)\eta!}, \varsigma \in \mathbb{C}. \tag{6}$$

Using (6), the normalized form of the imaginary error function "nierf" is defined by

$$nierf(\varsigma) = \frac{\sqrt{\pi}\varsigma}{2} ierf(\sqrt{\varsigma}) = \varsigma + \sum_{\eta=2}^{\infty} \frac{\varsigma^{\eta}}{(2\eta-1)(\eta-1)!}, \varsigma \in \mathbb{C}.$$

and utilizing the convolution product "*", we define

$$I\phi(\varsigma) = (nierf * \phi)(\varsigma) = \varsigma + \sum_{\eta=2}^{\infty} \frac{d_{\eta}}{(2\eta-1)(\eta-1)!} \varsigma^{\eta}, \tag{7}$$

where $\phi \in \mathcal{S}$ is of the form (3).

For $\mathfrak{a}_1$, $\mathfrak{a}_2 \in \mathcal{A}$ analytic in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$ if there is a function $\theta(\varsigma)$ that is holomorphic in $\mathfrak{U}$ with $\theta(0) = 0$ and $|\theta(\varsigma)| < 1$, such that $\mathfrak{a}_1(\varsigma) = \mathfrak{a}_2(\theta(\varsigma)), \varsigma \in \mathfrak{U}$. This is symbolised as

$$\mathfrak{a}_1 \prec \mathfrak{a}_2 \, or \, \mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma), \varsigma \in \mathfrak{U}.$$

In this case, if $\mathfrak{a}_2 \in \mathcal{S}$, then

$$\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma) \Leftrightarrow \mathfrak{a}_1(0) = \mathfrak{a}_2(0) \quad and \quad \mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U}).$$

The Bernoulli polynomials form a distinguished family of polynomials with notable mathematical properties, GFT is a branch of complex analysis concerned with the geometric behavior of analytic functions, and the imaginary error function is a special complex-valued function arising in various applied and theoretical contexts. These three concepts are related but represent different ideas in mathematics. When combined, they can be used to study complex analytic functions and their geometric behavior, especially in relation to conformal mappings and univalent functions. We may direct readers to [21,23] for certain investigations that combine the ideas of GFT and the error function. For some studies that

integrate the concepts of GFT and Bernoulli polynomials, we might refer readers to [24–27]. In [28–30], intriguing investigations are conducted by fusing the concepts of Bernoulli polynomials, GFT, and the imaginary function.

In recent years, considerable attention has been devoted to the study of functions belonging to specific subfamilies of Σ that are associated with well-known special polynomials. For such subfamilies, where the functions are subordinate to special polynomials, several researchers have established coefficient bounds and the FSF $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ (see [31–34]).

Building on this foundation, Swamy and Kala [29] introduced a comprehensive subclass of bi-univalent functions defined through an intricate relationship involving the imaginary error function and Bernoulli polynomials. Their definition uniquely employs a linear combination of two expressions, each with denominators formed by linear combinations of related terms, thereby creating a nuanced and versatile framework for analysis. This construction enables the encapsulation of a wide range of known function classes as special cases, thereby showcasing its generality. The class's adaptability has made it particularly valuable in extending classical results and exploring new directions in Geometric Function Theory.

Furthermore, Swamy et al. [30] explored specific bi-univalent function subfamilies connected to the imaginary error function and Bernoulli polynomials. Within these subclasses, they successfully derived initial coefficient bounds and established Fekete–Szegö inequalities, while also elucidating important connections to prior results in the literature. Notably, the subclasses examined serve as a unifying framework that generalizes numerous earlier results, thereby reinforcing the foundational role of these special functions in bi-univalent function theory. This work not only deepens theoretical understanding but also opens avenues for further generalizations and applications in complex analysis.

