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On Wijsman f_ρ -Statistical Convergence of Order α of Modulus Functions

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Abstract

In the present paper, we introduce and investigate the concepts of Wijsman f_ρ -statistical convergence of order α and Wijsman strong f_ρ -convergence of order α . These notions are defined as natural generalizations of classical statistical convergence and Wijsman convergence, incorporating the tools of modulus functions and natural density through the function f . We provide a detailed analysis of their structural properties, including inclusion relations, basic characterizations, and illustrative examples. Furthermore, we establish the inclusion relations between Wijsman f_ρ -statistical convergence and Wijsman strong f_ρ -convergence of order α , showing conditions under which one implies the other. These notions are different in general, while coinciding under certain restrictions on the function f , the parameter α , and the sequence ρ . The results obtained not only extend some well-known findings in the literature but also open up new directions for further study in the theory of statistical convergence and its applications to analysis and approximation theory.

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1. Introduction

The concept of summability is essential for understanding the convergence behavior of sequences and series, especially those that diverge in the classical sense. A foundational contribution to this theory was made by Cesàro in 1890 through his paper *Sur la multiplication des séries* [1], where he introduced a summability method that later came to bear his name. This pioneering idea paved the way for more generalized summability techniques. Among these, strong summability emerged as a significant extension, allowing for the analysis of sequences that fail to converge in the usual sense, as elaborated by Soomer and Tali [2]. The further generalization known as strong p -summability incorporates a parameter p , providing finer control over convergence behavior. These methods have proven useful across several mathematical disciplines, including functional analysis and number theory. In 1951, Fast introduced the concept of statistical convergence, offering an alternative viewpoint within the broader framework of summability theory [3]. The concept of convergence forms the foundation of analysis and functional analysis. A generalization of this concept, statistical convergence, which is based on the idea of the natural density of positive integers, plays a crucial role in summability theory and functional analysis. The idea of statistical convergence was initially proposed by Zygmund [4] in 1935,

where it was referred to as almost convergence. Steinhaus [5] developed this idea independently of Fast [3] in 1951. In addition to summability theory, it has played a significant role—under various names—in Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory, and Banach spaces. In 1959, Schoenberg [6] redefined statistical convergence and presented it as a method of summability. Salát [7] examined several topological properties of statistical convergence for sequences of real numbers. Fridy [8] introduced the notion of statistical Cauchyness and proved that it is equivalent to statistical convergence. In 1988, Connor [9] demonstrated that there exists a strong relationship between the concept of statistical convergence and strong p -Cesàro summability. Çolak [10] introduced a generalized form of statistical convergence involving a real parameter α , where $0 < \alpha \leq 1$. Based on Çolak's approach [10], the concept of statistical convergence was applied to the theory of probability by Akbaş and Işık [11,12]. In 2014, Aizpuru et al. [13] defined the concept of f -statistical convergence, using an unbounded modulus function. Building on the framework proposed by Aizpuru et al. [13], Bhardwaj and Dhawan [14] employed the concept of f -density to formulate the f -statistical convergence of sequences. Over time, substantial effort has been devoted to extending the concept of statistical convergence and devising new related summability methods. Some important studies on statistical convergence can be found in [15–22].

The concept of convergence of number sequences has been extended to the convergence of set-valued sequences by many authors. One such extension is the Wijsman convergence, which is considered in this study. Nuray and Rhoades [23] extended the concept of Wijsman convergence of set valued sequences to statistical convergence and provided some fundamental theorems. Ulusu et al. [24,25] defined the concept of Wijsman lacunary statistical convergence based on lacunary sequences and explored its relationship with Wijsman statistical convergence. Bhardwaj et al. [26] have worked on the applications of this concept to specific sequences and its generalization. As a result, Wijsman statistical convergence has played a key role in the development of functional analysis and sequence theory.

This paper introduces the new notions of Wijsman statistical convergence and Wijsman strong summability of order α via unbounded modulus functions, establishes their fundamental properties and strict inclusion relations, and demonstrates that these concepts extend and generalize several classical results. The framework proposed is significant because it unifies and generalizes several existing notions of statistical convergence in metric spaces, many of which arise as special cases of our definitions. This comprehensive approach not only clarifies the relationships among different convergence concepts but also provides a solid foundation for further developments and applications in functional analysis, approximation theory, and related areas. In this way, our work fills notable gaps in the literature and opens up new directions for future research.

This study is divided into the following sections. The subsequent section provides the necessary preliminaries, including classical results related to density, statistical convergence, and strong p -Cesàro summability. In Section 3, we define and explore the notions of Wijsman f_ρ -statistical convergence and Wijsman f_ρ -summability of order α , where f is an unbounded modulus function and ρ is a non-decreasing sequence of positive real numbers diverging to infinity. Additionally, we examine the connections between these two concepts.

2. Definitions and Preliminaries

We begin this section by exploring the concepts of natural density and statistical convergence. We denote the set of natural numbers by $\mathbb{N}$.

Statistical convergence, which generalizes classical convergence, is based on the natural density of subsets of the natural numbers and is precisely defined as follows:

The natural density of a subset $D \subset \mathbb{N}$ is given by

$$d(D) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{j \leq n : j \in D\}|,$$

where $|\{j \leq n : j \in D\}|$ denotes the count of elements in D that are less than or equal to n . Clearly, the density $d(D)$ equals zero when D is finite.

A sequence $z = (z_j)$ is said to be statistically convergent to r if, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{j \leq n : |z_j - r| \geq \epsilon\}| = 0.$$

If a sequence $z = (z_j)$ is statistically convergent to r , we denote this by $S - \lim z_j = r$.

The function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus function if it satisfies the following conditions: $f(z) = 0$ if and only if $z = 0$, $f(z + v) \leq f(z) + f(v)$ for $z, v \geq 0$, f is right-continuous at 0, and f is an increasing function. Moreover, f is continuous on the interval $[0, \infty)$ when it is defined as a modulus function [27].

