

Article

Fixed-Point Results in Elliptic-Valued Metric Spaces with Applications

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Abstract: The primary objective of this research is to investigate the notion of elliptic-valued metric spaces and prove new fixed-point theorems for diverse generalized contractions. Our contributions extend several existing results in the field. To underscore the novelty of our main result, we provide a concrete example. Additionally, we showcase the practical relevance of our primary theorem by solving a Urysohn integral equation.

Keywords: fixed point; Kannan contraction; Chatterjea contraction; Dass and Gupta-type contraction; elliptic-valued metric space; Urysohn integral equation

MSC: 46S40; 47H10; 54H25

1. Introduction

The real number system forms the foundation of classical mathematics, encompassing numbers that can represent quantities along a continuous line, including both rational and irrational numbers. These numbers are used extensively in everyday life and scientific computation. Building upon this foundational system, mathematicians have extended the concept to include complex numbers. The complex number system, with its fundamental unit $i^2 = -1$, was first explored by Italian mathematicians Cardano and Bombelli [1] in the 16th century. Since then, numerous mathematicians have modified this unit to create various number systems. Clifford [2] introduced the hyperbolic number system, characterized by $i^2 = 1$, which has found applications in solving mechanical problems. Another modification, $i^2 = 0$, led to the development of dual numbers by [3], with applications in fields such as kinematics, robotics, and virtual reality. In recent years, these number systems have been generalized by allowing $i^2 = p$, where p is a real number. This generalized number system is categorized as elliptic for $p < 0$, parabolic (or dual) for $p = 0$, and hyperbolic for $p > 0$.

A metric space is a fundamental concept in mathematics introduced by M. Fréchet [4] in 1906. It provides a framework for defining and studying distances between elements of a set. Formally, a metric space consists of a set along with a metric function that assigns a non-negative real number to each pair of points, satisfying properties such as non-negativity, symmetry, and the triangle inequality. Extending this core concept, Azam et al. [5] introduced the notion of complex-valued metric spaces (C-VMSs), which generalizes traditional metric spaces. In this generalized framework, the metric takes values in the set of complex numbers instead of non-negative real numbers, while still adhering to appropriately modified versions of the standard metric axioms. This extension broadens the applicability of metric spaces in various mathematical and applied contexts. Further advancing this idea, Ahmed et al. [6] introduced quaternion-valued metric spaces



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(Q-VMSs), where the metric takes values in the set of quaternions. This innovative extension incorporates the non-commutative properties of quaternions, enabling new insights and applications in higher-dimensional analysis and theoretical studies. In a similar vein, Ozturk et al. [7] introduced elliptic-valued metric spaces (E-VMSs), where the metric takes values in the set of elliptic numbers, an extension of real numbers that includes elements from the elliptic function theory. Elliptic numbers are closely related to the complex analysis of elliptic curves and offer a broader range of values compared to standard real or complex numbers. In E-VMSs, the distance between two points is not only defined by a scalar quantity but also involves the properties of elliptic functions. This approach provides new avenues for exploring geometric properties in spaces with complex, cyclic, or periodic structures, making it particularly valuable in areas such as number theory, algebraic geometry, and theoretical physics. Later, Ghosh et al. [8] extended this concept further by introducing elliptic-valued b -metric spaces (E-VbMSs).

On the other hand, Banach [9] was the pioneer researcher in fixed point (FP) theory who introduced the notion of contraction and established a fundamental FP theorem. Subsequently, Kannan [10] proposed a different contractive condition and proved a corresponding FP theorem. Chatterjea [11], building on Kannan's work, modified the contractive condition and obtained another FP result. While Banach's contraction mapping requires continuity, Kannan's and Chatterjea's results do not impose this restriction. Fisher [12], a significant contributor to the field, introduced rational contractions, extending the applicability of Banach's theorem. In 1975, Dass and Gupta [13] further generalized Banach's contraction by proposing a new rational contractive condition. For a deeper understanding of this topic, please refer to [14–18].

In this research article, we explore the E-VMSs and present new FP theorems for Kannan-, Chatterjea-, Dass- and Gupta-type contractions. Our results extend some previously known findings in the area. To emphasize the importance of our main theorem, we offer a detailed example. Additionally, we showcase the practical relevance of our central theorem by solving a Urysohn integral equation.

2. Preliminaries

Let us recall some important notations and definitions that we shall need for the rest of the paper. See [1] for more information. Let $\mathbb{E}_p$ be the set comprising all elliptic numbers, defined as follows:

$$\mathbb{E}_p = \left\{ z = \nu + i\omega : \nu, \omega \in \mathbb{R}, i^2 = p < 0 \right\}.$$

For an elliptic number $z = \nu + i\omega$, the component ν is referred to as the real part of z , while ω is termed the imaginary part of z .

The fundamental operations for elliptic numbers, including addition, scalar multiplication, elliptic multiplication, and conjugation, are defined as follows:

The addition of two elliptic numbers, $z_1 = \nu_1 + i\omega_1$ and $z_2 = \nu_2 + i\omega_2$, is defined as

$$z_1 + z_2 = \nu_1 + \nu_2 + i(\omega_1 + \omega_2).$$

The multiplication of an elliptic number $z_1 \in \mathbb{E}_p$ by a scalar $\lambda \in \mathbb{R}$ is defined as

$$\lambda z_1 = \lambda(\nu_1 + i\omega_1) = \lambda\nu_1 + i\lambda\omega_1.$$

Moreover, the product of two elliptic numbers z_1 and z_2 within $\mathbb{E}_p$ under elliptic multiplication is given by

$$z_1 z_2 = (\nu_1 + i\omega_1)(\nu_2 + i\omega_2) = (\nu_1\nu_2 + p\omega_1\omega_2) + i(\nu_1\omega_2 + \nu_2\omega_1).$$

The conjugate of an elliptic number z is represented as $\bar{z}$ and is given by

$$\bar{z} = \operatorname{Re}(z) - \operatorname{Im}(z) = \nu - i\omega.$$

Lastly, the norm of an elliptic number z is defined as

$$\|z\|_{\mathbb{E}} = \sqrt{z\bar{z}} = \sqrt{\nu^2 - p\omega^2}.$$

It is clear that $\mathbb{E}_p$ forms a two-dimensional vector space over the field $\mathbb{R}$, equipped with addition and scalar multiplication. The set $\mathbb{E}_p$ possesses the necessary properties to qualify as a field. Consequently, a one-to-one correspondence can be established between $\mathbb{E}_p$ and $\mathbb{R}^2$, allowing each elliptic number $z = \nu + i\omega$ to be uniquely represented in the (standard) plane. This plane is referred to as the elliptic plane, where the distance between two elliptic numbers $z_1 = (\nu_1, \omega_1)$ and $z_2 = (\nu_2, \omega_2)$ is defined as

$$\|z_1 - z_2\|_{\mathbb{E}} = \sqrt{(\nu_1 - \nu_2)^2 - p(\omega_1 - \omega_2)^2},$$

where $(\nu_1 - \nu_2)^2 - p(\omega_1 - \omega_2)^2 > 0$ and $p < 0$.

In this elliptic geometry, the collection of all points situated a unit distance from the origin traces out an ellipse, as described by the equation $\nu^2 - p\omega^2 = 1$ (cf. [1]).

