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Weighted Integral Operators from the $\mathcal{H}_v^\infty$ Space to the $\mathcal{B}_\mu$ Space on Cartan–Hartogs Domains

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Abstract

In this paper, necessary and sufficient conditions are established for the boundedness and compactness of a weighted integral operator when acting between the weighted Bloch space $\mathcal{B}_\mu$ and the weighted Hardy space $\mathcal{H}_v^\infty$ on the Cartan–Hartogs domain.

Keywords: Cartan–Hartogs domain; weighted integral operator; weighted Bers space; weighted Bloch space; boundedness and compactness

MSC: 32A27; 47B33

1. Introduction

In the study of multivariable complex function theory, operators represent important mathematical objects that have attracted extensive scholarly attention. In recent years, significant progress has been made, particularly in the analysis of operators associated with holomorphic functions defined on domains such as the unit disk, the unit ball, bounded symmetric domains, and Hua domains. Researchers have obtained substantial results regarding properties such as boundedness, compactness, and essential norms. These developments not only provide valuable tools to related research areas but also highlight the promising future growth and applicability of the theory of several complex variables.

For example, on the unit disk, we obtain the following:

$$D = \{z \in \mathbb{C} \mid |z| < 1\},$$

notable studies include the following: Mohan and Venku [1] proved that, on a certain special domain, the set of all bounded radial operators coincides with that of all radial operators. Ueki [2] established the necessary and sufficient conditions for the boundedness and compactness of Li–Stević operators from the weighted Bergman space to the β-Zygmund space, and estimated the essential norm of such operators. Gong [3] characterized the non-negative measure defined on the unit disk for which the sublinear operator from the Hardy space to the Lebesgue space is bounded or compact. References [4,5] present certain favorable properties of the Volterra and sublinear operators on Bergman spaces. Panteris [6] established three necessary and sufficient conditions under which integral operators possess closed range on Hardy, BMOA, and Besov spaces. Finally, Qian and Zhu [7] investigated the boundedness, compactness, and essential norm of integral operators from Morrey-type spaces to weighted BMOA spaces.

On the unit ball, we obtain the following:

$$B_n = \{z \in \mathbb{C}^n \mid |z| < 1\},$$

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where $z = (z_1, z_2, z_3, \cdots, z_n) \in \mathbb{C}^n$ and $|z| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$, the following contributions are notable: Zhang [8] derived necessary conditions for bounded linear integral operators mapping from the weighted Lebesgue space $L^p(B_n, dv_t)$ into $L^q(B_n, dv_t)$. David [9] investigated the boundedness and compactness of Hankel operators between different weighted Bergman spaces. Dogan [10] proved that the Hermitian square of the Toeplitz and Hankel operators is compact and related to the transformation of the operator. Stevich [11] investigated the boundedness and compactness of integral operators from mixed norm spaces on the unit ball, while Zhu [12] studied the same but from Privalov spaces to Bloch-type and little Bloch-type spaces. Both Liang and Zhou [13] and Ren [14] explored the boundedness and compactness of certain integral-type operators, with the former focused on those acting from $F(p, q, s)$ spaces to mixed-norm spaces, and the latter from weighted Bergman–Privalov spaces to Bloch-type spaces, on the unit ball. Finally, Stević and Ueki [15] established conditions for the boundedness and compactness of integral-type operators acting between weighted-type spaces on the unit ball.

On the polydisk, we obtain the following:

$$D^n(r) = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid |z_1| < r_1, |z_2| < r_2, \cdots, |z_n| < r_n\},$$

where $r = (r_1, r_2, \cdots, r_n)$ is a positive radius vector, and each component is a positive value, the following studies have been conducted: Arkady [16] derived the spectrum and essential spectra of weighted automorphisms of the polydisk algebra. Nikolaos [17] studied the random Gram matrices generated by random sequences and then used the random matrix theory to achieve Random Carleson sequences for the Hardy space on the polydisk. Bhattacharyya [18] established a commutant lifting theorem along with a general result on the polydisk. Lastly, Stevich [19] studied the boundedness and compactness of integral-type operators from a mixed norm space to a Bloch-type space on the polydisk.

In 1935, Cartan showed that bounded symmetric domains in $\mathbb{C}^n$ are necessarily homogeneous and provided a complete classification of all irreducible cases, which are grouped into six classes [20]:

$$\Re_{\mathrm{I}}(m, n) := \{Z \in \mathbb{C}^{mn} : I - Z\overline{Z}' > 0\},$$

$$\Re_{\mathrm{II}}(p) := \{Z \in \mathbb{C}^{\frac{p(p+1)}{2}} : I - Z\overline{Z}' > 0, Z = Z'\},$$

$$\Re_{\mathrm{III}}(q) := \{Z \in \mathbb{C}^{\frac{q(q-1)}{2}} : I - Z\overline{Z}' > 0, Z = -Z'\},$$

$$\Re_{\mathrm{IV}}(n) := \{z \in \mathbb{C}^n : 1 + |zz'|^2 - 2z\bar{z} > 0, 1 - |zz'|^2 > 0\}.$$

Here, m, n, p, and q are all positive integers; Z represents an $m \times n$ complex matrix; $p \times p$ is the symmetric square matrix; $q \times q$ is the skew-symmetric square matrix; n indicates the dimensional complex vector; $\overline{Z}$ represents the conjugate of Z; and Z' represents the transpose of Z. By carefully examining the first type of classical domain, it is evident that it reduces to a unit ball when m = 1. Furthermore, when m = n = 1, the domain corresponds to the unit disk.