Let f be an unbounded modulus function. The f -density of a set $D \subset \mathbb{N}$ is defined by

$$d^f(D) = \lim_{n \rightarrow \infty} \frac{f(|\{j \leq n : j \in D\}|)}{f(n)}$$

if the limit exists [13]. Here and in what follows, we assume that f is an unbounded modulus function unless stated otherwise.

Let $\alpha \in (0, 1]$. The α -density of $D \subset \mathbb{N}$ is given by

$$d_\alpha(D) = \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{j \leq n : j \in D\}|$$

provided that this limit exists. Here and hereafter, we assume that α is a real number such that $0 < \alpha \leq 1$, unless otherwise stated. It is important to note that when $\alpha = 1$, the α -density coincides with the natural density. However, as observed with f -density, the identity $d(D) + d(\mathbb{N} - D) = 1$ does not generally hold when natural density is replaced by α -density for any $\alpha \in (0, 1)$. Additionally, similar to the case of f -density, the α -density of any finite set remains zero [28].

The sequence $z = (z_j)$ is said to be f -statistically convergent to r , or S^f -convergent to r , if for every $\epsilon > 0$,

$$d^f(\{j \in \mathbb{N} : |z_j - r| \geq \epsilon\}) = 0,$$

that is,

$$\lim_{n \rightarrow \infty} \frac{1}{f(n)} f(|\{j \leq n : |z_j - r| \geq \epsilon\}|) = 0,$$

and we denote this as $S^f - \lim z_j = r$ [13].

A number sequence $z = (z_j)$ is said to be statistically convergent of order α to r , or S_α -convergent to r , if

$$d_\alpha(\{j \in \mathbb{N} : |z_j - r| \geq \epsilon\}) = 0,$$

i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{j \leq n : |z_j - r| \geq \epsilon\}| = 0,$$

for every $\epsilon > 0$.

We denote this as $S_\alpha - \lim z_j = r$. The set of all statistically convergent sequences of order α is denoted by S_α . When $\alpha = 1$, statistical convergence of order α reduces to the usual statistical convergence [10].

The f_α -density of $D \subset \mathbb{N}$ and f -statistical convergence of a sequence of order α are defined as follows:

The f_α -density of $D \subset \mathbb{N}$ is defined by

$$d_\alpha^f(D) = \lim_{n \rightarrow \infty} \frac{1}{f(n^\alpha)} f(|\{j \leq n : j \in D\}|),$$

provided the limit exists [28].

A sequence $z = (z_j)$ is said to be f -statistically convergent of order α to r , or S_α^f -convergent to r , if for every $\epsilon > 0$,

$$d_\alpha^f(\{j \in \mathbb{N} : |z_j - r| \geq \epsilon\}) = 0,$$

that is,

$$\lim_{n \rightarrow \infty} \frac{1}{f(n^\alpha)} f(\{j \leq n : |z_j - r| \geq \epsilon\}) = 0.$$

In this case, we write $S_\alpha^f - \lim z_j = r$ [28].

Let (Z, σ) be a metric space, and consider non-empty closed subsets B and B_j of the metric space Z . The distance $d(z, B)$ is defined as [23]

$$d(z, B) = \inf_{w \in B} \sigma(z, w)$$

If $\sup_j d(z, B_j) < \infty$ (for each $z \in Z$), then we say that the sequence $\{B_j\}$ is bounded.

A sequence $\{B_j\}$ is said to be Wijsman ρ -statistically convergent of order α (denoted as WS_ρ^α -convergent) to B if, for every $\epsilon > 0$ and $z \in Z$, the following condition holds:

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n^\alpha} |\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}| = 0.$$

Here and in what follows, $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to infinity, satisfying the following conditions:

$$\limsup_n \frac{\rho_n}{n} < \infty, \Delta \rho_n = \rho_{n+1} - \rho_n = O(1) \quad (1)$$

In this case, we write $B_j \rightarrow B \left(S \left[W_\rho^\alpha \right] \right)$. The set of all sequences $\{B_j\}$ that are Wijsman ρ -statistically convergent of order α to B is denoted by $S \left[W_\rho^\alpha \right]$ [29].

3. Main Results

Our work begins with the introduction of two new definitions

Definition 1. Let (Z, σ) be a metric space, and let $\rho = (\rho_n)$ be as above. Let B and B_j be non-empty closed subsets of Z . The sequence $\{B_j\}$ is said to be Wijsman f_ρ -statistically convergent of order α to B , if

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) = 0.$$

We denote by $\left[W_\rho^\alpha, f \right]$ the set of all Wijsman f_ρ -statistically convergent of order α .

For special choices of the modulus function f , the parameter α and the sequence $\rho = (\rho_n)$, we obtain one of the following special cases. For example:

If $f(z) = z$, then we write $S[W_\rho^\alpha]$ instead of $S[W_\rho^\alpha, f]$, which was defined and studied by Aral et al. [29].

If $f(z) = z$ and $\alpha = 1$, then we write $S[W_\rho]$ instead of $S[W_\rho^\alpha, f]$, which was defined and studied by Aral et al. [30].

If $f(z) = z$, $\alpha = 1$, and $\rho_n = n$ ($n \in \mathbb{N}$), then we write $S[W]$ instead of $S[W_\rho^\alpha, f]$, which was defined and studied by Nuray and Rhoades [23].

If $\rho_n = n$ ($n \in \mathbb{N}$), then we write $S[W^\alpha, f]$ instead of $S[W_\rho^\alpha, f]$.

If $\alpha = 1$, then we write $S[W_\rho, f]$ instead of $S[W_\rho^\alpha, f]$.