Inspired by these recent advancements, and driven by the quest for deeper understanding of the interplay between special functions and the theory of bi-univalent functions, we introduce two new subclasses of Σ, denoted by $\mathfrak{D}_\Sigma(\varrho, x)$ and $\mathfrak{T}_\Sigma(\varrho, x)$. We define these subclasses in terms of the imaginary error function, nierf, due to its appealing analytic and structural properties. Specifically, nierf is an entire function with a well-known series expansion that allows for explicit coefficient comparison and analytic continuation. Moreover, its close relation to the Gaussian integral makes it a natural candidate when working with integral transforms and symmetric functions in the complex domain. Utilizing nierf provides a unifying framework that generalizes several previously studied subclasses under a common analytic umbrella. This choice also leads to tractable results in terms of coefficient estimates and functional inequalities, thereby enriching the ongoing exploration within this dynamic field of research.

Definition 1. *Let $\varrho \geq 0$. If $\phi \in \Sigma$ satisfies*

$$\frac{\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))''}{I\phi(\varsigma)} \prec \mathfrak{B}(x, \varsigma), \tag{8}$$

and

$$\frac{w(I\psi(w))' + \varrho w^2(I\psi(w))''}{I\psi(w)} \prec \mathfrak{B}(x, w), \tag{9}$$

then we say that $\phi \in \mathfrak{D}_\Sigma(\varrho, x)$, where $\mathfrak{B}(x, \varsigma)$ is defined as in (1), $\psi(w) = \phi^{-1}(w)$ is as stated in (4), and $\varsigma, w \in \mathfrak{U}$.

Definition 2. *Let $\varrho \geq 0$. If $\phi \in \Sigma$ satisfies*

$$\frac{(\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))'')'}{(I\phi(\varsigma))'} \prec \mathfrak{B}(x,\varsigma), \tag{10}$$

and

$$\frac{(w(I\psi(w))' + \varrho w^2(I\psi(w))'')'}{(I\psi(w))'} \prec \mathfrak{B}(x,w), \tag{11}$$

then we say that $\phi \in \mathfrak{T}_\Sigma(\varrho,x)$, where $\mathfrak{B}(x,\varsigma)$ is defined as in (1), $\psi(w) = \phi^{-1}(w)$ is as stated in (4), and $\varsigma, w \in \mathfrak{U}$.

The remainder of this paper is organized as follows. In Section 2, we establish bounds for the first two Maclaurin's coefficients and FSF$|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for functions in the classes $\mathfrak{D}_\Sigma(\varrho,x)$ and $\mathfrak{T}_\Sigma(\varrho,x)$. Section 3 highlights relevant connections between special cases and the main results presented. Finally, Section 4 presents several concluding remarks.

2. Principal Findings

Section 2 commences with the derivation of bounds for the first two Maclaurin's coefficients and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for functions belonging to the classes $\mathfrak{D}_\Sigma(\varrho,x)$ and $\mathfrak{T}_\Sigma(\varrho,x)$.

Theorem 1. *If a function $\phi \in \sigma$ belongs to the family $\mathfrak{D}_\Sigma(\varrho,x)$ with $\varrho \geq 0$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{5|2x-1|}}{\sqrt{2|(4+17\varrho)(2x-1)^2 - 10(1+2\varrho)^2(x^2-x+\frac{1}{6})|}}, \tag{12}$$

$$|d_3| \leq \frac{5|2x-1|}{2(1+3\varrho)} + \left(\frac{3(2x-1)}{2(1+2\varrho)}\right)^2, \tag{13}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{2(1+3\varrho)} & ; |1-\xi| \leq \urcorner \\ \frac{45|2x-1|^3|1-\xi|}{2|(4+17\varrho)(2x-1)^2 - 10(1+2\varrho)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \urcorner, \end{cases} \tag{14}$$

where $\xi \in \mathbb{R}$ and

$$\urcorner = \left|\frac{(4+17\varrho)(2x-1)^2 - 10(1+2\varrho)^2(x^2-x+\frac{1}{6})}{9(1+3\varrho)(2x-1)^2}\right|. \tag{15}$$