However, Cartan's classification includes certain exceptional domains [21], and the analysis associated with these domains is highly intricate, making direct applications difficult. Hua focused his research on the first four classes, commonly referred to as the classical domains. On the bounded symmetric domains, Issa [22] determined the eigenvalues of several fundamental k-invariant Toeplitz-type operators and showed that the compactness of these operators is independent of weights under certain conditions. Furthermore, Allen and Colonna [23] investigated norm estimates for weighted composition operators mapping from the Hardy space to the Bloch space.

Building upon Hua's contributions, Yin Weiping extended this framework by introducing four classes of Cartan–Hartogs domains. The first class is defined as follows:

$$Y_I(N,m,n,s) = \left\{ W \in \mathbb{C}^N, Z \in \Re_I(m,n) : |W|^2 < \det(I - Z\overline{Z}')^s \right\}.$$

Here, $\Re_I(m,n)$ denotes the first classical domain, where N, m, and n are positive integers, and $s > 0$. For the sake of simplicity, we will denote $Y_I(N,m,n,s)$ as Y_I in the subsequent discussion. Subsequently, Yin further developed the concept by introducing Hua domains, which encompass Cartan–Hartogs domains, Cartan–Egg domains, Hua domains, generalized Hua domains, and the Hua construction [24].

At present, research within the Hua domain is becoming increasingly comprehensive. For example, Jiang and Li [25] investigated four types of weighted composition operators on the Hua Luogeng domain. Su, Li, and Wang [26] examined the boundedness and compactness of operators in Bloch spaces under varying weight conditions. Wang and Su [27] established necessary and sufficient conditions for the boundedness and compactness of weighted composition operators on generalized Hua domains of the fourth kind. Liu and Liu [28] studied the product of weighted composition operators and radial derivative operators from the Bloch-type space into Bers-type space on the fourth Loo-Keng Hua domain. Su and Wu [29] introduced weighted Bloch spaces over a class of generalized Cartan–Hartogs domains and explored the necessary and sufficient conditions for the boundedness and compactness of composition operators within these spaces.

The applications of weighted integral operators are extremely extensive. Zenaw [30] established the spectra and eigenvectors for a class of generalized Volterra-type integral operators acting on Fock spaces over the complex plane. Leveraging these spectral findings as a foundational tool, the author further analyzed the dynamical behaviors of the aforementioned operators and presented a rigorous characterization of certain power-bounded integral operators. Stević [31] derived explicit formulas for the norm and essential norm of an integral-type operator mapping from the logarithmic Bloch space to the Bloch-type space over the unit ball. As key applications of these formulas, rigorous characterizations of the boundedness and compactness of the aforementioned integral-type operator were provided. Bardac and Vlada [32] introduced new integral operators and established geometric characterizations of the starlikeness and convexity properties associated with these operators. They also provided a concise illustrative example to demonstrate the application of the proposed operators. In the work of Liang and Wang [33], the authors present several new equivalent characterizations for the boundedness of the differences of integral-type operators from the α-Bloch space to the β-Bloch–Orlicz space on the unit disk. The authors also estimate the essential norms of these operator differences in terms of the n-th power of the induced analytic self-maps on the unit disk, and this estimation can provide novel and interesting compactness criteria. Du and Li [34] investigated the boundedness and compactness of the generalized Volterra integral operator acting on weighted Bergman spaces with doubling weights over the unit disk. As a key application of their findings, they further characterized the Schatten class membership of such generalized Volterra integral operators. Yang and Yuan [35] conducted a systematic investigation on a class of integral operators over the unit ball and proved that such operators preserve the core properties of Carleson measures within the Mobius-invariant Besov space. On this basis, as a significant application of the operator theory established above, they further derived the estimation for the distance from Bloch-type functions to the space of holomorphic functions. This result extends the classical Jones formula and provides a more general theoretical framework for related research in complex function spaces.

Since the inception of Hua domains, research centered on these domains has long been a focal point of scholarly attention in complex analysis, encompassing fundamental topics such as the Bergman problem, rigidity theorems, comparison theorems, and extremal problems. To illustrate, Dong, Li, and Treuer [36] obtained bounded solutions to the equation $\bar{\partial}u = f$ on classical Cartan domains by imposing a stronger assumption on f, which has facilitated the resolution of numerous related problems. Similarly, such $\bar{\partial}$-equations can also be identified on Hua domains.

Zhang [37] derived an explicit expression for the Bergman kernel function on the second-type Hua construction; Feng [38] established rigidity results for properly holomorphic mappings over Hua domains; and Wang and Su [39] generalized Hua domains to a more abstract and general framework, further conducting systematic investigations into operator theory within this extended domain structure. Nevertheless, despite the substantial achievements made by numerous researchers in the field of complex analysis, studies on integral operators over Hua domains remain relatively scarce. Therefore, investigating problems related to integral operators on Hua domains holds considerable significance.

Currently, the study of integral operators over the Hua domain remains unexplored, suggesting that new findings in this area could significantly contribute to the advancement of complex function theory.