In the special case $\alpha = 1$ and $\rho_n = n$ ($n \in \mathbb{N}$), we write $S[W, f]$ instead of $S[W_\rho^\alpha, f]$.

The following example is important in showing that there exists a sequence of sets that is *Wijsman f_ρ -statistically convergent of order α* ; that is, the set $S[W_\rho^\alpha, f]$ is non-empty.

Define the sequence $\{B_j\}$ by

$$B_j = \begin{cases} \{\sqrt{j}\}, & j = n^2 \text{ for some } n \in \mathbb{N}. \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $(\mathbb{R}, d)$ be the metric space with usual metric, $\rho = (\rho_n) > n$, $\alpha > \frac{1}{2}$, and $f(z) = z^p$, with $0 < p \leq 1$. Then for every $\epsilon > 0$, we have

$$\frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) \leq \frac{f(\sqrt{n})}{f(\rho_n^\alpha)} = \frac{n^{\frac{p}{2}}}{n^{p\alpha}} \leq n^{p(\frac{1}{2} - \alpha)} \rightarrow 0.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) = 0,$$

which implies that $B_j \rightarrow \{0\} \left(S[W_\rho^\alpha, f] \right)$.

Every *Wijsman convergent* sequence is also *Wijsman f_ρ -statistically convergent of order α* for any unbounded modulus f and $\alpha \in (0, 1]$. However, as the following example shows, a *Wijsman f_ρ -statistically convergent sequence of order α* need not be *Wijsman convergent*.

Consider the sequence $\{B_j\}$ defined by

$$B_j = \begin{cases} \{j\}, & \text{if } j \text{ is a square} \\ \{0\}, & \text{otherwise} \end{cases}.$$

Then $B \in S[W_\rho^\alpha, f]$ ($\frac{1}{2} < \alpha \leq 1$), but it not *Wijsman convergent* when $f(z) = z^p$ with $0 < p \leq 1$.

Theorem 1. Let (Z, σ) be a metric space, f an unbounded modulus function, and let $B, B_j \subset Z$ ($j \in \mathbb{N}$) be non-empty closed subsets. For any $0 < \alpha \leq \beta \leq 1$, the inclusion $S[W_\rho^\alpha, f] \subset S[W_\rho^\beta, f]$ holds strictly. Here and hereafter, α and β will represent two real numbers such that $0 < \alpha \leq \beta \leq 1$.

Proof. Suppose $B_j \rightarrow B \left(S[W_\rho^\alpha, f] \right)$. If $\alpha \leq \beta$, then $n^\alpha \leq n^\beta$. Since f is increasing, we have $f(n^\alpha) \leq f(n^\beta)$. Moreover, since $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers, it follows that

$$\frac{1}{f(\rho_n^\beta)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) \leq \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|)$$

for every $\epsilon > 0$. This gives $S[W_\rho^\alpha, f] \subset S[W_\rho^\beta, f]$.

To show that the inclusion is strict consider the sequence $\{B_j\}$ defined by

$$B_j = \begin{cases} (u, v) \in \mathbb{R}^2, u^2 + (v - 1)^2 = j^2, & \text{if } j \text{ is a square} \\ \{0, 0\}, & \text{otherwise} \end{cases}.$$

Then $\{B_j\} \in S[W_\rho^\beta, f]$ ($\frac{1}{2} < \beta \leq 1$), but $\{B_j\} \notin S[W_\rho^\alpha, f]$ ($0 < \alpha \leq \frac{1}{2}$) when $f(z) = z$. $\square$

Definition 2. Let (Z, σ) be a metric space, and let $\rho = (\rho_n)$ be as above. Let B and B_j be non-empty closed subsets of Z . The sequence $\{B_j\}$ is said to be Wijsman strongly $f_\rho^{(p)}$ -summable of order α to B if, for each sequence $p = (p_j)$ of strictly positive real numbers and for each $z \in Z$,

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} = 0.$$

We denote by $N[W_\rho^\alpha, f, (p)]$ the set of all Wijsman strongly $f_\rho^{(p)}$ -summable sequences of order α . In this context, we write $B_j \rightarrow B$ ($N[W_\rho^\alpha, f, (p)]$), or equivalently, $N[W_\rho^\alpha, f, (p)] - \lim B_j = B$.

When $p_j = p$ for all $j \in \mathbb{N}$, we write $N[W_\rho^\alpha, f, p]$ instead of $N[W_\rho^\alpha, f, (p)]$.

When $p_j = 1$ for all $j \in \mathbb{N}$, we write $N[W_\rho^\alpha, f]$ instead of $N[W_\rho^\alpha, f, (p)]$.

When $p_j = 1$ for all $j \in \mathbb{N}$ and $f(z) = z$ we write $N[W_\rho^\alpha]$ instead of $N[W_\rho^\alpha, f, (p)]$ (as introduced by Aral et al. [29]).

When $p_j = 1$ for all $j \in \mathbb{N}$, $f(z) = z$, and $\alpha = 1$ we write $N[W_\rho]$ instead of $N[W_\rho^\alpha, f, (p)]$ (as introduced by Aral et al. [30]).

In particular, when $p_j = 1$ for all $j \in \mathbb{N}$ and $f(z) = z$, $\alpha = 1$, and $\rho_n = n$ (for all $n \in \mathbb{N}$) we write $N[W]$ instead of $N[W_\rho^\alpha, f, (p)]$ (as introduced by Nuray and Rhoades [23]).

Theorem 2. Let (Z, σ) be a metric space, f be an unbounded modulus function and let $B, B_j \subset Z$ ($j \in \mathbb{N}$) be non-empty closed subsets. For any $0 < \alpha \leq \beta \leq 1$, the inclusion $N[W_\rho^\alpha, f, p] \subset N[W_\rho^\beta, f, p]$ holds strictly.