Henceforth, let θ represent the zero vector in the space $\mathbb{E}_p$. A partial order $\preceq$ on $\mathbb{E}_p$ is defined as follows:

For $z_1 = \nu_1 + i\omega_1, z_2 = \nu_2 + i\omega_2 \in \mathbb{E}_p$, define $z_1 \preceq z_2$ if and only if

$$\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

Hence, $z_1 \preceq z_2$ if any of the following conditions holds:

- (i) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (ii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (iii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (iv) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

Specifically, the notation $z_1 \prec z_2 (z_1 \neq z_2)$ will be used when any of the conditions (i), (ii), or (iii) are satisfied, while the notation $z_1 \prec z_2$ will be used exclusively when condition (iii) is satisfied. Some fundamental properties of the partial order $\preceq$ on $\mathbb{E}_p$ are outlined as follows:

- (1) If $\theta \preceq z_1 \prec z_2$, then $\|z_1\|_{\mathbb{E}} < \|z_2\|_{\mathbb{E}}$,
- (2) $z_1 \preceq z_2$ is equivalent to $z_2 - z_1 \succeq \theta$,
- (3) If $z_1 \preceq z_2$ and $z_2 \preceq z_3$, then $z_1 \preceq z_3$,
- (4) If $z_1 \preceq z_2$ and $\lambda > 0, (\lambda \in \mathbb{R})$, then $\lambda z_1 \preceq \lambda z_2$,
- (5) $\theta \preceq z_1$ and $\theta \preceq z_2$ do not necessarily imply that $\theta \preceq z_1 z_2$.

In 2011, Azam et al. [5] introduced the concept of a C-VMS in the following manner.

Definition 1 ([5]). Consider $\mathcal{W} \neq \emptyset$ and a function $d_c : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{C}$ satisfying the following properties:

- (c₁) : $\theta \preceq d_c(l, \sigma)$, and $d_c(l, \sigma) = \theta \Leftrightarrow l = \sigma$,
 - (c₂) : $d_c(l, \sigma) = d_c(\sigma, l)$,
 - (c₃) : $d_c(l, \sigma) \preceq d_c(l, \omega) + d_c(\omega, \sigma)$,
- for all $l, \sigma, \omega \in \mathcal{W}$, then $(\mathcal{W}, d_c)$ is C-VMS.

Following the approach of Azam et al. [5], Ozturk et al. [7] defined a new class of metric spaces, namely E-VMSs, in this way.

Definition 2 ([7]). Let $\mathcal{W} \neq \emptyset$ and $d_e : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{E}_p$ be a function satisfying the following axioms:

- (E₁) : $\theta \preceq d_e(l, \sigma)$, and $d_e(l, \sigma) = \theta \Leftrightarrow l = \sigma$,
 - (E₂) : $d_e(l, \sigma) = d_e(\sigma, l)$,
 - (E₃) : $d_e(l, \sigma) \preceq d_e(l, \omega) + d_e(\omega, \sigma)$,
- for all $l, \sigma, \omega \in \mathcal{W}$, then $(\mathcal{W}, d_e)$ is called an E-VMS.

Example 1 ([7]). Consider the set of elliptic numbers, denoted by $\mathcal{W} = \mathbb{E}_p$. We define the elliptic metric $d_e : \mathbb{E}_p \times \mathbb{E}_p \rightarrow \mathbb{E}_p$ as follows:

$$d_e(z_1, z_2) = \sqrt{(v_1 - v_2)^2 - p(\omega_1 - \omega_2)^2},$$

for $(v_1 - v_2)^2 - p(\omega_1 - \omega_2)^2 > 0$ and for any two elliptic numbers $z_1 = v_1 + i\omega_1, z_2 = v_2 + i\omega_2$ in $\mathbb{E}_p$ and $i^2 = p < 0$, then $(\mathbb{E}_p, d_e)$ satisfies the properties of an E-VMS.

In an E-VMS $(\mathcal{W}, d_e)$, a point $l \in \mathcal{W}$ is considered an e-interior point of a subset A of $\mathcal{W}$ if there exists a non-zero elliptic number $\theta \prec r \in \mathbb{E}_p$ such that

$$B(l, r) = \{\sigma \in \mathcal{W} : d_e(l, \sigma) \prec r\} \subseteq A,$$

where $B(l, r)$ denotes an open ball centered at l with radius r . In this way, a closed ball $\overline{B(l, r)}$ is defined as follows:

$$\overline{B(l, r)} = \{\sigma \in \mathcal{W} : d_e(l, \sigma) \preceq r\}.$$

In the set $\mathcal{W}$, a point l is considered a limit point of a subset A of $\mathcal{W}$ if, for any non-zero elliptic number $\theta \prec r \in \mathbb{E}_p$, we have

$$(B(l, r) - \{l\}) \cap A \neq \emptyset.$$

A subset A of $\mathcal{W}$ is said to be open if every point in A is an interior point of A . Conversely, a subset A of $\mathcal{W}$ is termed closed if every limit point of A is also a member of A . The collection

$$\Omega = \{B(l, r) : l \in \mathcal{W} \text{ and } \theta \prec r \in \mathbb{E}_p\},$$

forms a sub-basis for a Hausdorff topology on $\mathcal{W}$.

Definition 3 ([7]). Consider a sequence $\{l_n\}_{n \in \mathbb{N}}$ in an E-VMS $(\mathcal{W}, d_e)$.

(i) The sequence $\{l_n\}_{n \in \mathbb{N}}$ is said to converge to a point $l \in \mathcal{W}$ if, for any non-zero elliptic number $\theta \prec \delta \in \mathbb{E}_p$, there exists a natural number $n_0 \in \mathbb{N}$ such that $d_e(l_n, l) \prec \delta, \forall n > n_0$. This is denoted by $l_n \rightarrow l$ as $n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} l_n = l$.

(ii) A sequence $\{l_n\}_{n \in \mathbb{N}}$ in $(\mathcal{W}, d_e)$ is called an Cauchy sequence if, for any non-zero elliptic number $\theta \prec \delta$, there exists a natural number $n_0 \in \mathbb{N}$ such that $d_e(l_n, l_{n+m}) \prec \delta$, for all $n > n_0$ and $m \in \mathbb{N}$.

(iii) An E-VMS $(\mathcal{W}, d_e)$ is said to be complete if every Cauchy sequence in $\mathcal{W}$ converges to a point in $\mathcal{W}$.

The following lemmas are needed:

Lemma 1 ([7]). Consider an E-VMS $(\mathcal{W}, d_e)$ and a sequence $\{l_n\}_{n \in \mathbb{N}}$ in $\mathcal{W}$. Then, the sequence $\{l_n\}_{n \in \mathbb{N}}$ converges to $l \in \mathcal{W}$ if and only if the elliptic norm of $d_e(l_n, l)$ approaches zero as n tends to infinity, that is,

$$\|d_e(l_n, l)\|_{\mathbb{E}} \rightarrow 0,$$

as $n \rightarrow \infty$.

Lemma 2 ([7]). Consider an E-VMS $(\mathcal{W}, d_e)$ and a sequence $\{l_n\}_{n \in \mathbb{N}}$ in $\mathcal{W}$. The sequence $\{l_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if the elliptic norm of $d_e(l_n, l_{n+m})$ approaches zero as n tends to infinity, for any natural number m , that is,

$$\|d_e(l_n, l_{n+m})\|_{\mathbb{E}} \rightarrow 0,$$

as $n \rightarrow \infty$, where $m \in \mathbb{N}$.

Remark 1 ([7]). In an E-VMS, the limit of a convergent sequence is unique.

3. Main Results

In the following theorem, we generalize the Kannan-type FP theorem from [10] to the context of E-VMSs.

