

Mappings of a Bounded Dirichlet Integral: The Modulus Method

Mihai Cristea

Faculty of Mathematics and Computer Sciences, University of Bucharest, Str. Academiei 14,
R-010014 Bucharest, Romania; mcristea@fmi.unibuc.ro

Abstract: We study the geometric properties of some classes of mappings for which an inverse Poletsky modular inequality holds. In these classes of mappings, we give some extensions of the theorems of Lindelöf and Fatou from the classical complex analysis. We also find some conditions for the existence of injective minimizers for mappings of biconformal energy.

Keywords: generalizations of quasiregular mappings; Riemannian manifolds; mappings of finite conformal energy; mappings of finite distortion

MSC: 30C65; 35J25

1. Introduction

Throughout this paper, X, Y are locally compact and locally connected metric measure spaces endowed with Borel regular measures μ and ν , which are finite on compact sets. Also, X is Ahlfors n -regular and ν is doubling and $D \subset X$ is an open subset of X . We set by d the distance on X and Y . If $f : D \rightarrow Y$ is a mapping and $x \in D$, we set

$$J_f(x) = \limsup_{r \rightarrow 0} \frac{\nu(f(B(x, r)))}{\mu(B(x, r))},$$

and we set

$$l(x, f) = \liminf_{y \rightarrow x} \frac{d(f(y), f(x))}{d(x, y)} \text{ and } L(x, f) = \limsup_{y \rightarrow x} \frac{d(f(y), f(x))}{d(y, x)}.$$

We say that $f : D \rightarrow Y$ is of finite conformal energy if $\int_D L(x, f)^n d\mu < \infty$. If $p > 1$ and $D \subset \mathbb{R}^n$ is open, we set $W_{loc}^{1,p}(D, \mathbb{R}^n)$, the Sobolev space of all mappings $f : D \rightarrow \mathbb{R}^n$, which are locally in L^p , together with their first-order distributional derivatives. Usually, if $X = \mathbb{R}^n$, a mapping $f : D \rightarrow \mathbb{R}^n$ is of finite conformal energy (or of a bounded Dirichlet integral) if $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$ and $\int_D |f'(x)|^n dx < \infty$. If f is a.e. differentiable, then $L(x, f) = |f'(x)|$ a.e. and both definitions coincide in this case. The following example shows that our definition is more general, since we find a mapping $f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a.e. differentiable such that $\int_\Omega |f'(x, y, z)|^3 d\mu_3 < \infty$ and $f \notin W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$.

We denote by μ_n the Lebesgue measure in $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is Lebesgue measurable and $f : A \rightarrow \mathbb{R}^n$ is Lebesgue integrable, we set $\int_A f d\mu_n = \int_A f(x) dx$.

Example 1. Let $E \subset [0, 1]$ be a Cantor set such that $\mu_1(E) > 0$ and let $g : [0, 1] \rightarrow [0, \mu_1(\mathbb{C}E)]$ be given by $g(x) = \int_0^x \chi_{\mathbb{C}E}(t) dt$ for every $x \in [0, 1]$. We see that g is absolutely continuous and



Academic Editor: Nak Eun Cho

Received: 10 November 2024

Revised: 22 December 2024

Accepted: 25 December 2024

Published: 31 December 2024

Citation: Cristea, M. Mappings of a Bounded Dirichlet Integral: The Modulus Method. *Axioms* **2025**, *14*, 28. <https://doi.org/10.3390/axioms14010028>

Copyright: © 2024 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

strictly increasing and this inverse $h : [0, \mu_1(\mathbb{C}E)] \rightarrow [0, 1]$ is not absolutely continuous, is a.e. differentiable and $h'(t) = 1$ a.e. Let $f : B(0, 1) \rightarrow \mathbb{C}$, $f(z) = z^n$ for every $z \in B(0, 1)$, suppose that $n \geq 2$ and let $\Omega = (0, \mu_1(\mathbb{C}E)) \times B(0, 1)$ and $F : \Omega \rightarrow \mathbb{R}^3$, $F(x_1, x_2, x_3) = (h(x_1), f(x_2, x_3))$. Then, $\mu_3(B_F) = 0$, $\mu_3(F(B_F)) = 0$, $F \notin W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$, $\int_{\Omega} |F'(x, y, z)|^3 d\mu_3 < \infty$ and F satisfies condition (N^{-1}) .

Here, if $f : D \rightarrow Y$ is a mapping, we set $B_f = \{x \in D \mid f \text{ is not a local homeomorphism at } x\}$ and we say that f satisfies condition (N) if $\nu(f(A)) = 0$ whenever $A \subset X$ is such that $\mu(A) = 0$ and we say that f satisfies condition (N^{-1}) if $\mu(f^{-1}(A)) = 0$ whenever $A \subset Y$ is such that $\nu(A) = 0$.

Let $D \subset X$. Let $A(D)$ be the set of all nonconstant path families in D and if $\Gamma \in A(D)$, and we set $F(\Gamma) = \{\rho : D \rightarrow [0, \infty] \text{ Borel functions} \mid \int_{\gamma} \rho ds \geq 1 \text{ for every } \gamma \in \Gamma \text{ locally rectifiable}\}$.

If $p > 1$ and $\omega : D \rightarrow [0, \infty]$ is μ measurable and finite a.e., we define the p -modulus of weight ω by

$$M_{\omega}^p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \omega(x) \rho(x)^p d\mu \text{ if } \Gamma \in A(D).$$

If $F(\Gamma) = \emptyset$, we set $M_{\omega}^p(\Gamma) = 0$. If $\omega = 1$, we set

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho(x)^p d\mu \text{ if } \Gamma \in A(D).$$

One of the basic tools in studying quasiregular mappings is the modular inequality of Poletsky

$$M_n(f(\Gamma)) \leq KM_n(\Gamma) \text{ for every } \Gamma \in A(D). \quad (1)$$

We recommend the monographs [1–4] to the reader for more information about quasiregular mappings.

Several generalizations of quasiregular mappings were introduced in the last 30 years. The most important is the class of mappings of finite distortion (see monographs [5,6] for further information), and a Poletsky modular inequality holds in this class of mappings (see [7,8] or [9]). A mapping $f : D \rightarrow \mathbb{R}^n$ is of finite distortion if $n \geq 2$, $D \subset \mathbb{R}^n$ is open, $f \in W_{loc}^{1,1}(D, \mathbb{R}^n)$, $J_f \in L_{loc}^1(D)$ and there exists $K : D \rightarrow [0, \infty]$ measurable and finite a.e. such that $|f'(x)|^n \leq K(x)J_f(x)$ a.e. and we set the outer dilatation

$$K_0(x, f) = \frac{|f'(x)|^n}{J_f(x)} \text{ if } J_f(x) \neq 0, K_0(x, f) = 1 \text{ if } J_f(x) = 0.$$

Here, $f'(x)$ is the matrix formed by the partial derivatives of f at x and $J_f(x) = \det f'(x)$ for a.e. $x \in D$ and we use this definition of the Jacobian J_f for mappings of finite distortion.

Martio proposed the study of mappings distinguished by moduli inequalities, first on $\mathbb{R}^n$ and then on metric measure spaces. Such mappings satisfy a generalized Poletsky modular inequality of the type

$$M_q(f(\Gamma)) \leq M_{\omega}^p(\Gamma) \text{ for every } \Gamma \in A(D). \quad (2)$$

In this class of mappings, Montel- and Picard-type theorems, boundary extension results, equicontinuity results and estimates of the modulus of continuity were given (see [7,8,10–27]) and we recommend the monographs [22,28] to the reader for further information.

If $f : D \rightarrow Y$ is a mapping and $L(x, f, r) = \sup_{y \in S(x, r)} d(f(x), f(y))$, $l(x, f, r) = \inf_{y \in S(x, r)} d(f(x), f(y))$ for $x \in X$ and $r > 0$, the linear dilatation $H(\cdot, f) : D \rightarrow [0, \infty]$ is defined by $H(x, f) = \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{l(x, f, r)}$ for every $x \in D$.

A mapping $f : D \rightarrow Y$ between general metric measure spaces is quasiregular if it is continuous, open and discrete, and there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in D$. This is the metric definition of quasiregularity (see [29–39]). In some particular cases, a Poletsky modular inequality holds for quasiregular mappings defined on general metric measure spaces.

As in the classical case, even on very general metric measure spaces X and Y , a mapping $f : D \rightarrow Y$ which is continuous, open and discrete is quasiregular if and only if it is geometrically quasiregular, i.e., $M_n(\Gamma) \leq K \int_Y N(y, f, G) \rho^n(y) dv$ for every open set $G \subset\subset D$, every $\Gamma \in A(G)$ and every $\rho \in F(f(\Gamma))$. This important property inspired the study of some classes of mappings satisfying a generalized inverse Poletsky modular inequality (see [12,28,40]), like

$$M_p(\Gamma) \leq \gamma_G(M_{\omega_G}^q(f(\Gamma))) \text{ for every } \Gamma \in A(D) \text{ and every } G \subset\subset D. \quad (3)$$

Here, $p, q > 1$, $f : D \rightarrow Y$ is continuous, open and discrete, $\omega_G : Y \rightarrow [0, \infty]$ is ν measurable and finite a.e. and $\gamma_G : (0, \infty) \rightarrow (0, \infty)$ is increasing and $\lim_{t \rightarrow 0} \gamma_G(t) = 0$ for every $G \subset\subset D$.

The class of mappings satisfying a relation of type (3) is strictly larger than the class of quasiregular mappings, where relation (3) holds for a particular weight $\omega_G(y) = N(y, f, G)$ for every $G \subset\subset D$ and every $y \in Y$. This thing is shown by the function $F : \Omega \rightarrow \mathbb{R}^3$ from Example 1. We see that $F \notin W_{loc}^{1,1}(\Omega, \mathbb{R}^3)$ (and hence F is not quasiregular), and since $\int_{\Omega} |F'(x)|^3 dx < \infty$, we see from Lemma 1 that there exists $\omega \in L^1(\mathbb{R}^3)$ such that $M_3(\Gamma) \leq M_{\omega}^3(F(\Gamma))$ for every $\Gamma \in A(\Omega)$. In fact, F is not a mapping of finite distortion.