Proof. The proof is similar to that of Theorem 1. To show that the inclusion is strict, consider the sequence $\{B_j\}$ defined by

$$B_j = \begin{cases} \{1, 1\}, & \text{if } j \text{ is a square} \\ \{0, 0\}, & \text{otherwise} \end{cases}$$

Let $\rho_n = n$, $p = 1$. Since $f(0) = 0$, we have

$$\frac{1}{n^\beta} \sum_{j=1}^n f(|d(z, B_j) - d(z, \{0, 0\})|) \leq \frac{\sqrt{n}}{n^\beta} f(1) = n^{\frac{1}{2}-\beta} f(1) \rightarrow 0 (n \rightarrow \infty)$$

for $\frac{1}{2} < \beta \leq 1$.

On the other hand,

$$\frac{1}{n^\alpha} \sum_{j=1}^n f(|d(z, B_j) - d(z, \{0, 0\})|) \geq \frac{\sqrt{n} - 1}{n^\alpha} f(1) \rightarrow \infty (n \rightarrow \infty)$$

for $0 < \alpha \leq \frac{1}{2}$.

Thus, $\{B_j\} \in N[W_\rho^\beta, f]$ for $\frac{1}{2} < \beta \leq 1$, but $\{B_j\} \notin N[W_\rho^\alpha, f]$ $0 < \alpha \leq \frac{1}{2}$. $\square$

The results stated in Theorems 1 and 2 are quite general. For special choices of the modulus function f , the parameter α , and the sequence $\rho = (\rho_n)$, we obtain the following special cases:

Corollary 1.

- (i) The inclusions $N[W_\rho^\alpha, p] \subset N[W_\rho^\beta, p]$ and $S[W_\rho^\alpha] \subset S[W_\rho^\beta]$ hold strictly, for $0 < \alpha \leq \beta \leq 1$,
- (ii) The inclusions $N[W_\rho^\alpha, f, p] \subset N[W_\rho, f, p]$ and $S[W_\rho^\alpha, f] \subset S[W_\rho, f]$ hold strictly, for $0 < \alpha \leq 1$,
- (iii) The inclusions $N[W^\alpha, f, p] \subset N[W^\beta, f, p]$ and $S[W^\alpha, f] \subset S[W^\beta, f]$ hold strictly, for $0 < \alpha \leq \beta \leq 1$,
- (iv) The inclusions $N[W_\rho^\alpha] \subset N[W_\rho]$ and $S[W_\rho^\alpha] \subset S[W_\rho]$ hold strictly, for $0 < \alpha \leq 1$,
- (v) The inclusion $N[W_\rho^\alpha, p] \subset N[W_\rho, p]$ hold strictly, for $0 < \alpha \leq 1$,
- (vi) The inclusion $N[W_\rho^\alpha, f] \subset N[W_\rho^\beta, f]$ hold strictly, for $0 < \alpha \leq \beta \leq 1$.

Theorem 3. Let (Z, σ) be a metric space, f an unbounded modulus function, and let $B, C, B_j, C_j \subset Z$ ($j \in \mathbb{N}$) be non-empty closed subsets. For $0 < \alpha \leq 1$, we have

- (i) If $c \in \mathbb{R}$ and $S[W_\rho^\alpha, f] - \lim B_j = B$, then $S[W_\rho^\alpha, f] - \lim cB_j = cB$.
- (ii) If $S[W_\rho^\alpha, f] - \lim B_j = B$ and $S[W_\rho^\alpha, f] - \lim C_j = C$, then $S[W_\rho^\alpha, f] - \lim (B_j + C_j) = B + C$.
- (iii) If $c \in \mathbb{R}$ and $N[W_\rho^\alpha, f] - \lim B_j = B$, then $N[W_\rho^\alpha, f] - \lim cB_j = cB$.
- (iv) If $N[W_\rho^\alpha, f] - \lim B_j = B$ and $N[W_\rho^\alpha, f] - \lim C_j = C$, then $N[W_\rho^\alpha, f] - \lim (B_j + C_j) = B + C$.

Proof. (i) Let $c \in \mathbb{R}$ and suppose that $S[W_\rho^\alpha, f] - \lim B_j = B$. If $c = 0$, the result is immediate. Now assume $c \neq 0$. For any $\epsilon > 0$, the result follows directly from the following equality

$$\frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(cz, cB_j) - d(cz, cB)| \geq \epsilon\}|) = \frac{1}{f(\rho_n^\alpha)} f\left(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \frac{\epsilon}{|c|}\}|\right).$$

(ii) Next, assume $S[W_\rho^\alpha, f] - \lim B_j = B$ and $S[W_\rho^\alpha, f] - \lim C_j = C$. For any $\epsilon > 0$, we can write

$$\begin{aligned} & |\{j \leq n : |d(z, B_j + C_j) - d(z, B + C)| \geq \epsilon\}| \\ & \leq |\{j \leq n : |d(z, B_j) - d(z, B)| \geq \frac{\epsilon}{2}\}| + |\{j \leq n : |d(z, C_j) - d(z, C)| \geq \frac{\epsilon}{2}\}|. \end{aligned}$$

By the subadditivity of the modulus function f ,

$$\begin{aligned} & \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j + C_j) - d(z, B + C)| \geq \epsilon\}|) \\ & \leq \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \frac{\epsilon}{2}\}|) + \frac{1}{f(\rho_n^\alpha)} f(|\{j \leq n : |d(z, C_j) - d(z, C)| \geq \frac{\epsilon}{2}\}|). \end{aligned}$$

Since both terms on the right-hand side tend to zero by assumption, it follows that

$$S[W_\rho^\alpha, f] - \lim (B_j + C_j) = B + C.$$

The proofs of (iii) and (iv) are analogous to those of (i) and (ii), respectively. $\square$