3.1. Kannan's Fixed-Point Theorem

Theorem 1. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exists a constant $\hbar \in [0, \frac{1}{2})$ such that

$$d_e(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar(d_e(l, \mathcal{G}l) + d_e(\sigma, \mathcal{G}\sigma)), \quad (1)$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has a unique FP.

Proof. Choose an arbitrary point l_0 from the set $\mathcal{W}$. Define a sequence $\{l_n\}$ recursively as follows:

$$l_{n+1} = \mathcal{G}l_n.$$

By the inequality (1), we have

$$\begin{aligned} d_e(l_n, l_{n+1}) &= d_e(\mathcal{G}l_{n-1}, \mathcal{G}l_n) \preceq \hbar(d_e(l_{n-1}, \mathcal{G}l_{n-1}) + d_e(l_n, \mathcal{G}l_n)) \\ &= \hbar(d_e(l_{n-1}, l_n) + d_e(l_n, l_{n+1})), \end{aligned}$$

which implies

$$d_e(l_n, l_{n+1}) \preceq \frac{\hbar}{1-\hbar} d_e(l_{n-1}, l_n). \quad (2)$$

Let $\lambda = \frac{\hbar}{1-\hbar} < 1$. Then, the above inequality gives

$$d_e(l_n, l_{n+1}) \preceq \lambda d_e(l_{n-1}, l_n),$$

which implies that

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}}. \quad (3)$$

Similarly, by (1), we have

$$\begin{aligned} d_e(l_{n-1}, l_n) &= d_e(\mathcal{G}l_{n-2}, \mathcal{G}l_{n-1}) \preceq \hbar(d_e(l_{n-2}, \mathcal{G}l_{n-2}) + d_e(l_{n-1}, \mathcal{G}l_{n-1})) \\ &= \hbar(d_e(l_{n-2}, l_{n-1}) + d_e(l_{n-1}, l_n)), \end{aligned}$$

which implies

$$d_e(l_{n-1}, l_n) \preceq \frac{\hbar}{1-\hbar} d_e(l_{n-2}, l_{n-1}). \quad (4)$$

Let $\lambda = \frac{\hbar}{1-\hbar} < 1$. Then, the above inequality gives

$$d_e(l_{n-1}, l_n) \preceq \lambda d_e(l_{n-2}, l_{n-1}),$$

which implies

$$\|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}}. \quad (5)$$

Now, from the inequalities (3) and (5), we have

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda^2 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} \leq \dots \leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}},$$

for each natural number n . Now, for any natural numbers m and n such that $m > n$, it follows that

$$\begin{aligned} \|d_e(l_n, l_m)\|_{\mathbb{E}} &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \|d_e(l_{n+2}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \dots + \|d_e(l_{m-1}, l_m)\|_{\mathbb{E}} \\ &\leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}} + \lambda^{n+1} \|d_e(l_0, l_1)\|_{\mathbb{E}} + \dots + \lambda^{m-1} \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left(\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1} \right) \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left[\frac{\lambda^n}{1-\lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}}, \end{aligned}$$

and so

$$\|d_e(l_n, l_m)\|_{\mathbb{E}} \leq \left[\frac{\lambda^n}{1-\lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

By Lemma (2), the sequence $\{l_n\}$ is Cauchy. Since $\mathcal{W}$ is complete, there exists a point l^* such that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$. To show that l^* is FP of $\mathcal{G}$, we note that, by (1),

$$\begin{aligned} \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \|d_e(\mathcal{G}l_n, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \hbar(d_e(l_n, \mathcal{G}l_n) + d_e(l^*, \mathcal{G}l^*)) \\ &= \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \hbar(d_e(l_n, l_{n+1}) + d_e(l^*, \mathcal{G}l^*)). \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality and using the fact that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$, we obtain

$$\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq \hbar \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}},$$

which is equivalent to

$$(1 - \hbar) \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq 0,$$

and since $(1 - \hbar) \neq 0$, we have $\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} = 0$. Thus, $l^* = \mathcal{G}l^*$. To establish the uniqueness of l^* , suppose that there exists another FP l' of $\mathcal{G}$. Then,

$$l' = \mathcal{G}l',$$

but $l^* \neq l'$. Now, from (1), we have

$$\begin{aligned} \|d_e(l^*, l')\|_{\mathbb{E}} &= \|d_e(\mathcal{G}l^*, \mathcal{G}l')\|_{\mathbb{E}} \\ &\leq \hbar(\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} + \|d_e(l', \mathcal{G}l')\|_{\mathbb{E}}) \\ &= 0, \end{aligned}$$

which implies that $l^* = l'$; thus, the FP of $\mathcal{G}$ is unique. $\square$

3.2. Chatterjea's Fixed-Point Theorem

The following theorem extends the Chatterjea-type FP theorem, as presented in [11], to the framework of E-VMSs.

Theorem 2. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exists a constant $\hbar \in [0, \frac{1}{2})$ such that

$$d_e(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar(d_e(l, \mathcal{G}\sigma) + d_e(\sigma, \mathcal{G}l)), \quad (6)$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has a unique FP.

Proof. Consider an arbitrary point l_0 in the set $\mathcal{W}$. Construct a sequence $\{l_n\}$ as follows:

$$l_{n+1} = \mathcal{G}l_n.$$

By (6), we have

$$\begin{aligned} d_e(l_n, l_{n+1}) &= d_e(\mathcal{G}l_{n-1}, \mathcal{G}l_n) \preceq \hbar(d_e(l_{n-1}, \mathcal{G}l_n) + d_e(l_n, \mathcal{G}l_{n-1})) \\ &= \hbar(d_e(l_{n-1}, l_{n+1})) \\ &\preceq \hbar(d_e(l_{n-1}, l_n) + d_e(l_n, l_{n+1})), \end{aligned}$$

which implies

$$d_e(l_n, l_{n+1}) \preceq \frac{\hbar}{1-\hbar} d_e(l_{n-1}, l_n). \quad (7)$$

Let $\lambda = \frac{\hbar}{1-\hbar} < 1$. Then, the above inequality gives

$$d_e(l_n, l_{n+1}) \preceq \lambda d_e(l_{n-1}, l_n),$$

which implies that

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}}. \quad (8)$$

Similarly by (6), we have

$$\begin{aligned} d_e(l_{n-1}, l_n) &= d_e(\mathcal{G}l_{n-2}, \mathcal{G}l_{n-1}) \preceq \hbar(d_e(l_{n-2}, \mathcal{G}l_{n-1}) + d_e(l_{n-1}, \mathcal{G}l_{n-2})) \\ &= \hbar(d_e(l_{n-2}, l_n)) \\ &\preceq \hbar(d_e(l_{n-2}, l_{n-1}) + d_e(l_{n-1}, l_n)), \end{aligned}$$

which implies

$$d_e(l_{n-1}, l_n) \preceq \frac{\hbar}{1-\hbar} d_e(l_{n-2}, l_{n-1}). \quad (9)$$