The following class of mappings of finite conformal energy satisfies an inverse Poletsky modular inequality:

Theorem 1. Let $n \geq 2$, $q > 1$, let $f : D \rightarrow Y$ be continuous such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$, f satisfies condition (N^{-1}) and $\int_D L(z, f)^q d\mu < \infty$. Then, there exists $\omega \in L^1(Y)$ such that $M_q(\Gamma) \leq M_{\omega}^q(f(\Gamma))$ for every $\Gamma \in A(D)$.

Theorem 1 extends Lemma 1 from [17] given for open, discrete mappings between Riemannian n -manifolds. Our result is more general and is valid for non-open mappings, as shown in the example given by $f : B^2 \rightarrow B^2$, $f(x, y) = (x, y)$ if $y > 0$, $f(x, y) = (x - y)$ if $y < 0$.

The aim of this paper is to study the geometric properties of such mappings. Let us consider the following class of mappings of finite distortion:

$$f \in W_{loc}^{1,n}(D, \mathbb{R}^n) \text{ such that } K_0(\cdot, f) \in L_{loc}^p(D), \text{ where } p > n - 1 \text{ if } n \geq 3, p = 1 \text{ if } n = 2. \quad (4)$$

We see from [41,42] that such mappings $f : D \rightarrow \mathbb{R}^n$ are open, discrete and satisfy condition (N^{-1}) and that $\mu_n(B_f) = 0$, $\mu_n(f(B_f)) = 0$. Using Theorem 1, we see that for every $G \subset\subset D$ there exists $\omega_G \in L^1(\mathbb{R}^n)$ such that

$$M_n(\Gamma) \leq M_{\omega_G}^n(f(\Gamma)) \text{ for every } \Gamma \in A(G). \quad (5)$$

Using Hölder's inequality, we can replace n by some $n - 1 < q < n$ in (5). We note that relation (5) is the first inverse Poletsky modular inequality given for an important class

of mappings of finite distortion. We will establish, in a future paper, further geometric properties of this class of mappings using the modulus method.

A significant part of mathematical analysis is the study of geometric properties of mappings. In our paper, we establish some geometric properties of mappings of a bounded Dirichlet integral using the fact that such mappings satisfy an inverse Poletsky modular inequality.

First, we study the modulus of continuity of such mappings and we extend in this way Theorem 7.5.2 in [6] given for mappings $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$.

The main objective of this paper is to generalize important theorems from complex analysis and quasiregular mappings in the class of mappings of a bounded Dirichlet integral. These extensions are consistent due to the generality of such mappings and due to the fact that these extensions are established on general metric measure spaces. However, the main spaces we have in mind are Riemannian n -manifolds.

We extend in Theorem 3 the known theorems of Fatou and F. and M. Riesz from complex analysis.

Another important theorem from classical complex analysis is the Lindelöf-type result from Theorem 5 and Theorem 6, which extends some results of Vuorinen (see [4,43–45]).

We prove in Theorem 7 a pointwise η -quasisymmetry property of some homeomorphisms $f : D \rightarrow D_f$ between Riemannian n -manifolds satisfying a direct Poletsky modular inequality.

We say that a mapping $f : D \rightarrow Y$ is quasimonotone if there exists a constant $L > 0$ (the coefficient of quasimonotonicity) such that for every set $K \subset A \subset\subset D$ with K compact and A open, it results that $d(f(K)) \leq Ld(f(\partial A))$. If $L = 1$, we say that f is monotone. If $Y = \mathbb{R}^n$ and f is open and continuous, then f is monotone.

Let B^n be the unit ball from $\mathbb{R}^n$. An important theorem from classical complex analysis is the theorem of Fatou, which says that if $f : B^2 \rightarrow \mathbb{C}$ is a bounded analytic mapping, then f has a.e. radial limits. Another theorem from classical complex analysis is the theorem of F. and M. Riesz, which says that if $f : B^2 \rightarrow \mathbb{C}$ is a bounded analytic mapping and $w \in \mathbb{C}$, then, if $E_w = \{y \in S(0, 1) \mid \lim_{t \rightarrow 1} f(ty) = w\}$, it results that $\mathcal{H}^1(E_w) = 0$. Here, if $p \geq 1$, we denote by $\mathcal{H}^p$ the p -dimensional Hausdorff measure.

We say that a mapping $f : B^n \rightarrow \mathbb{R}^n$ has a nontangential limit at a point $x \in \partial B^n$ if for every finite cone Γ and Γ_1 with a vertex at x such that $\bar{\Gamma} \setminus \{x\} \subset \Gamma_1 \subset B^n$, it results that there exists $\lim_{\substack{y \rightarrow x \\ y \in \Gamma}} f(y) \in \mathbb{R}^n$. Also, the mapping $f : B^n \rightarrow \mathbb{R}$ is monotone (in the sense of Lebesgue) if

$$\max_{x \in \bar{G}} f(x) = \max_{x \in \partial G} f(x) \text{ and } \min_{x \in \bar{G}} f(x) = \min_{x \in \partial G} f(x) \text{ for every } G \subset\subset B^n.$$

Using the modulus method, Manfredi and Villamor showed in [42,46,47] that if $n - 1 < p \leq n$, $f : B^n \rightarrow \mathbb{R}^n$ has monotone components, $f \in W_{loc}^{1,p}(D, \mathbb{R}^n)$ and $\int_{B^n} |f'(x)|^p dx < \infty$, then f has nontangential limits with the possible exception of a set of zero p -capacity (see [46,47]). Also, Miklyukov showed in [48] that if $f : B^n \rightarrow \mathbb{R}^n$ is a bounded quasiregular mapping and $\int_{B^n} |f'(x)|^n dx < \infty$, then f has a.e. nontangential limits.

It is not yet known whether a bounded quasiregular mapping $f : B^n \rightarrow \mathbb{R}^n$, $n \geq 3$ has at least a radial limit. Using the modulus method, an important result related to the theorems of Fatou and M. and F. Riesz was obtained by Martio and Rickman in [49]. They showed that if $f : B^n \rightarrow \mathbb{R}^n$ is quasiregular and there exist $C > 0$ and $0 < \alpha < n - 1$ such that $\int_{\mathbb{R}^n} N(y, f, B(0, r)) \leq C(1 - r)^\alpha$ for every $0 < r < 1$, then f has a.e. radial limits, and if $w \in \mathbb{R}^n$ and $E_w = \{y \in S(0, 1) \mid \lim_{t \rightarrow 1} f(ty) = w\}$, they showed that $\mathcal{H}^{n-1}(E_w) = 0$. Akkinen showed in [50] that if $f : B^n \rightarrow \mathbb{R}^n$ is a mapping of finite distortion with exponentially

integrable outer distortion and there exist $C > 0$ and $0 < \alpha < n - 1$ such that $\int_{B^n} J_f(x) dx < C(1 - r)^{-\alpha}$ for every $0 < r < 1$, then f has a.e. radial limits.

We partially extend these results in Theorem 3 for mappings $f : D \rightarrow Y$ with Y complete and having doubling measure ν and such that there exists a homeomorphism $\Phi : \overline{B}^n \rightarrow \overline{D}$ such that $l(z, \Phi) \geq m > 0$ and $L(z, \Phi) < M < \infty$ for every $z \in \overline{B}^n$. Due to the generality of homeomorphism Φ and of spaces X and Y , we obtain a rather general version of the theorems of Fatou and F. and M. Riesz. A version for a particular type of homeomorphism Φ and for open, discrete mappings $f : D \rightarrow Y$ between Riemannian n -manifolds is given in [17].

Let $D \subset \mathbb{R}^n$ be open, $q > 1$ and $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$. We set

$$K_{I,q}(x, f) = \frac{J_f(x)}{l(f'(x))^q} \text{ if } J_f(x) \neq 0, K_{I,q}(x, f) = 0 \text{ otherwise.}$$

Here, if $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, we set $l(A) = \inf_{|x|=1} |A(x)|$. If, in addition, $f : D \rightarrow D_f$ is a homeomorphism, then f is a.e. differentiable and satisfies condition (N). Let $Z_f = \{x \in D \mid f \text{ is differentiable at } x \text{ and } J_f(x) = 0\}$ and $A_f = \{x \in D \mid f \text{ is not differentiable at } x\}$. Using Sard's lemma from [51], we see that $\mu_n(f(Z_f)) = 0$ and then $\mu_n(f(A_f \cup Z_f)) = 0$. It results that if $g : D_f \rightarrow D$ is the inverse of f , then g is a.e. differentiable and $J_g(y) \neq 0$ a.e. Using the change of variable formulae, we see that

$$\int_{D_f} |g'(y)|^q dy = \int_{D \setminus A_f} |g'(f(x))|^q |J_f(x)| dx = \int_D K_{I,q}(x, f) dx.$$

It results that $\int_D K_{I,q}(x, f) dz < \infty$ if and only if $\int_D |g'(y)| dy < \infty$. If, in addition, f satisfies condition (N^{-1}) , then f is a.e. differentiable and $J_f(x) \neq 0$ a.e.

The chordal distance between two points $a, b \in \mathbb{R}^n$ is the number

$$q(a, b) = |a - b|(1 + |a|^2)^{-\frac{1}{2}}(1 + |b|^2)^{-\frac{1}{2}} \text{ if } a, b \in \mathbb{R}^n, q(a, \infty) = (1 + |a|^2)^{-\frac{1}{2}} \text{ if } a \in \mathbb{R}^n.$$

We shall apply our results to the mappings of finite conformal energy. Such mappings are homeomorphisms $f : D \rightarrow D_f$ between two domains from $\mathbb{R}^n$ such that $f \in W_{loc}^{1,n}(D, D_f)$ and $\int_D |f'(x)|^n dx < \infty$. We set $E_D(f) = \int_D |f'(x)|^n dx$ the energy of f and $E_{t,D}^q(f) = \int_D (|f'(x)|^n + K_{I,q}(x, f)) dx$ the total energy of f and usually we take $q = n$.

