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Three-Way Approximations with Covering-Based Rough Set

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Abstract

In order to approximate an undefinable set of objects by using the extensions in OE-concept lattices, this study combines three-way concept analysis with covering-based rough set and introduces an innovative approach for managing uncertain information and decision-making. This approach employs the minimal neighborhood of the maximal description, which is determined by meet-irreducible elements, to define the lower and upper of an undefinable set. On this basis, we formalize the concepts of lower and upper approximation OE-concepts and propose a three-way approximation optimization algorithm. Experimental results demonstrate the effectiveness and efficiency of our algorithm.

Keywords: covering-based rough set; the maximal description; the lower and upper approximations; optimization

MSC: 06-08

1. Introduction

Rough set is a powerful tool for data analysis and knowledge discovery. Initially introduced by the Polish scholar Pawlak [1] in 1982. Since its inception, rough set theory has undergone substantial advancements in both theoretical extensions and practical applications. For instance, Stepaniuk and Skowron [2] explored a three-way rough set-based approach for decision granule approximation and developed a novel approximation space to optimize the characterization of compound decision granules. Janusz and Ślęzak [3] proposed rough set methods for attribute clustering and selection in which greedy heuristics are incorporated to reduce computation times and generate multiple decision reducts. Inspired by the concepts of lower and upper approximations in rough set theory, Siminski and Wakulicz-Deja [4] proposed an extended forward inference algorithm. This method enhances classical forward inference by enabling the process to continue after an inference failure through the interactive supplementation of missing facts. Rough set is established upon the indiscernibility relations that exist within the universe of discourse. However, the indiscernibility relations may sometimes restrict the practical applications of rough set [5]. To overcome this limitation, Zakowsk [6] proposed covering-based rough sets by utilizing cover relations instead of indiscernibility relations. To address issues of the minimal and maximal descriptions of covering-based rough sets, Wang et al. [7] devised matrix approaches that facilitate these descriptions and introduced two types of covering information reductions under these descriptions. Zhu [8] further enriched the field by identifying four distinct types of covering-based rough sets and investigated their topological properties, which has been instrumental in understanding the connections among different covering-based rough sets. Moreover, Zhang et al. [9] built the equivalences between the four types



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of covering-based rough sets and traditional rough set, and also presented sufficient and necessary conditions for these equivalences. Yao et al. [10] presented a framework for approximating covering-based rough sets by constructing four distinct neighborhoods, six novel coverings, and two sub-systems. They also proposed approximation operators based on elements, granules, and subsystems relevant to covering-based rough sets and discussed the relationships between these operators and existing ones. Janicki et al. [11] defined optimal approximation in the context of similarity measures and provided an algorithm that employs the Jaccard Index as an optimal approximation operator.

Formal Concept Analysis (FCA), introduced by Wille in 1982, has evolved into a valuable method utilized in data mining, information retrieval, knowledge discovery, and various other fields [12–17]. Some scholars have expanded FCA by introducing multiple types of formal concepts, which significantly enriches the original FCA. In 2014, Qi et al. [18] established three-way operators and their inverses and introduced two distinct types of three-way concepts: the object-induced concept and the attribute-induced concept. Furthermore, they integrated these concepts within the framework of three-way decision to construct three-way concept lattices. Zhang et al. [19] proposed a novel algorithm, termed 3WOC, which can directly and quickly generate all OEO concepts with lower time complexity. The introduction of the three-way concept lattice has since prompted widespread research interests across multiple scholarly fields including lattice construction [20–24], lattice reduction [25–27], rule acquisition derived from lattice analysis [28–30], and conceptual knowledge [31–33].

Some researchers have incorporated rough sets into formal concept analysis to investigate the uncertainty present in FCA [34–36]. Yao et al. [37] utilized approximation operators to represent and elucidate classical concept lattices and studied three distinct types of concept lattices: object-oriented, attribute-oriented, and complement-oriented concept lattices. Furthermore, Yao [34] developed a set of operators drawing from both lattice theory and set theory and investigated the interconnections between these operators. Building on these foundations, Monhanty [38] generalized approximation operators on the formal concepts and represented undefinable sets by employing two definable sets in a concept context. Mao et al. [39] proposed the variable precision rough set approximations within concept lattices informed by FCA, thereby facilitating the approximation of undefinable sets and analyzing trends in the changes of the β -upper and lower approximations. Recently, some researchers have traditionally constructed a complete concept lattice before employing approximation operators to approximately characterize conceptual knowledge. However, this type of approach overlooks the fact that constructing a concept lattice is an NP-hard problem. Existing approximation methods require the computation of all concepts, which results in significant computational time, especially when dealing with massive datasets. If approximate characterization can be achieved without building the entire concept lattice, the computational cost of mining conceptual knowledge would be significantly reduced. Therefore, lowering the time complexity of the computational process is a meaningful research task. This study aims to optimize the computational process of approximation methods by improving knowledge representation, thereby decreasing computational time. This improvement lays the foundation for a more efficient and concise knowledge representation in subsequent three-way concept lattices.

Building on the foundational work reference [39], this paper introduces an innovative approach for developing approximation operators within the framework of three-way concept lattices. The primary contribution of this work is to extend the framework of the object-induced three-way concept. The organization of the rest of this paper is outlined below. Section 2 provides an overview of the foundational concepts of formal concept analysis. Sections 3 and 4 elaborate on the extended framework of the OE concept: Section 3 defines

the concept of maximal description and introduces the lower and upper approximation OE concepts; Section 4 proposes an optimization algorithm for approximation operators and analyzes its time complexity. Section 5 provides experimental validation of the proposed algorithm. Finally, conclusions are drawn in Section 6.

2. Preliminaries

This section will provide an overview of essential definitions and properties involved in this paper.

Definition 1 ([40]). Let L be a lattice. For $\forall k_1, k_2 \in L$, if $k_3 \in L$, $k_3 = k_1 \wedge k_2$, and $k_3 \neq 0$ implies that $k_3 = k_1$ or $k_3 = k_2$, then k_3 is meet-irreducible.

Definition 2 ([41]). A formal context $K = (P, Q, R)$ is composed of two sets, P and Q , along with a relation R that links elements of P to those of Q . The elements of P are referred to as the objects, while the elements of Q are known as the attributes.

To indicate that an object $p \in P$ is related to an attribute $q \in Q$, we use the notation pRq or $(p, q) \in R$.

For an object $p \in P$, we denote the object intent $\{q \in Q | pRq\}$ simply as p^* instead of $\{p\}^*$. Accordingly, we define $q^* := \{p \in P | pRq\}$ to represent the attribute extent of the attribute q .

Let $Z \subseteq P$ and $B \subseteq Q$. The following defines the pair of operators used in formal concept analysis:

$$\begin{aligned} Z^* &= \{q | q \in Q, \forall p \in Z, pRq\}, \\ B^* &= \{p | p \in P, \forall q \in B, pRq\}. \end{aligned}$$

A formal concept is defined as a pair (Z, B) where $Z \subseteq P, B \subseteq Q$, and it satisfies $Z^* = B$ and $B^* = Z$. The notation $L(K)$ represents the set of all concepts associated with the context K .