Let $\lambda = \frac{\hbar}{1-\hbar} < 1$. Then, the above inequality gives

$$d_e(l_{n-1}, l_n) \preceq \lambda d_e(l_{n-2}, l_{n-1}),$$

which implies

$$\|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}}. \quad (10)$$

Now, from the inequalities (8) and (10), we have

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda^2 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} \leq \dots \leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}},$$

for every natural number n . Now, for any two natural numbers m, n with $m > n$, we have

$$\begin{aligned} \|d_e(l_n, l_m)\|_{\mathbb{E}} &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \|d_e(l_{n+2}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \dots + \|d_e(l_{m-1}, l_m)\|_{\mathbb{E}} \\ &\leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}} + \lambda^{n+1} \|d_e(l_0, l_1)\|_{\mathbb{E}} + \dots + \lambda^{m-1} \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left(\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1} \right) \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left[\frac{\lambda^n}{1 - \lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}}, \end{aligned}$$

and so

$$\|d_e(l_n, l_m)\|_{\mathbb{E}} \leq \left[\frac{\lambda^n}{1 - \lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

As a consequence of Lemma (2), the sequence $\{l_n\}$ is a Cauchy sequence. Given the completeness of $\mathcal{W}$, there exists a point l^* such that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$. To prove that l^* is an FP of $\mathcal{G}$, we observe from (6) that

$$\begin{aligned} d_e(l^*, \mathcal{G}l^*) &\leq d_e(l^*, l_{n+1}) + d_e(l_{n+1}, \mathcal{G}l^*) \\ &= d_e(l^*, l_{n+1}) + d_e(\mathcal{G}l_n, \mathcal{G}l^*) \\ &\leq d_e(l^*, l_{n+1}) + \hbar(d_e(l_n, \mathcal{G}l^*) + d_e(l^*, \mathcal{G}l_n)) \\ &= d_e(l^*, l_{n+1}) + \hbar(d_e(l_n, \mathcal{G}l^*) + d_e(l^*, l_{n+1})), \end{aligned}$$

which implies

$$\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \hbar \|d_e(l_n, \mathcal{G}l^*)\|_{\mathbb{E}} + \hbar \|d_e(l^*, l_{n+1})\|_{\mathbb{E}}.$$

Letting $n \rightarrow \infty$ in the above inequality and using the fact that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$, we obtain

$$\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq \hbar \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}},$$

which is equivalent to

$$(1 - \hbar) \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq 0,$$

and since $(1 - \hbar) \neq 0$, we have $\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} = 0$. Thus, $l^* = \mathcal{G}l^*$. To show that l^* is the unique FP of $\mathcal{G}$, assume that there exists another point l' such that

$$l' = \mathcal{G}l',$$

but $l^* \neq l'$. Now, from (1), we have

$$\begin{aligned} d_e(l^*, l') &= d_e(\mathcal{G}l^*, \mathcal{G}l') \\ &\leq \hbar(d_e(l^*, \mathcal{G}l') + d_e(l', \mathcal{G}l^*)) \\ &= \hbar(d_e(l^*, l') + d_e(l', l^*)), \end{aligned}$$

which implies

$$\|d_e(l^*, l')\|_{\mathbb{E}} \leq \hbar \|d_e(l^*, l')\|_{\mathbb{E}} + \hbar \|d_e(l', l^*)\|_{\mathbb{E}},$$

which follows

$$(1 - 2\hbar) \|d_e(l^*, l')\|_{\mathbb{E}},$$

since $(1 - 2\hbar) \neq 0$, so $\|d_e(l^*, l')\|_{\mathbb{E}} = 0$, hence $l^* = l'$. Thus, the FP of $\mathcal{G}$ is unique. $\square$

3.3. Dass and Gupta's Fixed-Point Theorem

Theorem 3. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 + \hbar_3 + 2\hbar_4 + 2\hbar_5 < 1$ such that

$$\begin{aligned} d_e(\mathcal{G}l, \mathcal{G}\sigma) &\preceq \hbar_1 d_e(l, \sigma) + \hbar_2 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}l)}{1 + d_e(l, \sigma)} \\ &+ \hbar_3 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)} + \hbar_4 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)} \\ &+ \hbar_5 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}l)}{1 + d_e(l, \sigma)}, \end{aligned} \quad (11)$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has a FP.

Proof. Let l_0 be any point in $\mathcal{W}$ and the sequence $\{l_n\}$ be defined by

$$l_{n+1} = \mathcal{G}l_n.$$

By (11), we have

$$\begin{aligned} d_e(l_n, l_{n+1}) &= d_e(\mathcal{G}l_{n-1}, \mathcal{G}l_n) \preceq \hbar_1 d_e(l_{n-1}, l_n) \\ &+ \hbar_2 \frac{(1 + d_e(l_{n-1}, \mathcal{G}l_{n-1}))d_e(l_{n-1}, \mathcal{G}l_{n-1})}{1 + d_e(l_{n-1}, l_n)} \\ &+ \hbar_3 \frac{(1 + d_e(l_{n-1}, \mathcal{G}l_{n-1}))d_e(l_n, \mathcal{G}l_n)}{1 + d_e(l_{n-1}, l_n)} \\ &+ \hbar_4 \frac{(1 + d_e(l_{n-1}, \mathcal{G}l_{n-1}))d_e(l_{n-1}, \mathcal{G}l_n)}{1 + d_e(l_{n-1}, l_n)} \\ &+ \hbar_5 \frac{(1 + d_e(l_{n-1}, \mathcal{G}l_{n-1}))d_e(l_n, \mathcal{G}l_{n-1})}{1 + d_e(l_{n-1}, l_n)} \\ &= \hbar_1 d_e(l_{n-1}, l_n) \\ &+ \hbar_2 \frac{(1 + d_e(l_{n-1}, l_n))d_e(l_{n-1}, l_n)}{1 + d_e(l_{n-1}, l_n)} \\ &+ \hbar_3 \frac{(1 + d_e(l_{n-1}, l_n))d_e(l_n, l_{n+1})}{1 + d_e(l_{n-1}, l_n)} \\ &+ \hbar_4 \frac{(1 + d_e(l_{n-1}, l_n))d_e(l_{n-1}, l_{n+1})}{1 + d_e(l_{n-1}, l_n)} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_e(l_n, l_{n+1}) &\preceq \hbar_1 d_e(l_{n-1}, l_n) + \hbar_2 d_e(l_{n-1}, l_n) + \hbar_3 d_e(l_n, l_{n+1}) \\ &+ \hbar_4 d_e(l_{n-1}, l_{n+1}). \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} d_e(l_n, l_{n+1}) &\preceq \hbar_1 d_e(l_{n-1}, l_n) + \hbar_2 d_e(l_{n-1}, l_n) + \hbar_3 d_e(l_n, l_{n+1}) \\ &+ \hbar_4 d_e(l_{n-1}, l_n) + \hbar_4 d_e(l_n, l_{n+1}), \end{aligned}$$

which implies that

$$d_e(l_n, l_{n+1}) \preceq \hbar_1 d_e(l_{n-1}, l_n) + \hbar_2 d_e(l_{n-1}, l_n) + \hbar_3 d_e(l_n, l_{n+1}) \\ + \hbar_4 d_e(l_{n-1}, l_n) + \hbar_4 d_e(l_n, l_{n+1}).$$

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \hbar_1 \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} + \hbar_2 \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \\ + \hbar_3 \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \hbar_4 \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} + \hbar_4 \|d_e(l_n, l_{n+1})\|_{\mathbb{E}}.$$

It follows that

$$(1 - \hbar_3 - \hbar_4) \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq (\hbar_1 + \hbar_2 + \hbar_4) \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}},$$

that is,

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \frac{\hbar_1 + \hbar_2 + \hbar_4}{1 - \hbar_3 - \hbar_4} \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}}. \quad (12)$$