2. Preliminaries

We investigate the integral operator defined on the first type of Cartan–Hartogs domains as follows:

$$Q_{\phi}^{g} f(Z,W) = \int_0^1 f(\phi(tZ,tW)) g(tZ,tW) \frac{dt}{t},$$

where $f, g \in H(Y_I)$ with $g(0,0) = 0$, and ϕ is a holomorphic self-mapping on Y_I. Here, $H(Y_I)$ denotes the space of holomorphic functions on Y_I.

Throughout this paper, C represents a constant that does not depend on the function f under consideration. Its value may vary at different instances.

We define the space as follows:

$$\mathcal{H}_v^{\infty}(Y_I) = \{f \in H(Y_I) : \|f\|_{v,\infty} < +\infty\}, v > 0,$$

equipped with the norm, written as follows:

$$\|f\|_{v,\infty} = \sup_{(Z,W)\in Y_I} \left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^v |f(Z,W)|. \tag{1}$$

It is straightforward to verify that $\mathcal{H}_v^{\infty}(Y_I)$ is a Banach space.

Here, we provide an example of $f(Z,W)$:
For any fixed (Z_0, W_0), set the following:

$$f(Z,W) = \frac{1}{\left(\det\left(I - Z\overline{Z_0}'\right)^s - \langle W, W_0\rangle\right)^v};$$

then, $f(Z,W)$ is holomorphic, and according to Lemma 1, written as follows:

$$\begin{aligned}
&\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^v|f(Z,W)|\\
&=\frac{(\det(I-Z\overline{Z}')^s-|W|^2)^v}{|\det(I-Z\overline{Z_0}')^s-\langle W,W_0\rangle|^v}\\
&\leq\frac{(\det(I-Z\overline{Z}')^s-|W|^2)^v}{\left[\frac{1}{2}(\det(I-Z\overline{Z}')^s-|W|^2+\det(I-Z_0\overline{Z_0}')^s-|W_0|^2)\right]^v}\\
&\leq\frac{2^v(\det(I-Z\overline{Z}')^s-|W|^2)^v}{(\det(I-Z\overline{Z}')^s-|W|^2)^v}\\
&=2^v,
\end{aligned}$$

so, we obtain the following:

$$\|f\|_{v,\infty}=\sup_{(Z,W)\in Y_I}\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^v|f(Z,W)|\leq 2^v.$$

A function $f\in H(Y_I)$ is called a weighted Bloch function if it satisfies the following:

$$\|f\|_{\mathcal{B}_\mu}:=\sup_{(Z,W)\in Y_I}\left[\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right]^\mu|\mathcal{R}f(Z,W)|<\infty,$$

where $\mathcal{R}f(Z,W)$ denotes the radial derivative of $f(Z,W)$, $\mu>0$. The set of all such functions constitutes the weighted Bloch space on Y_I, denoted by $\mathcal{B}_\mu(Y_I)$. The norm on this space is defined as follows:

$$\|f\|_\mu=|f(0,0)|+\|f\|_{\mathcal{B}_\mu}.$$

In the particular case when $\mu=1$, the space $\mathcal{B}_\mu$ reduces to the classical Bloch space $\mathcal{B}$.

Lemma 1 ([39]). *Let $(Z,W),(X,Y)\in Y_I$ and $0<ms\leq 1$. Then, the following is written:*

$$\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)+\left(\det\left(I-X\overline{X}'\right)^s-|Y|^2\right)\leq 2\left|\det\left(I-Z\overline{X}'\right)^s-\langle W,Y\rangle\right|.$$

Lemma 2 ([39]). *Let $(Z,W),(X,Y)\in Y_I$ and $0<ms\leq 1$. Then, the following is written:*

$$\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)\left(\det\left(I-X\overline{X}'\right)^s-|Y|^2\right)\leq\left|\det\left(I-Z\overline{X}'\right)^s-\langle W,Y\rangle\right|^2.$$

Lemma 3. *For all $(Z,W)\in Y_I$ and $f\in\mathcal{H}_v^\infty(Y_I)$, we have the following:*

$$|f(Z,W)|\leq\frac{\|f\|_{v,\infty}}{[\det(I-Z\overline{Z}')^s-|W|^2]^v}$$

Proof. The proof follows directly from Formula (1). □

Lemma 4 ([39]). *Given $0<ms\leq 1$, and $(Z,W)\in Y_I$. Let f be a holomorphic function on $\mathcal{B}_\mu(Y_I)$. Then there exists a constant C such that we obtain the following:*

$$|f(Z,W)|\leq\begin{cases}C\|f\|_\mu & 0<\mu<1,\\ C\|f\|_\mu\ln\frac{2}{\det\left(I-Z\overline{Z}'\right)^s-|W|^2} & \mu=1,\\ C\|f\|_\mu\frac{1}{\left[\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right]^{\mu-1}} & \mu>1.\end{cases}$$

Lemma 5. *Let $f, g \in H(Y_I)$, and let ϕ be a holomorphic self-mapping of Y_I. Suppose that $g(0,0) = 0$. Then, for $(Z, W) \in Y_I$, the following identity holds:*

$$\mathcal{R}Q_\phi^g f(Z,W) = f(\phi(Z,W))g(Z,W). \tag{2}$$

Proof. By definition, the operator Q_ϕ^g acts on a function f as follows:

$$Q_\phi^g f(Z,W) = \int_0^1 f(\phi(tZ,tW))g(tZ,tW)\frac{dt}{t}.$$

Therefore, we obtain the following:

$$\mathcal{R}Q_\phi^g f(Z,W) = \mathcal{R}\int_0^1 f(\phi(tZ,tW))g(tZ,tW)\frac{dt}{t}.$$