Theorem 4. If $\liminf_{t \rightarrow \infty} \frac{f(t)}{t} > 0$ and $p > 1$, then $N[W_\rho^\alpha, f, p] = N[W_\rho^\alpha, f, [p]]$, where $N[W_\rho^\alpha, f, [p]]$ denotes the family of sequences such that

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n |d(z, B_j) - d(z, B)|^p = 0.$$

Proof. If $\liminf_{t \rightarrow \infty} \frac{f(t)}{t} > 0$, then there exists a constant $m > 0$ such that $f(t) \geq mt$ for $t > 0$. Hence,

$$\frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^p \geq \frac{m^p}{f(\rho_n^\alpha)} \sum_{j=1}^n |d(z, B_j) - d(z, B)|^p$$

On the other hand, since the inequality $f(t) \leq 2f(1)\delta^{-1}t$ is satisfied for any modulus function f , any real number $0 < \delta < 1$; and, for each $t \geq \delta$, we obtain

$$\begin{aligned} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n |d(z, B_j) - d(z, B)|^p &\geq \frac{1}{f(\rho_n^\alpha)} \sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| \geq \delta}}^n |d(z, B_j) - d(z, B)|^p \\ &\geq \frac{1}{f(\rho_n^\alpha)} \sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| \geq \delta}}^n \left| \frac{f(|d(z, B_j) - d(z, B)|)}{2f(1)\delta^{-1}} \right|^p \\ &\geq \frac{1}{f(\rho_n^\alpha)} \frac{\delta^p}{f(1)^p 2^p} \sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| \geq \delta}}^n [f(|d(z, B_j) - d(z, B)|)]^p. \end{aligned}$$

Therefore, $N[W_\rho^\alpha, f, p] = N[W_\rho^\alpha, f, [p]]$. $\square$

Theorem 5. Let $f(uv) \geq mf(u)f(v)$ for all $u \geq 0, v \geq 0$ with $m > 0$ (a constant), and suppose that $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > 0$. Then $N[W_\rho^{\alpha, f}] \subset S[W_\rho^\beta, f]$ and $N[W_\rho^{\alpha, f}]$ is a proper subset of $S[W_\rho^\beta, f]$, where $N[W_\rho^{\alpha, f}]$ denotes the family of sequences such that

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n^\alpha} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)] = 0.$$

Proof. For $\epsilon > 0$, we can write

$$\begin{aligned} \frac{1}{\rho_n^\alpha} \sum_{j=1}^n f(|d(z, B_j) - d(z, B)|) &\geq \frac{1}{\rho_n^\alpha} f\left(\sum_{j=1}^n |d(z, B_j) - d(z, B)|\right) \\ &\geq \frac{1}{\rho_n^\alpha} f\left(\sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| \geq \epsilon}}^n |d(z, B_j) - d(z, B)|\right) \\ &\geq \frac{1}{\rho_n^\alpha} f\left(\left|\left\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\right\}\right| \epsilon\right) \\ &\geq \frac{m}{\rho_n^\alpha} f\left(\left|\left\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\right\}\right| \epsilon\right) f(\epsilon) \\ &\geq \frac{m}{\rho_n^\beta} f\left(\left|\left\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\right\}\right| \epsilon\right) f(\epsilon) \\ &\geq mf(\epsilon) \frac{f(\rho_n^\beta)}{\rho_n^\beta} \frac{1}{f(\rho_n^\beta)} f\left(\left|\left\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\right\}\right| \epsilon\right). \end{aligned}$$

Accordingly, it follows that $N[W_\rho^{\alpha, f}] \subseteq S[W_\rho^\beta, f]$.

To show that the inclusion is proper, we need to find a sequence belonging to $S[W_\rho^\beta, f]$ but not to $N[W_\rho^{\alpha, f}]$. Define $\{B_j\}$ by

$$B_j = \begin{cases} \{\sqrt{j}\}, & j = r^2 \\ \{0\}, & \text{otherwise} \end{cases}$$

and let $Z = \mathbb{R}$, $\sigma(z, y) = |z - y|$, $\rho_n = n$ and $f(z) = z$. For $\epsilon > 0$, $z > 0$ and $\frac{1}{2} < \alpha \leq 1$, we have

$$\frac{1}{\rho_n^\alpha} |\{j \leq n : |d(z, B_j) - d(z, \{0\})| \geq \epsilon\}| \leq n^{\frac{1}{2}-\alpha} \rightarrow 0 \quad (n \rightarrow \infty),$$

so $B_j \rightarrow 0(S[W_\rho^\alpha, f])$.

On the other hand, for $z > 0$ and $0 \leq \alpha \leq \frac{1}{2}$,

$$\frac{1}{\rho_n^\alpha} \sum_{j=1}^n |d(z, B_j) - d(z, \{0\})| \leq \frac{\sqrt{n}(\sqrt{n}+1)}{2n^\alpha} \rightarrow 0,$$

hence, $B_j \rightarrow 0(N[W_\rho^\alpha, f])$. $\square$

Corollary 2. Let f be an unbounded modulus function such that $f(uv) \geq mf(u)f(v)$ for all $u \geq 0$, $v \geq 0$, $m > 0$ (where m is a constant), and suppose that $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > 0$. Then $N[W_\rho^{\alpha, f}] \subseteq S[W_\rho^\alpha, f]$ and $N[W_\rho^f] \subseteq S[W_\rho, f]$.

Theorem 6. If $\rho_n > n$, then $N[W_\rho] \subset N[W_\rho^f]$.