If $f^{-1} \in W_{loc}^{1,n}(D_f, D)$ and $\int_{D_f} |(f^{-1})'(y)|^n dy < \infty$, we say that f is of biconformal energy. Such mappings form an important class with connections to mathematical models of Nonlinear Elasticity (see [52–54]). The general task in mathematical models of elasticity is to find deformations of the smallest energy $E_D(f)$ or of the smallest total energy $E_{t,D}^q(f)$. These deformations are usually found as weak limits of minimizing sequences. However, in the limiting process, we may lose the injectivity of f , and we refer to this incident as permanent damage in the material. We accept the weak limits of energy-minimizing sequences of homeomorphisms as legitimate deformations and the minimal energy (usually attained) can be strictly smaller than the infimum energy among homeomorphisms.

We are interested to find injective minimizers of the energy $E_M^q(D)$ and of the total energy $E_t^q(D)$ in some classes of deformations.

Let $D \subset \mathbb{R}^n$ be a domain, $a, b \in D$, $a \neq b$, $n - 1 < q \leq n$, $r > 0$, $H_D^q = \{f \in W_{loc}^{1,n}(D, D_f) \text{ homeomorphisms such that } f^{-1} \in W_{loc}^{1,q}(D_f, D), E_{t,D}^q(f) < \infty \text{ and } q(f(a), f(b)) \geq r\}$ and let $H_{D,M}^q = \{f \in H_D^q \mid E_{t,D}^q(f) < M\}$. We set the total en-

$$\text{ergy } E_t^q(D) = \inf_{f \in H_{D,D}^q} E_{t,D}^q(f) \text{ and } E_{t,M}^q(D) = \inf_{f \in H_{D,M}^q} E_{t,D}^q(f) \text{ and the energy } E_M^q(D) = \inf_{f \in H_{D,M}^q} E_D(f).$$

We apply the preceding results in Theorem 8 and we find some conditions for the existence of injective minimizers for mappings of biconformal energy in connection with the seminal paper of Ball [55].

2. Notations and Definitions

We say that a set $E \subset \mathbb{R}^n$ is a cap of a sphere $S(x, t)$ in $\mathbb{R}^n$ if $E = H \cap S(x, t)$ where H is an open half-space in $\mathbb{R}^n$. We extend now some results from [12] and Theorem 6.4 from [43].

We say that a mapping $f : D \rightarrow Y$ is open if it carries open sets into open sets, we say that it is closed if it carries closed sets into closed sets and we say that it is discrete if either $f^{-1}(y) = \emptyset$ or $f^{-1}(y)$ is a discrete subset of D . We set $G \subset\subset D$ if G is open, $\overline{G}$ is compact and $\overline{G} \subset D$.

Let $f : D \rightarrow Y$ be a mapping. We say that f is η -quasisymmetric if $\eta : (0, \infty) \rightarrow (0, \infty)$ is increasing and $\lim_{t \rightarrow 0} \eta(t) = 0$ and $\frac{d(f(y), f(x))}{d(f(z), f(x))} \leq \eta\left(\frac{d(y, x)}{d(z, x)}\right)$ for every distinct point $x, y, z \in D$. Such a continuous, open, discrete mapping is quasiregular, since $H(x, f) \leq \eta(1)$ for every $x \in D$. If $X = Y = \mathbb{R}^n$, every quasiconformal mapping $f : D \rightarrow f(D)$ is locally quasymmetric.

If $\omega : D \rightarrow [0, \infty]$ is μ measurable and A is a Borel subset of D , we set $\int_A \omega(z) d\mu = \int_A \omega(z) d\mu / \mu(A)$, and if $x \in D$, we set $\omega_x = \limsup_{r \rightarrow 0} \int_{B(x, r)} \omega(z) d\mu$.

If $E, F \subset \overline{D}$, we set $\Delta(E, F, D) = \{\gamma : [0, 1] \rightarrow \overline{D} \text{ path } | \gamma(0) \in E, \gamma(1) \in F \text{ and } \gamma((0, 1)) \subset D\}$. A domain $A \subset \mathbb{R}^n$ is a ring if $\mathbb{C}A$ has exactly two components, and if these components of $\mathbb{C}A$ are C_0 and C_1 , we denote $A = R(C_0, C_1)$ and we set $\Gamma_A = \Delta(C_0, C_1, A)$. Given $r > 0$, we let $\Phi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ such that C_0 contains the origin and a point a such that $|a| = 1$ and C_1 contains ∞ and a point b such that $|b| = r$ and we denote $\mathcal{H}_n(r) = \inf M_n(\Gamma_A)$ over all rings $A \in \Phi_n(r)$. The function $\mathcal{H}_n(r) : (0, \infty) \rightarrow (0, \infty)$ is decreasing and $\lim_{r \rightarrow 0} \mathcal{H}_n(r) = \infty$, $\lim_{r \rightarrow \infty} \mathcal{H}_n(r) = 0$.

If $A = R(C_0, C_1)$ and $a, b \in C_0, c, \infty \in C_1$, then $M_n(\Gamma_A) \geq \mathcal{H}_n\left(\frac{|c-a|}{|b-a|}\right)$ (see [3]).

Given $0 < r \leq 1, q > n - 1$ and $0 < t \leq 1$, we let $\Psi_{n,q}(r, t)$ be the set of all rings $A = R(C_0, C_1)$ such that $q(C_0) \geq r, q(C_1) \geq r, C_1$ is unbounded, $q(C_0, C_1) \leq t$ and we set $\lambda_{n,q}(r, t) = \inf M_q(\Gamma_A)$ over all rings $A \in \Psi_{n,q}(r, t)$. Then, the function $\lambda_{n,q} : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ is increasing in r and decreasing in t and $\lim_{t \rightarrow 0} \lambda_{n,q}(r, t) = \infty$ for every fixed $r \in (0, 1]$ (see [3,56]).

Here, if $A, B \subset \mathbb{R}^n$, we set $q(A)$ as the diameter of A corresponding to the chordal distance in $\mathbb{R}^n$ and we set $q(A, B)$ as the distance between the sets A and B corresponding to the chordal distance in $\mathbb{R}^n$.

Let $E_1, \dots, E_m, \dots$ be a sequence of sets in $\mathbb{R}^n$. The kernel $\text{Ker}_{j \rightarrow \infty} E_j$ of this sequence is the set of all points in $\mathbb{R}^n$, which have a neighborhood that is contained in all but a finite number of sets E_j , i.e., $\text{Ker}_{j \rightarrow \infty} E_j = \bigcup_{k=1}^{\infty} \text{Int} \bigcap_{j=k}^{\infty} E_j$.

Let X be a Riemannian n -manifold and $x \in X$. A set $D \subset X$ is a normal neighborhood of x if there exist $r > 0$ and a C^1 diffeomorphism $\Phi : \overline{B}(0, r) \rightarrow \overline{B}(x, r)$ such that $\Phi(0) = x, \Phi(S(0, t)) = S(x, t)$ for every $0 < t \leq r$ and $D = B(x, r)$. Here, $B(0, r)$ is a ball from $\mathbb{R}^n$ endowed with the Euclidean distance and $B(x, r)$ is the ball in X endowed with the geodesic distance. We see from [57] that if X is a Riemannian n -manifold and $x \in X$, then X has a fundamental system of normal neighborhoods.

Let $\gamma : [a, b] \rightarrow X$ be rectifiable. We set $|\gamma| = \text{Im}\gamma$ and if $\rho : X \rightarrow [0, \infty]$ is a Borel function, we set $\int_{\gamma} \rho ds = \int_0^{l(\gamma)} \rho(\gamma^0(t)) dt$, where $\gamma^0 : [0, l(\gamma)] \rightarrow X$ is such that $\gamma = \gamma^0 \circ s_{\gamma}$ and s_{γ} is the length function of γ . If $\gamma : [a, b] \rightarrow X$ is locally rectifiable, we set $\int_{\gamma} \rho ds = \sup_{\beta} \int_{\beta} \rho ds$, where the supremum is taken over closed rectifiable subpaths of γ .

If $\Gamma_1, \Gamma_2 \in A(D)$, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma_1 \in \Gamma_1$ has a subpath $\gamma_2 \in \Gamma_2$. If $\Gamma_1 > \Gamma_2$, $p > 1$ and $\omega : D \rightarrow [0, \infty]$ is measurable and finite a.e., then $M_{\omega}^p(\Gamma_1) \leq M_{\omega}^p(\Gamma_2)$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $|x| = (\sum_{i=1}^n x_i^2)^{-\frac{1}{2}}$.

A metric measure space X is Ahlfors n -regular if there exists a constant $C > 0$ such that $\frac{r^n}{C} \leq \mu(B(x, r)) \leq Cr^n$ for every ball $B(x, r) \subset X$.

A domain $D \subset X$ is finitely connected at a boundary point $x \in \partial D$ if there exists $(U_m)_{m \in \mathbb{N}}$ a fundamental system of neighborhoods of x such that $U_m \cap D$ has a finite number of components for every $m \in \mathbb{N}$.

We denote by $\bar{Y}$ the Alexandrov compactification of Y .

We set $C_{x,a,b} = B(x, b) \setminus \bar{B}(x, a)$ if $x \in X$ and $0 < a < b$ and $\Gamma_{x,a,b} = \Delta(\bar{B}(x, a), \mathbb{C}B(x, b), C_{x,a,b})$ for $x \in X$ and $0 < a < b$.

3. Preliminaries

As in Lemma 2.5 in [8], we prove the following:

Lemma 1. Let X, Y be Ahlfors n -spaces, $p > 1$, $D \subset X$, $G \subset Y$ be open and let $f : D \rightarrow G$ be a homeomorphism such that there exist $m, M > 0$ such that $L(z, f) \leq M$ and $l(z, f) > m$ for every $z \in D$. Then, there exists $C > 0$ such that

$$M_p(\Gamma) \leq CM_p(f(\Gamma)) \text{ for every } \Gamma \in A(D). \quad (6)$$

Remark 1. If X, Y are Riemannian n -manifolds, $D \subset X$, $G \subset Y$ are domains and $f : \bar{D} \rightarrow \bar{G}$ is a C^1 diffeomorphism, then there exist $m, M > 0$ such that $L(z, f) \leq M$ and $l(z, f) \geq m$ for every $z \in D$. If $X = Y = \mathbb{R}^n$, the condition (6) with $p = n$ holds if $f : D \rightarrow G$ is quasiconformal.