Proof. Let $\phi \in \mathfrak{D}_\Sigma(\varrho,x)$. Then, using the subordinations (8) and (9), we can write

$$\frac{\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))''}{(I\phi(\varsigma))'} = \mathfrak{B}(x,\mathfrak{l}(\varsigma)), \tag{16}$$

and

$$\frac{w(I\psi(w))' + \varrho w^2(I\psi(w))''}{I\psi(w)} = \mathfrak{B}(x,\mathfrak{m}(w)), \tag{17}$$

where the functions

$$\mathfrak{l}(\varsigma) = \sum_{j=1}^{\infty} \mathfrak{l}_j \varsigma^j, \varsigma \in \mathfrak{U} \quad \textit{and} \quad \mathfrak{m}(w) = \sum_{j=1}^{\infty} \mathfrak{m}_j w^j, \, w \in \mathfrak{U} \tag{18}$$

satisfy the property (see [8])

$$|\mathfrak{l}_j| \leq 1, \textit{ and } |\mathfrak{m}_j| \leq 1 \, (j \in \mathbb{N}). \tag{19}$$

By employing standard mathematical techniques with Equation (7), Equation (16) can be rewritten as

$$\frac{\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))''}{(I\phi(\varsigma))'} = 1 + \left(\frac{1}{3}\right)(1+2\varrho)d_2\varsigma + \left(\frac{1}{5}(1+3\varrho)d_3 - \frac{1}{9}(1+2\varrho)d_2^2\right)\varsigma^2 + \cdots, \tag{20}$$

and

$$\mathfrak{B}(x, \mathfrak{l}(\varsigma)) = 1 + B_1(x)\mathfrak{l}_1\varsigma + \left(B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2\right)\varsigma^2 + \cdots. \tag{21}$$

By employing standard mathematical techniques with Equation (7), Equation (17) can be rewritten as

$$\frac{w(I\psi(w))' + \varrho w^2(I\psi(w))''}{I\psi(w)} = 1 - \left(\frac{1}{3}\right)(1+2\varrho)d_2 w + \left(\frac{1}{5}(1+3\varrho)(2d_2^2 - d_3) - \frac{1}{9}(1+2\varrho)d_2^2\right)w^2 + \cdots, \tag{22}$$

and

$$\mathfrak{B}(x, \mathfrak{m}(w)) = 1 + B_1(x)\mathfrak{m}_1 w + \left(B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2\right)w^2 + \cdots. \tag{23}$$

By matching coefficients of corresponding powers in Equations (20) and (21), and in light of (16), we can deduce the following conclusions:

$$\frac{1}{3}(1+2\varrho)d_2 = B_1(x)\mathfrak{l}_1, \tag{24}$$

and

$$\frac{1}{5}(1+3\varrho)d_3 - \frac{1}{9}(1+2\varrho)d_2^2 = B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2. \tag{25}$$

Similarly, by matching coefficients of corresponding powers in Equations (22) and (23), and in view of (17), we arrive at the following conclusions:

$$-\frac{1}{3}(1+2\varrho)d_2 = B_1(x)\mathfrak{m}_1, \tag{26}$$

and

$$\frac{1}{5}(1+3\varrho)(2d_2^2 - d_3) - \frac{1}{9}(1+2\varrho)d_2^2 = B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2. \tag{27}$$

From (24) and (26), we get

$$\mathfrak{l}_1 = -\mathfrak{m}_1, \tag{28}$$

and

$$\frac{2}{9}(1+2\varrho)^2 d_2^2 = (\mathfrak{l}_1^2 + \mathfrak{m}_1^2)B_1^2(x). \tag{29}$$

Addition of (25) and (27) yields

$$\left(\frac{4}{45} + \frac{17}{45}\varrho\right)2d_2^2 = B_1(x)(\mathfrak{l}_2 + \mathfrak{m}_2) + \frac{B_2(x)}{2}(\mathfrak{l}_1^2 + \mathfrak{m}_1^2). \tag{30}$$

Replacing $\mathfrak{l}_1^2 + \mathfrak{m}_1^2$ from (29) in (30), we obtain

$$d_2^2 = \frac{45B_1^3(x)(\mathfrak{l}_2 + \mathfrak{m}_2)}{(4+17\varrho)2B_1^2(x) - 5(1+2\varrho)^2 B_2(x)}. \tag{31}$$

By applying Equation (2) to $B_1(x)$ and $B_2(x)$, together with Equation (19) applied to $\mathfrak{l}_2$ and $\mathfrak{m}_2$, we obtain the inequality (12).