Proof. Let $\epsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \leq t \leq \delta$. Set

$$M_n = \frac{1}{\rho_n} \sum_{j=1}^n |d(z, B_j) - d(z, B)|.$$

Since $f(t) \leq 2f(1)\delta^{-1}t$, we can write

$$\begin{aligned} \frac{1}{\rho_n} \sum_{j=1}^n f(|d(z, B_j) - d(z, B)|) &= \frac{1}{\rho_n} \sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| \leq \delta}}^n f(|d(z, B_j) - d(z, B)|) \\ &\quad + \frac{1}{\rho_n} \sum_{\substack{j=1 \\ |d(z, B_j) - d(z, B)| > \delta}}^n f(|d(z, B_j) - d(z, B)|) \\ &\leq \frac{n}{\rho_n} \epsilon + 2f(1)\delta^{-1}M_n \\ &\leq \epsilon + 2f(1)\delta^{-1}M_n. \end{aligned}$$

Hence, we have $N[W_\rho] \subset N[W_\rho^f]$. $\square$

Theorem 7. Let $f(uv) \geq mf(u)f(v)$ for all $u \geq 0$, $v \geq 0$, $m > 0$ (where m is a constant) and $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > 0$. Let $\rho_n > n$ and suppose that the sequence $\{B_j\}$ is bounded, then $S[W_\rho^f] \subset N[W_\rho^f]$.

Proof. Suppose that $B_j \rightarrow B\left(S\left[W_\rho^f\right]\right)$ and the sequence $\{B_j\}$ is bounded. Then we can find a constant M such that $|d(z, B_j) - d(z, B)| \leq M$. Now, for a given $\epsilon > 0$ and $z \in Z$, we have

$$\begin{aligned} \frac{1}{\rho_n} \sum_{j=1}^n f(|d(z, B_j) - d(z, B)|) &= \frac{1}{\rho_n} \sum_{j=1}^n f(|d(z, B_j) - d(z, B)|) \\ &\quad + \frac{1}{\rho_n} \sum_{j=1}^n f(|d(z, B_j) - d(z, B)|) \\ &\leq f(M) \frac{1}{\rho_n} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) + \frac{n}{\rho_n} f(\epsilon) \\ &\leq f(M) \frac{1}{\rho_n} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) + f(\epsilon). \end{aligned}$$

Hence, we conclude that $B_j \rightarrow B\left(N\left[W_\rho^f\right]\right)$. $\square$

Theorem 8. If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^\alpha)}{f(n^\alpha)} > 0$, then $N[W^\alpha, f, (p)] \subset N[W_\rho^\beta, f, (p)]$.

Proof. If $B_j \rightarrow B(N[W^\alpha, f, (p)])$, then we have

$$\lim_{n \rightarrow \infty} \frac{1}{f(n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} = 0.$$

For $0 < \alpha \leq \beta \leq 1$, we have $n^\alpha \leq n^\beta$. Since f is increasing, we can write $f(n^\alpha) \leq f(n^\beta)$. Since $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers, we have

$$\begin{aligned} \frac{1}{f(n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} &\geq \frac{1}{f(n^\beta)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} \\ &\geq \frac{f(\rho_n^\alpha)}{f(n^\beta)} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} \\ &\geq \frac{f(\rho_n^\alpha)}{f(n^\beta)} \frac{1}{f(\rho_n^\beta)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain $B_j \rightarrow B\left(N\left[W_\rho^\beta, f, (p)\right]\right)$. $\square$

Corollary 3. (i) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^\alpha)}{f(n^\alpha)} > 0$, then $N[W^\alpha, f, (p)] \subset N[W_\rho^\alpha, f, (p)]$ for $0 < \alpha \leq 1$,

(ii) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n)}{f(n^\alpha)} > 0$, then $N[W^\alpha, f, (p)] \subset N[W_\rho, f, (p)]$ for $0 < \alpha \leq 1$,

(iii) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^\alpha)}{f(n^\alpha)} > 0$, then $N[W^\alpha, f, (p)] \subset N[W_\rho^\alpha, f, (p)]$.

Proof. We will briefly prove only (i); the others follow similarly. If we take $\beta = \alpha$ in Theorem 8, then the condition " $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^\alpha)}{f(n^\alpha)} > 0$ " reduces to the condition " $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^\alpha)}{f(n^\alpha)} > 0$ " and " $N[W^\alpha, f, (p)] \subset N[W_\rho^\beta, f, (p)]$ " becomes " $N[W^\alpha, f, (p)] \subset N[W_\rho^\alpha, f, (p)]$ ". $\square$

Theorem 9. Let f be an unbounded modulus function, and let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha \leq \beta \leq 1$. Assume $\liminf_{n \rightarrow \infty} \frac{\rho_n}{n} > 1$ and $\lim_{t \rightarrow \infty} \frac{f(t^\alpha)}{t^\alpha} = v > 0$ ($v \in \mathbb{R}$). Then the inclusion $S[W^\alpha, f] \subset S[W_\rho^\beta, f]$ is proper.

Proof. First, consider $\liminf_{n \rightarrow \infty} \frac{\rho_n}{n} > 1$. Then there exists $\lambda > 0$ such that $\frac{\rho_n}{n} \geq 1 + \lambda$ for large n , which implies

$$\frac{\rho_n}{n} \geq 1 + \lambda \Rightarrow \left(\frac{\rho_n}{n}\right)^\alpha \geq (1 + \lambda)^\alpha.$$

Assuming $S[W^\alpha, f] - \lim B_j = B$, for any $\epsilon > 0$ there exists a sufficiently large n such that

$$\begin{aligned} & \frac{1}{f(n^\alpha)} f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|) \\ &= \frac{f(\rho_n^\alpha)}{\rho_n^\alpha} \frac{n^\alpha}{f(n^\alpha)} \frac{\rho_n^\alpha}{n^\alpha} \frac{f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|)}{f(\rho_n^\alpha)} \\ &\geq \frac{f(\rho_n^\alpha)}{\rho_n^\alpha} \frac{n^\alpha}{f(n^\alpha)} (1 + \lambda)^\alpha \frac{f(|\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}|)}{f(\rho_n^\beta)}. \end{aligned}$$

To show that the inclusion is proper, consider the following example.