As in Lemma 2.6 in [8], we prove the following:

Lemma 2. Let X be a Riemannian n -manifold, let $x \in X$ and let D be a normal neighborhood of x such that there exist $r_x > 0$ and a C^1 diffeomorphism $\Phi : \bar{B}(0, r_x) \rightarrow \bar{D} = \bar{B}(x, r_x)$ and let $0 < a < b < r_x$ and $n - 1 < q \leq n$. Then, if $E, F \subset D$ are such that $S(x, t) \cap E \neq \emptyset$, $S(x, t) \cap F \neq \emptyset$ for every $a < t < b$, it results that

$$M_n(\Delta(E, F, C_{x,a,b})) \geq C(n, \Phi) \ln\left(\frac{b}{a}\right) \text{ if } q = n,$$

$$M_q(\Delta(E, F, C_{x,a,b})) \geq C(n, q, \Phi)(b^{n-q} - a^{n-q}) \text{ if } n - 1 < q < n.$$

Lemma 3. Let $n \geq 2$, let $n - 1 < q \leq n$, let X, Y be Riemannian n -manifolds, let $D \subset X$ a domain and let $f_j : D \rightarrow D_j$ be homeomorphisms such that $f_j \rightarrow f$ and the family $(f_j)_{j \in \mathbb{N}}$ is equicontinuous. Suppose that there exists $\omega_j : D \rightarrow [0, \infty]$ μ measurable and finite μ a.e. such that $M_q(f_j(\Gamma)) \leq M_{\omega_j}^q(\Gamma)$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$ and that for every $x \in D$ there exist $M_x > 0$ and $U_x \in V(x)$ such that $\int_{U_x} \omega_j(z) d\mu \leq M_x$ for every $j \in \mathbb{N}$. Then, either f is constant on D or $f : D \rightarrow f(D)$ is a homeomorphism.

Proof. Let us show first that every point $x \in D$ has a neighborhood U_x such that either f is constant on U_x or f is injective on U_x . Suppose that this thing is false. Then, there exist $U_x \in \mathcal{V}(x)$ as before and a C^1 diffeomorphism $\Phi : \mathbb{R}^n \rightarrow U_x$ and distinct points $a_1, a_2, a_3 \in \Phi(B^n)$ such that $f(a_1) \neq f(a_2)$ and $f(a_2) = f(a_3)$. Since the family $(f_j)_{j \in \mathbb{N}}$ is equicontinuous, we can also suppose that there exists $V_x \in \mathcal{V}(f(x))$ such that $f_j(U_x) \subset V_x$ for every $j \in \mathbb{N}$ and that there exists a C^1 diffeomorphism $\psi : B^n \rightarrow V_x$. Let $x_k \in B^n$ be such that $\Phi(x_k) = a_k$ for $k = 1, 2, 3$. Let J_0 be an arc joining x_1 with x_2 in B^n and let J_1 be an arc joining x_3 with some point $x_4 \in S(0, 1)$ and such that $J_0 \cap J_1 = \emptyset$. Let $A = R(J_0, J_1 \cup \mathbb{C}B^n)$ and let $F_j = \psi^{-1} \circ f_j \circ \Phi$ for every $j \in \mathbb{N}$. Then, $F_j : \mathbb{R}^n \rightarrow F_j(\mathbb{R}^n)$ is a homeomorphism, and $F_j(\Gamma_A)$ is a ring $A_j = (C_{0j}, C_{1j})$, where $C_{0j} = F_j(J_0)$ and $C_{1j} = F_j(J_1) \cup \mathbb{C}F_j(B^n)$ for every $j \in \mathbb{N}$. Since $f(a_1) \neq f(a_2)$, there exists $j_0 \in \mathbb{N}$ and $0 < \rho < q(\mathbb{C}B^n)$ such that $q(F_j(J_0)) > \rho$ for every $j \geq j_0$ and we can take $j_0 = 1$. Let $d(\Phi(J_0), \Phi(J_1 \cup \mathbb{C}B^n)) = \delta > 0$. Since $F_j(B^n) \subset B^n$ for $j \in \mathbb{N}$, we see that $\mathbb{C}F_j(B^n) \supset \mathbb{C}B^n$ and then $q(C_{1j}) \geq q(\mathbb{C}B^n) > \rho > 0$ if $j \in \mathbb{N}$.

Let $r_j = \min\{q(C_{0j}), q(C_{1j})\}$ and $t_j = q(C_{0j}, C_{1j})$ for every $j \in \mathbb{N}$. Then, $r_j \geq \rho$ for every $j \in \mathbb{N}$ and

$$\begin{aligned} \lambda_{n,q}(\rho, t_j) &\leq \lambda_{n,q}(r_j, t_j) \leq M_q(F_j(\Gamma_A)) = M_q(\psi^{-1} \circ f_j \circ \Phi(\Gamma_A)) \leq \\ &\leq CM_q(f_j(\Phi(\Gamma_A))) \leq CM_{\omega_j}^q(\Phi(\Gamma_A)) \leq \frac{C}{\delta^q} \int_{U_x} \omega_j(z) d\mu \leq \frac{CM_x}{\delta^q} \end{aligned}$$

for every $j \in \mathbb{N}$. Since $\lambda_{n,q}(\rho, t_j) \rightarrow \infty$ if $j \rightarrow \infty$, we reached a contradiction. We therefore proved that for every point $x \in D$ there exists $U_x \in \mathcal{V}(x)$ such that either f is constant on U_x or f is injective on U_x .

Let $E_1 = \{x \in D \mid \text{there exists } U_x \in \mathcal{V}(x) \text{ such that } f \text{ is constant on } U_x\}$ and $E_2 = \{x \in D \mid \text{there exists } U_x \in \mathcal{V}(x) \text{ such that } f \text{ is injective on } U_x\}$. Then, $D = E_1 \cup E_2$, E_1 and E_2 are open and we proved that $E_1 \cap E_2 = \emptyset$.

Suppose that f is not injective on D and let $x_1, x_2 \in D$, $x_1 \neq x_2$ be such that $f(x_1) = f(x_2)$ and let $\delta = d(x_1, x_2) > 0$. Let $U_1 = B(x_1, \frac{\delta}{4})$, $U_2 = B(x_2, \frac{\delta}{4})$. Then, $\overline{U_1} \cap \overline{U_2} = \emptyset$, and since f_j is a homeomorphism, we see that $f_j(\overline{U_1})$ and $f_j(\overline{U_2})$ are compact disjoint sets. Let $0 < \epsilon < \frac{\delta}{4}$ and let $d_j : [0, 1] \rightarrow Y$ be a geodesic joining $f_j(x_1)$ with $f_j(x_2)$ for every $j \in \mathbb{N}$. Then, $f_j(x_2) \notin f_j(\overline{U_1})$ and $d_j(0) = f_j(x_1)$, $d_j(1) = f_j(x_2)$. Let $t_j \in (0, 1)$ be such that $d_j(t_j) \in \partial f_j(B(x_1, \epsilon)) = f_j(S(x_1, \epsilon))$ for every $j \in \mathbb{N}$ and let $x_j \in S(x_1, \epsilon)$ be such that $d_j(t_j) = f_j(x_j)$ for every $j \in \mathbb{N}$ and can suppose that $x_j \rightarrow x_\epsilon \in S(x_1, \epsilon)$ and then $f_j(x_j) \rightarrow f(x_\epsilon)$. On the other side, $d(f_j(x_j), f_j(x_2)) = d(d_j(t_j), d_j(1)) < d(d_j(0), d_j(1)) = d(f_j(x_1), f_j(x_2)) \rightarrow 0$ and hence $f(x_\epsilon) = f(x_2) = f(x_1)$. We found in every sphere $S(x_1, \epsilon)$ points x_ϵ such that $f(x_1) = f(x_\epsilon)$ and hence $x_1 \notin E_2$. It results that $x_1 \in E_1$ and hence $D = E_1$.

We proved that either f is injective on D or f is constant on D . $\square$

Lemma 4. Let $q > 1$, $D \subset \mathbb{R}^n$ be open and $f_m \in W_{loc}^{1,q}(D, \mathbb{R}^n)$ such that $f_m \rightarrow f$ uniformly on the compact subsets of D . Then, $f_m \rightarrow f$ weakly in $W_{loc}^{1,q}(D, \mathbb{R}^n)$ and $\int_D |f'(x)|^q dx \leq \liminf_{m \rightarrow \infty} \int_D |f'_m(x)|^q dx$.

Proof. Since $f_m \rightarrow f$ uniformly on the compact subsets of D , it results that also $f \in W_{loc}^{1,q}(D, \mathbb{R}^n)$ and $\int_F |f'_m(x) - f'(x)|^q dx \rightarrow 0$ if $m \rightarrow \infty$ for every fixed compact set $F \subset D$. We can now suppose that $I = \liminf_{m \rightarrow \infty} \int_D |f'_m(x)|^q dx < \infty$ and also that $\int_D |f'(x)|^q dx > 0$. Let $F \subset \subset D$ be such that $\int_F |f'(x)|^q dx > 0$ and let $\epsilon > 0$. Let $\delta_\epsilon > 0$ be such that $\int_A |f'(x)|^q dx <$

$\frac{\epsilon}{3}$ if $A \subset F$ and $\mu_n(A) < \delta_\epsilon$ and let $m_\epsilon \in \mathbb{N}$ be such that $\int_D |f'_m(x)|^q dx \leq I + \frac{\epsilon}{3}$ and

$$\int_F |f'_m(x) - f'(x)|^q dx \leq \left(\frac{\epsilon}{3q \left(\int_F |f'(x)|^q dx \right)^{\frac{q-1}{q}}} \right)^q \text{ if } m \geq m_\epsilon. \text{ Using Riesz's theorem and taking}$$

a subsequence if necessary, we can suppose that $|f'_m(x)| \rightarrow |f'(x)|$ a.e. Using Egorov's theorem, we find $F_\epsilon \subset F$ measurable such that $\mu_n(F \setminus F_\epsilon) < \delta_\epsilon$ and $|f'_m(x)| \rightarrow |f'(x)|$ uniformly on F_ϵ .

