

Article

Similarity of Overlap Functions and Robustness of Fuzzy Reasoning

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Abstract: The overlap function has been extensively utilized across various fields. In this paper, we introduce the concepts of the similarity and δ -equality of overlap functions to measure the degree of similarity between two overlap functions. Subsequently, we examine the δ -equality of several operations on overlap functions, including meet, join, and weighted sum, to assess how these operations maintain the similarity. Finally, we discuss the robustness of fuzzy reasoning for FMP, FMT, and FHS models based on the δ -equality of the overlap functions.

Keywords: similarity; overlap functions; grouping functions; robustness; fuzzy reasoning

MSC: 03E72

1. Introduction

Fuzzy reasoning is crucial in the field of fuzzy systems, attracting substantial interest and producing valuable results across multiple disciplines [1,2]. Within the spectrum of fuzzy reasoning methods, fuzzy conjunction and fuzzy implication are frequently employed as essential components [2,3]. Among these, overlap functions have emerged as a category of fuzzy conjunctions that are utilized in fuzzy reasoning processes [4,5].

Bustince et al. [6,7], as well as Fodor and Keresztfalvi [8], highlighted that the associativity of t-norms is not always required in certain applications, such as decision-making processes and classification tasks. Motivated by this observation, Bustince et al. [6] introduced the concept of overlap functions, a subclass of aggregation operators which do not necessarily adhere to the property of associativity. Dai et al. [9] introduced the concept of (O, N)-difference operators based on overlap functions. Du et al. [10] developed some equivalence operators based on overlap and grouping functions. Dimuro and Bedregal [11] introduced the concept of the residual implication induced by overlap functions. Dimuro et al. [12–14] developed the (G,N)- and (O,G,N)-operations further. Qiao [15] studied the residual implication derived from overlap functions on finite scales. Fan et al. [16] established fuzzy β -covering relations and fuzzy β -covering rough set models based on overlap functions. Han et al. [17] developed multigranulation fuzzy probabilistic rough sets derived from overlap functions. Li et al. [18] introduced the concept of (O, G)-granular, variable-precision fuzzy rough sets. Qiao [19] introduced the concept of (IO, O)-fuzzy rough sets. Zhang et al. [20] studied variable-precision fuzzy rough sets based on overlap functions. Each of these innovations has contributed to the theoretical toolbox available for manipulating and analyzing fuzzy data, facilitating more refined modeling of uncertainty and variability in various fields. In the field of image processing, for instance, certain properties of the overlap functions are studied and used to image thresholding [21]. For



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classification tasks, the overlap function has led to novel methods [4,22–24]. Similarly, in clustering, overlap-function-based similarity measures are constructed and used in the environment of expert assessments and datasets [25]. In the realm of multi-attribute decision-making, multi-granulation fuzzy rough sets based on overlap functions are established and used to handle decision-making problems [26,27]. Additionally, (I, O)-fuzzy rough sets based on overlap functions have been applied to feature selection and image edge extraction [28].

This paper addresses the problem of whether one overlap function can serve as a substitute for another. More specifically, we investigate whether replacing one overlap function with another that is highly similar will yield results that are also similar. To analyze this problem, it is necessary to establish an appropriate notion for the similarity of overlap functions. In this context, we define the degree of similarity between two overlap functions, which serves as a quantitative measure of their similarity. Building on this concept, we introduce the concept of δ -equality for the overlap functions, which allows us to evaluate “how close” two overlap functions are in a precise way. This leads to the question of whether operations on overlap functions are preserved under δ -equality. For example, if overlap function A is δ_1 -equal to overlap function B , and overlap function C is δ_2 -equal to overlap function D , how similar are the results of $A \circ C$ and $B \circ D$, where $\circ$ is some operation of the overlap functions? This paper explores these questions by investigating the relationships between similarity, δ -equality, and the preservation of operations on overlap functions.

It is noteworthy that the notion of the δ -equality between fuzzy sets has previously been established [29–31]. Expanding on this idea, Cai [32] explored the robustness of fuzzy reasoning by addressing the following question:

Problem 1. *Would replacing one fuzzy set with another highly similar fuzzy set yield similar reasoning results?*

Inspired by this line of inquiry, the present study extends this investigation into the domain of fuzzy reasoning with overlap functions. Specifically, our aim is to address the following question:

Problem 2. *If an overlap function was substituted with another closely overlap function, would the resulting reasoning outcomes also exhibit a high degree of similarity?*

Based on this idea, the robustness of fuzzy reasoning with overlap functions will be investigated. Note that the robustness of fuzzy reasoning have been extensively investigated by other scholars [32–38]. However, these studies focus on Problem 1, which generally involves evaluating the robustness of fuzzy reasoning when the input values are varied. In this paper, we focus on Problem 2, which offers a fresh perspective on the robustness of fuzzy reasoning.

This paper is organized as follows: Section 2 reviews some of the necessary concepts and presents several lemmas to be used in this paper. Section 3 introduces the similarity of the overlap functions. Section 4 studies the robustness of fuzzy reasoning based on the similarity of the overlap functions. Section 5 presents the conclusion of our research.

2. Preliminaries

This section recalls the concepts of overlap functions (see [6]), grouping functions (see [7]), and fuzzy implications (see [1]).

Definition 1 ([6]). *An overlap function is a function $\phi : [0, 1]^2 \rightarrow [0, 1]$ if for all $\xi, \eta \in [0, 1]$, it satisfies*

- ($\phi 1$) *Commutativity*;
- ($\phi 2$) *Continuity*;
- ($\phi 3$) *Monotonicity*;
- ($\phi 4$) $\phi(\xi, \varrho) = 0 \iff \xi \cdot \varrho = 0$;
- ($\phi 5$) $\phi(\xi, \varrho) = 1 \iff \xi = \varrho = 1$.