We denote the object concept as $OC(K) = (p^{**}, p^*)$ and the attribute concept as $AC(K) = (q^*, q^{**})$.

Definition 3 ([18]). Let $K = (P, Q, R)$ be a formal context with $Z \subseteq P$ and $B \subseteq Q$. The pair of negative operators is defined as follows:

$$\begin{aligned} Z^{\bar{*}} &= \{q | q \in Q, \forall p \in Z, pR^c q\}, \\ B^{\bar{*}} &= \{p | p \in P, \forall q \in B, pR^c q\}. \end{aligned}$$

Here, $R^c = (P \times Q) - R$.

Definition 4 ([18]). Let $K = (P, Q, R)$ be a formal context and $Z \subseteq P, B \subseteq Q$. The three-way operators are defined as follows:

$$\begin{aligned} Z^{\triangleleft} &= (Z^*, Z^{\bar{*}}), \\ B^{\triangleleft} &= (B^*, B^{\bar{*}}). \end{aligned}$$

Dually, for $Z_1, Z_2 \subseteq P$ and $B_1, B_2 \subseteq Q$, we define their inverses by

$$\begin{aligned} (Z_1, Z_2)^{\triangleright} &= \{b \in Q | b \in Z_1^*, b \in Z_2^{\bar{*}}\} = Z_1^* \cap Z_2^{\bar{*}}, \\ (B_1, B_2)^{\triangleright} &= \{z \in P | z \in B_1^*, z \in B_2^{\bar{*}}\} = B_1^* \cap B_2^{\bar{*}}. \end{aligned}$$

Proposition 1 ([18]). Let $K = (P, Q, R)$ be a formal context. For $Z, Z_1, Z_2 \subseteq P$, and $B, B_1, B_2, C, C_1, C_2 \subseteq Q$, the following properties hold:

- (1) $Z \subseteq Z^{\triangleleft \triangleright}$
- (2) $(B, C) \subseteq (B, C)^{\triangleright \triangleleft}$,
- (3) $Z_1 \subseteq Z_2 \Rightarrow Z_2^{\triangleleft} \subseteq Z_1^{\triangleleft}$,
- (4) $(B_1, C_1) \subseteq (B_2, C_2) \Rightarrow (B_2, C_2)^{\triangleright} \subseteq (B_1, C_1)^{\triangleright}$,
- (5) $Z^{\triangleleft} = Z^{\triangleleft \triangleright \triangleleft}$,
- (6) $(B, C)^{\triangleright} = (B, C)^{\triangleright \triangleleft \triangleright}$,
- (7) $Z \subseteq (B, C)^{\triangleright} \Leftrightarrow (B, C) \subseteq Z^{\triangleleft}$,
- (8) $(Z_1 \cup Z_2)^{\triangleleft} = Z_1^{\triangleleft} \cap Z_2^{\triangleleft}$,
- (9) $((B_1, C_1) \cup (B_2, C_2))^{\triangleright} = (B_1, C_1)^{\triangleright} \cap (B_2, C_2)^{\triangleright}$,
- (10) $(Z_1 \cap Z_2)^{\triangleleft} \supseteq Z_1^{\triangleleft} \cup Z_2^{\triangleleft}$,
- (11) $((B_1, C_1) \cap (B_2, C_2))^{\triangleright} \supseteq (B_1, C_1)^{\triangleright} \cup (B_2, C_2)^{\triangleright}$.

Definition 5 ([18]). Let $K = (P, Q, R)$ be a formal context. The pair $(Z, (B, C))$ is referred to as an object-induced three-way concept (abbreviated as an OE concept) if and only if $Z^{\triangleleft} = (B, C)$ and $(B, C)^{\triangleright} = Z$. The set Z is termed the extension of the concept $(Z, (B, C))$ (shortened to OEL_P), and the set of all OE concepts is known as the object-induced three-way concept lattice of $K = (P, Q, R)$, shortened to $OEL(K)$.

We denote the meet-irreducible elements in $OEL(K)$ by $M(OEL)$ and $M_P(OEL)$ for the extension of the meet-irreducible elements $M(OEL)$.

We denote the attribute OE concept as $(q^*, q^{**}, q^{*\bar{*}})$ or $(q^{\bar{*}}, q^{\bar{*}*}, q^{\bar{*}\bar{*}})$. If $(Z, (B, C)) \in M(OEL)$, then $\exists a \in Q, (Z, (B, C)) = (q^*, q^{**}, q^{*\bar{*}})$ or $(q^{\bar{*}}, q^{\bar{*}*}, q^{\bar{*}\bar{*}})$. That is, the meet-irreducible element in $OEL(K)$ must be the attribute OE concept.

Definition 6 ([39]). Let $L(P, Q, R)$ be a concept lattice. For $Y \subseteq P$ and $0 \leq \beta < 0.5$, $\left(\underline{apr}_{\beta} Y, (\underline{apr}_{\beta} Y)^*\right)$ is referred to as the lower approximation concept, while $\left(\overline{apr}_{\beta} Y, (\overline{apr}_{\beta} Y)^*\right)$ is designated as the upper approximation concept, where the lower approximation of Y is defined as

$$\underline{apr}_{\beta} Y = \bigcup \{Z | h(Y, Z) \leq \beta, \exists B \in Q, (Z, B) \in L(P, Q, R)\},$$

and the upper approximation of Y is defined as

$$\overline{apr}_{\beta} Y = \bigcup \{Z | h(Y, Z) < 1 - \beta, \exists B \in Q, (Z, B) \in L(P, Q, R)\}.$$

Here, $h(Y, Z) = 1 - \frac{|Y \cap Z|}{|Y|}$.

3. Three-Way Approximations

Definition 7. Let $K = (P, Q, R)$ be a formal context. For any $y \in P$, the definition of the maximal description of y is given by

$$MD(y) = \{ Z \subseteq P \mid y \in Z \wedge (Z, (B, C)) \in M(OEL) \wedge \forall (Z_1, (B_1, C_1)) \in M(OEL) \\ [(y \in Z_1 \wedge Z_1 \subseteq P \wedge (Z, (B, C)) \preceq (Z_1, (B_1, C_1))) \Rightarrow Z = Z_1] \}. \quad (1)$$

$\cap \{Z | Z \in MD(y)\}$ is referred to as the minimal neighborhood of the maximal description of y , denoted as $\cap MD(y)$.

To clarify Definition 7, we illustrate it through Example 1.

Example 1. Table 1 presents a formal context K and Figure 1 depicts the details of its $OEL(K)$.