Let us define the auxiliary function, written as follows:

$$H(tZ,tW) = \frac{f(\phi(tZ,tW))g(tZ,tW)}{t}.$$

Without loss of generality, suppose that $\phi = (\phi_{11}, \ldots, \phi_{mn}, \phi_{W_1}, \ldots, \phi_{W_N})$. Then, we compute the partial derivative of $H(tZ,tW)$ with respect to z_{pq} as follows:

$$\frac{\partial H}{\partial z_{pq}} = \frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial \phi_{ij}}\cdot\frac{\partial \phi_{ij}}{\partial t z_{pq}}\cdot t + \sum_{d=1}^{N}\frac{\partial f}{\partial \phi_{W_d}}\cdot\frac{\partial \phi_{W_d}}{\partial t z_{pq}}\cdot t\right)g(tZ,tW) + \left(\frac{\partial g}{\partial t z_{pq}}\cdot t\right)f(\phi(tZ,tW))}{t},$$

for $1 \le p \le m$ and $1 \le q \le n$.

Similarly, for $1 \le k \le N$, the partial derivative of $H(tZ,tW)$ with respect to W_k is given by the following:

$$\frac{\partial H}{\partial W_k} = \frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial \phi_{ij}}\cdot\frac{\partial \phi_{ij}}{\partial t W_k}\cdot t + \sum_{d=1}^{N}\frac{\partial f}{\partial \phi_{W_d}}\cdot\frac{\partial \phi_{W_d}}{\partial t W_k}\cdot t\right)g(tZ,tW) + \left(\frac{\partial g}{\partial t W_k}\cdot t\right)f(\phi(tZ,tW))}{t}.$$

By the chain rule, we obtain the following:

$$\frac{d}{dt}(f(\phi(tZ,tW))g(tZ,tW)) = \sum_{1\le k\le m}\sum_{1\le l\le n}\frac{\partial H}{\partial z_{kl}}z_{kl} + \sum_{d=1}^{N}\frac{\partial H}{\partial W_d}W_d.$$

and

$$\begin{aligned}\mathcal{R}Q_\phi^g f(Z,W) = \int_0^1 \Bigg(&\sum_{s=1}^{m}\sum_{t=1}^{n}\Bigg[\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial \phi_{ij}}\cdot\frac{\partial \phi_{ij}}{\partial t z_{st}} + \sum_{d=1}^{N}\frac{\partial f}{\partial \phi_{W_d}}\cdot\frac{\partial \phi_{W_d}}{\partial t z_{st}}\right)g(tZ,tW)z_{st}\\ &+ \frac{\partial g}{\partial t z_{st}}\cdot f(\phi(tZ,tW))z_{st}\Bigg]\\ &+ \sum_{k=1}^{N}\Bigg[\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial \phi_{ij}}\cdot\frac{\partial \phi_{ij}}{\partial t W_k} + \sum_{d=1}^{N}\frac{\partial f}{\partial \phi_{W_d}}\cdot\frac{\partial \phi_{W_d}}{\partial t W_k}\right)g(tZ,tW)W_k\\ &+ \frac{\partial g}{\partial t W_k}\cdot f(\phi(tZ,tW))W_k\Bigg]\Bigg)dt.\end{aligned}$$

The following can thus be derived:

$$\begin{aligned}\mathcal{R}Q_{\phi}^{g}f(Z,W) &= \int_0^1 \frac{d(f(\phi(tZ,tW))g(tZ,tW))}{dt}dt \\ &= f(\phi(Z,W))g(Z,W) - f(\phi(0,0))g(0,0) \\ &= f(\phi(Z,W))g(Z,W).\end{aligned}$$

This completes the proof. □

Lemma 6. *Let $g \in H(Y_I)$ with $g(0,0) = 0$, and let ϕ be a holomorphic self-mapping of Y_I. Then, the integral operator $Q_{\phi}^{g} : \mathcal{H}_v^{\infty}(Y_I) \to \mathcal{B}_{\mu}(Y_I)$ is compact if and only if Q_{ϕ}^{g} is bounded. Moreover, for every bounded sequence $\{f_j\}_{j\geq 1}$ in $\mathcal{H}_v^{\infty}(Y_I)$ that converges uniformly to 0 on every compact subset of Y_I as $j \to \infty$, the following holds:*

$$\lim_{j\to\infty} \|Q_{\phi}^{g}f_j\|_{\mu} = 0.$$

Proof. Suppose that $Q_{\phi}^{g} : \mathcal{H}_v^{\infty}(Y_I) \to \mathcal{B}_{\mu}(Y_I)$ is compact. It is thus doubtlessly a bounded operator. Let $\{f_n\}_{n\geq 1}$ be a bounded sequence in $\mathcal{H}_v^{\infty}(Y_I)$ and $f_n \to 0$ uniformly on compact subsets of Y_I as $n \to \infty$.