Subtracting Equation (27) from (25) and applying (28) yields the bound for $|d_3|$:

$$d_3 = d_2^2 + \frac{5B_1(x)(\mathfrak{l}_2 - \mathfrak{m}_2)}{2(1+3\varrho)}. \tag{32}$$

If we replace d_2^2 from (29) in (32), we obtain

$$d_3 = \frac{9B_1^2(x)(\mathfrak{l}_1^2 + \mathfrak{m}_2^2)}{2(1+2\varrho)^2} + \frac{5B_1(x)(\mathfrak{l}_2 - \mathfrak{m}_2)}{2(1+3\varrho)}. \tag{33}$$

Inequality (13) is derived from (33) through the application of (2) and (19).

Finally, we find the bound on $|d_3 - \xi d_2^2|$ using the values of d_2^2 and d_3 from (31) and (32), respectively. Consequently, we obtain

$$|d_3 - \xi d_2^2| = 5|B_1(x)|\left|\left(\frac{1}{2(1+3\varrho)} + \mathcal{V}(\xi, x)\right)\mathfrak{l}_2 - \left(\frac{1}{2(1+3\varrho)} - \mathcal{V}(\xi, x)\right)\mathfrak{m}_2\right|.$$

Clearly,

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|B_1(x)|}{1+3\varrho} & ; \ |\mathcal{V}(\xi, x)| \leq \frac{1}{2(1+3\varrho)} \\ 10|B_1(x)||\mathcal{V}(\xi, x)| & ; \ |\mathcal{V}(\xi, x)| \geq \frac{1}{2(1+3\varrho)}. \end{cases} \tag{34}$$

We derive the inequality (14) from (34), where $\urcorner$ is defined as in (15). □

The following result is obtained by setting $\xi = 1$ in Theorem 2:

Corollary 1. *If a function $\phi \in \Sigma$ belongs to the family $\mathfrak{D}_\Sigma(\varrho, x)$, $\varrho \geq 0$, then $|d_3 - d_2^2| \leq \frac{5|2x-1|}{2(1+3\varrho)}$.*

Theorem 2. *If a function $\phi \in \Sigma$ belongs to the family $\mathfrak{T}_\Sigma(\varrho, x)$ with $\varrho \geq 0$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{\frac{5|2x-1|}{2}}}{\sqrt{|(7+41\varrho)(2x-1)^2 - 40(1+2\varrho)^2(x^2 - x + \frac{1}{6})|}}, \tag{35}$$

$$|d_3| \leq \left(\frac{3(2x-1)}{4(1+2\varrho)}\right)^2 + \frac{5|2x-1|}{6(1+3\varrho)}, \tag{36}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{6(1+3\varrho)} & ; |1-\xi| \leq \mathcal{Z} \\ \frac{45|2x-1|^3\,|1-\xi|}{2|(7+41\varrho)(2x-1)^2 - 40(1+2\varrho)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \mathcal{Z}, \end{cases} \tag{37}$$

where $\xi \in \mathbb{R}$ and

$$\mathcal{Z} = \left|\frac{(7+41\varrho)(2x-1)^2 - 40(1+2\varrho)^2(x^2 - x + \frac{1}{6})}{27(1+3\varrho)(2x-1)^2}\right|. \tag{38}$$

Proof. Let $\phi \in \mathfrak{T}_\Sigma(\varrho, x)$. Then we can write the subordinations (10) and (11) explicitly as

$$\frac{(\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))'')'}{(I\phi(\varsigma))'} = \mathfrak{B}(x, \mathfrak{l}(\varsigma)), \tag{39}$$

and

$$\frac{(w(I\psi(w))' + \varrho w^2(I\psi(w))'')'}{(I\psi(w))'} = \mathfrak{B}(x, \mathfrak{m}(w)), \tag{40}$$

where the functions $\mathfrak{l}(\varsigma)$ and $\mathfrak{m}(w)$ are of the form (18) with the property (19).