Table 1. The formal context (P, Q, R) .

	a	b	c	d	e
y_1	✓	✓			
y_2	✓	✓	✓	✓	
y_3		✓	✓	✓	
y_4			✓		
y_5			✓		✓
y_6				✓	

By Definition 1 and Definition 5, we can identify the following meet-irreducible elements in $OEL(K)$:

$$\begin{aligned}
 &(y_1y_2, (ab, e)) & (y_3y_4y_5y_6, (\emptyset, a)) \\
 &(y_1y_2y_3, (b, e)) & (y_4y_5y_6, (\emptyset, ab)) \\
 &(y_2y_3y_4y_5, (c, \emptyset)) & (y_1y_6, (\emptyset, ce)) \\
 &(y_2y_3y_6, (d, e)) & (y_1y_4y_5, (\emptyset, d)) \\
 &(y_5, (ce, abd)) & (y_1y_2y_3y_4y_6, (\emptyset, e))
 \end{aligned}$$

For $y_1 \in P$, then

$$\begin{aligned}
 MD(y_1) &= (\{y_1y_4y_5\}, \{y_1y_2y_3y_4y_6\}), \\
 \bigcap MD(y_1) &= \{y_1y_4y_5\} \cap \{y_1y_2y_3y_4y_6\} = \{y_1y_4\}.
 \end{aligned}$$

The extension of the concept in FCA can be considered analogous to the definable set in a rough set, but they are not precisely equivalent [38]. When a given set of objects serves as the extension of an OE concept, it is deemed definable. Conversely, set that does not correspond to any OE concept's extension being classified as undefinable. In order to describe an undefinable set of objects $Y \subseteq P$ by using the extensions in the OE concept lattice, we introduce formal definitions of the lower and upper approximation OE concepts, which are elaborated upon below:

Definition 8. Let $K = (P, Q, R)$ be a formal context. For a set of objects $Y \subseteq P$ and $0.5 < \alpha \leq 1$, the lower approximation of Y is defined by

$$\underline{apr}_\alpha Y = \bigcup \{y | c(Y \cap MD(y)) \geq \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\},$$

and the upper approximation of Y is defined by

$$\overline{apr}_\alpha Y = \bigcup \{y | c(Y \cap MD(y)) > 1 - \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\},$$

where $c(Y \cap MD(y)) = \frac{|Y \cap MD(y)|}{|MD(y)|}$.

We call $\left(\underline{apr}_\alpha Y, ((\underline{apr}_\alpha Y)^*, (\underline{apr}_\alpha Y)^{\bar{*}})\right)$ and $\left(\overline{apr}_\alpha Y, ((\overline{apr}_\alpha Y)^*, (\overline{apr}_\alpha Y)^{\bar{*}})\right)$ the lower approximation OE concept and upper approximation OE concept, respectively.

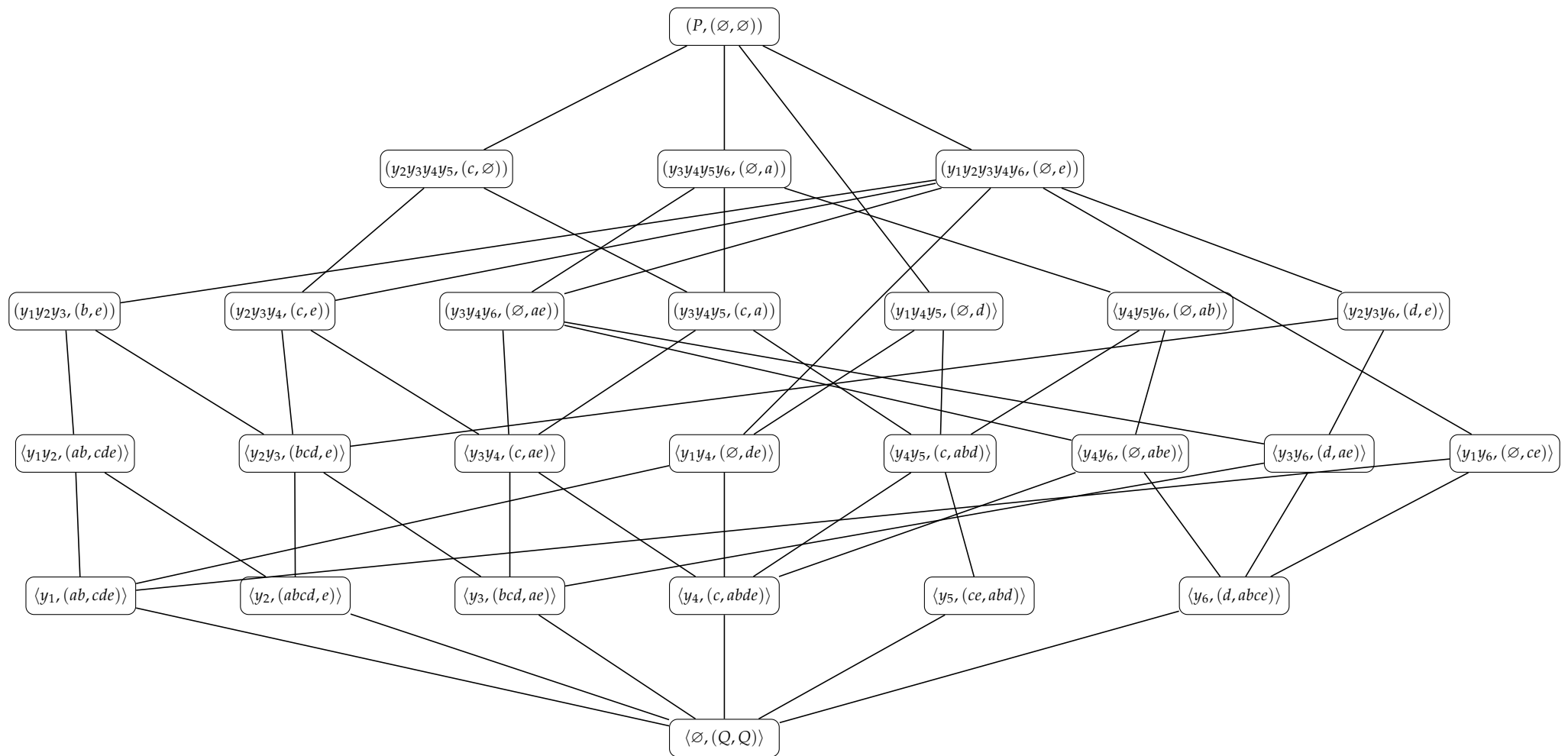


Figure 1. The $OEL(K)$ in Example 1.

Example 2 (building upon Example 1). Let $Y = \{y_2y_4y_5\} \subseteq P$ and $\alpha = 0.8$.
The maximal description is as follows:

$$\begin{aligned} MD(y_1) &= (\{y_1y_4y_5\}, \{y_1y_2y_3y_4y_6\}) \\ MD(y_2) &= (\{y_2y_3y_4y_5\}, \{y_1y_2y_3y_4y_6\}) \\ MD(y_3) &= (\{y_3y_4y_5y_6\}, \{y_2y_3y_4y_5\}, \{y_1y_2y_3y_4y_6\}) \\ MD(y_4) &= (\{y_3y_4y_5y_6\}, \{y_2y_3y_4y_5\}, \{y_1y_4y_5\}, \{y_1y_2y_3y_4y_6\}) \\ MD(y_5) &= (\{y_3y_4y_5y_6\}, \{y_2y_3y_4y_5\}, \{y_1y_4y_5\}) \\ MD(y_6) &= (\{y_3y_4y_5y_6\}, \{y_1y_2y_3y_4y_6\}) \end{aligned}$$