Denote with Φ the set of all of the overlap functions. Let $\phi_1, \phi_2 \in \Phi$, and the order of ϕ_1 and ϕ_2 is defined by

$$\phi_1 \leq \phi_2 \text{ if and only if } \phi_1(\xi, \varrho) \leq \phi_2(\xi, \varrho), \forall \xi, \varrho \in [0, 1]. \quad (1)$$

The meet and join operations for ϕ_1 and ϕ_2 are, respectively, defined by

$$(\phi_1 \vee \phi_2)(\xi, \varrho) = \max\{\phi_1(\xi, \varrho), \phi_2(\xi, \varrho)\}, \forall \xi, \varrho \in [0, 1], \quad (2)$$

$$(\phi_1 \wedge \phi_2)(\xi, \varrho) = \min\{\phi_1(\xi, \varrho), \phi_2(\xi, \varrho)\}, \forall \xi, \varrho \in [0, 1]. \quad (3)$$

Let $\phi_1, \phi_2, \dots, \phi_n \in \Phi$; their union and intersection are, respectively, defined by

$$\left(\bigcup_{i=1}^n \phi_i\right)(\xi, \varrho) = \sup_i \{\phi_i(\xi, \varrho)\}, \forall \xi, \varrho \in [0, 1] \quad (4)$$

$$\left(\bigcap_{i=1}^n \phi_i\right)(\xi, \varrho) = \inf_i \{\phi_i(\xi, \varrho)\}, \forall \xi, \varrho \in [0, 1]. \quad (5)$$

Their weighted sum is defined by

$$\left(\sum_{i=1}^n w_i \phi_i\right)(\xi, \varrho) = w_1 \phi_1(\xi, \varrho) + w_2 \phi_2(\xi, \varrho) + \dots + w_n \phi_n(\xi, \varrho), \forall \xi, \varrho \in [0, 1], \quad (6)$$

where $(w_i)_i^n \in [0, 1]^n$ and $\sum_{i=1}^n w_i = 1$.

Example 1. Some examples of overlap functions are

- $\phi_p(\xi, \varrho) = \xi^p \varrho^p$, where $p > 0$ (see Example 3.1 of [39]);
- $\phi_{mp}(\xi, \varrho) = \min(\xi^p, \varrho^p)$, where $p > 0$ (see Example 1 of [7] or Example 3.7 of [39]);
- $\phi_{Mp}(\xi, \varrho) = 1 - \max((1 - \xi)^p, (1 - \varrho)^p)$, where $p > 0$ (see Example 2.2 of [12]).

Definition 2 ([7]). A grouping function is a function $\Omega : [0, 1]^2 \rightarrow [0, 1]$ if, for all $\xi, \varrho \in [0, 1]$, it satisfies

- ($\Omega 1$) *Commutativity*;
- ($\Omega 2$) *Continuity*;
- ($\Omega 3$) *Monotonicity*;
- ($\Omega 4$) $\Omega(\xi, \varrho) = 0 \iff \xi = \varrho = 0$;
- ($\Omega 5$) $\Omega(\xi, \varrho) = 1 \iff \xi = 1 \text{ or } \varrho = 1$.

Example 2. Some examples of grouping functions are

- $\Omega_p(\xi, \varrho) = 1 - (1 - \xi)^p (1 - \varrho)^p$, where $p > 0$ (see Example 4.1 of [39]);
- $\Omega_{mp}(\xi, \varrho) = 1 - \min((1 - \xi)^p, (1 - \varrho)^p)$, where $p > 0$ (see Example 2.2 of [12]);
- $\Omega_{Mp}(\xi, \varrho) = \max(\xi^p, \varrho^p)$, where $p > 0$ (see Example 2 of [7] or Example 2.2 of [12]).

Definition 3 ([3]). A fuzzy implication is a function $\rightarrow: [0, 1]^2 \rightarrow [0, 1]$ such that it decreases in the first variable and increases in the second, and $1 \rightarrow 0 = 0, 0 \rightarrow 0 = 0 \rightarrow 1 = 1 \rightarrow 1 = 1$.

Denote the set of all fuzzy implications as $\mathcal{I}$.

The following lemmas will be used in the rest of this paper.

Lemma 1. Let $\xi_1, \xi_2, \dots, \xi_n \in [0, 1]$, $w = (w_i)_i^n \in [0, 1]^n$, and $\sum_{i=1}^n w_i = 1$; then,

$$\sum_{i=1}^n (w_i \xi_i) \leq \max_i \xi_i. \quad (7)$$

Proof. Let $r = \max_i \xi_i$; then,

$$\begin{aligned} \sum_{i=1}^n (w_i \xi_i) &\leq \sum_{i=1}^n (w_i r) \\ &= r \cdot \left(\sum_{i=1}^n w_i \right) \\ &= r. \end{aligned}$$

□

Lemma 2 ([29]). Let φ, χ be bounded, real-valued functions on a set A . Then,

$$\left| \sup_{\xi \in A} \varphi(\xi) - \sup_{\xi \in A} \chi(\xi) \right| \leq \sup_{\xi \in A} |\varphi(\xi) - \chi(\xi)|, \quad (8)$$

$$\left| \inf_{\xi \in A} \varphi(\xi) - \inf_{\xi \in A} \chi(\xi) \right| \leq \sup_{\xi \in A} |\varphi(\xi) - \chi(\xi)|. \quad (9)$$

Lemma 3. Let φ, χ be bounded, real-valued, bivariate functions on a set $A \times B$. Then,

$$\left| \sup_{(\xi, \varrho) \in A \times B} \varphi(\xi, \varrho) - \sup_{(\xi, \varrho) \in A \times B} \chi(\xi, \varrho) \right| \leq \sup_{(\xi, \varrho) \in A \times B} |\varphi(\xi, \varrho) - \chi(\xi, \varrho)|, \quad (10)$$

$$\left| \inf_{(\xi, \varrho) \in A \times B} \varphi(\xi, \varrho) - \inf_{(\xi, \varrho) \in A \times B} \chi(\xi, \varrho) \right| \leq \sup_{(\xi, \varrho) \in A \times B} |\varphi(\xi, \varrho) - \chi(\xi, \varrho)|. \quad (11)$$

Proof. Let $\sup_{(\xi, \varrho) \in A \times B} \varphi(\xi, \varrho) = r$ and $\sup_{(\xi, \varrho) \in A \times B} \chi(\xi, \varrho) = s$. In case I, $r = s$, the first inequality clearly holds. In case II, $r > s$, let $\epsilon > 0$ be given. Then, $(\xi_0, \varrho_0) \in A \times B$ applies such that $r - \epsilon < \varphi(\xi_0, \varrho_0)$. Since $\chi(\xi_0, \varrho_0) < s, r - s < \varphi(\xi_0, \varrho_0) - \chi(\xi_0, \varrho_0) + \epsilon$. Since ϵ is arbitrary, we have $r - s \leq \sup_{(\xi, \varrho) \in A \times B} |\varphi(\xi, \varrho) - \chi(\xi, \varrho)|$. Thus, the first inequality holds. For case III, $r < s$, the proof is similar to the proof for case II.