By applying a few basic mathematical manipulations based on Equation (7), we can express Equation (39) in the following form:

$$\frac{(\varsigma(I\phi(\varsigma))' + \varrho\varsigma^2(I\phi(\varsigma))'')'}{(I\phi(\varsigma))'} = 1 + \left(\frac{2}{3}\right)(1+2\varrho)d_2\varsigma + 2\left(\frac{3}{10}(1+3\varrho)d_3 - \frac{2}{9}(1+2\varrho)d_2^2\right)\varsigma^2 + \cdots, \tag{41}$$

and

$$\mathfrak{B}(x, \mathfrak{l}(\varsigma)) = 1 + B_1(x)\mathfrak{l}_1\varsigma + \left(B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2\right)\varsigma^2 + \cdots. \tag{42}$$

By applying a few basic mathematical manipulations based on Equation (7), we can express Equation (40) in the following form:

$$\frac{(w(I\psi(w))' + \varrho w^2(I\psi(w))'')'}{(I\psi(w))'} = 1 - \left(\frac{2}{3}\right)(1+2\varrho)d_2 w + 2\left(\frac{3}{10}(1+3\varrho)(2d_2^2 - d_3) - \frac{2}{9}(1+2\varrho)d_2^2\right)w^2 + \cdots, \tag{43}$$

and

$$\mathfrak{B}(x, \mathfrak{m}(w)) = 1 + B_1(x)\mathfrak{m}_1 w + \left(B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2\right)w^2 + \cdots. \tag{44}$$

By invoking Equation (39), we can deduce the following by contrasting terms of equal degree in Equations (41) and (42):

$$\frac{2}{3}(1+2\varrho)d_2 = B_1(x)\mathfrak{l}_1, \tag{45}$$

and

$$2\left(\frac{3}{10}(1+3\varrho)d_3 - \frac{2}{9}(1+2\varrho)d_2^2\right) = B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2. \tag{46}$$

In a similar manner, invoking Equation (40), we can deduce the following by contrasting terms of equal degree in Equations (43) and (44):

$$-\frac{2}{3}(1+2\varrho)d_2 = B_1(x)\mathfrak{m}_1 \tag{47}$$

and

$$2\left(\frac{3}{10}(1+3\varrho)(2d_2^2 - d_3) - \frac{2}{9}(1+2\varrho)d_2^2\right) = B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2. \tag{48}$$

From (45) and (47), we obtain

$$\mathfrak{l}_1 = -\mathfrak{m}_1, \tag{49}$$

and

$$\frac{8}{9}(1+2\varrho)^2 d_2^2 = (\mathfrak{l}_1^2 + \mathfrak{m}_1^2)B_1^2(x). \tag{50}$$

Addition of (46) and (48) yields

$$\left(\frac{7}{45} + \frac{41}{45}\varrho\right)2d_2^2 = B_1(x)(\mathfrak{l}_2 + \mathfrak{m}_2) + \frac{B_2(x)}{2}(\mathfrak{l}_1^2 + \mathfrak{m}_1^2). \tag{51}$$

Replacing $\mathfrak{l}_1^2 + \mathfrak{m}_1^2$ from (50) in (51), we obtain

$$2d_2^2 = \frac{45B_1^3(x)(\mathfrak{l}_2 + \mathfrak{m}_2)}{(7+41\varrho)B_1^2(x) - 10(1+2\varrho)^2 B_2(x)}. \tag{52}$$

Utilizing Equation (2) for $B_1(x)$ and $B_2(x)$, and applying (19) to $\mathfrak{l}_2$ and $\mathfrak{m}_2$, yields the inequality (35).