If $\|Q_{\phi}^{g}f_n\|_{\mu} \nrightarrow 0$ as $n \to \infty$, then there exists a subsequence $\{f_{n_j}\}_{j\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that we obtain the following:

$$\inf_{j\in N} \|Q_{\phi}^{g}f_{n_j}\|_{\mu} > 0.$$

Since Q_{ϕ}^{g} is compact, there exists a subsequence of the bounded sequence $\{f_{n_j}\}_{j\geq 1}$ (without loss of generality, still written by $\{f_{n_j}\}_{j\geq 1}$), and $f \in \mathcal{B}_{\mu}(Y_I)$ such that we obtain the following:

$$\lim_{j\to\infty} \|Q_{\phi}^{g}f_{n_j} - f\|_{\mu} = 0.$$

When $\mu > 1$, from Lemma 4, the following holds:

$$|(Q_{\phi}^{g}f_{n_j} - f)(Z,W)| \leq \frac{\|Q_{\phi}^{g}f_{n_j} - f\|_{\mu}}{(\det(I - Z\overline{Z}')^s - |W|^2)^{\mu-1}}.$$

Let K be a compact subspace of Y_I that contains zero points. Then $\det(I - Z\overline{Z}')^s - |W|^2$ has a positive lower realm on K.

Thus, $Q_{\phi}^{g}f_{n_j} - f \to 0$ uniformly on K. From this, for arbitrary $\varepsilon > 0, \exists N_1 > 0$, whenever $j > N_1$, we have the following:

$$|Q_{\phi}^{g}f_{n_j}(Z,W) - f(Z,W)| < \varepsilon, \text{ for all } (Z,W) \in K.$$

Notice that $g(tZ,tW)/t$ is holomorphic on Y_I, and thus, it is bounded on K. Let the following be true:

$$M = \max_{(Z,W)\in K, t\in[0,1]} \frac{|g(tZ,tW)|}{t}.$$

Since $f_{n_j} \to 0$ uniformly on compact subsets of Y_I as $j \to \infty$, there also exists a positive integer N_2, written as follows:

$$\begin{aligned} |Q_\phi^g f_{n_j}(Z,W)| &= \left| \int_0^1 f_{n_j}(\phi(tZ,tW)) g(tZ,tW) \frac{\mathrm{d}t}{t} \right| \\ &\leq \int_0^1 \left| f_{n_j}(\phi(tZ,tW)) \right| |g(tZ,tW)| \frac{\mathrm{d}t}{t} \\ &< M\varepsilon \end{aligned}$$

for $(Z,W) \in K$ whenever $j > N_2$.

Let $N = \max\{N_1, N_2\}$, and whenever $j > N$, we have the following:

$$\begin{aligned} |f(Z,W)| &\leq \left| Q_\phi^g f_{n_j}(Z,W) - f(Z,W) \right| + \left| Q_\phi^g f_{n_j}(Z,W) \right| \\ &\leq (M+1)\varepsilon, \ \ \forall (Z,W) \in K. \end{aligned}$$

From the arbitrariness of ε, we obtain $f(Z,W) \equiv 0$, $\forall (Z,W) \in K$. By the uniqueness theorem of the analytic function, we have $f(Z,W) \equiv 0$, $\forall (Z,W) \in Y_I$. This shows that $\lim_{j\to\infty} \|Q_\phi^g f_{n_j}\|_\mu = 0$, which contradicts with $\inf_{j\in N} \|Q_\phi^g f_{n_j}\|_\mu > 0$.

When $0 < \mu \leq 1$, the desired conclusion can be proven by similar methods.

Conversely, suppose that $\{f_n\}_{n\geq 1}$ is a bounded sequence in $\mathcal{H}_v^\infty(Y_I)$, then for all n, $\|f_n\|_{v,\infty} \leq D_1$. From Lemma 3, it is obvious that $\{f_n\}_{n\geq 1}$ is uniformly bounded on compact subsets of Y_I. By Montel's theorem, there exists a subsequence $\{f_{n_j}\}_{j\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that $f_{n_j} \to f$ uniformly on every compact subset of Y_I, and $f \in H(Y_I)$. So, $f_{n_j} - f \to 0$ uniformly on every compact subset of Y_I.

For any fixed $(Z_0, W_0) \in Y_I$, there exists a compact subset $K_{(Z_0,W_0)}$ of Y_I such that $(Z_0, W_0) \in K_{(Z_0,W_0)}$. Then, there also exists a $J_0 > 0$, and whenever $j > J_0$, we have the following:

$$|f_{n_j}(Z,W) - f(Z,W)| < 1, \ \ \forall\, (Z,W) \in K_{(Z_0,W_0)}.$$

It suffices to obtain the following:

$$\begin{aligned} &(\det(I - Z_0\overline{Z_0}')^s - |W_0|^2)^v |f(Z_0,W_0)| \\ &\leq (\det(I - Z_0\overline{Z_0}')^s - |W_0|^2)^v |f(Z_0,W_0) - f_{n_j}(Z_0,W_0)| \\ &+ (\det(I - Z_0\overline{Z_0}')^s - |W_0|^2)^v |f_{n_j}(Z_0,W_0)| \\ &\leq 1 + D_1. \end{aligned}$$

For all $(Z,W) \in Y_I$, $(\det(I - Z\overline{Z}')^s - |W|^2)^v |f(Z,W)| \leq 1 + D_1$. From this, we have $\|f\|_{v,\infty} \leq 1 + D_1$.