The second inequality can be obtained according to the fact that $\inf_{(\xi, \varrho) \in A \times B} \varphi(\xi, \varrho) = -\sup_{(\xi, \varrho) \in A \times B} (-\varphi(\xi, \varrho))$. □

3. Similarity of Overlap Functions

Initially, a definition for the similarity of n -order functions is provided.

Definition 4. Let $f_1, f_2: [0, 1]^n \rightarrow [0, 1]$ be two n -order functions, with $\mathbf{a} = (a_1, a_2, \dots, a_n) \in [0, 1]^n$; the degree of similarity $S(f_1, f_2)$ of f_1 and f_2 is defined by

$$S(f_1, f_2) = 1 - \sup_{\mathbf{a} \in [0, 1]^n} |f_1(\mathbf{a}) - f_2(\mathbf{a})|. \quad (12)$$

For $\delta \in [0, 1]$, we say that f_1 and f_2 are δ -equal if $S(f_1, f_2) \geq \delta$, denoted as $f_1 = (\delta)f_2$.

Subsequently, this notion is examined in more detail with respect to the overlap functions.

Definition 5. Let $\phi_1, \phi_2 \in \Phi$ be two overlap functions; the degree of similarity $S(\phi_1, \phi_2)$ of ϕ_1 and ϕ_2 is then defined by

$$S(\phi_1, \phi_2) = 1 - \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) - \phi_2(\xi, \varrho)|. \quad (13)$$

For $\delta \in [0, 1]$ we say that ϕ_1 and ϕ_2 are δ -equal ($\phi_1 = (\delta)\phi_2$ in symbols) if $S(\phi_1, \phi_2) \geq \delta$.

Example 3. Consider the overlap functions $\phi_1(a, b) = \min(a^2, b^2)$ and $\phi_2(a, b) = a^2 \cdot b^2$. As shown in Figure 1, the error function $E_1(a, b)$ between ϕ_1 and ϕ_2 is

$$E_1(a, b) = |\min(a^2, b^2) - a^2 \cdot b^2|. \quad (14)$$

Clearly, the degree of similarity $S(\phi_1, \phi_2)$ between ϕ_1 and ϕ_2 is

$$S(\phi_1, \phi_2) = 1 - \sup_{a, b \in [0,1]} E_1(a, b) = 0.75$$

when $a = b = \frac{\sqrt{2}}{2}$.

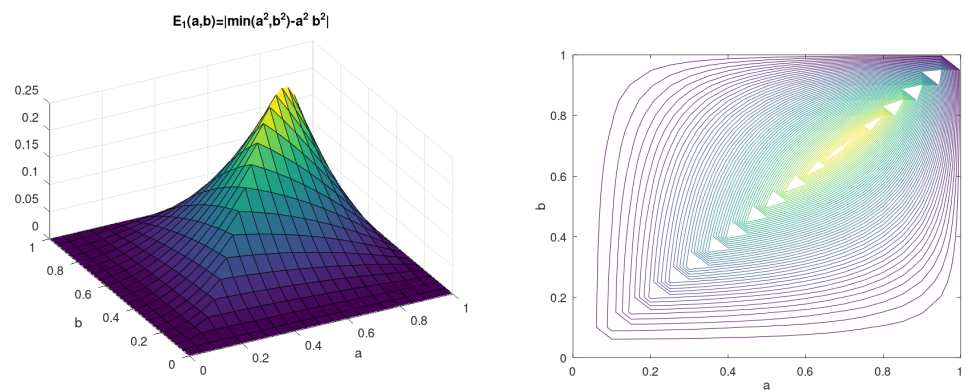


Figure 1. Characteristics of $E_1(a, b)$ and its contour line.

Example 4. Consider the overlap functions $\phi_1(a, b) = \min(a^2, b^2)$ and $\phi_2(a, b) = \min(a^3, b^3)$. As shown in Figure 2, the error function $E_2(a, b)$ between ϕ_1 and ϕ_2 is

$$E_2(a, b) = |\min(a^2, b^2) - \min(a^3, b^3)|. \quad (15)$$

Clearly, the degree of similarity $S(\phi_1, \phi_2)$ between ϕ_1 and ϕ_2 is

$$S(\phi_1, \phi_2) = 1 - \sup_{a, b \in [0,1]} E_2(a, b) = \frac{23}{27},$$

when $a = b = \frac{2}{3}$.

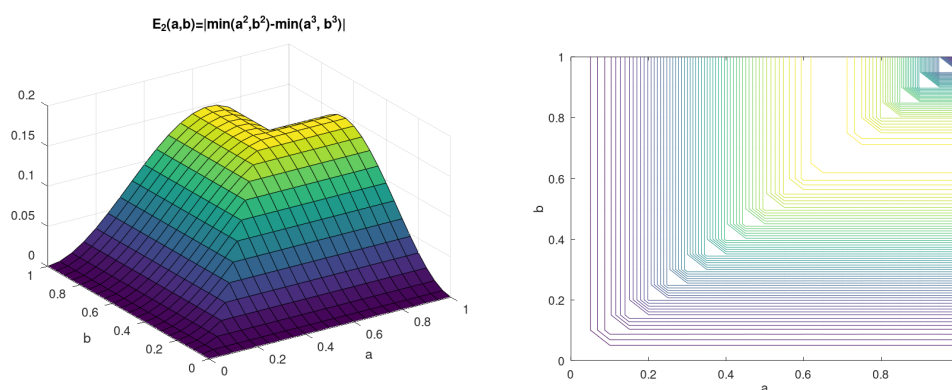


Figure 2. Characteristics of $E_2(a, b)$ and its contour line.