The minimal neighborhood of the maximal description is given as follows:

$$\begin{aligned} \cap MD(y_1) &= \{y_1y_4\} & \cap MD(y_2) &= \{y_2y_3y_4\} \\ \cap MD(y_3) &= \{y_3y_4\} & \cap MD(y_4) &= \{y_4\} \\ \cap MD(y_5) &= \{y_4y_5\} & \cap MD(y_6) &= \{y_3y_4y_6\} \end{aligned}$$

and

$$\begin{aligned} c(Y | \cap MD(y_1)) &= \frac{1}{2} & c(Y | \cap MD(y_2)) &= \frac{2}{3} \\ c(Y | \cap MD(y_3)) &= \frac{1}{2} & c(Y | \cap MD(y_4)) &= 1 \\ c(Y | \cap MD(y_5)) &= 1 & c(Y | \cap MD(y_6)) &= \frac{1}{3} \end{aligned}$$

Hence,

$$\begin{aligned} \underline{apr}_{0.8}Y &= (\{y_4\} \cup \{y_5\})^{\triangleleft \triangleright} = \{y_4y_5\} \\ \left(\underline{apr}_{0.8}Y, ((\underline{apr}_{0.8}Y)^*, (\underline{apr}_{0.8}Y)^{\bar{*}}) \right) &= (y_4y_5, (c, abd)) \end{aligned}$$

and

$$\begin{aligned} \overline{apr}_{0.8}Y &= (\{y_1\} \cup \{y_2\} \cup \{y_3\} \cup \{y_4\} \cup \{y_5\} \cup \{y_6\})^{\triangleleft \triangleright} = \{y_1y_2y_3y_4y_5y_6\} \\ \left(\overline{apr}_{0.8}Y, ((\overline{apr}_{0.8}Y)^*, (\overline{apr}_{0.8}Y)^{\bar{*}}) \right) &= (P, (\emptyset, \emptyset)). \end{aligned}$$

Example 2 illustrates a specific scenario in which the statement “If $0.5 < \alpha \leq 1$, then $\underline{apr}_\alpha Y \subseteq Y \subseteq \overline{apr}_\alpha Y$ ” holds. However, the general applicability of this statement may be questionable. In order to clarify this, Example 3 shows that the aforementioned statement is not universally valid.

Example 3 (building upon Example 2). Let $Y = \{x_1x_2x_3x_4\}$.

The lower and upper approximation OE concepts are presented below:

$$\begin{aligned} \underline{apr}_{0.8}Y &= (\{y_1\} \cup \{y_2\} \cup \{y_3\} \cup \{y_4\})^{\triangleleft \triangleright} = \{y_1y_2y_3y_4y_6\}, \\ \left(\underline{apr}_{0.8}Y, ((\underline{apr}_{0.8}Y)^*, (\underline{apr}_{0.8}Y)^{\bar{*}}) \right) &= (y_1y_2y_3y_4y_6, (\emptyset, e)), \end{aligned}$$

and

$$\overline{apr_{0.8}}Y = (\{y_1\} \cup \{y_2\} \cup \{y_3\} \cup \{y_4\} \cup \{y_5\} \cup \{y_6\})^{\triangleleft \triangleright} = \{y_1 y_2 y_3 y_4 y_5 y_6\} \\ \left(\overline{apr_{0.8}}Y, ((\overline{apr_{0.8}}Y)^*, (\overline{apr_{0.8}}Y)^*) \right) = (P, (\emptyset, \emptyset)).$$

Thus,

$$\underline{apr_{0.8}}Y \not\subseteq Y \subseteq \overline{apr_{0.8}}Y.$$

Theorem 1. For $0.5 < \alpha \leq 1$ and $Y, Y_1, Y_2 \subseteq P$, the following properties are valid:

- (1) $\underline{apr_{\alpha}}\emptyset = \emptyset$,
- (2) $\overline{apr_{\alpha}}\emptyset = \emptyset$,
- (3) $\underline{apr_{\alpha}}P = P$,
- (4) $\overline{apr_{\alpha}}P = P$,
- (5) $\underline{apr_{\alpha}}Y \subseteq \overline{apr_{\alpha}}Y$,
- (6) If $Y_1 \subseteq Y_2 \Rightarrow \underline{apr_{\alpha}}Y_1 \subseteq \underline{apr_{\alpha}}Y_2$,
- (7) If $Y_1 \subseteq Y_2 \Rightarrow \overline{apr_{\alpha}}Y_1 \subseteq \overline{apr_{\alpha}}Y_2$,
- (8) $\underline{apr_{\alpha}}(Y_1 \cap Y_2) \subseteq (\underline{apr_{\alpha}}Y_1 \cap \underline{apr_{\alpha}}Y_2)$,
- (9) $\overline{apr_{\alpha}}(Y_1 \cap Y_2) \subseteq (\overline{apr_{\alpha}}Y_1 \cap \overline{apr_{\alpha}}Y_2)$,
- (10) $(\underline{apr_{\alpha}}Y_1 \cup \underline{apr_{\alpha}}Y_2) \subseteq \underline{apr_{\alpha}}(Y_1 \cup Y_2)$,
- (11) $(\overline{apr_{\alpha}}Y_1 \cup \overline{apr_{\alpha}}Y_2) \subseteq \overline{apr_{\alpha}}(Y_1 \cup Y_2)$.

Proof. (1) Since $0.5 < \alpha \leq 1$ and $c(\emptyset | \cap MD(y)) = 0$, then

$$\underline{apr_{\alpha}}\emptyset = \bigcup \{y | c(\emptyset | \cap MD(y)) \geq \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\} \\ = \bigcup \{y | 0 \geq \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\}.$$

Hence, $\underline{apr_{\alpha}}\emptyset = \emptyset$ is proven.

(2) Similarly to (1), $\overline{apr_{\alpha}}\emptyset = \emptyset$ can be proven.

(3) Since $0.5 < \alpha \leq 1$ and $c(P | \cap MD(y)) = 1$, then

$$\underline{apr_{\alpha}}P = \bigcup \{y | c(P | \cap MD(y)) \geq \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\} \\ = \bigcup \{y | 1 \geq \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\}.$$

Hence, $\underline{apr_{\alpha}}P = P$ is proven.

(4) Similarly to (3), we can prove that $\overline{apr_{\alpha}}P = P$.

(5) By Definition 8, $\underline{apr_{\alpha}}Y \subseteq \overline{apr_{\alpha}}Y$ holds.

(6) Since $Y_1 \subseteq Y_2$, then $c(Y_2 | \cap MD(y)) \geq c(Y_1 | \cap MD(y))$. By Definition 8, $\underline{apr_{\alpha}}Y_1 \subseteq \underline{apr_{\alpha}}Y_2$ is proven.

(7) Similarly to (6), $\overline{apr_{\alpha}}Y_1 \subseteq \overline{apr_{\alpha}}Y_2$ can be proven.