Similarly, by (11), we have

$$d_e(l_{n-1}, l_n) = d_e(\mathcal{G}l_{n-2}, \mathcal{G}l_{n-1}) \preceq \hbar_1 d_e(l_{n-2}, l_{n-1}) \\ + \hbar_2 \frac{(1 + d_e(l_{n-2}, \mathcal{G}l_{n-2}))d_e(l_{n-2}, \mathcal{G}l_{n-2})}{1 + d_e(l_{n-2}, l_{n-1})} \\ + \hbar_3 \frac{(1 + d_e(l_{n-2}, \mathcal{G}l_{n-2}))d_e(l_{n-1}, \mathcal{G}l_{n-1})}{1 + d_e(l_{n-2}, l_{n-1})} \\ + \hbar_4 \frac{(1 + d_e(l_{n-2}, \mathcal{G}l_{n-2}))d_e(l_{n-2}, \mathcal{G}l_{n-1})}{1 + d_e(l_{n-2}, l_{n-1})} \\ + \hbar_5 \frac{(1 + d_e(l_{n-2}, \mathcal{G}l_{n-2}))d_e(l_{n-1}, \mathcal{G}l_{n-2})}{1 + d_e(l_{n-2}, l_{n-1})} \\ = \hbar_1 d_e(l_{n-2}, l_{n-1}) \\ + \hbar_2 \frac{(1 + d_e(l_{n-2}, l_{n-1}))d_e(l_{n-2}, l_{n-1})}{1 + d_e(l_{n-2}, l_{n-1})} \\ + \hbar_3 \frac{(1 + d_e(l_{n-2}, l_{n-1}))d_e(l_{n-1}, l_n)}{1 + d_e(l_{n-2}, l_{n-1})} \\ + \hbar_4 \frac{(1 + d_e(l_{n-2}, l_{n-1}))d_e(l_{n-2}, l_n)}{1 + d_e(l_{n-2}, l_{n-1})}$$

which is equivalent to

$$d_e(l_{n-1}, l_n) \preceq \hbar_1 d_e(l_{n-2}, l_{n-1}) + \hbar_2 d_e(l_{n-2}, l_{n-1}) + \hbar_3 d_e(l_{n-1}, l_n) \\ + \hbar_4 d_e(l_{n-2}, l_n).$$

By the triangle inequality, we have

$$d_e(l_{n-1}, l_n) \preceq \hbar_1 d_e(l_{n-2}, l_{n-1}) + \hbar_2 d_e(l_{n-2}, l_{n-1}) + \hbar_3 d_e(l_{n-1}, l_n) \\ + \hbar_4 d_e(l_{n-2}, l_{n-1}) + \hbar_4 d_e(l_{n-1}, l_n),$$

which implies that

$$\|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \hbar_1 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} + \hbar_2 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} \\ + \hbar_3 \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} + \hbar_4 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} + \hbar_4 \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}}.$$

It follows that

$$(1 - \hbar_3 - \hbar_4) \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq (\hbar_1 + \hbar_2 + \hbar_4) \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}}$$

that is,

$$\|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \frac{\hbar_1 + \hbar_2 + \hbar_4}{1 - \hbar_3 - \hbar_4} \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}}. \quad (13)$$

Hence, by (12) and (13), we have

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda^2 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}}.$$

Thus, a sequence $\{l_n\}$ in $\mathcal{W}$ can be constructed inductively as follows $l_{n+1} = \mathcal{G}l_n$ satisfying

$$\|d_e(l_n, l_{n+1})\|_{\mathbb{E}} \leq \lambda \|d_e(l_{n-1}, l_n)\|_{\mathbb{E}} \leq \lambda^2 \|d_e(l_{n-2}, l_{n-1})\|_{\mathbb{E}} \leq \dots \leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}},$$

for every natural number n . Now, for $m > n$, we obtain

$$\begin{aligned} \|d_e(l_n, l_m)\|_{\mathbb{E}} &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \|d_e(l_{n+2}, l_m)\|_{\mathbb{E}} \\ &\leq \|d_e(l_n, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, l_{n+2})\|_{\mathbb{E}} + \dots + \|d_e(l_{m-1}, l_m)\|_{\mathbb{E}} \\ &\leq \lambda^n \|d_e(l_0, l_1)\|_{\mathbb{E}} + \lambda^{n+1} \|d_e(l_0, l_1)\|_{\mathbb{E}} + \dots + \lambda^{m-1} \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left(\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1} \right) \|d_e(l_0, l_1)\|_{\mathbb{E}} \\ &\leq \left[\frac{\lambda^n}{1 - \lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}}, \end{aligned}$$

and so

$$\|d_e(l_n, l_m)\|_{\mathbb{E}} \leq \left[\frac{\lambda^n}{1 - \lambda} \right] \|d_e(l_0, l_1)\|_{\mathbb{E}} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Lemma (2) implies that $\{l_n\}$ is a Cauchy sequence. The completeness of $\mathcal{W}$ ensures the existence of a point l^* such that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$. To show that l^* is an FP of $\mathcal{G}$, we note that, by (11),

$$\begin{aligned} \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \|d_e(l_{n+1}, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \|d_e(\mathcal{G}l_n, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &\leq \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \left(\begin{aligned} &\frac{\hbar_1 \|d_e(l_n, l^*)\|_{\mathbb{E}}}{\|(1 + d_e(l_n, \mathcal{G}l_n))\|_{\mathbb{E}}} + \frac{\hbar_2 \|d_e(l_n, \mathcal{G}l_n)\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \\ &+ \frac{\hbar_3 \|(1 + d_e(l_n, \mathcal{G}l_n))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &+ \frac{\hbar_4 \|(1 + d_e(l_n, \mathcal{G}l_n))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l_n, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &+ \frac{\hbar_5 \|(1 + d_e(l_n, \mathcal{G}l_n))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l^*, \mathcal{G}l_n)\|_{\mathbb{E}} \end{aligned} \right) \\ &= \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} + \left(\begin{aligned} &\frac{\hbar_1 \|d_e(l_n, l^*)\|_{\mathbb{E}}}{\|(1 + d_e(l_n, l_{n+1}))\|_{\mathbb{E}}} + \frac{\hbar_2 \|d_e(l_n, l_{n+1})\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \\ &+ \frac{\hbar_3 \|(1 + d_e(l_n, l_{n+1}))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &+ \frac{\hbar_4 \|(1 + d_e(l_n, l_{n+1}))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l_n, \mathcal{G}l^*)\|_{\mathbb{E}} \\ &+ \frac{\hbar_5 \|(1 + d_e(l_n, l_{n+1}))\|_{\mathbb{E}}}{\|1 + d_e(l_n, l^*)\|_{\mathbb{E}}} \|d_e(l^*, l_{n+1})\|_{\mathbb{E}} \end{aligned} \right). \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality and using the fact that $l_n \rightarrow l^* \in \mathcal{W}$ as $n \rightarrow \infty$, we obtain

$$\|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq (\hbar_3 + \hbar_4) \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}},$$

which is equivalent to

$$(1 - \hbar_3 - \hbar_4) \|d_e(l^*, \mathcal{G}l^*)\|_{\mathbb{E}} \leq 0,$$

and since $(1 - \hbar_3 - \hbar_4) \neq 0$, we have $\|d_e(l^*, Gl^*)\|_{\mathbb{E}} = 0$. Thus, $l^* = Gl^*$. $\square$