The similarity of the overlap functions has the following properties.

Theorem 1. For any overlap functions $\phi_1, \phi_2, \phi_3 \in \Phi$, the following hold:

- (1) $S(\phi_1, \phi_2) \in [0, 1]$;
- (2) $\phi_1 = \phi_2$ if and only if $S(\phi_1, \phi_2) = 1$;
- (3) If $\phi_1 \leq \phi_2 \leq \phi_3$, then $S(\phi_1, \phi_2) \geq S(\phi_1, \phi_3)$ and $S(\phi_2, \phi_3) \geq S(\phi_1, \phi_3)$;
- (4) $S(\phi_1, \phi_2) + S(\phi_2, \phi_3) - 1 \leq S(\phi_1, \phi_3)$.

Proof. (1)–(3) are obvious.

(4) Since

$$\begin{aligned} & \sup_{\xi, \varrho \in [0, 1]} |\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)| \\ &= \sup_{\xi, \varrho \in [0, 1]} |\phi_1(\xi, \varrho) - \phi_2(\xi, \varrho) + \phi_2(\xi, \varrho) - \phi_3(\xi, \varrho)| \\ &= \sup_{\xi, \varrho \in [0, 1]} (|\phi_1(\xi, \varrho) - \phi_2(\xi, \varrho)| + |\phi_2(\xi, \varrho) - \phi_3(\xi, \varrho)|) \\ &\leq \sup_{\xi, \varrho \in [0, 1]} |\phi_1(\xi, \varrho) - \phi_2(\xi, \varrho)| + \sup_{\xi, \varrho \in [0, 1]} |\phi_2(\xi, \varrho) - \phi_3(\xi, \varrho)|, \end{aligned}$$

then

$$\begin{aligned} & 1 - \sup_{\xi, \varrho \in [0, 1]} |\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)| \\ &\geq (1 - \sup_{\xi, \varrho \in [0, 1]} |\phi_1(\xi, \varrho) - \phi_2(\xi, \varrho)|) + (1 - \sup_{\xi, \varrho \in [0, 1]} |\phi_2(\xi, \varrho) - \phi_3(\xi, \varrho)|) - 1. \end{aligned}$$

Thus, $S(\phi_1, \phi_3) \geq S(\phi_1, \phi_2) + S(\phi_2, \phi_3) - 1$. $\square$

Remark 1. From the above Theorem 1(4),

$$\phi_1 = (\delta_1)\phi_2, \phi_2 = (\delta_2)\phi_3 \implies \phi_1 = (\max\{\delta_1 + \delta_2 - 1, 0\})\phi_3. \quad (16)$$

Theorem 2. For any overlap functions $\phi_1, \phi_2, \phi_3, \phi_4 \in \Phi$, it holds that

$$S(\phi_1 \vee \phi_2, \phi_3 \vee \phi_4) \geq \min\{S(\phi_1, \phi_3), S(\phi_2, \phi_4)\}. \quad (17)$$

Proof. According to Lemmas 2 and 3,

$$\begin{aligned} & \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) \vee \phi_2(\xi, \varrho) - \phi_3(\xi, \varrho) \vee \phi_4(\xi, \varrho)| \\ & \leq \sup_{\xi, \varrho \in [0,1]} (|\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)| \vee |\phi_2(\xi, \varrho) - \phi_4(\xi, \varrho)|) \\ & \leq \max \left\{ \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)|, \sup_{\xi, \varrho \in [0,1]} |\phi_2(\xi, \varrho) - \phi_4(\xi, \varrho)| \right\}, \end{aligned}$$

then

$$\begin{aligned} & 1 - \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) \vee \phi_2(\xi, \varrho) - \phi_3(\xi, \varrho) \vee \phi_4(\xi, \varrho)| \\ & \geq 1 - \max \left\{ \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)|, \sup_{\xi, \varrho \in [0,1]} |\phi_2(\xi, \varrho) - \phi_4(\xi, \varrho)| \right\} \\ & = \min \left\{ 1 - \sup_{\xi, \varrho \in [0,1]} |\phi_1(\xi, \varrho) - \phi_3(\xi, \varrho)|, 1 - \sup_{\xi, \varrho \in [0,1]} |\phi_2(\xi, \varrho) - \phi_4(\xi, \varrho)| \right\}. \end{aligned}$$

Thus, $S(\phi_1 \vee \phi_2, \phi_3 \vee \phi_4) \geq \min \{S(\phi_1, \phi_3), S(\phi_2, \phi_4)\}$. $\square$

Theorem 3. For any overlap functions $\phi_1, \phi_2, \phi_3, \phi_4 \in \Phi$, it holds that

$$S(\phi_1 \wedge \phi_2, \phi_3 \wedge \phi_4) \geq \min \{S(\phi_1, \phi_3), S(\phi_2, \phi_4)\}. \quad (18)$$

Proof. It is similar to Theorem 2. $\square$

Remark 2. The two theorems above demonstrate that

$$\phi_1 = (\delta)\phi_3, \phi_2 = (\delta)\phi_4 \implies (\phi_1 \wedge \phi_2) = (\delta)(\phi_3 \wedge \phi_4), \quad (19)$$

$$\phi_1 = (\delta)\phi_3, \phi_2 = (\delta)\phi_4 \implies (\phi_1 \vee \phi_2) = (\delta)(\phi_3 \vee \phi_4). \quad (20)$$