Let $n_\epsilon \geq m_\epsilon$ be such that $|f'(x)| - |f'_m(x)| < \epsilon$ for every $x \in F_\epsilon$ and every $m \geq n_\epsilon$. Let $A_\epsilon(x) = \langle q|f'(x)|^{q-2}f'(x), f'_m(x) - f'(x) \rangle$ if $x \in F_\epsilon$ and $m \geq n_\epsilon$. We have

$$\begin{aligned} |f'_m(x)|^q - |f'(x)|^q &\geq q \min\{|f'_m(x)|, |f'(x)|\}^{q-1} (|f'_m(x)| - |f'(x)|) \geq \\ &\geq q|f'(x)|^{q-1} \frac{1}{|f'(x)|} \langle f'(x), f'_m(x) - f'(x) \rangle = A_\epsilon(x) \end{aligned}$$

if $x \in F_\epsilon$ and $m \geq n_\epsilon$. Using Hölder's inequality, we have

$$\begin{aligned} \int_{F_\epsilon} |A_\epsilon(x)| dx &\leq q \left(\int_{F_\epsilon} |f'(x)|^q dx \right)^{\frac{q-1}{q}} \left(\int_{F_\epsilon} |f'_m(x) - f'(x)|^q dx \right)^{\frac{1}{q}} \leq \\ &\leq q \left(\int_F |f'(x)|^q dx \right)^{\frac{q-1}{q}} \left(\int_F |f'_m(x) - f'(x)|^q dx \right)^{\frac{1}{q}} \leq \frac{\epsilon}{3} \end{aligned}$$

if $m \geq n_\epsilon$. Then, $\int_F |f'(x)|^q dx = \int_{F_\epsilon} |f'(x)|^q dx + \int_{F \setminus F_\epsilon} |f'(x)|^q dx \leq \int_{F_\epsilon} |f'_m(x)|^q dx + \int_{F_\epsilon} |A_\epsilon(x)| dx + \int_{F \setminus F_\epsilon} |f'(x)|^q dx \leq I + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = I + \epsilon$ if $m \geq n_\epsilon$.

Letting $\epsilon \rightarrow 0$ and $F \nearrow D$, the theorem is proved. $\square$

Let X be a metric space, $A \subset X$ and $k, r > 0$. We set $\mathcal{H}_r^k(A) = \inf \sum_{i=1}^\infty d(B_i)^k$, where the infimum is taken over all balls B_i such that $A \subset \bigcup_{i=1}^\infty B_i$ and $d(B_i) < r$ for every $i \in \mathbb{N}$.

We set $\mathcal{H}^k(A) = \lim_{r \rightarrow 0} \mathcal{H}_r^k(A)$.

Lemma 5. Let X, Y be Ahlfors n -regular, let $U \subset X$, let $V \subset Y$ be open and let $\Phi : \overline{U} \rightarrow \overline{V}$ be a homeomorphism such that there exist $m, M > 0$ such that $l(z, \Phi) > m$ and $L(z, \Phi) < M$ for every $z \in \overline{U}$. Then, if $A \subset \overline{U}$ is such that $\mathcal{H}^k(A) = 0$, it results that $\mathcal{H}^k(\Phi(A)) = 0$ and the mapping $\Phi|_U : U \rightarrow V$ satisfies condition (N) and condition (N^{-1}) .

Proof. Let $A \subset \overline{U}$ be such that $\mathcal{H}^k(A) = 0$ and let $r > 0, \epsilon > 0$. We can find a covering $A \subset \bigcup_{i=1}^\infty B(x_i, r_i)$ such that $d(\Phi(x), \Phi(x_i)) \leq Md(x, x_i)$ for every $x \in A \cap B(x_i, r_i)$, $r_i < r$ for every $i \in \mathbb{N}$ and $\sum_{i=1}^\infty (2r_i)^k < \epsilon$. Then, $\Phi(A) \subset \bigcup_{i=1}^\infty \Phi(B(x_i, r_i) \cap A) \subset \bigcup_{i=1}^\infty B(\Phi(x_i), Mr_i)$ and $\sum_{i=1}^\infty (2Mr_i)^k \leq M^k \sum_{i=1}^\infty (2r_i)^k \leq M^k \epsilon$. It results that $\mathcal{H}_{Mr}^k(\Phi(A)) \leq M^k \epsilon$ and letting $\epsilon \rightarrow 0$ and then $r \rightarrow 0$ we find that $\mathcal{H}^k(\Phi(A)) = 0$.

Let now $A \subset U$ be such that $\mu(A) = 0$ and let $\epsilon > 0$. Since μ is Borel regular, we find $Q \subset U$ open such that $A \subset Q \subset U$ and $\mu(Q) < \epsilon$. We see that $\Phi(B(x, r)) \subset B(\Phi(x), Mr)$ and $B(x, r) \subset \Phi^{-1}(B(\Phi(x), Mr))$ for every $x \in A$ and $r > 0$ small enough. Let $C_1, C_2 > 0$ be such that $\frac{r^n}{C_1} \leq \mu(B(x, r)) \leq C_1 r^n$ for every ball $B(x, r) \subset X$ and $\frac{r^n}{C_2} \leq \nu(B(y, r)) \leq C_2 r^n$

for every ball $B(y, r) \subset Y$. We cover the set A with balls $B(x, r_x)$ such that $B(x, r_x) \subset Q$, $B(\Phi(x), 5Mr_x) \subset \Phi(Q)$ and then $\Phi(A) \subset \Phi(\bigcup_{x \in A} B(x, r_x)) \subset \bigcup_{x \in A} B(\Phi(x), Mr_x)$.

Using the covering lemma from page 30 in [35], we find disjoint balls $B(\Phi(x_i), Mr_i)$, $i \in \mathbb{N}$ such that $\bigcup_{x \in A} B(\Phi(x), Mr_x) \subset \bigcup_{i=1}^{\infty} B(\Phi(x_i), 5Mr_i)$. Then, $\Phi(A) \subset \bigcup_{i=1}^{\infty} B(\Phi(x_i), 5Mr_i)$ and hence $\nu(\Phi(A)) \leq \nu(\bigcup_{i=1}^{\infty} B(\Phi(x_i), 5Mr_i)) \leq \sum_{i=1}^{\infty} \nu(B(\Phi(x_i), 5Mr_i)) \leq C_2(5M)^n \sum_{i=1}^{\infty} r_i^n$. We see that also the sets $\Phi^{-1}(B(\Phi(x_i), Mr_i))$, $i \in \mathbb{N}$ are disjoint, and since $B(x_i, r_i) \subset \Phi^{-1}(B(\Phi(x_i), Mr_i))$ for every $i \in \mathbb{N}$, it results that the balls $B(x_i, r_i)$, $i \in \mathbb{N}$ are disjoint subsets of Q and $\mu(\bigcup_{i=1}^{\infty} B(x_i, r_i)) \leq \mu(Q) \leq \epsilon$. We find that

$$\frac{1}{C_1} \sum_{i=1}^{\infty} r_i^n \leq \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) = \mu(\bigcup_{i=1}^{\infty} B(x_i, r_i)) \leq \mu(Q) \leq \epsilon.$$

We have $\nu(\Phi(A)) \leq C_2(5M)^n \sum_{i=1}^{\infty} r_i^n \leq C_1 C_2(5M)^n \epsilon$.

Letting $\epsilon \rightarrow 0$, we see that $\nu(\Phi(A)) = 0$ and we proved that $\Phi|U : U \rightarrow V$ satisfies condition (N). In the same way, we prove that $\Phi|U : U \rightarrow V$ satisfies condition (N^{-1}) . $\square$

Lemma 6. Let $p > 1$, let X be a metric measure space, let $x \in X$ and let $\omega : X \rightarrow [0, \infty]$ be μ measurable and finite μ a.e. such that $\omega_x < \infty$. Then, there exists $r_x > 0$ such that

$$M_{\omega}^p(\Gamma_{x,a,b}) \leq \frac{C \epsilon^p \omega_x \sum_{k=1}^{\infty} \frac{1}{k^p}}{(\ln \ln(\frac{be}{a}))^n} \text{ for every } 0 < a < b < be < r_x.$$

Proof. We follow the ideas from Lemma 2.2 in [16]. $\square$

A measure μ in a metric space X is called doubling if the balls have a finite and positive measure and there exists a constant C such that $\mu(2B) \leq C\mu(B)$ for all balls $B \subset X$.

Lemma 7. Let (X, μ) be a metric measure space and let μ be doubling. Let $D \subset \subset X$ and let $f : \overline{D} \rightarrow \overline{D}_f \subset Y$ be a homeomorphism satisfying condition (N) and let $g : Y \rightarrow [0, \infty]$ be a Borel function. Then, $\int_A g(f(x)) J_f(x) d\mu = \int_{f(A)} g(y) dv$ for every Borel set $A \subset D$.