Example 2. Let $\mathcal{W} = E_{-2} = \{z = l + i\sigma : l, \sigma \in \mathbb{R}, l^2 = p = -2 < 0\}$ and $d_e : E_{-2} \times E_{-2} \rightarrow E_{-2}$ is defined as

$$d_e(z_1, z_2) = \|z_1 - z_2\|_{\mathbb{E}} = \sqrt{(l_1 - l_2)^2 - 2(\sigma_1 - \sigma_2)^2},$$

for $(l_1 - l_2)^2 - 2(\sigma_1 - \sigma_2)^2 > 0$ and for two elliptic numbers $z_1 = l_1 + i\sigma_1$ and $z_2 = l_2 + i\sigma_2$. Then, (E_{-1}, d_e) is a complete E-VMS. Define a mapping $\mathcal{G} : E_{-1} \rightarrow E_{-1}$ by $\mathcal{G}(z) = \frac{z}{2}$. Now, $\mathcal{G}(z_1) = \frac{1}{2}(l_1 + i\sigma_1)$ and $\mathcal{G}(z_2) = \frac{1}{2}(l_2 + i\sigma_2)$. Then,

$$\begin{aligned} d_e(\mathcal{G}z_1, \mathcal{G}z_2) &= \|\mathcal{G}z_1 - \mathcal{G}z_2\|_{\mathbb{E}} \\ &= \sqrt{\left(\frac{l_1 - l_2}{2}\right)^2 - 2\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2} \\ &= \frac{1}{2}\sqrt{(l_1 - l_2)^2 - 2(\sigma_1 - \sigma_2)^2} \\ &= \frac{1}{2}d_e(z_1, z_2). \end{aligned}$$

Therefore, the mapping $\mathcal{G}(z) = \frac{z}{2}$ satisfies the contractive condition (11) with $\hbar_1 = \frac{1}{2}$ and any values of $\hbar_2, \hbar_3, \hbar_4, \hbar_5$ in $[0, 1)$ such that $\hbar_1 + \hbar_2 + \hbar_3 + 2\hbar_4 + 2\hbar_5 < 1$ and $0 + 0i$ is an FP of mapping $\mathcal{G}$.

Corollary 1. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 + \hbar_3 + 2\hbar_4 < 1$ such that

$$\begin{aligned} d_e(\mathcal{G}l, \mathcal{G}\sigma) &\preceq \hbar_1 d_e(l, \sigma) + \hbar_2 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}l)}{1 + d_e(l, \sigma)} \\ &\quad + \hbar_3 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)} + \hbar_4 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)}, \end{aligned}$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has an FP.

Proof. Set $\hbar_5 = 0$ in Theorem 3. $\square$

Corollary 2. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2, \hbar_3 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 + \hbar_3 < 1$ such that

$$\begin{aligned} d_e(\mathcal{G}l, \mathcal{G}\sigma) &\preceq \hbar_1 d_e(l, \sigma) + \hbar_2 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}l)}{1 + d_e(l, \sigma)} \\ &\quad + \hbar_3 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)}, \end{aligned}$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has an FP.

Proof. Take $\hbar_4 = \hbar_5 = 0$ in the Theorem 3. $\square$

Corollary 3. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 < 1$ such that

$$d_e(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar_1 d_e(l, \sigma) + \hbar_2 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}l)}{1 + d_e(l, \sigma)},$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has an FP.

Proof. Take $\hbar_3 = \hbar_4 = \hbar_5 = 0$ in Theorem 3. $\square$

Theorem 4. Let $(\mathcal{W}, d_e)$ be a complete E-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 + \hbar_3 + 2\hbar_4 + 2\hbar_5 < 1$ such that

$$\begin{aligned} d_e(\mathcal{G}l, \mathcal{G}\sigma) \preceq & \hbar_1 d_e(l, \sigma) + \hbar_2 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}l)}{1 + d_e(l, \sigma)} \\ & + \hbar_3 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)} + \hbar_4 \frac{(1 + d_e(l, \mathcal{G}l))d_e(l, \mathcal{G}\sigma)}{1 + d_e(l, \sigma)} \\ & + \hbar_5 \frac{(1 + d_e(l, \mathcal{G}l))d_e(\sigma, \mathcal{G}l)}{1 + d_e(l, \sigma)}, \end{aligned} \quad (14)$$

for $l_0, l, \sigma \in \overline{B(l_0, r)}$ and there exists $0 \prec r \in \mathbb{E}_p$ such that

$$d_e(l_0, \mathcal{G}l_0) \preceq (1 - \lambda) r \quad (15)$$

for all $l, \sigma \in \mathcal{W}$ and $\lambda = \frac{\hbar_1 + \hbar_2 + \hbar_4}{1 - \hbar_3 - \hbar_4} < 1$, then $\mathcal{G}$ has an FP.

Proof. Let l_0 be any point in $\mathcal{W}$ and the sequence $\{l_n\}$ be defined by

$$l_{n+1} = \mathcal{G}l_n,$$

for all $n \in \mathbb{N}$. First, we show that $l_n \in \overline{B(l_0, r)}$, for all $n \in \mathbb{N}$. Using inequality (15), we have

$$d_e(l_0, l_1) = d_e(l_0, \mathcal{G}l_0) \preceq (1 - \lambda) r \preceq r.$$

It follows that $l_1 \in \overline{B(l_0, r)}$. Let $l_2, l_3, \dots, l_{j-1} \in \overline{B(l_0, r)}$. By (14), we have

$$\begin{aligned} d_e(l_j, l_{j+1}) &= d_e(\mathcal{G}l_{j-1}, \mathcal{G}l_j) \preceq \hbar_1 d_e(l_{j-1}, l_j) \\ &+ \hbar_2 \frac{(1 + d_e(l_{j-1}, \mathcal{G}l_{j-1}))d_e(l_{j-1}, \mathcal{G}l_{j-1})}{1 + d_e(l_{j-1}, l_j)} \\ &+ \hbar_3 \frac{(1 + d_e(l_{j-1}, \mathcal{G}l_{j-1}))d_e(l_j, \mathcal{G}l_j)}{1 + d_e(l_{j-1}, l_j)} \\ &+ \hbar_4 \frac{(1 + d_e(l_{j-1}, \mathcal{G}l_{j-1}))d_e(l_{j-1}, \mathcal{G}l_j)}{1 + d_e(l_{j-1}, l_j)} \\ &+ \hbar_5 \frac{(1 + d_e(l_{j-1}, \mathcal{G}l_{j-1}))d_e(l_j, \mathcal{G}l_{j-1})}{1 + d_e(l_{j-1}, l_j)} \\ &= \hbar_1 d_e(l_{j-1}, l_j) \\ &+ \hbar_2 \frac{(1 + d_e(l_{j-1}, l_j))d_e(l_{j-1}, l_j)}{1 + d_e(l_{j-1}, l_j)} \\ &+ \hbar_3 \frac{(1 + d_e(l_{j-1}, l_j))d_e(l_j, l_{j+1})}{1 + d_e(l_{j-1}, l_j)} \\ &+ \hbar_4 \frac{(1 + d_e(l_{j-1}, l_j))d_e(l_{j-1}, l_{j+1})}{1 + d_e(l_{j-1}, l_j)} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_e(l_j, l_{j+1}) \preceq & \hbar_1 d_e(l_{j-1}, l_j) + \hbar_2 d_e(l_{j-1}, l_j) + \hbar_3 d_e(l_j, l_{j+1}) \\ & + \hbar_4 d_e(l_{j-1}, l_{j+1}). \end{aligned}$$