Theorem 4. For any overlap functions $\phi_{11}, \phi_{12}, \dots, \phi_{1n}, \phi_{21}, \phi_{22}, \dots, \phi_{2n} \in \Phi$, it holds that

$$S\left(\bigcap_{i=1}^n \phi_{1i}, \bigcap_{i=1}^n \phi_{2i}\right) \geq \inf_i \{S(\phi_{1i}, \phi_{2i})\}, \quad (21)$$

$$S\left(\bigcup_{i=1}^n \phi_{1i}, \bigcup_{i=1}^n \phi_{2i}\right) \geq \inf_i \{S(\phi_{1i}, \phi_{2i})\}. \quad (22)$$

Proof. They are derived from theorems 2 and 3 and mathematical induction. $\square$

Theorem 5. Let $\phi_1, \phi_2 \in \Phi$. If Ω_1 and Ω_2 , respectively, are the dual grouping functions of ϕ_1 and ϕ_2 , i.e.,

$$\Omega_1(\xi, \varrho) = 1 - \phi_1(1 - \xi, 1 - \varrho), \quad (23)$$

$$\Omega_2(\xi, \varrho) = 1 - \phi_2(1 - \xi, 1 - \varrho), \quad (24)$$

then

$$S(\phi_1, \phi_2) = S(\Omega_1, \Omega_2). \quad (25)$$

Proof. First, according to the definitions of Ω_1 and Ω_2 ,

$$\begin{aligned} & \sup_{\xi, \varrho \in [0,1]} |\Omega_1(\xi, \varrho) - \Omega_2(\xi, \varrho)| \\ &= \sup_{\xi, \varrho \in [0,1]} (|(1 - \phi_1(1 - \xi, 1 - \varrho)) - (1 - \phi_2(1 - \xi, 1 - \varrho))|) \\ &= \sup_{\xi, \varrho \in [0,1]} (|\phi_1(1 - \xi, 1 - \varrho) - \phi_2(1 - \xi, 1 - \varrho)|). \end{aligned}$$

As the range of both (ξ, ϱ) is $[0, 1]^2$, the range of $(1 - \xi, 1 - \varrho)$ is also $[0, 1]^2$. Letting $1 - \xi = s, 1 - \varrho = t$, then

$$\sup_{\xi, \varrho \in [0,1]} (|\phi_1(1 - \xi, 1 - \varrho) - \phi_2(1 - \xi, 1 - \varrho)|) = \sup_{s, t \in [0,1]} (|\phi_1(s, t) - \phi_2(s, t)|).$$

Thus $S(\phi_1, \phi_2) = S(\Omega_1, \Omega_2)$. $\square$

Example 5. Consider the grouping functions $\Omega_1(a, b) = 1 - \min\{(1 - a)^2, (1 - b)^2\}$ and $\Omega_2(a, b) = 1 - (1 - a)^2(1 - b)^2$. Ω_1 and Ω_2 , respectively, are the dual grouping functions of $\phi_1(a, b) = \min(a^2, b^2)$ and $\phi_2 = a^2 \cdot b^2$ in Example 3.

$$S(\Omega_1, \Omega_2) = 1 - \sup_{a, b \in [0,1]} |\min((1 - a)^2, (1 - b)^2) - (1 - a)^2 \cdot (1 - b)^2| = 0.75,$$

when $a = b = 1 - \frac{\sqrt{2}}{2}$. Note that from Example 3, $S(\phi_1, \phi_2) = 0.75$. Thus, $S(\Omega_1, \Omega_2) = S(\phi_1, \phi_2)$.

Theorem 6. Let $\phi_1, \phi_2 \in \Phi$ be two overlap functions, $0 < k < +\infty$, if two functions are defined by

$$\phi_1^k(\xi, \varrho) = \phi_1(\xi^k, \varrho^k), \quad (26)$$

$$\phi_2^k(\xi, \varrho) = \phi_2(\xi^k, \varrho^k). \quad (27)$$

Then,

$$S(\phi_1, \phi_2) = S(\phi_1^k, \phi_2^k). \quad (28)$$

Proof. As the range of (ξ, ϱ) is $[0, 1]^2$, the range of (ξ^k, ϱ^k) is also $[0, 1]^2$. Letting $\xi^k = s, \varrho^k = t$, then

$$\begin{aligned} & \sup_{\xi, \varrho \in [0,1]} |\phi_1^k(\xi, \varrho) - \phi_2^k(\xi, \varrho)| \\ &= \sup_{\xi, \varrho \in [0,1]} (|\phi_1(\xi^k, \varrho^k) - \phi_2(\xi^k, \varrho^k)|) \\ &= \sup_{s, t \in [0,1]} (|\phi_1(s, t) - \phi_2(s, t)|). \end{aligned}$$

Thus, $S(\phi_1, \phi_2) = S(\phi_1^k, \phi_2^k)$. $\square$

Theorem 7. For any overlap functions $\phi_{11}, \phi_{12}, \dots, \phi_{1n}, \phi_{21}, \phi_{22}, \dots, \phi_{2n} \in \Phi$, it holds that

$$S\left(\sum_{i=1}^n w_i \phi_{1i}, \sum_{i=1}^n w_i \phi_{2i}\right) \geq \min_i \{S(\phi_{1i}, \phi_{2i})\}. \quad (29)$$

where $(w_i)_i^n \in [0, 1]^n$ and $\sum_{i=1}^n w_i = 1$.

Proof. According to Lemma 1,

$$\begin{aligned} & \sup_{\xi, \varrho \in [0,1]} \left| \sum_{i=1}^n w_i \phi_{1i}(\xi, \varrho) - \sum_{i=1}^n w_i \phi_{2i}(\xi, \varrho) \right| \\ &= \sup_{\xi, \varrho \in [0,1]} \left| \sum_{i=1}^n w_i (\phi_{1i}(\xi, \varrho) - \phi_{2i}(\xi, \varrho)) \right| \\ &\leq \sup_{\xi, \varrho \in [0,1]} \left| \max_i (\phi_{1i}(\xi, \varrho) - \phi_{2i}(\xi, \varrho)) \right| \\ &\leq \max_i \left(\sup_{\xi, \varrho \in [0,1]} |\phi_{1i}(\xi, \varrho) - \phi_{2i}(\xi, \varrho)| \right), \end{aligned}$$

then

$$\begin{aligned} & 1 - \sup_{\xi, \varrho \in [0,1]} \left| \sum_{i=1}^n w_i \phi_{1i}(\xi, \varrho) - \sum_{i=1}^n w_i \phi_{2i}(\xi, \varrho) \right| \\ &\geq 1 - \max_i \left(\sup_{\xi, \varrho \in [0,1]} |\phi_{1i}(\xi, \varrho) - \phi_{2i}(\xi, \varrho)| \right) \\ &= \min_i \left(1 - \sup_{\xi, \varrho \in [0,1]} |\phi_{1i}(\xi, \varrho) - \phi_{2i}(\xi, \varrho)| \right). \end{aligned}$$