Subtracting (48) from (46) and incorporating (49), we establish the bound on $|d_3|$:

$$d_3 = d_2^2 + \frac{5B_1(x)(\mathfrak{l}_2 - \mathfrak{m}_2)}{6(1+3\varrho)}. \tag{53}$$

If we replace d_2^2 from (50) in (53), we obtain

$$d_3 = \frac{9B_1^2(x)(\mathfrak{l}_1^2 + \mathfrak{m}_2^2)}{8(1+2\varrho)^2} + \frac{5B_1(x)(\mathfrak{l}_2 - \mathfrak{m}_2)}{6(1+3\varrho)}. \tag{54}$$

We can deduce the inequality (36) from Equation (54) by applying Equation (2) and the inequalities in (19).

Finally, we determine the bound on $|d_3 - \xi d_2^2|$ using the values of d_2^2 and d_3 from Equations (52) and (53), respectively. Consequently, we obtain

$$|d_3 - \xi d_2^2| = \frac{5}{2}|B_1(x)|\left|\left(\frac{1}{3(1+3\varrho)} + \mathcal{V}(\xi, x)\right)\mathfrak{l}_2 - \left(\frac{1}{3(1+3\varrho)} - \mathcal{V}(\xi, x)\right)\mathfrak{m}_2\right|.$$

Clearly,

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|B_1(x)|}{3(1+3\varrho)} & ; \ |\mathcal{V}(\xi, x)| \leq \frac{1}{3(1+3\varrho)} \\ 5|B_1(x)||\mathcal{V}(\xi, x)| & ; \ |\mathcal{V}(\xi, x)| \geq \frac{1}{3(1+3\varrho)}. \end{cases} \tag{55}$$

Inequality (37) follows from (55), with $\mathcal{Z}$ as defined in Equation (38). □

The following is obtained by setting $\xi = 1$ in Theorem 2:

Corollary 2. *If a function $\phi \in \Sigma$ belongs to the family $\mathfrak{T}_\Sigma(\varrho, x)$, $\varrho \geq 0$, then $|d_3 - d_2^2| \leq \frac{5|2x-1|}{6(1+3\varrho)}$.*

Remark 1. *In deriving the coefficient bounds, we adopt the standard assumptions $|\mathfrak{l}(\varsigma)| \leq 1$ and $|\mathfrak{m}(w) \leq 1|$ for the Schwarz functions $\mathfrak{l}(\varsigma)$ and $\mathfrak{m}(w)$. As is typical in the study of bi-univalent function classes, these estimates are not sharp, and no extremal functions are presently known. This reflects a general feature of the field, where sharp bounds remain elusive and continue to be an open area of investigation.*

3. Specific Instances

The following instance is obtained by specializing the parameter ϱ in Theorem 1.

Example 1. *If $\varrho = 0$, then we have $\mathfrak{D}_\Sigma(0, x) \equiv \mathfrak{S}_\Sigma(x)$, a subclass of functions $\phi \in \Sigma$ satisfying*

$$\frac{\varsigma(I\phi(\varsigma))'}{I\phi(\varsigma)} \prec \mathfrak{B}(x, \varsigma),$$

and

$$\frac{w(I\psi(w))'}{I\psi(w)} \prec \mathfrak{B}(x, w),$$

where $\mathfrak{B}(x, \varsigma)$ is defined as in (1), $\psi(w) = \phi^{-1}(w)$ is stated as in (4), and $\varsigma, w \in \mathfrak{U}$.

Corollary 3. *If a function $\phi \in \Sigma$ belongs to the family $\mathfrak{G}_\Sigma(x)$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{5|2x-1|}}{2\sqrt{|2(2x-1)^2 - 5(x^2 - x + \frac{1}{6})|}},$$

$$|d_3| \leq \frac{5|2x-1|}{2} + \left(\frac{3(2x-1)}{2}\right)^2,$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{2} & ; |1-\xi| \leq \daleth_1 \\ \frac{45|2x-1|^3\,|1-\xi|}{4|2(2x-1)^2-5(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth_1, \end{cases}$$

where

$$\daleth_1 = \frac{2}{9}\left|\frac{2(2x-1)^2 - 5(x^2-x+\frac{1}{6})}{(2x-1)^2}\right|,$$

and $\xi \in \mathbb{R}$.