By the triangle inequality, we have

$$d_e(l_j, l_{j+1}) \preceq \hbar_1 d_e(l_{j-1}, l_j) + \hbar_2 d_e(l_{j-1}, l_j) + \hbar_3 d_e(l_j, l_{j+1}) \\ + \hbar_4 d_e(l_{j-1}, l_j) + \hbar_4 d_e(l_j, l_{j+1}),$$

which implies that

$$d_e(l_j, l_{j+1}) \preceq \hbar_1 d_e(l_{j-1}, l_j) + \hbar_2 d_e(l_{j-1}, l_j) + \hbar_3 d_e(l_j, l_{j+1}) \\ + \hbar_4 d_e(l_{j-1}, l_j) + \hbar_4 d_e(l_j, l_{j+1}),$$

which is equivalent to

$$d_e(l_j, l_{j+1}) \preceq \frac{\hbar_1 + \hbar_2 + \hbar_4}{1 - \hbar_3 - \hbar_4} d_e(l_{j-1}, l_j) = \lambda d_e(l_{j-1}, l_j). \quad (16)$$

Similarly, by (14), we have

$$d_e(l_{j-1}, l_j) = d_e(\mathcal{G}l_{j-2}, \mathcal{G}l_{j-1}) \preceq \hbar_1 d_e(l_{j-2}, l_{j-1}) \\ + \hbar_2 \frac{(1 + d_e(l_{j-2}, \mathcal{G}l_{j-2})) d_e(l_{j-2}, \mathcal{G}l_{j-2})}{1 + d_e(l_{j-2}, l_{j-1})} \\ + \hbar_3 \frac{(1 + d_e(l_{j-2}, \mathcal{G}l_{j-2})) d_e(l_{j-1}, \mathcal{G}l_{j-1})}{1 + d_e(l_{j-2}, l_{j-1})} \\ + \hbar_4 \frac{(1 + d_e(l_{j-2}, \mathcal{G}l_{j-2})) d_e(l_{j-2}, \mathcal{G}l_{j-1})}{1 + d_e(l_{j-2}, l_{j-1})} \\ + \hbar_5 \frac{(1 + d_e(l_{j-2}, \mathcal{G}l_{j-2})) d_e(l_{j-1}, \mathcal{G}l_{j-2})}{1 + d_e(l_{j-2}, l_{j-1})} \\ = \hbar_1 d_e(l_{j-2}, l_{j-1}) \\ + \hbar_2 \frac{(1 + d_e(l_{j-2}, l_{j-1})) d_e(l_{j-2}, l_{j-1})}{1 + d_e(l_{j-2}, l_{j-1})} \\ + \hbar_3 \frac{(1 + d_e(l_{j-2}, l_{j-1})) d_e(l_{j-1}, l_j)}{1 + d_e(l_{j-2}, l_{j-1})} \\ + \hbar_4 \frac{(1 + d_e(l_{j-2}, l_{j-1})) d_e(l_{j-2}, l_j)}{1 + d_e(l_{j-2}, l_{j-1})}$$

which is equivalent to

$$d_e(l_{j-1}, l_j) \preceq \hbar_1 d_e(l_{j-2}, l_{j-1}) + \hbar_2 d_e(l_{j-2}, l_{j-1}) + \hbar_3 d_e(l_{j-1}, l_j) \\ + \hbar_4 d_e(l_{j-2}, l_j).$$

By the triangle inequality, we have

$$d_e(l_{j-1}, l_j) \preceq \hbar_1 d_e(l_{j-2}, l_{j-1}) + \hbar_2 d_e(l_{j-2}, l_{j-1}) + \hbar_3 d_e(l_{j-1}, l_j) \\ + \hbar_4 d_e(l_{j-2}, l_{j-1}) + \hbar_4 d_e(l_{j-1}, l_j).$$

which implies that

$$d_e(l_{j-1}, l_j) \preceq \frac{\hbar_1 + \hbar_2 + \hbar_4}{1 - \hbar_3 - \hbar_4} d_e(l_{j-2}, l_{j-1}) = \lambda d_e(l_{j-2}, l_{j-1}). \quad (17)$$

By (16) and (17), we have

$$d_e(l_j, l_{j+1}) \preceq \lambda d_e(l_{j-1}, l_j) \preceq \lambda^2 d_e(l_{j-2}, l_{j-1}) \preceq \dots \preceq \lambda^j d_e(l_0, l_1).$$

Now,

$$\begin{aligned} d_e(l_0, l_{j+1}) &\preceq d_e(l_0, l_1) + d_e(l_1, l_2) + \dots + d_e(l_j, l_{j+1}) \\ &\preceq d_e(l_0, l_1) + \lambda d_e(l_0, l_1) + \dots + \lambda^j d_e(l_0, l_1) \\ &= [1 + \lambda + \dots + \lambda^j] d_e(l_0, l_1) \\ &\preceq [1 + \lambda + \dots + \lambda^j] (1 - \lambda) r \\ &\preceq \frac{(1 - \lambda^{j+1})}{(1 - \lambda)} (1 - \lambda) r \\ &\preceq (1 - \lambda^{j+1}) r \\ &\preceq r. \end{aligned}$$

Then, $l_{j+1} \in \overline{B(l_0, r)}$. Hence, $l_n \in \overline{B(l_0, r)}$ for all $n \in \mathbb{N}$. Following a similar approach to that employed in Theorem 3, it can be demonstrated that there exists an $l^* \in \overline{B(l_0, r)}$ such that $l^* = \mathcal{G}l^*$. $\square$

4. Fixed-Point Results in Complex-Valued Metric Spaces

If we take $p = -1$ in Definition 2, then the concept of E-VMS is reduced to C-VMS and we derive the following results.

As an immediate outcome of Theorem 1, we derive a Kannan-type FP result [10] within the framework of complex-valued metric spaces.

Corollary 4. Let $(\mathcal{W}, d_c)$ be a complete C-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exists a constant $\hbar \in [0, \frac{1}{2})$ such that

$$d_c(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar(d_c(l, \mathcal{G}l) + d_c(\sigma, \mathcal{G}\sigma)),$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has a unique FP.

Now, we derive a result which advances the Chatterjea-type FP theorem [11], by reformulating it for C-VMSs.

Corollary 5. Let $(\mathcal{W}, d_c)$ be a complete C-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exists a constant $\hbar \in [0, \frac{1}{2})$ such that

$$d_c(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar(d_c(l, \mathcal{G}\sigma) + d_c(\sigma, \mathcal{G}l)),$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has a unique FP.

As a direct consequence of Corollary 3, we obtain a result similar to that of Sithikul et al. [16].

Corollary 6. Let $(\mathcal{W}, d_c)$ be a complete C-VMS and let $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$. Assume that there exist the constants $\hbar_1, \hbar_2 \in [0, 1)$ satisfying $\hbar_1 + \hbar_2 < 1$ such that

$$d_c(\mathcal{G}l, \mathcal{G}\sigma) \preceq \hbar_1 d_c(l, \sigma) + \hbar_2 \frac{(1 + d_c(l, \mathcal{G}l))d_c(l, \mathcal{G}l)}{1 + d_c(l, \sigma)},$$

for all $l, \sigma \in \mathcal{W}$, then $\mathcal{G}$ has an FP.

5. Applications

Integral equations involve unknown functions as part of an integral expression. These equations are prevalent in various disciplines, including physics, engineering, and economics. A prominent application of integral equations is in the study of equations defined by integrals. In the present section, we shall discuss the solution of the Urysohn integral equation.