Thus, $S\left(\sum_{i=1}^n w_i \phi_{1i}, \sum_{i=1}^n w_i \phi_{2i}\right) \geq \min_i \{S(\phi_{1i}, \phi_{2i})\}$. $\square$

Example 6. It is easy to verify that $S(\phi_1, \phi_2) = 0.5$ for $\phi_1(a, b) = \min(a, b)$ and $\phi_2(a, b) = ab$. And from Example 3, $S(\phi_3, \phi_4) = 0.75$ for $\phi_3(a, b) = \min(a^2, b^2)$ and $\phi_4(a, b) = a^2 \cdot b^2$.

Now, considering the overlap functions

$$\phi_5(a, b) = \frac{\phi_1(a, b) + \phi_3(a, b)}{2} = \frac{\min(a, b) + \min(a^2, b^2)}{2} \quad (30)$$

$$\phi_6(a, b) = \frac{\phi_2(a, b) + \phi_4(a, b)}{2} = \frac{ab + a^2 \cdot b^2}{2} \quad (31)$$

As shown in Figure 3, the error function $E_3(a, b)$ between ϕ_5 and ϕ_6 is

$$E_4(a, b) = \left| \frac{\min(a^2, b^2) + \min(a, b) - (a^2 \cdot b^2 + a \cdot b)}{2} \right|. \quad (32)$$

Clearly, the degree of similarity $S(\phi_5, \phi_6)$ between ϕ_5 and ϕ_6 is

$$S(\phi_5, \phi_6) = 1 - \sup_{a, b \in [0,1]} E_4(a, b) = \frac{1}{2\sqrt[3]{4}} - \frac{1}{2\sqrt[3]{4^4}} \approx 0.7637$$

when $a = b = \frac{1}{\sqrt[3]{4}}$.

Thus, it is easy to verify that $S(\phi_5, \phi_6) \geq \min\{S(\phi_1, \phi_2), S(\phi_3, \phi_4)\}$.

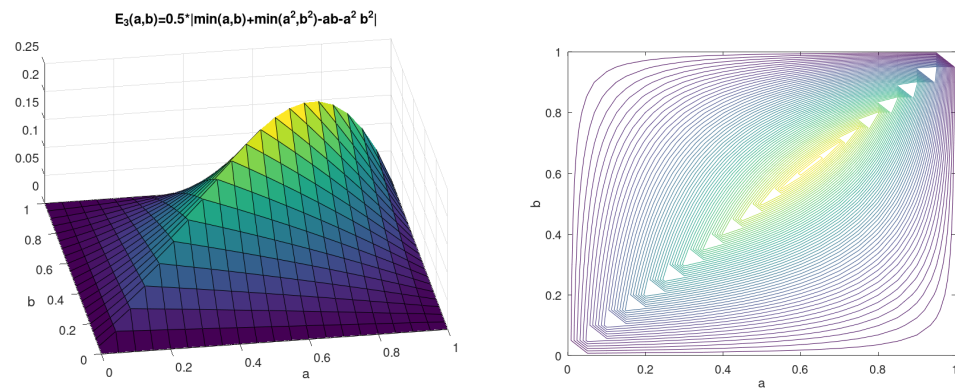


Figure 3. Characteristics of $E_3(a, b)$ and its contour line.

4. Robustness of Fuzzy Reasoning with Overlap Functions

4.1. Fuzzy Modus Ponens

Fuzzy modus ponens (FMP) is a basic inference rule in fuzzy reasoning. It states the following:

Premise 1: If τ is T , then λ is Λ ;

Premise 2: τ is T^* ;

Conclusion: λ is Λ^* .

Here, τ, λ are objects; T, T^* are fuzzy sets on the universe X ; and Λ, Λ^* are the fuzzy sets on universe Y . Zadeh introduced the compositional rule of inference (CRI) for executing FMP. The CRI solution for the FMP model is presented as $\Lambda^* = T^* \star (T \rightarrow \Lambda)$, or, more precisely, as

$$\Lambda^*(\lambda) = \sup_{\tau \in X} (T^*(\tau) \star (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \lambda \in Y, \quad (33)$$

where $\star$ is a t-norm and $\rightarrow \in \mathcal{I}$.

We study the CRI method based on the overlap functions (O-CRI for short). The O-CRI solution for the FMP model is presented as $\Lambda^* = \phi(T^*, T \rightarrow \Lambda)$, or, more precisely,

$$\Lambda^*(\lambda) = \sup_{\tau \in X} \phi(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \lambda \in Y, \quad (34)$$

where $\phi \in \Phi$ and $\rightarrow \in \mathcal{I}$.

In this section, the following problem is discussed: if the overlap function ϕ was substituted with another closely overlap function ϕ' in the above fuzzy reasoning method, would the resulting reasoning outcomes also exhibit a high degree of similarity?

The problem can be formally described as follows: Suppose $\phi_1 = (\delta)\phi_2$; then, what is the relationship between Λ_1^* and Λ_2^* , where Λ_i^* is the O-CRI solution of the FMP model $\phi_i(T^*, T \rightarrow \Lambda) \Rightarrow \Lambda_i^*$ for $i = 1, 2$?

Theorem 8. Suppose that $(\phi_1, \rightarrow)$ and $(\phi_2, \rightarrow)$ are used in FMP, respectively, that is,

$$\Lambda^*(\lambda) = \sup_{\tau \in X} \phi_1(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \lambda \in Y. \quad (35)$$

$$\Lambda'^*(\lambda) = \sup_{\tau \in X} \phi_2(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \lambda \in Y. \quad (36)$$

If $S(\phi_1, \phi_2) \geq \delta$ then $S(\Lambda^*, \Lambda'^*) \geq \delta$.