(8) Since $(Y_1 \cap Y_2) \subseteq Y_1$ and $(Y_1 \cap Y_2) \subseteq Y_2$, then

$$c(Y_1 | \cap MD(y)) \geq c((Y_1 \cap Y_2) | \cap MD(y)), \\ c(Y_2 | \cap MD(y)) \geq c((Y_1 \cap Y_2) | \cap MD(y)),$$

We then obtain $\underline{apr}_\alpha(Y_1 \cap Y_2) \subseteq \underline{apr}_\alpha Y_1, \underline{apr}_\alpha(Y_1 \cap Y_2) \subseteq \underline{apr}_\alpha Y_2$. Therefore, $\underline{apr}_\alpha(Y_1 \cap Y_2) \subseteq (\underline{apr}_\alpha Y_1 \cap \underline{apr}_\alpha Y_2)$ can be proven.

(9) Similarly to (8), $\overline{apr}_\alpha(Y_1 \cap Y_2) \subseteq (\overline{apr}_\alpha Y_1 \cap \overline{apr}_\alpha Y_2)$ can be proven.

(10) According to (6), since $Y_1 \subseteq (Y_1 \cup Y_2), Y_2 \subseteq (Y_1 \cup Y_2)$, then $\underline{apr}_\alpha Y_1 \subseteq \underline{apr}_\alpha(Y_1 \cup Y_2)$ and $\underline{apr}_\alpha Y_2 \subseteq \underline{apr}_\alpha(Y_1 \cup Y_2)$. Therefore, $(\underline{apr}_\alpha Y_1 \cup \underline{apr}_\alpha Y_2) \subseteq \underline{apr}_\alpha(Y_1 \cup Y_2)$ can be proven.

(11) The same as (10), $(\overline{apr}_\alpha Y_1 \cup \overline{apr}_\alpha Y_2) \subseteq \overline{apr}_\alpha(Y_1 \cup Y_2)$ can be proven. $\square$

Subsequently, we discuss further properties of varying thresholds for the lower and upper approximations.

Theorem 2. Let $K = (P, Q, R)$ be a formal context. For $Y \subseteq P$ and $0.5 < \alpha_1 < \alpha_2 \leq 1$, the following properties are true:

(1) $\underline{apr}_{\alpha_2} Y \subseteq \underline{apr}_{\alpha_1} Y$,

(2) $\overline{apr}_{\alpha_1} Y \subseteq \overline{apr}_{\alpha_2} Y$.

Proof. (1) According to Definition 8, since $0.5 < \alpha_1 < \alpha_2 \leq 1$, then $c(Y| \cap MD(y)) \geq \alpha_2$ and $c(Y| \cap MD(y)) \geq \alpha_1$. Therefore, $\underline{apr}_{\alpha_2} Y \subseteq \underline{apr}_{\alpha_1} Y$ is proven.

(2) Since $0.5 < \alpha_1 < \alpha_2 \leq 1$, we then obtain $1 - \alpha_2 < 1 - \alpha_1$; then, $c(Y| \cap MD(y)) > 1 - \alpha_2$ and $c(Y| \cap MD(y)) > 1 - \alpha_1$. Therefore, $\overline{apr}_{\alpha_1} Y \subseteq \overline{apr}_{\alpha_2} Y$ is proven. $\square$

Theorem 3. Let $K = (P, Q, R)$ be a formal context. For $Y \subseteq P$ and $0.5 < \alpha \leq 1$, then

(1) $\underline{apr}_\alpha(Y^c) = \bigcup \{ y \mid c(Y^c| \cap MD(y)) > 1 - \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL) \},$

(2) $\overline{apr}_\alpha(Y^c) = \bigcup \{ y \mid c(Y| \cap MD(y)) < \alpha, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL) \}.$

where Y^c denotes the complement of Y .

Proof. For $\forall y \in P$, it holds that

$$c(Y^c| \cap MD(y)) = 1 - c(Y| \cap MD(y)).$$

(1) If $c(Y^c| \cap MD(y)) \geq \alpha$, then $1 - c(Y| \cap MD(y)) \geq \alpha$, which is equivalent to $c(Y| \cap MD(y)) \leq 1 - \alpha$. Hence, (1) can be proven.

(2) Similarly, if $c(Y^c| \cap MD(y)) > 1 - \alpha$, then $c(Y^c| \cap MD(y)) > 1 - \alpha$, which is equivalent to $c(Y| \cap MD(y)) < \alpha$. Thus (2) can be proved. $\square$

According to Definition 8, the establishment of the concept model introduced in this paper depends on the parameter α . As a special case, we will examine the unique characteristics of the proposed model when $\alpha = 1$.

Theorem 4. Let $OEL(P, Q, R)$ be an OE concept lattice. For a subset of objects $Y \subseteq P$, if $\alpha = 1$, then

(1) $\underline{apr}_1 Y = \bigcup \{ y|Y \cap MD(y) \subseteq Y, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL) \},$

(2) $\overline{apr}_1 Y = \bigcup \{ y|Y \cap MD(y) \neq \emptyset, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL) \}.$

Proof. (1) For $\alpha = 1$, since $c(Y| \cap MD(y)) \geq 1$ and $c(Y| \cap MD(y)) \in [0, 1]$, we get $|Y \cap MD(y)| = |MD(y)|$; thus, $MD(y) \subseteq Y$. By Definition 8, then

$$\begin{aligned}\underline{apr}_1 Y &= \bigcup \{y \mid c(Y \mid \cap MD(y)) \geq 1, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\} \\ &= \bigcup \{y \mid \cap MD(y) \subseteq Y, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\}\end{aligned}$$

(2) Similarly, $\overline{apr}_1 Y = \bigcup \{y \mid Y \cap (\cap MD(y)) \neq \emptyset, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\}$ can be proven. $\square$

Assume that $0.5 < \alpha \leq 1$; then, $\underline{apr}_1 Y \subseteq \underline{apr}_\alpha Y$ and $\overline{apr}_\alpha Y \subseteq \overline{apr}_1 Y$ hold, which is consistent with Theorem 2.

Theorem 5. Let $OEL(K)$ be an OE concept lattice. For $Y \subseteq P$, then $\underline{apr}_1 Y \subseteq Y^{\triangleleft \triangleright}$.

Proof. Let $y_j \in P, E_j = \{y_j\}, j = 1, 2, \dots, |P|, E = (\cup E_j)$. By Theorem 4, then

$$\underline{apr}_1 Y = \bigcup \{y \mid (\cap MD(y) \subseteq Y), y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\}.$$

Then, $E_j = \{y_j\} \subseteq Y$. By Proposition 1(3) and Proposition 1(4), $E_j^{\triangleleft \triangleright} \subseteq Y^{\triangleleft \triangleright}$.

According to Proposition 1(1), it can easily seen that $E_j \subseteq E_j^{\triangleleft \triangleright}$; then, $E_j \subseteq E_j^{\triangleleft \triangleright} \subseteq Y^{\triangleleft \triangleright}, (\cup E_j) \subseteq Y^{\triangleleft \triangleright}$. Furthermore, by Proposition 1(3) and Proposition 1(5), $Y^{\triangleleft \triangleright \triangleleft} = Y^{\triangleleft} \subseteq (\cup E_j)^{\triangleleft}, (\cup E_j)^{\triangleleft \triangleright} = E^{\triangleleft \triangleright} \subseteq Y^{\triangleleft \triangleright}$. Therefore, $\underline{apr}_1 Y \subseteq Y^{\triangleleft \triangleright}$. $\square$

Theorem 6. Let $OEL(P, Q, R)$ be an OE concept lattice. For $Y \subseteq P$, then $Y^{\triangleleft \triangleright} \subseteq \overline{apr}_1 Y$.