Hence, $f \in \mathcal{H}_v^\infty(Y_I)$ and $f_{n_j} - f \in \mathcal{H}_v^\infty(Y_I)$.

Further, we have the following:

$$\lim_{j\to\infty} \|Q_\phi^g(f_{n_j} - f)\|_\mu = \lim_{j\to\infty} \|Q_\phi^g f_{n_j} - Q_\phi^g f\|_\mu = 0,$$

which shows that $Q_\phi^g : \mathcal{H}_v^\infty(Y_I) \to \mathcal{B}_\mu(Y_I)$ is compact. □

3. Main Results

Theorem 1. *Let $g \in H(Y_I)$ with $g(0,0) = 0$, and let ϕ be a holomorphic self-mapping on Y_I. Denote that $\phi(Z,W) =: (Z_\phi, W_\phi)$. $0 < ms \leq 1$. Then, we obtain the following:*

$$\frac{(\det(I - Z\overline{Z}')^s - |W|^2)^\mu}{(\det(I - Z_\phi\overline{Z_\phi}')^s - |W_\phi|^2)^v}|g(Z,W)| < \infty \tag{3}$$

if and only if the operator $Q_\phi^g : \mathcal{H}_v^\infty(Y_I) \to \mathcal{B}_\mu(Y_I)$ is a bounded integral operator.

Proof. "$\Longrightarrow$" Suppose that (3) holds. Then, for all $f \in \mathcal{H}_v^\infty(Y_I)$, we have the following:

$$\|f\|_{v,\infty} = \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^v |f(Z,W)| < \infty. \tag{4}$$

So, we obtain the following:

$$\begin{aligned}
\left\|Q_\phi^g f\right\|_\mu &= \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu \left|\mathcal{R}Q_\phi^g f(Z,W)\right| \\
&= \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu |f(\phi(Z,W))||g(Z,W)| \\
&\leq \sup_{(Z,W)\in Y_I}\frac{\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu}{\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^s - |W_\phi|^2\right)^v}|g(Z,W)| \\
&\quad \times \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^s - |W_\phi|^2\right)^v |f(\phi(Z,W))| \\
&\leq \|f\|_{v,\infty}\sup_{(Z,W)\in Y_I}\frac{\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu}{\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^s - |W_\phi|^2\right)^v}|g(Z,W)| \\
&\leq C\|f\|_{v,\infty}.
\end{aligned}$$

Therefore, $Q_\phi^g : \mathcal{H}_v^\infty(Y_I) \to \mathcal{B}_\mu(Y_I)$ is a bounded integral operator.

"$\Longleftarrow$" Suppose $\left\|Q_\phi^g f\right\|_\mu \leq C\|f\|_{v,\infty}$ for all $f \in \mathcal{H}_v^\infty(Y_I)$. For $(X_0, Y_0) \in Y_I$, construct

$$f_0(Z,W) = \frac{\left(\det\left(I - X_0\overline{X_0}'\right)^s - |Y_0|^2\right)^v}{\left(\det\left(I - Z\overline{X_0}'\right)^s - \langle W, Y_0\rangle\right)^{2v}}. \tag{5}$$

From Lemma 3,

$$\|f_0\|_{v,\infty} = \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^v \frac{\left(\det\left(I - X_0\overline{X_0}'\right)^s - |Y_0|^2\right)^v}{\left|\det\left(I - Z\overline{X_0}'\right)^s - \langle W, Y_0\rangle\right|^{2v}} \leq 1.$$

Thus, $f_0(Z,W) \in \mathcal{H}_v^\infty(Y_I)$. Then, we have the following:

$$\begin{aligned}
C &\geq \left\| Q_\phi^g f_0 \right\|_\mu \\
&= \sup_{(Z,W)\in Y_I} \left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu \left|\mathcal{R} Q_\phi^g f_0(Z,W)\right| \\
&= \sup_{(Z,W)\in Y_I} \left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu |f_0(\phi(Z,W))||g(Z,W)| \\
&\geq \left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu |f_0(\phi(Z,W))||g(Z,W)| \\
&= \left(\det\left(I - Z\overline{Z}'\right)^s - |W|^2\right)^\mu |g(Z,W)| \frac{\left(\det\left(I - X_0\overline{X_0}'\right)^s - |Y_0|^2\right)^v}{\left|\det\left(I - Z_\phi\overline{X_0}'\right)^s - \langle W_\phi, Y_0\rangle\right|^{2v}}
\end{aligned}$$

Let $(X_0, Y_0) = (Z_\phi, W_\phi)$. Then, we have the following:

$$\infty > C \geq \frac{(\det(I - Z\overline{Z}')^s - |W|^2)^\mu}{(\det(I - Z_\phi\overline{Z_\phi}')^s - |W_\phi|^2)^v} |g(Z,W)|. \tag{6}$$

The proof is thus completed. □

If $s = m = 1$, then Y_I becomes a unit ball: $B_n = \{z \in \mathbb{C}^n \mid |z| < 1\}$, and thus, Theorem 1 simplifies to a widely recognized result, as follows.

Corollary 1. *Let $g \in H(B_n)$, with $g(0) = 0$, and let ϕ be a holomorphic self-mapping on B_n. Then, we obtain the following:*

$$\frac{\left(1 - |z|^2\right)^\mu}{\left(1 - |\phi(z)|^2\right)^v} \cdot |g(z)| < \infty \tag{7}$$

if and only if the operator $Q_\phi^g : \mathcal{H}_v^\infty(B_n) \to \mathcal{B}_\mu(B_n)$ is a bounded integral operator.