Let $Z = \mathbb{R}$, $B = \{0\}$, $\rho_n = 2n$, $f(z) = z$ and $0 < \alpha = \frac{1}{k} < \beta < 1$ ($k \in \mathbb{N}$).

Let $J = \{m^{1/\alpha} : m \in \mathbb{N}\} = \{m^k : m \in \mathbb{N}\}$ and consider a sequence $\{B_j\}$ defined as follows:

$$B_j = \begin{cases} \{1\}, & \text{if } j \in J \\ \{0\}, & \text{if } j \notin J \end{cases}.$$

Then, for $z = 0$ we have

$$|d(0, B_j) - d(0, B)| = \begin{cases} 1, & \text{if } j \in J \\ 0, & \text{if } j \notin J \end{cases}.$$

Since $J = \{m^k\}$, for $\epsilon = 1/2$ we obtain

$$|\{j \leq n : |d(0, B_j) - d(0, B)| \geq 1/2\}| = |J \cap \{1, \dots, n\}| \sim n^\alpha = [n^\alpha] \sim n^\alpha.$$

Then, for $\alpha < \beta$ we have

$$\frac{1}{f(\rho_n^\beta)} f(|\{j \leq n : |d(0, B_j) - d(0, B)| \geq 1/2\}|) = \frac{[n^\alpha]}{(2n)^\beta} \rightarrow 0 \quad (n \rightarrow \infty)$$

and

$$\frac{1}{f(n^\alpha)} f\left(\left|\left\{j \leq n : |d(0, B_j) - d(0, B)| \geq \frac{1}{2}\right\}\right|\right) = \frac{[n^\alpha]}{n^\alpha} \rightarrow 1 \quad (n \rightarrow \infty).$$

Hence, $\{B_j\} \in S[W_\rho^\beta, f] \setminus S[W^\alpha, f]$. $\square$

Remark 1. In the following Theorem, we obtain the same result as in Theorem 8 by changing the conditions on the function f and the sequence $\rho = (\rho_n)$.

Theorem 10. Let f be an unbounded modulus function, and let $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. If $\liminf_{n \rightarrow \infty} \frac{\rho_n}{n} > 1$ and $\lim_{t \rightarrow \infty} \frac{f(t^\alpha)}{t^\alpha} = v > 0$ ($v \in \mathbb{R}$), then $N[W^\alpha, f, (p)] \subset N[W_\rho^\beta, f, (p)]$.

Proof. If $B_j \rightarrow B(N[W^\alpha, f, (p)])$, then we have

$$\lim_{n \rightarrow \infty} \frac{1}{f(n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} = 0.$$

Since $\liminf_{n \rightarrow \infty} \frac{\rho_n}{n} > 1$ and $\lim_{t \rightarrow \infty} \frac{f(t^\alpha)}{t^\alpha} > 0$, the proof follows from the following inequality using the argument from Theorem 9.

$$\begin{aligned} & \frac{1}{f(n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} \\ &= \frac{f(\rho_n^\alpha)}{\rho_n^\alpha} \frac{n^\alpha}{f(n^\alpha)} \frac{\rho_n^\alpha}{n^\alpha} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j} \\ &\geq \frac{f(\rho_n^\alpha)}{\rho_n^\alpha} \frac{n^\alpha}{f(n^\alpha)} (1 + \lambda)^\alpha \frac{1}{f(\rho_n^\beta)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_j}. \end{aligned}$$

□

Theorem 11. If $\lim_{j \rightarrow \infty} p_j > 0$, then $N[W_\rho^\alpha, f, (p)] - \lim_{j \rightarrow \infty} B_j = B$ converges uniquely.

Proof. Consider $\lim_{j \rightarrow \infty} p_j = u > 0$. Suppose $N[W_\rho^\alpha, f, (p)] - \lim_{j \rightarrow \infty} B_j = B_1$ and $N[W_\rho^\alpha, f, (p)] - \lim_{j \rightarrow \infty} B_j = B_2$. Then,

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B_1)|)]^{p_j} = 0,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B_2)|)]^{p_j} = 0.$$

By the definition of f , we obtain

$$\begin{aligned} & \frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_1) - d(z, B_2)|)]^{p_j} \\ &\leq \frac{U}{f(\rho_n^\alpha)} \left(\sum_{j=1}^n [f(|d(z, B_j) - d(z, B_1)|)]^{p_j} + \sum_{j=1}^n [f(|d(z, B_j) - d(z, B_2)|)]^{p_j} \right) \\ &= \frac{U}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B_1)|)]^{p_j} + \frac{U}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B_2)|)]^{p_j}, \end{aligned}$$

where $\sup_j p_j = T$ and $U = \max(1, 2^{T-1})$. Therefore

$$\frac{1}{f(\rho_n^\alpha)} \sum_{j=1}^n [f(|d(z, B_1) - d(z, B_2)|)]^{p_k} = 0.$$

Since $\lim p_j = u$, it follows that $B_1 - B_2 = 0$. Hence, the limit is unique. □

Theorem 12. Let $\rho = (\rho_n)$ and $\rho' = (\rho'_n)$ be two sequences such that $\rho_n \leq \rho'_n$ for all $n \in \mathbb{N}$, and let α_1, α_2 be two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If

$$\liminf_{n \rightarrow \infty} \frac{f(\rho_n^{\alpha_1})}{f(\rho_n^{\alpha_2})} > 0, \quad (2)$$

then

- (i) $N[W_{\rho'}^{\alpha_2}, f, (p)] \subset N[W_{\rho}^{\alpha_1}, f, (p)],$
- (ii) $S[W_{\rho'}^{\alpha_2}, f] \subset S[W_{\rho}^{\alpha_1}, f].$