Proof. Let $y_j \in P, E_j = \{y_j\}, j = 1, 2, \dots, |P|$. By Theorem 4, we obtain

$$\overline{apr}_1 Y = \bigcup \{y \mid Y \cap (\cap MD(y)) \neq \emptyset, y \in Z, Z \subseteq P, \exists B, C \in Q, (Z, (B, C)) \in M(OEL)\},$$

Then, $Y \subseteq E = \cup \{y_j\}$. By Proposition 1(3) and Proposition 1(4), we have $Y^{\triangleleft \triangleright} \subseteq E^{\triangleleft \triangleright}$. Therefore, $Y^{\triangleleft \triangleright} \subseteq \overline{apr}_1 Y$. $\square$

4. Approximating Undefinable Sets in a Three-Way Concept

In this section, we present an optimization algorithm (three-way approximation optimization algorithm, TAO) to approximate undefinable sets.

4.1. Proposed Algorithm

The purpose of the TAO algorithm is to identify the lower and upper approximations for an undefinable set of objects. Initially, a maximal description is established on the basis of meet-irreducible elements to create its minimal neighborhood set. Afterwards, the covering-based rough set is employed to derive the the lower and upper approximations.

The TAO algorithm is made up of three parts: Algorithm 1, which is tasked to find the minimal neighborhood of the maximal description; Algorithm 2, which computes the lower approximation; and Algorithm 3, which calculates the upper approximation.

Reducing redundant information is a key objective of the TAO algorithm. To gauge the effectiveness of the TAO algorithm in handling redundant information, we employ the redundancy ratio and reduction ratio for quantitative measurement.

Definition 9 (redundancy ratio).

$$R_{lu} = \lambda R_{lower} + (1 - \lambda) R_{upper}.$$

where

$$R_{upper} = \frac{|\overline{apr}_\alpha(Y) - Y|}{|P|}, R_{lower} = \frac{|\underline{apr}_\alpha(Y) - Y|}{|P|} \quad \text{and} \quad \lambda \in [0, 1].$$

R_{upper} , the upper approximation redundancy ratio, measures the diversity between the upper approximation and the undefinable set Y . R_{lower} , the lower approximation redundancy ratio, measures the difference between the lower approximation and the undefinable set Y . Moreover, a higher R_{lu} indicates stronger data redundancy, which may degrade the algorithm's performance. Additionally, when $\lambda = 1$, R_{lu} depends entirely on R_{lower} , and when $\lambda = 0$, R_{lu} depends entirely on R_{upper} .

Definition 10 (reduction ratio, RR).

$$RR = 1 - \frac{C_1}{C_2}.$$

The reduction ratio (RR) serves to evaluate the comparative efficacy of concept reduction between two algorithms. Here, C_1 denotes the count of concepts generated by Algorithm 1 and C_2 represents the count of concepts generated by Algorithm 2. Mathematically, the value of RR lies within the interval $[0, 1)$. A higher reduction ratio implies that Algorithm 1 outperforms Algorithm 2 in generating fewer concepts. Specifically, when $C_1 = C_2$, this indicates that Algorithm 1 exhibits no reduction effect compared to Algorithm 2, resulting in $RR = 0$. Conversely, as RR approaches 1, this signifies that Algorithm 1 achieves a substantial reduction in the number of concepts, highlighting its superior concept reduction capability.

Algorithm 1 Determining the minimal neighborhood of the maximal description

Input: the formal context $K = (P, Q, R)$

Output: $\bigcap MD(y_j)$

```

1: while  $y_j \in P, (j = 1, 2, \dots, |P|)$  do
2:   according to Definition 5, find out the set of extension of meet-irreducible elements
    $M(OEL)$ , denote it as  $M_P(OEL)$ .
3: end while
4: while  $y_j \in P, (j = 1, 2, \dots, |P|)$  do
5:   if  $y_j \in Z_s, Z_t, Z_s, Z_t \subseteq P$  then
6:     for  $Z_s \in M_P(OEL), (s = 1, 2, \dots, m)$  do
7:       for  $Z_t \in M_P(OEL), (t = 1, 2, \dots, m)$  do
8:         Compute the description  $MD(y_j)$  by Equation (1).
9:       end for
10:    end for
11:  end if
12: end while
13: Generate the minimal neighborhood of the maximal description  $\bigcap MD(y_j)$ .
14: return  $\bigcap MD(y_j)$ 
```

Algorithm 2 Computing the lower approximation

Input: $\bigcap MD(y_j), Y = \{y_1 \cdots y_n\}, n \leq |P|, E_j = \{y_j\}$, threshold α , $r_0 = c(Y \cap MD(y_0)) = 1$

Output: $apr_\alpha Y = \bigcup E^{<\triangleright}$.

```

1:  $j = 1, \overline{E} = \emptyset, r_j = r_0$ .
2: if  $j < |P| + 1$  then
3:   if  $r_j \geq \alpha$  then
4:      $\overline{E} = \bigcup E_j, j = j + 1$ 
5:   else
6:      $\overline{E} = E, j = j + 1$ 
7:   end if
8: else
9:    $apr_\alpha Y = \bigcup E^{<\triangleright}$ 
10: end if
11: return  $apr_\alpha Y = \bigcup E^{<\triangleright}$ 
```

Algorithm 3 Calculating the upper approximation

Input: $\cap MD(x_j), Y = \{y_1 \cdot \cdot y_n\}, n \leq |P|, E_j = \{y_j\}, \text{threshold } \alpha, r_0 = c(Y \cap MD(y_0)) = 1$
Output: $\overline{apr}_\alpha Y = \bigcup E^{<\triangleright}$

```

1:  $j = 1, E = \emptyset, r_j = r_0.$ 
2: if  $j < |U| + 1$  then
3:   if  $r_j > 1 - \alpha$  then
4:      $E = \bigcup E_j, \quad j = j + 1$ 
5:   else
6:      $E = E, \quad j = j + 1$ 
7:   end if
8: else
9:    $\overline{apr}_\alpha Y = \bigcup E^{<\triangleright}$ 
10: end if
11: return  $\overline{apr}_\alpha Y = \bigcup E^{<\triangleright}$ 

```

4.2. Algorithm Complexity Analysis

Let $K = (P, Q, R)$ be a formal context. While meet-irreducible elements are classified as attribute concepts, not all attribute concepts are necessarily meet-irreducible elements [40]; therefore, $|M_P(OEL)| \leq 2 * |Q|$. Since the statement that object OE concepts are indeed OE concepts holds true, $|P| \leq |OEL(K)| \leq 2^{|P|}$.