Proof. Let $\nu_f(A) = \nu(f(A))$ for every Borel set $A \subset D$. Then, ν_f is a finite Borel measure, and since f satisfies condition (N), we see that $\nu_f \leq \mu$. Using Radon–Nikodym’s theorem, we find $h \in L^1(f(D))$ such that $\nu_f(A) = \int_A h(x) d\mu$ for every Borel set $A \subset D$. Using Lebesgue’s differentiation theorem (see Theorem 1.8 in [32]), we see that in every Lebesgue point x of h we have

$$J_f(x) = \limsup_{r \rightarrow 0} \frac{\nu(f(B(x, r)))}{\mu(B(x, r))} = \limsup_{r \rightarrow 0} \frac{\nu_f(B(x, r))}{\mu(B(x, r))} = \limsup_{r \rightarrow 0} \int_{B(x, r)} h(z) d\mu = h(x).$$

Then, $\nu(f(A)) = \nu_f(A) = \int_A h(x) d\mu = \int_A J_f(x) d\mu$ for every Borel set $A \subset D$. Using now standard arguments, we see that $\int_A g(f(x)) J_f(x) d\mu = \int_{f(A)} g(y) dv$ for every Borel set $A \subset D$. $\square$

4. The Main Results

Proof of Theorem 1. Let $D_k \subset \subset D \setminus B_f$ be such that $\overline{D}_k \subset D_{k+1}$ for every $k \in \mathbb{N}$ and $D \setminus B_f = \bigcup_{k=1}^{\infty} D_k$. Let $k \in \mathbb{N}$ be fixed. Since $\mu(B_f) = 0$, f is open on $D_k \setminus B_f$ and every ball $B(x, r)$ in Y with $r > 0$ has a positive measure and $\nu(f(B_f)) = 0$, we see that $f(D_k) \setminus f(B_f) \neq \emptyset$. Let $y \in f(D_k) \setminus f(B_f)$. Since f is a local homeomorphism on $\overline{D}_k$ and $\overline{D}_k$ is compact, we see that $f^{-1}(y) \cap \overline{D}_k$ is finite and we can find $V \in \mathcal{V}(y)$ and open sets $U_i \subset \subset D \setminus B_f$ such that $f^{-1}(V) \cap \overline{D}_k \subset \bigcup_{i=1}^m U_i \subset D_{k+1}$ and $f|_{U_i} : U_i \rightarrow V$ is a homeomorphism for every $i = 1, \dots, m$. Since ν is doubling, we can use Vitali's covering theorem (see Theorem 1.6 in [32]) and we find $H_k \subset Y$ with $\nu(H_k) = 0$, disjoint balls V_i such that $f(D_k) \setminus H_k = \bigcup_{i=1}^{\infty} V_i$ and disjoint sets U_{ij} such that $f|_{U_{ij}} : U_{ij} \rightarrow V_i$ is a homeomorphism, $D_k \setminus f^{-1}(H_k) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{j(i)} U_{ij}$, and if we denote by $g_{ij} : V_i \rightarrow U_{ij}$, $g_{ij} = (f|_{U_{ij}})^{-1}$ for $i \in \mathbb{N}$, $j = 1, \dots, j(i)$, we can also suppose that $J_{g_{ij}}(y) < \infty$ for every $y \in V_i \setminus H_k$ and for $i \in \mathbb{N}$ and $j = 1, \dots, j(i)$. Since f satisfies condition (N^{-1}) , we see that $\mu(f^{-1}(H_k)) = 0$.

$$\begin{aligned} \text{Let } \omega_k : Y \rightarrow [0, \infty], \omega_k(y) &= \sum_{i=1}^{\infty} \chi_{V_i}(y) \sum_{j=1}^{j(i)} J_{g_{ij}}(y) L(g_{ij}(y), f)^q \text{ if } y \in f(D_k) \setminus H_k, \\ \omega_k(y) &= 0 \text{ otherwise. Using Lemma 7, we have } \int_{f(D_k)} \omega_k(y) d\nu = \sum_{i=1}^{\infty} \int_{V_i} \omega_k(y) d\nu = \\ \sum_{i=1}^{\infty} \int_{V_i} \sum_{j=1}^{j(i)} J_{g_{ij}}(y) L(g_{ij}(y), f)^q d\nu &= \sum_{i=1}^{\infty} \sum_{j=1}^{j(i)} \int_{U_{ij}} L(z, f)^q d\mu \leq \int_{D_k} L(z, f)^q d\mu \text{ and we find that} \\ \int_{f(D_k)} \omega_k(y) d\nu &\leq \int_{D_k} L(z, f)^q d\mu \end{aligned} \quad (7)$$

If $\Gamma \in A(D)$ is such that $F(\Gamma) = \emptyset$, then $M_q(\Gamma) = 0$ and the theorem is proved. We can suppose that $F(\Gamma) \neq \emptyset$.

Let $\Gamma \in A(D)$ and $\Delta = \{\gamma \in \Gamma \mid \gamma \text{ is locally rectifiable and } f \circ \gamma^0 \text{ is absolutely continuous}\}$. Using Fuglede's theorem from [15], we see that $M_q(\Gamma) = M_q(\Delta)$. Let $\eta \in F(f(\Gamma))$ and $\rho : X \rightarrow [0, \infty]$ be defined by $\rho(z) = \eta(f(z)) L(z, f)$ if $z \in D$, $\rho(z) = 0$ otherwise. We see from [15] that $\rho \in F(\Delta)$, and using Lemma 7, we have:

$$\begin{aligned} \int_{D_k} \rho(z)^q d\mu &= \int_{D_k} \eta(f(z))^q L(z, f)^q d\mu \leq \sum_{i=1}^{\infty} \sum_{j=1}^{j(i)} \int_{U_{ij}} \eta(f(z))^q L(z, f)^q d\mu = \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{j(i)} \int_{V_i} \eta(f(g_{ij}(y)))^q L(g_{ij}(y), f)^q J_{g_{ij}}(y) d\nu = \\ &= \sum_{i=1}^{\infty} \int_{V_i} \eta(y)^q \chi_{V_i}(y) \sum_{j=1}^{j(i)} L(g_{ij}(y), f)^q J_{g_{ij}}(y) d\nu = \\ &= \int_Y \eta(y)^q \omega_k(y) d\nu. \end{aligned}$$

We proved that

$$\int_{D_k} \rho(z)^q d\mu \leq \int_Y \eta(y)^q \omega_k(y) d\nu \quad (8)$$

Let $\omega : Y \rightarrow [0, \infty]$, $\omega(y) = \sum_{x \in f^{-1}(y)} J_{g_x}(y) L(g_x(y), f)^q$ if $y \in f(D) \setminus f(B_f)$, $\omega(y) = 0$ otherwise. Here, if $x \in f^{-1}(y)$, we set by g_x a local inverse of f such that $g_x(y) = x$ and we see that $\omega_k = \omega|_{D_k} \nu$ a.e. and $\omega_k \nearrow \omega$ ν a.e. Using the fact that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$ and letting $k \rightarrow \infty$ in (7) and (8), we obtain that

$$\int_{f(D)} \omega(y) d\nu \leq \int_D L(z, f)^q d\mu < \infty \text{ and } M_q(\Gamma) = M_q(\Delta) \leq \int_D \rho(z)^q d\mu \leq \int_Y \eta(y)^q \omega(y) d\nu.$$

Since $\eta \in F(f(\Gamma))$ was arbitrarily chosen, we find that $M_q(\Gamma) \leq M_\omega^q(f(\Gamma))$ for every $\Gamma \in A(D)$ and that $\omega \in L^1(Y)$. $\square$

Remark 2. Let us take in Theorem 1 the measure ν_ω given by $\nu_\omega(A) = \int_A \omega(x) d\mu$ for every Borel set $A \subset Y$. Since $\omega \in L^1(Y)$, we see that $\nu_\omega \leq \mu$, and if $\eta : Y \rightarrow [0, \infty]$ is a Borel function and $A \subset Y$ is a Borel set, then

$$\int_A \omega(x) \eta(x)^q d\mu = \int_A \eta(x)^q d\nu_\omega.$$

Since $M_q(\Gamma) \leq M_\omega^q(f(\Gamma))$ for every $\Gamma \in A(D)$, it results that

$$M_q(\Gamma) \leq M_q(f(\Gamma)) \text{ for every } \Gamma \in A(D). \quad (9)$$

Here, we take the measure μ on X and the measure ν_ω on Y .

Remark 3. The idea in the proof of Theorem 1 comes from the old methods in the book of Väisälä [3]. Here, if D, G are domains in $\mathbb{R}^n$ and $f : D \rightarrow G$ is a homeomorphism satisfying condition (N), the Jacobian is defined by

$$J_f(x) = \lim_{r \rightarrow 0} \frac{\mu_n(f(B(x, r)))}{\mu_n(B(x, r))}$$

which exists a.e. and J_f is measurable and

$$\mu_n(f(A)) = \int_A J_f(x) dx$$

for every Borel set $A \subset D$ (see page 85 in [3]). Using standard arguments, we see that

$$\int_A g(f(x)) J_f(x) dx = \int_{f(A)} g(y) dy$$

for every positive Borel function g and every Borel set $A \subset D$.

Suppose now that D, G are domains in $\mathbb{R}^n$, let $q > 1$ and let $f : D \rightarrow G$ be a homeomorphism such that its inverse $g : G \rightarrow D$ satisfies condition (N) and $\int_D L(x, f)^q dx < \infty$. Let $\omega : \mathbb{R}^n \rightarrow [0, \infty]$, $\omega(y) = J_g(y) L(g(y), f)^q$ if $y \in G$, $\omega(y) = 0$ otherwise.

We see that $\int_{\mathbb{R}^n} \omega(y) dy = \int_G J_g(y) L(g(y), f)^q dy = \int_D L(x, f)^q dx < \infty$ and hence $\omega \in L^1(\mathbb{R}^n)$.

Let $\Gamma \in A(D)$, $\Delta = \{\gamma \in \Gamma | f \circ \gamma^0 \text{ is absolutely continuous}\}$ and let $\eta \in F(f(\Gamma))$. Let $\rho : \mathbb{R}^n \rightarrow [0, \infty]$, $\rho(x) = \eta(f(x)) L(x, f)$ if $x \in D$, $\rho(x) = 0$ otherwise and let $\gamma \in \Delta$. Then,

using Theorem 5.3 in [3], we have $1 \leq \int_{f \circ \gamma} \eta ds \leq \int_{\gamma} \eta(f(x))L(x, f)ds = \int_{\gamma} \rho ds$ and hence $\rho \in F(\Delta)$.

We have

$$\begin{aligned} M_q(\Gamma) &= M_q(\Delta) \leq \int_D \rho(x)^q dx = \int_D \eta(f(x))^q L(x, f)^q dx = \\ &= \int_G \eta(f(g(y)))^q L(g(y), f)^q J_g(y) dy = \int_G \eta(y)^q \omega(y) dy. \end{aligned}$$

Since $\eta \in F(f(\Gamma))$ was arbitrarily chosen, we see that

$$M_q(\Gamma) \leq M_{\omega}^q(f(\Gamma))$$

for every $\Gamma \in A(D)$.

Now, we see from Example 1 and Theorem 1 that we can take the homeomorphism $f : D \rightarrow G$ such that $f \notin W_{loc}^{1,1}(D, G)$.