Proof. The proof of (i) is obtained from the following inequality:

$$\frac{1}{f(\rho_n^{\alpha_2})} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_k} \geq \frac{f(\rho_n^{\alpha_1})}{f(\rho_n^{\alpha_2})} \frac{1}{f(\rho_n^{\alpha_1})} \sum_{j=1}^n [f(|d(z, B_j) - d(z, B)|)]^{p_k}.$$

(ii) Let $B_j \rightarrow B(S[W_{\rho'}^{\alpha_2}, f])$. For $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\rho_n \leq \rho'_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{1}{f(\rho_n^{\alpha_2})} f(\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}) \\ & \geq \frac{f(\rho_n^{\alpha_1})}{f(\rho_n^{\alpha_2})} \frac{1}{f(\rho_n^{\alpha_1})} f(\{j \leq n : |d(z, B_j) - d(z, B)| \geq \epsilon\}). \end{aligned}$$

Thus, we have $B_j \rightarrow B(S[W_{\rho}^{\alpha_1}, f])$. $\square$

Corollary 4. (i) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^{\alpha_1})}{f(\rho_n^{\alpha_2})} > 0$, then $N[W_{\rho'}^{\alpha_2}, f, (p)] \subset N[W_{\rho}^{\alpha_1}, f, (p)]$ and $S[W_{\rho'}^{\alpha_2}, f] \subset S[W_{\rho}^{\alpha_1}, f]$, for $0 < \alpha_1 \leq 1$,

(ii) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n)}{f(\rho_n^{\alpha_2})} > 0$, then $N[W_{\rho'}^{\alpha_2}, f, (p)] \subset N[W_{\rho}, f, (p)]$ and $S[W_{\rho'}^{\alpha_2}, f] \subset S[W_{\rho}, f]$,

(iii) If $\liminf_{n \rightarrow \infty} \frac{f(\rho_n)}{f(\rho_n^{\alpha_2})} > 0$, then $N[W_{\rho'}^{\alpha_2}, f, (p)] \subset N[W_{\rho}, f, (p)]$ and $S[W_{\rho'}^{\alpha_2}, f] \subset S[W_{\rho}, f]$ for $0 < \alpha_2 \leq 1$.

Proof. Briefly, we will prove only a part of (i); the others follow in a similar way. If we take $\alpha_2 = 1$ in (1), then the condition " $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^{\alpha_1})}{f(\rho_n^{\alpha_2})} > 0$ " reduces to the condition

" $\liminf_{n \rightarrow \infty} \frac{f(\rho_n^{\alpha_1})}{f(\rho_n)} > 0$ " and the relation " $N[W_{\rho'}^{\alpha_2}, f, (p)] \subset N[W_{\rho}^{\alpha_1}, f, (p)]$ " becomes " $N[W_{\rho'}, f, (p)] \subset N[W_{\rho}^{\alpha_1}, f, (p)]$ ". $\square$

4. Discussion

In this study, the notions of Wijsman statistical convergence and Wijsman strong summability of order α have been generalized by employing unbounded modulus functions and non-decreasing sequences ρ . The findings indicate that fundamental properties such as linearity, stability, and uniqueness of limits are preserved, while strict inclusion relations among different convergence classes are also established.

This generalized framework has potential applications in areas such as approximation theory, functional analysis, and probability theory. Different choices of modulus functions

(e.g., linear or nonlinear) or sequences ρ lead to distinct convergence behaviors, thereby enriching the theoretical scope of the study.

Nevertheless, the fact that some of the inclusion relations are strict highlights the necessity of careful selection of the parameters f , ρ , and α in practical applications. Future research may focus on extending these notions to Banach and Hilbert spaces, as well as exploring their adaptations to probabilistic and fuzzy settings, which may provide further theoretical insights and practical applications.

5. Conclusions

In this paper, we introduced and studied new notions of Wijsman statistical convergence and Wijsman strong summability of order α for sequences of closed sets in metric spaces. These concepts generalize classical Wijsman convergence by incorporating unbounded modulus functions and generalized summability methods.

We established several inclusion relations between these newly defined concepts, proving that the inclusions are strict under appropriate conditions on the parameters, the modulus function, and the sequence. Additionally, uniqueness of limits was demonstrated for the strongly summable sequences, and various properties such as linearity and stability under scalar multiplication and addition were confirmed.

Our results provide a broad framework for further exploration of statistical convergence concepts in metric spaces, opening paths for applications in analysis and approximation theory where such generalized convergence methods are essential. Future work may focus on several directions, including applications of these generalized convergence concepts to other areas of mathematics, such as functional analysis, optimization, and fixed point theory. Additionally, investigating the behavior of these convergence notions in more complex structures, such as Banach or Hilbert spaces, and extending them to probabilistic or fuzzy set settings could provide deeper insights. Another promising direction is to explore computational aspects and algorithmic implementations, potentially leading to practical methods for approximating set-valued mappings in applied problems. Overall, these avenues suggest that the concepts introduced here have significant potential for both theoretical development and practical applications.

The results obtained in this study are quite general. If the function f , the parameter α and the sequence $\rho = (\rho_n)$ are chosen in specific forms, our results reduce to several well-known results in the literature. For example:

- When $f(z) = z$, our results reduce to the concepts of Wijsman ρ -statistical convergence of order α studied in [29].
- When $f(z) = z$ and $\alpha = 1$, our results reduce to the concepts of Wijsman ρ -statistical convergence studied in [30].
- When $f(z) = z$, $\rho = (\rho_n)$ for all $n \in \mathbb{N}$ and $\alpha = 1$, our results reduce to the concepts of Wijsman statistical convergence studied in [23].
- Moreover, if we consider sequences of real numbers instead of sequences of sets, our results generalize the works of Salat [7], Fridy [8], Connor [9], Çolak [10], Aizpuru et al. [13], and Bhardwaj and Dhawan [14,28].

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