Theorem 5. Let $\mathcal{W} = C([a, b], \mathbb{R}^n)$, $a > 0$ and $d_e : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{E}_p$ be an E-VM given in this way:

$$d_e(l, \sigma) = \left(\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \right) e^{i \tan^{-1} a}$$

where $i^2 = p < 0$.

Let us consider the Urysohn integral equation

$$l(t) = \int_a^b K(t, s, l(s)) d_e s + g(t), \quad (18)$$

where $t \in [a, b] \subset \mathbb{R}$ and $l, g \in \mathcal{W}$. Assume that $K(\cdot, \cdot, \cdot) : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that

$$Q_l = \int_a^b K(t, s, l(s)) d_e s,$$

where $Q_l \in \mathcal{W}$, for each $l \in \mathcal{W}$. Suppose that there exist the constants $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5 \in [0, 1)$ with $\hbar_1 + \hbar_2 + \hbar_3 + 2\hbar_4 + 2\hbar_5 < 1$ such that

$$\begin{aligned} \|\mathcal{G}l(t) - \mathcal{G}\sigma(t)\| e^{i \tan^{-1} a} &\preceq \hbar_1 \|l(t) - \sigma(t)\| e^{i \tan^{-1} a} \\ &+ \hbar_2 \frac{\left(1 + \|l(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}\right) \cdot \|l(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}}{\left(1 + \|l(t) - \sigma(t)\| e^{i \tan^{-1} a}\right)} \\ &+ \hbar_3 \frac{\left(1 + \|l(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}\right) \cdot \|\sigma(t) - \mathcal{G}\sigma(t)\| e^{i \tan^{-1} a}}{\left(1 + \|l(t) - \sigma(t)\| e^{i \tan^{-1} a}\right)} \\ &+ \hbar_4 \frac{\left(1 + \|l(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}\right) \cdot \|l(t) - \mathcal{G}\sigma(t)\| e^{i \tan^{-1} a}}{\left(1 + \|l(t) - \sigma(t)\| e^{i \tan^{-1} a}\right)} \\ &+ \hbar_5 \frac{\left(1 + \|l(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}\right) \cdot \|\sigma(t) - \mathcal{G}l(t)\| e^{i \tan^{-1} a}}{\left(1 + \|l(t) - \sigma(t)\| e^{i \tan^{-1} a}\right)}. \end{aligned} \quad (19)$$

Then, the Urysohn integral Equation (18) has a solution.

Proof. First of all, we prove that $(\mathcal{W} = C([a, b], \mathbb{R}^n), d_e)$ is a complete E-VMS. For this, we have to satisfy the axioms of E-VMS.

$(E_1) : \theta \preceq d_e(l, \sigma)$, and $d_e(l, \sigma) = \theta \Leftrightarrow l = \sigma$. Since

$$d_e(l, \sigma) = \left(\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \right) e^{i \tan^{-1} a}.$$

As $\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \geq 0$, so $d_e(l, \sigma) \succeq \theta$. The elliptic value $e^{i \tan^{-1} a}$ ensures that $d_e(l, \sigma)$ lies in E_p . Also, $d_e(l, \sigma) = \theta$ implies

$$\max_{t \in [a, b]} \|l(t) - \sigma(t)\| = 0.$$

This occurs only when $l(t) = \sigma(t)$, for all $t \in [a, b]$. Thus, $l = \sigma$. Hence, $d_e(l, \sigma) = \theta \Leftrightarrow l = \sigma$.

$(E_2) : d_e(l, \sigma) = d_e(\sigma, l)$. By definition,

$$\|l(t) - \sigma(t)\| = \|\sigma(t) - l(t)\|, \text{ for all } t \in [a, b].$$

Therefore,

$$\max_{t \in [a, b]} \|l(t) - \sigma(t)\| = \max_{t \in [a, b]} \|\sigma(t) - l(t)\|.$$

Since $e^{i \tan^{-1} a}$ is independent of the order of l and σ , so $d_e(l, \sigma) = d_e(\sigma, l)$.

$(E_3) : d_e(l, \sigma) \preceq d_e(l, \omega) + d_e(\omega, \sigma)$. Since we have

$$d_e(l, \sigma) = \left(\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \right) e^{i \tan^{-1} a}.$$

Using the triangle inequality for norms,

$$\|l(t) - \sigma(t)\| \leq \|l(t) - \omega(t)\| + \|\omega(t) - \sigma(t)\|, \text{ for all } t \in [a, b].$$

Taking the maximum over $t \in [a, b]$, we have

$$\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \leq \max_{t \in [a, b]} \|l(t) - \omega(t)\| + \max_{t \in [a, b]} \|\omega(t) - \sigma(t)\|.$$

Multiplying by $e^{i \tan^{-1} a}$, we have

$$d_e(l, \sigma) \preceq d_e(l, \omega) + d_e(\omega, \sigma).$$

It is straightforward to verify that $(\mathcal{W} = C([a, b], \mathbb{R}^n), d_e)$ is complete. Now, we define the mapping $\mathcal{G} : \mathcal{W} \rightarrow \mathcal{W}$ by

$$\mathcal{G}l = Q_l + g(t).$$

Then,

$$d_e(\mathcal{G}l, \mathcal{G}\sigma) = \left(\max_{t \in [a, b]} \|\mathcal{G}l(t) - \mathcal{G}\sigma(t)\| \right) e^{i \tan^{-1} \alpha}$$

$$d_e(l, \sigma) = \left(\max_{t \in [a, b]} \|l(t) - \sigma(t)\| \right) e^{i \tan^{-1} \alpha}$$

$$d_e(l, \mathcal{G}l) = \left(\max_{t \in [a, b]} \|l(t) - \mathcal{G}l(t)\| \right) e^{i \tan^{-1} \alpha}$$

$$d_e(\sigma, \mathcal{G}\sigma) = \left(\max_{t \in [a, b]} \|\sigma(t) - \mathcal{G}\sigma(t)\| \right) e^{i \tan^{-1} \alpha}$$

$$d_e(l, \mathcal{G}\sigma) = \left(\max_{t \in [a, b]} \|l(t) - \mathcal{G}\sigma(t)\| \right) e^{i \tan^{-1} \alpha}$$

$$d_e(\sigma, \mathcal{G}l) = \left(\max_{t \in [a, b]} \|\sigma(t) - \mathcal{G}l(t)\| \right) e^{i \tan^{-1} \alpha}$$

for every $l, \sigma \in \mathcal{W}$. Thus, all the assumptions of Theorem 3 are satisfied and the mapping $\mathcal{G}$ has an FP, that is, there exists a point $l \in \mathcal{W}$ such that

$$l = \mathcal{G}l = Q_l + g(t).$$

Hence, the Urysohn integral Equation (18) has a solution. $\square$

6. Conclusions

In this study, we have explored the concept of E-VMs and established new FP theorems for various generalized contractions. Our findings extend several existing results in the field, contributing to the broader understanding of such spaces. To highlight the novelty of our main result, we have provided a concrete example that illustrates its significance. Furthermore, we demonstrate the practical application of our primary theorem by solving a Urysohn integral equation, showcasing its relevance in real-world problems.

7. Open Problems

Future research could explore the extension of FP and common FP theorems to multi-valued, fuzzy, and L -fuzzy mappings within the framework of E-VMs. Additionally, investigating differential and integral inclusions in the context of E-VMs presents a valuable direction. The findings of this study are expected to inspire further work and refinements by other researchers, potentially expanding the range of applications for these results.

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