Also, the modular inequality (9) implies that

$$M_q(\Gamma) \leq M_q(f(\Gamma)) \text{ for every } \Gamma \in A(D) \quad (10)$$

and we take the measure μ_n on D and the measure $(\mu_n)_{\omega}$ on G .

Theorem 2. Let $1 \leq n-1 < q \leq n$, let X, Y be Riemannian n -manifolds, let $f : D \rightarrow Y$ be continuous, open and discrete, satisfying condition (N^{-1}) and such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$ and $\int_D L(z, f)^q d\mu < \infty$. Then, there exists $r_x > 0$ such that

$$d(f(y), f(x))^n \leq \frac{C_1}{\ln(\frac{d(x,z)}{d(x,y)})} \int_{B(x, d(x,z))} L(w, f)^n d\mu \text{ and } 0 < d(x, y) < d(x, z) < r_x. \quad (11)$$

$$d(f(y), f(x))^q \leq \frac{C_2}{d(x, y)^{n-q}} \int_{B(x, 2d(x, y))} L(w, f)^q d\mu \text{ if } n-1 < q < n \text{ and } 2d(x, y) < r_x. \quad (12)$$

If $X = \mathbb{R}^n$, we can take $r_x = d(x, \partial D)$. The constant C_1 depends only on n and x , and the constant C_2 depends on n, q and x . If f is quasimonotone of constant L instead of being open and discrete, then C_1 depends on L, n and x and the constant C_2 depends on L, q, n and x .

Proof of Theorem 2. We see from Theorem 1 that there exists $\omega \in L^1(Y)$ such that $M_q(\Gamma) \leq M_{\omega}^q(f(\Gamma))$ for every $\Gamma \in A(D)$ and that $\int_{f(B)} \omega(y) d\nu \leq \int_B L(z, f)^q d\mu$ for every ball $B \subset D$.

Let $\Phi : \bar{B}(0, r_x) \rightarrow \bar{B}(x, r_x)$ be a C^1 diffeomorphism such that $\bar{B}(x, r_x) \subset D$, $\Phi(S(0, t)) = S(x, t)$ for every $0 < t < r_x$ and let $y, z \in X$ be such that $0 < d(x, y) < d(x, z) < r_x$. Suppose that f is quasimonotone. We can find for every $d(x, y) < t < d(x, z)$ some points $\alpha_t, \beta_t \in S(x, t)$ such that $df(\bar{B}(x, y)) \leq Ld(f(\alpha_t), f(\beta_t))$. Let $E = \{\alpha_t\}$, $F = \{\beta_t\}$ for $d(x, y) < t < d(x, z)$ and let $\Gamma = \Delta(E, F, C_{x, d(x, y), d(x, z)})$. Then, $\eta = \frac{L}{d(f(x), f(y))} \chi_{f(B(x, d(x, z)))} \in F(f(\Gamma))$. Here, L is the coefficient of quasimonotonicity of f .

If $q = n$, we see from Lemma 2 that

$$C(x, n) \ln\left(\frac{d(x, z)}{d(x, y)}\right) \leq M_n(\Gamma) \leq M_{\omega}^n(f(\Gamma)) \leq \frac{L^n}{d(f(x), f(y))^n} \int_{f(B(x, d(x, z)))} \omega(w) d\nu \leq$$

$$\leq \frac{L^n}{d(f(x), f(y))^n} \int_{B(x, d(x, z))} L(w, f)^n d\mu.$$

If $n - 1 < q < n$ and $2d(x, y) = d(x, z)$, we see from Lemma 2 that $C(x, n, q)(2^{n-q} - 1)d(x, y)^{n-q} = C(x, n, q)(d(x, z)^{n-q} - d(x, y)^{n-q}) \leq M_q(\Gamma) \leq M_\omega^q(f(\Gamma)) \leq \int_Y \omega(w)\eta(w)^q dv \leq$

$$\frac{L^q}{d(f(x), f(y))^q} \int_{f(B(x, d(x, z)))} \omega(w) dv \leq \frac{L^q}{d(f(x), f(y))^q} \int_{B(x, d(x, z))} L(w, f)^q d\mu.$$

Suppose now that f is open and discrete. In this case, we take the normal coordinates $\Phi : \bar{B}(0, r_x) \rightarrow \bar{B}(x, r_x)$ such that $f(\Phi(x, r_x)) \subset V_x$, where $V_x \in \mathcal{V}(f(x))$ and there exists a C^1 diffeomorphism $\psi : B^n \rightarrow V_x$ such that $\psi(0) = f(x)$. We can suppose that there exists $L_x > 0$ such that $\frac{d(a, b)}{L_x} \leq d(\psi^{-1}(a), \psi^{-1}(b)) \leq L_x d(a, b)$ for every $a, b \in V_x$. Let $F = \psi^{-1} \circ f : B(x, r_x) \rightarrow \mathbb{R}^n$. Then, F is well defined, is continuous, open, discrete and hence monotone and $L(w, F) \leq L_x L(w, f)$ for every $w \in B(x, r_x)$.

If $q = n$, using the preceding result, we have

$$\begin{aligned} \frac{d(f(y), f(x))^n}{L_x^n} &\leq d(F(y), F(x))^n \leq \frac{C(n, x)}{\ln\left(\frac{d(x, z)}{d(x, y)}\right)} \int_{B(x, d(x, z))} L(w, F)^n dw \leq \\ &\leq \frac{C(n, x)L_x^n}{\ln\left(\frac{d(x, z)}{d(x, y)}\right)} \int_{B(x, d(x, z))} L(w, f)^n d\mu. \end{aligned}$$

Also, if $n - 1 < q < n$, we have

$$\begin{aligned} \frac{d(f(y), f(x))^q}{L_x^q} &\leq d(F(y), F(x))^q \leq \frac{C(n, q, x)}{d(x, y)^{n-q}} \int_{B(x, 2d(x, y))} L(w, F)^q dw \leq \\ &\leq \frac{C(n, q, x)L_x^q}{d(x, y)^{n-q}} \int_{B(x, 2d(x, y))} L(w, f)^q d\mu. \end{aligned}$$

□

Also, the family W from Remark 5 is obviously equicontinuous on D .

Remark 4. Relation (11) is proved in Theorem 7.5.2 in [6] for mappings $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$ of finite dilatation. However, as Example 1 shows, relation (11) may be valid even for mappings $f \notin W_{loc}^{1,1}(D, \mathbb{R}^n)$.

Remark 5. If $W = \{f : D \rightarrow Y \mid f \text{ is continuous, has the coefficient of quasimonotonicity } L \text{ and satisfies condition } (N^{-1}), \mu(B_f) = 0, \nu(f(B_f)) = 0 \text{ and there exists a constant } M < \infty \text{ such that } \int_D L(x, f)^n d\mu \leq M\}$, then the family W is equicontinuous on D .

Theorem 3. Let Y be complete, let $D \subset X$ be a domain, let $x \in D$ be such that there exists a homeomorphism $\Phi : \bar{B}^n \rightarrow \bar{D}$ such that $\Phi(0) = x$ and there exist $m, M > 0$ such that $l(z, \Phi) > m$, $L(z, \Phi) < M$ for every $z \in \bar{B}^n$. Let us denote by $\gamma_y : [0, 1] \rightarrow Y$ the radial path joining x and y in D given by $\gamma_y(t) = \Phi(t\Phi^{-1}(y))$ for every $t \in [0, 1]$ and every $y \in \partial D = \Phi(S(0, 1))$. Let $f : D \rightarrow Y$ be continuous such that $\mu(B_f) = 0$, $\nu(f(B_f)) = 0$, f satisfies condition (N^{-1}) and there exists $q > 1$ such that $\int_D L(z, f)^q d\mu < \infty$. Then, there exists $\lim_{t \rightarrow 1} f(\gamma_y(t)) \in Y$ for $\mathcal{H}^{n-1}$ a.e. $y \in \partial D$, and if $E_w = \{y \in \partial D \mid \lim_{t \rightarrow 1} f(\gamma_y(t)) = w\}$, it results that $\mathcal{H}^{n-1}(E_w) = 0$ for every $w \in Y$.

Proof of Theorem 3. We use the ideas from Theorem 1 in [17] and we apply Theorem 1 instead of Lemma 1 in [17]. $\square$

Theorem 4. Let $D \subset \mathbb{R}^n$ be a domain, let $x \in D$ be such that there exists $\Phi : \overline{B}^n \rightarrow \overline{D}$ a C^1 diffeomorphism such that $\Phi(0) = x$ and let us denote by $\gamma_y : [0, 1] \rightarrow \mathbb{R}^n$ the radial path joining x and y in D given by $\gamma_y(t) = \Phi(t\Phi^{-1}(y))$ for every $t \in [0, 1]$ and every $y \in \partial D = \Phi(S(0, 1))$. Let $f : D \rightarrow \mathbb{R}^n$ be a mapping of finite distortion satisfying relation (4) and such that $\int_D |f'(z)|^n dz < \infty$. Then, there exists $\lim_{t \rightarrow 1} f(\gamma_y(t)) \in \mathbb{R}^n$ for $\mathcal{H}^{n-1}$ a.e. $y \in \partial D$, and if $E_w = \{y \in \partial D \mid \lim_{t \rightarrow 1} f(\gamma_y(t)) = w\}$, it results that $\mathcal{H}^{n-1}(E_w) = 0$ for every $w \in \mathbb{R}^n$.

Proof of Theorem 4. The mapping f satisfies the conditions from Theorem 1 and we apply Theorem 3. $\square$

Theorem 5. Let $n \geq 2, r > 0, D_1 \subset B(0, r) \subset \mathbb{R}^n$ be a domain such that $D_1 \cap S(0, t)$ is a cap of a sphere in $\mathbb{R}^n$ for every $0 < t \leq r$ and let D_2 be a domain such that $\overline{D}_2 \setminus \{0\} \subset D_1, 0 \in \overline{D}_2 \cap \overline{D}_1$ and there exists $0 < a < 1$ such that $B(y, a|y|) \subset D_1$ for every $y \in D_2$.