Proof. According to Lemma 2,

$$\begin{aligned} & \sup_{\lambda \in Y} |\Lambda^*(\lambda) - \Lambda^{*'}(\lambda)| \\ &= \sup_{\lambda \in Y} \left| \sup_{\tau \in X} \phi_1(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))) - \sup_{\tau \in X} \phi_2(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))) \right| \\ &\leq \sup_{\lambda \in Y} \sup_{\tau \in X} \left| \phi_1(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))) - \phi_2(T^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))) \right| \\ &\leq 1 - \delta. \end{aligned}$$

$$\text{Thus, } S(\Lambda^*, \Lambda^{*'}) = 1 - \sup_{\lambda \in Y} |\Lambda^*(\lambda) - \Lambda^{*'}(\lambda)| \geq \delta. \quad \square$$

From the above theorem, if we replace ϕ_1 with ϕ_2 such that $\phi_1 = (\delta)\phi_2$, then the corresponding two results $\Lambda^*, \Lambda^{*'}$ are still δ -equal.

Remark 3. The O-CRI method for the FMP model includes two operators, the overlap function ϕ and the fuzzy implication $\rightarrow$, and three data elements: the input T , the rule antecedent T^* , and rule consequent Λ . We focus only on the impact of slight perturbations in the overlap function ϕ on the inference results.

Example 7. Suppose that $X = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$, $Y = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, T, T^* are fuzzy sets on X , and Λ, Λ^* are fuzzy sets on Y . T, T^* and Λ are given as follows:

$$T = \frac{0.6}{\tau_1} + \frac{0.6}{\tau_2} + \frac{0.6}{\tau_3} + \frac{0.8}{\tau_4} + \frac{0.9}{\tau_5}, \quad (37)$$

$$T^* = \frac{0.8}{\tau_1} + \frac{0.8}{\tau_2} + \frac{0.9}{\tau_3} + \frac{0.9}{\tau_4} + \frac{0.9}{\tau_5}, \quad (38)$$

$$\Lambda = \frac{0.7}{\lambda_1} + \frac{0.6}{\lambda_2} + \frac{0.8}{\lambda_3} + \frac{0.9}{\lambda_4} + \frac{0.9}{\lambda_5}. \quad (39)$$

If $\phi_1(\xi, \varrho) = \min(\xi^2, \varrho^2)$ and $\xi \rightarrow \varrho = \min(1 - \xi + \varrho, 1)$ is used in FMP, then the conclusion Λ^* is

$$\Lambda^* = \frac{0.81}{\lambda_1} + \frac{0.81}{\lambda_2} + \frac{0.81}{\lambda_3} + \frac{0.81}{\lambda_4} + \frac{0.81}{\lambda_5}. \quad (40)$$

If $\phi_2(\xi, \varrho) = \min(\xi^3, \varrho^3)$ and $\xi \rightarrow \varrho = \min(1 - \xi + \varrho, 1)$ is used in FMP, then the conclusion $\Lambda^{*'}$ is

$$\Lambda^{*'} = \frac{0.729}{\lambda_1} + \frac{0.729}{\lambda_2} + \frac{0.729}{\lambda_3} + \frac{0.729}{\lambda_4} + \frac{0.729}{\lambda_5}. \quad (41)$$

Note that in Example 4, $S(\phi_1, \phi_2) = \frac{23}{27}$. From the above data, it holds that $S(\Lambda^*, \Lambda^{*'}) = 0.919 > \frac{23}{27}$.

4.2. Fuzzy Modus Tollens

Fuzzy modus tollens (FMT) is another basic inference rule in fuzzy reasoning. It states the following:

Premise 1: If τ is T , then λ is Λ ;

Premise 2: λ is Λ^* ;

Conclusion: τ is T^* .

Here, τ, λ are objects, T, T^* are fuzzy sets on X , and Λ, Λ^* are fuzzy sets on Y .

The O-CRI solution for the FMT model is presented as $T^* = \phi(\Lambda^*, T \rightarrow \Lambda)$ or more precisely as

$$T^*(\tau) = \sup_{\lambda \in Y} \phi(\Lambda^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \tau \in X, \quad (42)$$

where $\phi \in \Phi$ and $\rightarrow \in \mathcal{I}$.

The following problem is discussed: Suppose $\phi_1 = (\delta)\phi_2$; then, what is the relationship between T_1^* and T_2^* , where T_i^* is the O-CRI solution of the FMT model $\phi_i(\Lambda^*, T \rightarrow \Lambda) \Rightarrow T_i^*$ for $i = 1, 2$?

Theorem 9. Suppose that $(\phi_1, \rightarrow)$ and $(\phi_2, \rightarrow)$ are used in FMT, respectively, that is,

$$T^*(\tau) = \sup_{\lambda \in Y} \phi_1(\Lambda^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \tau \in X. \quad (43)$$

$$T^{*'}(\tau) = \sup_{\lambda \in Y} \phi_2(\Lambda^*(\tau), (T(\tau) \rightarrow \Lambda(\lambda))), \quad \forall \tau \in X. \quad (44)$$

If $S(\phi_1, \phi_2) \geq \delta$ then $S(T^*, T^{*'}) \geq \delta$.

Proof. It is similar to Theorem 8. $\square$

4.3. Fuzzy Hypothetical Syllogism

Fuzzy hypothetical syllogism (FHS) is also a basic inference rule in fuzzy reasoning. It states the following:

Premise 1: If τ is T , then λ is Λ ;

Premise 2: If λ is Λ^* , then z is E ;

Conclusion: τ is T^* , then z is E^* .

where τ, λ, z are objects; T, T^* is a fuzzy set on X ; Λ, Λ^* is a fuzzy set on Y ; and E, E^* is a fuzzy set on Z . The O-CRI solution for the FHS model is presented as $R^* = \phi(T \rightarrow \Lambda, \Lambda^* \rightarrow E)$, or, more precisely, as

$$R(\tau, z) = \sup_{\lambda \in Y} \phi(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)), \quad \forall \tau \in X, z \in Z, \quad (45)$$

where $\phi \in \Phi$ and $\rightarrow \in \mathcal{I}$.