By applying $\xi = 1$ in Corollary 3, the following is obtained:

Corollary 4. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{S}_\Sigma(x)$, *then* $|d_3 - d_2^2| \leq \frac{5|2x-1|}{2}$.

Example 2. *If* $\varrho = 1$, *then we have* $\mathfrak{D}_\Sigma(1,x) \equiv \mathfrak{T}_\Sigma(x)$, *a subclass of functions* $\phi \in \Sigma$ *satisfying*

$$\frac{\varsigma(I\phi(\varsigma))'}{I\phi(\varsigma)}\left(1 + \frac{\varsigma(I\phi(\varsigma))''}{(I\phi(\varsigma))'}\right) \prec \mathfrak{B}(x,\varsigma),$$

and

$$\frac{w(I\psi(w))'}{I\psi(w)}\left(1 + \frac{w(I\psi(w))''}{(I\psi(w))'}\right) \prec \mathfrak{B}(x,w),$$

where $\mathfrak{B}(x,\varsigma)$ *is defined as in (1),* $\psi(w) = \phi^{-1}(w)$ *is stated as in (4), and* $\varsigma, w \in \mathfrak{U}$.

Corollary 5. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{T}_\Sigma(x)$, *then*

$$|d_2| \leq \frac{|2x-1|\sqrt{15|2x-1|}}{\sqrt{2|7(2x-1)^2 - 30(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{5|2x-1|}{8} + \left(\frac{2x-1}{2}\right)^2,$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{8} & ; |1-\xi| \leq \daleth_2 \\ \frac{15|2x-1|^3\,|1-\xi|}{2|7(2x-1)^2-30(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth_2, \end{cases}$$

where

$$\daleth_2 = \frac{1}{12}\left|\frac{7(2x-1)^2 - 30(x^2-x+\frac{1}{6})}{(2x-1)^2}\right|,$$

and $\xi \in \mathbb{R}$.

By applying $\xi = 1$ in Corollary 5, the following is obtained:

Corollary 6. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{T}_\Sigma(x)$, *then* $|d_3 - d_2^2| \leq \frac{5|2x-1|}{8}$.

The following examples are obtained by specializing the parameter ϱ in Theorem 2.

Example 3. *If* $\varrho = 0$, *then we have* $\mathfrak{T}_\Sigma(0,x) \equiv \mathfrak{C}_\Sigma(x)$, *a subclass of functions* $\phi \in \Sigma$ *satisfying*

$$1 + \frac{\varsigma(I\phi(\varsigma))''}{(I\phi(\varsigma))'} \prec \mathfrak{B}(x,\varsigma),$$

and

$$1 + \frac{w(I\psi(w))''}{(I\psi(w))'} \prec \mathfrak{B}(x,w),$$

where $\mathfrak{B}(x,\varsigma)$ *is defined as in (1),* $\psi(w)=\phi^{-1}(w)$ *is stated as in (4), and* $\varsigma, w \in \mathfrak{U}$.

Corollary 7. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{C}_{\Sigma}(x)$, *then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{\frac{5|2x-1|}{2}}}{\sqrt{|7(2x-1)^2-40(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{5|2x-1|}{6}+\left(\frac{3(2x-1)}{4}\right)^2,$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{6} & ; |1-\xi| \leq \mathcal{Z}_1 \\ \frac{45|2x-1|^3\,|1-\xi|}{2|7(2x-1)^2-40(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \mathcal{Z}_1, \end{cases}$$

where

$$\mathcal{Z}_1=\left|\frac{7(2x-1)^2-40(x^2-x+\frac{1}{6})}{27(2x-1)^2}\right|,$$

and $\xi \in \mathbb{R}$.