Extensive in-depth research has been conducted on Equation (7); for further details, readers can refer to references [13,15].

Note: ① If $\{\phi(z) \mid z \in B_n\}$ is contained in a compact subset of B_n, that is, there exists $\delta > 0$ such that $1 - |\phi(z)|^2 \geq \delta > 0$ for all $z \in B_n$, then it is sufficient that $g(z)$ is a bounded holomorphic function on B_n with $g(0) = 0$. For example, consider $g(z) = z_1^2 + z_2^2 + \cdots + z_n^2$; in this case, since $|g(z)| \leq n$, we have the following:

$$\frac{\left(1 - |z|^2\right)^\mu}{\left(1 - |\phi(z)|^2\right)^v} \cdot |g(z)| < \frac{n}{\delta^v}.$$

② If $\phi(z)$ is a holomorphic automorphism of B_n and $\phi(a) = 0$, then it is well known that $1 - |\phi(z)|^2 = \dfrac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{\left|1 - z\overline{a}'\right|^2}$. In this case, Formula (7) becomes the following:

$$\frac{\left|1 - z\overline{a}'\right|^{2v}}{\left(1 - |a|^2\right)^v \left(1 - |z|^2\right)^{v-\mu}} |g(z)| < \infty.$$

When $0 \le v \le \mu$, we can consider that $g(z) = z_1 + z_2 + \cdots + z_n$; then, we obtain the following:

$$\frac{\left|1 - z\overline{a}'\right|^{2v}}{\left(1 - |a|^2\right)^{v}\left(1 - |z|^2\right)^{v-\mu}}|g(z)| \le \frac{n \cdot 2^{2v}}{\left(1 - |a|^2\right)^{v}}.$$

In fact, in this case, $g(z)$ can be chosen as any bounded holomorphic function on B_n, provided that it satisfies the condition $g(0) = 0$.

Theorem 2. *Let $g \in \mathcal{H}_{\mu}^{\infty}(Y_I)$ with $g(0,0) = 0$, and let ϕ be a holomorphic self-mapping of Y_I. Denote that $\phi(Z,W) =: (Z_\phi, W_\phi)$. $0 < ms \le 1$. Then, we obtain the following:*

$$\lim_{\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2 \to 0} \frac{\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}}{\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2\right)^{v}}|g(Z,W)| = 0 \tag{8}$$

if and only if the operator $Q_\phi^g : \mathcal{H}_v^{\infty} \to \mathcal{B}_\mu$ is compact.

Proof. "$\Longrightarrow$" Suppose that (8) holds. Then, for $\forall \varepsilon > 0$, $\exists \delta > 0$, such that when $\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2 < \delta$, we have

$$\frac{\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}}{\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2\right)^{v}}|g(Z,W)| < \varepsilon. \tag{9}$$

Let $\{f_j\}_{j\ge 1}$ be a bounded sequence in $\mathcal{H}_v^{\infty}$, and $\{f_j\}_{j\in N}$ converges uniformly to 0 on any compact subset of Y_I. Consider $E_{\phi,\delta} = \left\{\phi(Z,W) \mid \det(I - Z_\phi\overline{Z_\phi}')^s - |W_\phi|^2 \ge \delta\right\}$. Then, we obtain the following:

$$\begin{aligned}
\left\|Q_\phi^g f_j\right\|_\mu &= \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}\left|\mathcal{R}\left(Q_\phi^g f_j(Z,W)\right)\right| \\
&= \sup_{(Z,W)\in Y_I}\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}|f_j(\phi(Z,W))||g(Z,W)| \\
&\le \sup_{(Z,W)\in E_{\phi,\delta}}\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}|g(Z,W)||f_j(\phi(Z,W))| \\
&+ \sup_{(Z,W)\in Y_I\setminus E_{\phi,\delta}}\frac{\left(\det\left(I - Z\overline{Z}'\right)^{s} - |W|^2\right)^{\mu}}{\left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2\right)^{v}}|g(Z,W)| \\
&\cdot \left(\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2\right)^{v}|f_j(\phi(Z,W))| \\
&\le \sup_{(Z,W)\in E_{\phi,\delta}}\|g\|_{\mu,\infty}|f_j(\phi(Z,W))| + \sup_{(Z,W)\in Y_I\setminus E_{\phi,\delta}}\varepsilon \cdot \|f_j\|_{v,\infty}.
\end{aligned}$$

Therefore, we have $\lim_{j\to\infty}\left\|Q_\phi^g f_j\right\|_\mu = 0$. According to Lemma 6, Q_ϕ^g is a compact operator.

"$\Longleftarrow$" (i) Since $Q_\phi^g : \mathcal{H}_v^{\infty} \to \mathcal{B}_\mu$ is a compact operator, it is naturally a bounded operator. Similarly to the proof of Theorem 3 in reference [27], for all $(Z,W) \in Y_I$, there exists a $\delta \in (0,1)$ such that $\det\left(I - Z_\phi\overline{Z_\phi}'\right)^{s} - |W_\phi|^2 \ge \delta > 0$. Then, Equation (7) is naturally correct.