The following problem is discussed: Suppose $\phi_1 = (\delta)\phi_2$; then, what is the relationship between R_1^* and R_2^* , where R_i^* is the O-CRI solution of the FHS model $\phi_i(T \rightarrow \Lambda, \Lambda^* \rightarrow E) \Rightarrow R_i^*$ for $i = 1, 2$?

Theorem 10. Suppose that $(\phi_1, \rightarrow)$ and $(\phi_2, \rightarrow)$ are used in FHS, respectively, that is,

$$R(\tau, z) = \sup_{\lambda \in Y} \phi_1(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)), \quad \forall \tau \in X, z \in Z. \quad (46)$$

$$R'(\tau, z) = \sup_{\lambda \in Y} \phi_2(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)), \quad \forall \tau \in X, z \in Z. \quad (47)$$

If $S(\phi_1, \phi_2) \geq \delta$, then $S(R, R') \geq \delta$.

Proof. According to Lemmas 2 and 3,

$$\begin{aligned}
& \sup_{(\tau, z) \in X \times Z} |R(\tau, z) - R'(\tau, z)| \\
&= \sup_{(\tau, z) \in X \times Z} \left| \sup_{\lambda \in Y} \phi_1(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)) - \sup_{\lambda \in Y} \phi_2(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)) \right| \\
&\leq \sup_{(\tau, z) \in X \times Z} \sup_{\lambda \in Y} |\phi_1(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z)) - \phi_2(T(\tau) \rightarrow \Lambda(\lambda), \Lambda^*(\lambda) \rightarrow E(z))| \\
&\leq 1 - \delta.
\end{aligned}$$

$$\text{Thus, } S(R, R') = 1 - \sup_{(\tau, z) \in X \times Z} |R(\tau, z) - R'(\tau, z)| \geq \delta. \quad \square$$

Example 8. Suppose that $X = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$, $Y = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, $Z = (z_1, z_2, z_3, z_4, z_5)$, T, T^* are fuzzy sets on X , and Λ, Λ^* are fuzzy sets on Y . T, Λ, Λ^* , and E are given as follows:

$$T = \frac{0.6}{\tau_1} + \frac{0.6}{\tau_2} + \frac{0.6}{\tau_3} + \frac{0.8}{\tau_4} + \frac{0.9}{\tau_5}, \quad (48)$$

$$\Lambda = \frac{0.7}{\lambda_1} + \frac{0.6}{\lambda_2} + \frac{0.8}{\lambda_3} + \frac{0.9}{\lambda_4} + \frac{0.9}{\lambda_5}, \quad (49)$$

$$\Lambda^* = \frac{0.6}{\lambda_1} + \frac{0.6}{\lambda_2} + \frac{0.6}{\lambda_3} + \frac{0.9}{\lambda_4} + \frac{0.9}{\lambda_5}, \quad (50)$$

$$E = \frac{0.7}{z_1} + \frac{0.6}{z_2} + \frac{0.8}{z_3} + \frac{0.9}{z_4} + \frac{0.9}{z_5}. \quad (51)$$

If $\phi_1(\xi, \varrho) = \min(\xi^2, \varrho^2)$ and $\xi \rightarrow \varrho = \min(1 - \xi + \varrho, 1)$ is used in FMP, then the conclusion R is

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0.81 & 0.81 & 1 & 1 & 1 \end{bmatrix}.$$

If $\phi_2(\xi, \varrho) = \min(\xi^3, \varrho^3)$ and $\xi \rightarrow \varrho = \min(1 - \xi + \varrho, 1)$ is used in FMP, then the conclusion R' is

$$R' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0.729 & 0.729 & 1 & 1 & 1 \end{bmatrix}.$$

Note that in Example 4, $S(\phi_1, \phi_2) = \frac{23}{27}$. From the above data, $S(R, R') = 0.919 > \frac{23}{27}$.

5. Conclusions

The degree of similarity, denoted as $S(\phi_1, \phi_2)$, provides a quantifiable index of the resemblance between two overlap functions, ϕ_1 and ϕ_2 . This pivotal measurement provides the foundation for our comprehensive investigation into the response of fundamental operations, including meet, join, and weighted sum, on overlap functions in relation to their similarity. Our analysis demonstrated that the behavior of these operations maintains a remarkable level of similarity. Moreover, we discussed the robustness of fuzzy reasoning in terms of the similarity of the overlap functions. This study demonstrated that the O-CRI method for FMP, FMT, and FHS models maintains similarity when the overlap function is replaced with a closely overlap function, thereby exhibiting strong robustness.

This study on the robustness of fuzzy reasoning focuses solely on minor perturbations in the overlap function while keeping the implication operator unchanged, which presents

a limitation. Future research should explore more complex scenarios, such as when both the overlap function and the implication operator, as well as the rules, undergo changes, to assess the robustness of the fuzzy reasoning method.

For a robustness analysis, it is crucial to incorporate real-world data that include natural noise and variability, thereby validating the robustness of fuzzy reasoning methods. This is a useful avenue for future research.

Assessing the suitability of fuzzy reasoning, robustness is just one of the factors to consider. There are other aspects to consider, such as the logical foundation. Fuzzy reasoning should be consistent with fuzzy logic. It might be worthwhile to establish a sound logical basis for fuzzy reasoning that is grounded in overlap functions.

When evaluating the appropriateness of fuzzy reasoning for various applications, robustness emerges as a critical consideration, yet it is just one of the relevant factors. Beyond this, there are other aspects to consider, such as the logical foundation. The coherence between reasoning and logic cannot be overstated. Therefore, it is essential to consider the development of a sound logical framework for overlap-function-based fuzzy reasoning. This foundational work would not only increase the reliability of fuzzy reasoning but also provide a robust theoretical basis for its practical applications.

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