The primary operations of Procedure 1 are divided into two components: finding $M_P(OEL)$ and generating $\cap MD(y_j)$. The time complexity for the first component is $O(2 * |Q|)$ and the time complexity for the second component is $O(|P||M_P(OEL)|^2 + |P||M_P(OEL)|)$. Therefore, the time complexity of the whole of Procedure 1 is $O(2 * |Q| + |P||M_P(OEL)|^2 + |P||M_P(OEL)|) = O(|P||M_P(OEL)|^2) = O(4 * |P||Q|^2)$. The primary operations of Procedure 2 are dedicated to describing the lower approximation, with the overall time complexity of Procedure 2 being $O(|P|)$. Likewise, the complexity analysis of Procedure 3 parallels that of Procedure 2 and is not detailed here; thus, the time complexity of Procedure 3 also remains as $O(|P|)$. As a result, the overall time complexity of the entire algorithm is $O(4 * |P||Q|^2 + 2 * |P|)$.

The method in [39] demonstrates effective results in computing lower and upper approximations in classical concepts and can also be expanded to the three-way concept. The time complexity of the method [39] is composed of two parts: constructing all OE concepts and computing lower and upper approximations. This method involves creating $OEL(K)$ and subsequently performing a one-by-one comparison for all OE concepts. The set of all OE concepts, denoted as $OEL(K)$, is calculated by reference [22], with the time complexity of constructing $OEL(K)$ represented as $O(|OEL(K)||P||Q|^2)$. The second part could require, at most, $O(2 * |OEL(K)|)$ of time. Consequently, the overall time complexity of the method in [39] is $O(|OEL(K)||P||Q|^2 + 2 * |OEL(K)|)$.

The TAO algorithm offers the advantage of efficiently calculating the lower and upper approximations for large-scale formal contexts. Therefore, in such contexts, it is usually observed that $|OEL(K)| \geq 4$. As a result, the TAO algorithm exhibits lower complexity compared to the method in [39].

5. Experimental Results and Analysis

In order to verify the validity of the TAO algorithm, we compared it with the functionally similar method described in reference [39].

5.1. Datasets and Experimental Environment

We utilized five datasets (Surveillance, Wpbc, Wdbc, Sonar, CKD) from the UCI Machine Learning Repository (<https://archive.ics.uci.edu/>, accessed on 20 February 2025).

Due to constraints regarding running time and computer memory, we selected only partial data from the original datasets as formal contexts for the experiments. To ensure the scientific validity and reproducibility of the algorithm, we employed random sampling of the datasets, with the specific details presented in Table 2. For the Surveillance, Wpbc, Wdbc, Sonar, and CKD datasets, the attribute value, which could be normalized to the interval $[0, 1]$ using the Min–Max normalization method, was 0 if it was less than the object’s average across all attributes and 1 otherwise. The experimental environment consisted of a 64-bit Windows 11 operating system, an Intel Core i7-12700 2.10 GHz CPU, 16 GB of RAM, and Python 3.13.

Table 2. Description of the datasets.

	Datasets	Objects	Attributes
1	Surveillance	14	7
2	Wpbc	99	10
3	Wdbc	149	15
4	Sonar	79	15
5	CKD	232	24

5.2. Experimental Evaluation and Analysis

This subsection presents and analyzes the experimental results. To ensure the accuracy of the outcomes, each experiment was repeated 10 times, and the average values were recorded. The experiments measured the number of elements in the lower approximation and upper approximation and the memory usage for both the TAO algorithm and the method proposed in [39] across various thresholds, as detailed in Table 3, where L_1 and U_1 denote the lower and upper approximations derived from TAO, while L_2 and U_2 correspond to those of the method in [39]. Table 3 reveals that the memory usage of TAO is consistently much lower than that of the method in [39].

We applied the datasets to compare TAO with the method in reference [39]. The performance of these algorithms was evaluated by the redundancy ratio (with $\lambda = 0.5$ for balanced weighting), running time, and memory usage. Figure 2 presents a comparative analysis of the redundancy ratios between TAO and the method proposed in [39] across five datasets: Surveillance, Wpbc, Wdbc, Sonar, and CKD. Evidently, TAO exhibited a consistently lower redundancy ratio than the comparative method, underscoring its enhanced efficacy in reducing redundancy. Notably, for the Wpbc dataset, while the redundancy ratios coincided at $\alpha = 0.6$ and $\alpha = 0.7$ for both methods, TAO maintained superiority under other threshold values. Collectively, these results demonstrate that TAO outperforms the method [39] across diverse threshold settings. Figure 3 illustrates the running times of TAO and the method in [39] across the same datasets. The outcomes reveal that TAO achieves shorter execution durations on all evaluated datasets, confirming its superior computational efficiency.

TAO demonstrated significant advantages over the method in [39] in terms of redundancy ratio (Figure 2), running time (Figure 3), and memory usage (Table 3). Consequently, it is evident that TAO’s performance is superior.

The selected concept counts are presented in Table 4, where C_1 represents TAO’s counts and C_2 denotes those of the method in [39]. As shown, TAO not only yielded fewer selected concepts but also exhibited higher certainty compared to the approach in [39], with a lower redundancy ratio, running time, and memory usage. Thus, TAO’s selected concepts offer better representation and effectiveness.