Applying $\xi=1$ in Corollary 7 yields the following:

Corollary 8. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{C}_{\Sigma}(x)$, *then* $|d_3-d_2^2| \leq \frac{5|2x-1|}{6}$.

Example 4. *If* $\varrho=1$, *then we have* $\mathfrak{T}_{\Sigma}(1,x) \equiv \mathfrak{F}_{\Sigma}(x)$, *a subclass of functions* $\phi \in \Sigma$ *satisfying*

$$1+\frac{\varsigma(I\phi(\varsigma))''}{(I\phi(\varsigma))'}\left(1+\frac{(\varsigma^2(I\phi(\varsigma))'')'}{(\varsigma(I\phi((\varsigma))'}\right) \prec \mathfrak{B}(x,\varsigma),$$

and

$$1+\frac{w(I\psi(w))''}{(I\psi(w))'}\left(1+\frac{(w^2(I\psi(w))'')'}{(w(I\psi((w))'}\right) \prec \mathfrak{B}(x,w),$$

where $\mathfrak{B}(x,\varsigma)$ *is defined as in (1),* $\psi(w)=\phi^{-1}(w)$ *is stated as in (4), and* $\varsigma, w \in \mathfrak{U}$.

Corollary 9. *If a function* $\phi \in \Sigma$ *belongs to the family* $\mathfrak{F}_{\Sigma}(x)$, *then*

$$|d_2| \leq \frac{|2x-1|\sqrt{5|2x-1|}}{\sqrt{8|2(2x-1)^2-15(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{5|2x-1|}{24}+\left(\frac{2x-1}{4}\right)^2,$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{24} & ; |1-\xi| \leq \mathcal{Z}_2 \\ \frac{15|2x-1|^3\,|1-\xi|}{16|2(2x-1)^2-15(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \mathcal{Z}_2, \end{cases}$$

where

$$\mathcal{Z}_2=\frac{2}{9}\left|\frac{2(2x-1)^2-15(x^2-x+\frac{1}{6})}{(2x-1)^2}\right|,$$

and $\xi \in \mathbb{R}$.

Applying $\xi=1$ in Corollary 9 yields the following:

Corollary 10. *If a function $\phi \in \Sigma$ belongs to the family $\mathfrak{F}_{\Sigma}(x)$, then $|d_3 - d_2^2| \leq \frac{5|2x-1|}{24}$.*

4. Conclusions

In this study, we introduced two subclasses of analytic and bi-univalent functions, denoted by $\mathfrak{D}_{\Sigma}(\varrho, \varkappa)$ and $\mathfrak{T}_{\Sigma}(\varrho, x)$, which are subordinate to Bernoulli polynomials via the imaginary error function. We derived bounds for the first two Maclaurin coefficients $|d_2|$ and $|d_3|$ for functions within these defined Σ subfamilies. Additionally, we obtained estimates for the FSF $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for these subfamilies' functions. By specializing parameters, the results presented in Section 2 lead to various particular cases and corollaries. Finally, we remark that the introduced subfamilies offer promising avenues for further study, including higher-order Hankel determinant problems. By employing Bernoulli polynomials, one can continue to establish coefficient bounds and FSFs for new subclasses of Σ.

In essence, one obtains a powerful mathematical tool for approximating and solving complex problems involving fractional derivatives when one combines the ideas of the imaginary error function, Bernoulli polynomials, and fractional calculus. These concepts are frequently used in modeling phenomena with memory effects or non-integer order dynamics, where the imaginary error function adds a complex component to the solution that enables more nuanced analysis of oscillatory behaviors, and the Bernoulli polynomials provide a basis for function representation.

Also, combining the imaginary error function, Bernoulli polynomials, and q-calculus entails investigating the mathematical characteristics and connections among these seemingly separate ideas. This frequently involves number theory, complex analysis, and a particular type of calculus called q-calculus, in which derivatives are defined using a "q" parameter, producing intriguing extensions of standard calculus results.

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