(*ii*) If $\inf_{(Z,W)\in Y_I}\left\{\det\left(I-Z_\phi\overline{Z_\phi}'\right)^s-|W_\phi|^2\right\}=0$, then consider a sequence $\left(Z^j,W^j\right)\in Y_I$ such that $\left(Z^j_\phi,W^j_\phi\right)=\phi\left(Z^j,W^j\right)\to\partial Y_I$ as $j\to\infty$. Now, define a sequence of test functions $\{f_j\}_{j\geq 1}$ as follows:

$$f_j(Z,W)=\frac{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}{\left(\det\left(I-Z\overline{Z^j_\phi}'\right)^s-\langle W,W^j_\phi\rangle\right)^{2v}},\ j=1,2,\ldots. \tag{10}$$

Then,

$$\begin{aligned}|f_j(Z,W)|&=\frac{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}{\left|\left(\det\left(I-Z\overline{Z^j_\phi}'\right)^s-\langle W,W^j_\phi\rangle\right)\right|^{2v}}\\&\leq\frac{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}{\left(\frac{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)+\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)}{2}\right)^{2v}}\\&\leq\frac{2^{2v}\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}{\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^{2v}}.\end{aligned}$$

Let E be any compact subset of Y_I. Then, $\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)$ has a positive lower bound on E, and thus, $\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^{2v}$ also has a positive lower bound on E. Moreover, since the following is true:

$$\lim_{j\to\infty}\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)=0,$$

it follows that $\{f_j(Z,W)\}$ converges uniformly to 0 for $(Z,W)\in E$. As has been proven, $\|f_j\|_{v,\infty}\leq 1$. So, we obtain the following: $f_j(Z,W)\in\mathcal{H}^\infty_v$. Since $Q^g_\phi:\mathcal{H}^\infty_v\to\mathcal{B}_\mu$ is a compact operator, then $\lim_{j\to\infty}\left\|Q^g_\phi f_j\right\|_\mu=0$. So,

$$\begin{aligned}0\leftarrow\left\|Q^g_\phi f_j\right\|_\mu&=\sup_{(Z,W)\in Y_I}\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^\mu\left|\mathcal{R}Q^g_\phi f_j(Z,W)\right|\\&\geq\left(\det\left(I-Z^j\overline{Z^j}'\right)^s-|W^j|^2\right)^\mu\left|\mathcal{R}Q^g_\phi f_j(Z^j,W^j)\right|\\&=\left(\det\left(I-Z^j\overline{Z^j}'\right)^s-|W^j|^2\right)^\mu\left|f_j\left(\phi\left(Z^j,W^j\right)\right)\right|\left|g\left(Z^j,W^j\right)\right|\\&=\left(\det\left(I-Z^j\overline{Z^j}'\right)^s-|W^j|^2\right)^\mu\frac{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^{2v}}|g(Z^j,W^j)|\\&=\frac{\left(\det\left(I-Z^j\overline{Z^j}'\right)^s-|W^j|^2\right)^\mu}{\left(\det\left(I-Z^j_\phi\overline{Z^j_\phi}'\right)^s-|W^j_\phi|^2\right)^v}\cdot|g(Z^j,W^j)|.\end{aligned}$$

In conclusion, we have the following:

$$\lim_{\det\left(I-Z_\phi\overline{Z_\phi}'\right)^s-|W_\phi|^2\to 0}\frac{\left(\det\left(I-Z\overline{Z}'\right)^s-|W|^2\right)^\mu}{\left(\det\left(I-Z_\phi\overline{Z_\phi}'\right)^s-|W_\phi|^2\right)^v}|g(Z,W)|=0.$$

The proof is completed. □

On the unit ball, Theorem 2 simplifies to a widely recognized result, as follows.

Corollary 2. *Let $g\in H(B_n)$, with $g(0)=0$, and let ϕ be a holomorphic self-mapping on B_n. Then, the following is calculated:*

$$\lim_{\phi(z)\to 1}\frac{\left(1-|z|^2\right)^\mu}{\left(1-|\phi(z)|^2\right)^v}\cdot|g(z)|=0 \tag{11}$$

if and only if the operator $Q_\phi^g:\mathcal{H}_v^\infty(B_n)\to\mathcal{B}_\mu(B_n)$ is a compact integral operator.

Extensive in-depth research has been conducted on Equation (11), and readers can refer to references [13,15] for further details.

Note: If $\phi(z)$ is a holomorphic automorphism of B_n and $\phi(a)=0$, then it is well known that $1-|\phi(z)|^2=\dfrac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{\left|1-z\overline{a}'\right|^2}$. In this case, Formula (8) becomes the following:

$$\lim_{\phi(z)\to 1}\frac{\left(1-|z|^2\right)^\mu}{\left(1-|\phi(z)|^2\right)^v}\cdot|g(z)|=0$$

When $0\le v<\mu\le\infty$, we can take $g(z)=z_1+z_2+\cdots+z_n, z\in B_n$; then, the following is calculated:

$$\lim_{\phi(z)\to 1}\frac{\left|1-z\overline{a}'\right|^{2v}}{\left(1-|a|^2\right)^v\left(1-|z|^2\right)^{v-\mu}}|g(z)|\le\lim_{\phi(z)\to 1}\frac{n\cdot 2^{2v}\cdot\left(1-|z|^2\right)^{\mu-v}}{\left(1-|a|^2\right)^v}=0.$$

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