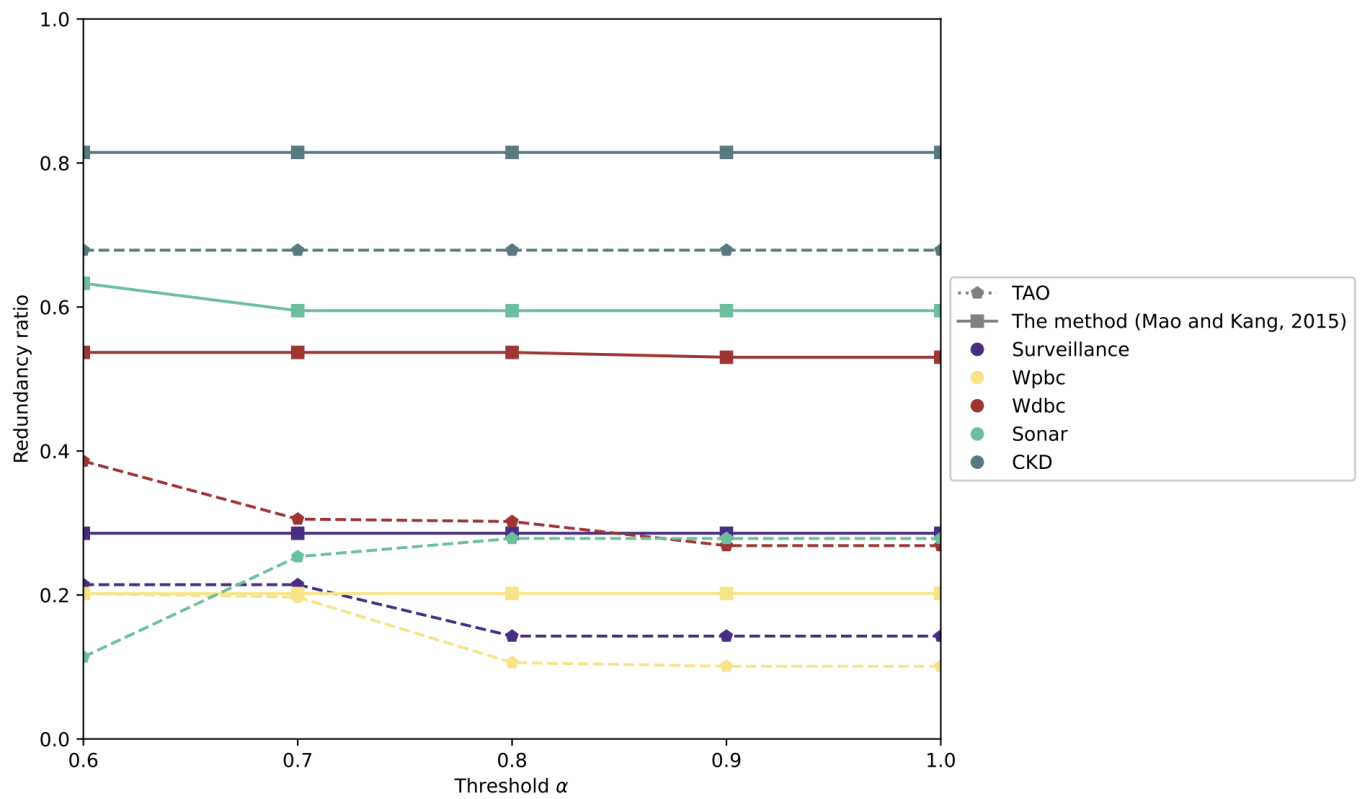


Figure 2. Comparison of redundancy ratio with different thresholds. Source: Mao and Kang (2015) [39].

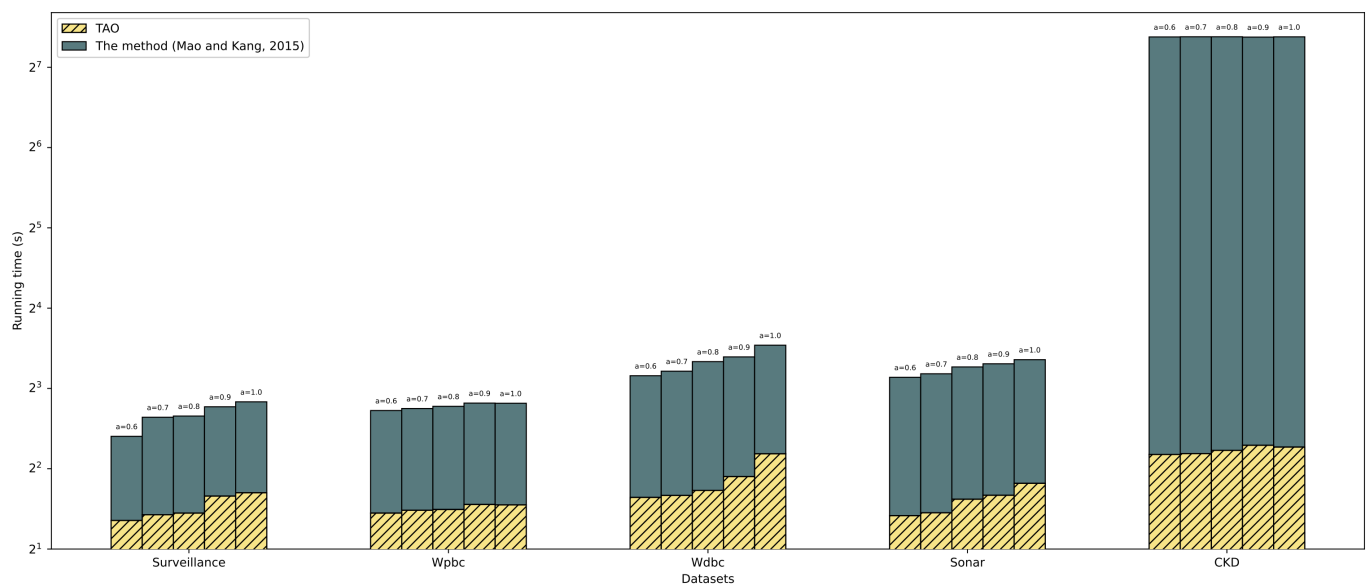


Figure 3. Comparison of running times with different thresholds for α . Source: Mao and Kang (2015) [39].

Table 3. Performance comparison with different thresholds for α .

Datasets	α	TAO			The Method in [39]		
		L_1	U_1	Memory (MB)	L_2	U_2	Memory (MB)
Surveillance	0.6	3	10	0.3015	14	14	10,349.0580
	0.7	3	14	0.3015	14	14	10,349.0604
	0.8	3	14	0.3016	14	14	10,349.0586
	0.9	3	14	0.3016	14	14	10,349.0587
	1.0	3	14	0.3015	14	14	10,349.0576
Wpbc	0.6	99	99	0.6436	99	99	11,999.6610
	0.7	98	99	0.6437	99	99	11,999.6803
	0.8	15	99	0.6442	99	99	11,999.6889
	0.9	0	99	0.6436	99	99	11,999.6736
	1.0	0	99	0.6437	99	99	11,999.7292
Wdbc	0.6	72	149	0.8624	149	149	11,807.6815
	0.7	29	149	0.8643	149	149	11,808.1439
	0.8	28	149	0.8659	149	149	11,808.1454
	0.9	7	149	0.8733	147	149	11,808.1494
	1.0	7	149	0.8743	147	149	11,808.1455
Sonar	0.6	13	19	0.4446	49	79	11,850.6182
	0.7	13	47	0.4445	42	79	11,850.8190
	0.8	13	51	0.4445	42	79	11,850.9866
	0.9	13	51	0.4446	42	79	11,851.4954
	1.0	13	51	0.4446	42	79	11,851.4988
CKD	0.6	196	232	0.7439	232	232	12,903.0880
	0.7	196	232	0.7439	232	232	12,906.9141
	0.8	196	232	0.7443	232	232	12,921.5596
	0.9	196	232	0.7444	232	232	12,940.7803
	1.0	196	232	0.7444	232	232	12,943.3284

Table 4. Comparison of selected concept counts.

Dataset	C_1	C_2	Reduction Ratio
Surveillance	14	103	86.4078%
Wpbc	99	980	89.8979%
Wdbc	149	5392	97.2366%
Sonar	79	8179	99.0341%
CKD	232	601,999	99.9615%

Overall, the experimental results demonstrate that TAO significantly outperforms the method in [39] in terms of redundancy ratio, running time, and memory usage. Compared to the approach in [39], TAO leverages a computing process that reduces the number of selected concepts, which are generally fewer. This reduction directly enhances computing efficiency and decreases memory usage.

6. Conclusions

In this study, we introduced novel lower and upper approximations by using meet-irreducible elements to accurately describe undefinable sets and presented the TAO algorithm, which reduces the counting overhead involved in approximating undefinable sets compared to the method in reference [39], which requires the construction of OE con-

cept lattices. The experimental results show that the TAO algorithm outperforms existing methods in terms of redundancy rate, running time, and memory usage.

In our future research, we will focus on the following two directions: (1) The work presented in this paper is grounded in OE concepts; considering the duality between OE concepts and AE concepts, we will further explore the approximation characterization of undefinable sets of attributes based on AE concepts. (2) The study was conducted in complete formal contexts, while incomplete formal contexts are quite common in practical applications. Therefore, our future research will involve studying the approximation of undefinable object sets in incomplete formal contexts.

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