Let $D \subset X$ and $x \in \partial D$ and suppose that there exists a homeomorphism $\Phi : \overline{B}(0, r) \rightarrow \overline{U}_r$, with $U_r \in \mathcal{V}(x)$ such that $\Phi(0) = x$ and there exist $m, M > 0$ such that $l(z, \Phi) > m, L(z, \Phi) < M$ for every $z \in \overline{B}(0, r)$ and let $G_1 = \Phi(D_1)$ be such that $G_1 \setminus \{x\} \subset D \cap U_r$ and let $G_2 = \Phi(D_2)$.

Let $f : D \rightarrow Y$ be quasimonotone and continuous, satisfying condition (N^{-1}) , such that $\mu(B_f) = 0, \nu(f(B_f)) = 0$ and $\int_{D \cap U_r} L(z, f)^n d\mu < \infty$ and suppose that there exists a path $\gamma : [0, 1] \rightarrow G_1$ such that $\lim_{t \rightarrow 1} \gamma(t) = x$ and $\lim_{t \rightarrow 1} f(\gamma(t)) = c \in Y$. Then, $\lim_{\substack{y \rightarrow x \\ y \in G_2}} f(y) = c$.

Proof of Theorem 5. Let $\delta > 0$ be such that $\Phi(D_1 \cap B(0, \delta)) \subset G_1 \cap U_r \subset D \cap U_r$. Let $0 < \alpha < \beta < \delta$ and let us take points $\alpha_t, \beta_t \in D_1 \cap S(0, t)$ for every $\alpha < t < \beta$ and let $\Gamma_t = \Delta(\alpha_t, \beta_t, D_1 \cap S(0, t))$ for $t \in (\alpha, \beta)$ and let $\Gamma = \bigcup_{t \in (\alpha, \beta)} \Gamma_t$. Let $\rho \in F(\Gamma)$ and we can suppose that $\rho = 0$ on $C_{0, \alpha, \beta} \setminus D_1$. We see from Theorem 10.12 in [3] that there exists a constant $C(n)$ depending only on n such that $M_{S(0, t)}^n(\Gamma_t) \geq \frac{C(n)}{t}$ for every $\alpha < t < \beta$. Here, $M_{S(0, t)}^n(\Gamma_t) = \inf_{\eta \in F(\Gamma_t) \cap S(0, t)} \int \eta^n(z) d_{S(0, t)}$ for every $\alpha < t < \beta$. Also, $\rho|_{S(0, t)} \in F(\Gamma_t)$ for every $t \in (\alpha, \beta)$. We have

$$C(n) \ln\left(\frac{\beta}{\alpha}\right) = C(n) \int_{\alpha}^{\beta} \frac{dt}{t} \leq \int_{\alpha}^{\beta} (M_{S(0, t)}^n(\Gamma_t)) dt \leq \int_{\alpha}^{\beta} \left(\int_{S(0, t)} \rho^n(z) d_{S(0, t)} \right) dt = \int_{D_1 \cap C_{0, \alpha, \beta}} \rho^n(z) dz$$

and hence

$$C(n) \ln\left(\frac{\beta}{\alpha}\right) \leq M_n(\Gamma) \quad (13)$$

Let $\gamma_1 = \Phi^{-1} \circ \gamma : [0, 1] \rightarrow D_1$. Then, γ_1 is a path, $\gamma_1(0) \in D_1$ and $\lim_{t \rightarrow 1} \gamma_1(t) = 0$. Let $F = f \circ \Phi : D_1 \cap B(0, \delta) \rightarrow Y$. Using Lemma 5, we see that F is quasimonotone, continuous and $\lim_{t \rightarrow 1} F(\gamma_1(t)) = c, \mu_n(B_F) = 0, \nu(F(B_F)) = 0$, F satisfies condition (N^{-1}) and $\int_{D_1 \cap B(0, \delta)} L(z, F)^n dz < \infty$. We see from Theorem 1 that there exists $\omega \in L^1(Y)$ such that $M_n(\Gamma) \leq M_{\omega}^n(F(\Gamma))$ for every $\Gamma \in A(D_1 \cap B(0, \delta))$ and $\int_{F(A)} \omega(y) dv \leq \int_A L(z, F)^n dz$

for every open set $A \subset D_1 \cap B(0, \delta)$. We show that $\lim_{\substack{y \rightarrow 0 \\ y \in D_2}} F(y) = c$. Indeed, otherwise

we find $\epsilon > 0$ and points $y_k \rightarrow 0$, $y_k \in D_2 \cap B(0, \frac{\delta}{2})$ such that $d(F(y_k), c) \geq 3\epsilon$ for every $k \in \mathbb{N}$, $Im\gamma_1 \cap S(0, t) \cap D_1 \neq \emptyset$ and we can suppose that $d(F(\gamma_1(t)), c) \leq \epsilon$ for every $0 < t < \delta$. We also suppose that $B(y_k, 2a|y_k|) \subset D_1 \cap B(0, \delta)$ for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since F is quasimonotone, we can find $L > 0$ and $\alpha_t, \beta_t \in D_1 \cap S(y_k, t)$ such that $d(F(y), F(y_k)) \leq Ld(F(\alpha_t), F(\beta_t))$ for every $a|y_k| < t < 2a|y_k|$ and every $y \in B(y_k, a|y_k|)$. Let $\eta_k = \frac{L}{d(F(y), F(y_k))} \chi_{F(B(y_k, 2a|y_k|))}$ and let us take $\Gamma = \bigcup_{a|y_k| < t < 2a|y_k|} \Delta(\alpha_t, \beta_t, S(y_k, t) \cap D_1)$.

Then, $\eta_k \in F(F(\Gamma))$, and as before, we have

$$\begin{aligned} C(n) \ln 2 &= C(n) \ln \left(\frac{2a|y_k|}{a|y_k|} \right) \leq M_n(\Gamma) \leq M_\omega^n(F(\Gamma)) \leq \int_Y \eta_k(z)^n \omega(z) dv \leq \\ &\leq \frac{L^n}{d(F(y), F(y_k))^n} \int_{F(B(y_k, 2a|y_k|))} \omega(z) dv \leq \frac{L^n}{d(F(y), F(y_k))^n} \int_{B(y_k, 2a|y_k|)} L(z, F)^n dz. \end{aligned}$$

We showed that $d(F(y), F(y_k))^n \leq \frac{L^n}{C(n) \ln 2} \int_{B(y_k, 2a|y_k|)} L(z, F)^n dz \rightarrow 0$ if $k \rightarrow \infty$.

We showed that there exists $k_\epsilon \in \mathbb{N}$ such that

$$d(F(y), F(y_k)) \leq \epsilon \text{ for every } y \in B(y_k, 2a|y_k|) \text{ and every } k \geq k_\epsilon \quad (14)$$

and we can take $k_\epsilon = 1$. Then, it results that

$$d(F(y), c) \geq 2\epsilon \text{ for every } y \in B(y_k, 2a|y_k|) \text{ and every } k \geq k_\epsilon. \quad (15)$$

Let $k \in \mathbb{N}$. We see that $S(0, t) \cap B(y_k, a|y_k|) \neq \emptyset$ for every $t \in ((1-a)|y_k|, (1+a)|y_k|)$, and since $Im\gamma_1 \cap D_1 \cap S(0, t) \neq \emptyset$ for every $t \in (0, \delta)$, we can find points $\alpha_t \in Im\gamma_1 \cap D_1 \cap S(0, t)$, $\beta_t \in D_1 \cap S(0, t) \cap B(y_k, a|y_k|)$ for every $t \in ((1-a)|y_k|, (1+a)|y_k|)$. Let $\alpha = (1-a)|y_k|$ and $\beta = (1+a)|y_k|$ and let us take in relation (13) the path family $\Gamma = \bigcup_{t \in (\alpha, \beta)} \Delta(\alpha_t, \beta_t, D_1 \cap S(0, t))$.

Since $d(F(\alpha_t), F(\beta_t)) \geq \epsilon$ for every $\alpha < t < \beta$, we see that $\rho_k = \frac{1}{\epsilon} \chi_{F(B(0, \delta) \cap D_1)} \in F(F(\Gamma))$.

We have

$$\begin{aligned} C(n) \ln \left(\frac{1+a}{1-a} \right) &\leq M_n(\Gamma) \leq M_\omega^n(F(\Gamma)) \leq \int_Y \rho_k(z)^n \omega(z) dv \leq \\ &\leq \frac{1}{\epsilon^n} \int_{F(B(0, \delta) \cap D_1)} \omega(z) dv \leq \frac{1}{\epsilon^n} \int_{B(0, \delta) \cap D_1} L(z, F)^n dz \rightarrow 0 \text{ if } \delta \rightarrow 0. \end{aligned}$$

We reached a contradiction and we therefore proved that $\lim_{\substack{y \rightarrow 0 \\ y \in D_2}} F(y) = c$ and hence

$$\lim_{\substack{y \rightarrow x \\ y \in G_2}} f(y) = c. \quad \square$$

Our theorem is a Lindelöf-type theorem for mappings of finite conformal energy. This theorem was inspired by the work of M. Vuorinen concerning nontangential limits of quasiregular mappings and angular limits of the mappings having a bounded Dirichlet integral and defined on open sets from $\mathbb{R}^n$ (see [4,43–45]). As in the corresponding theorem from classical complex analysis or from the theory of quasiregular mappings (see [43]), D_1 may be a cone centered at 0, direction d and angle $\varphi \in (0, \frac{\pi}{2})$ and D_2 a cone centered at

0, direction d and angle $0 < \psi < \varphi$. In our case, D_1 and D_2 may be a kind of spirallike domain and hence can be more general than cones in $\mathbb{R}^n$. The preceding theorem may be a theorem concerning angular limits over domains G_2 with $x \in \partial G_2$, which are images of a spirallike domain $D_2 \subset \mathbb{R}^n$ with $0 \in \partial D_2$ by a normal parametrization Φ of some manifold X . Due to the generality of domains G_1 and G_2 , our theorem is an extension of Lindelöf's theorem even in $\mathbb{R}^n$. We immediately prove the following:

Theorem 6. Let $n \geq 2, r > 0, D_1 \subset B(0, r) \subset \mathbb{R}^n$ be a domain such that $D_1 \cap S(0, t)$ is a cap of a sphere for every $0 < t \leq r$ and let D_2 be a domain such that $\overline{D_2} \setminus \{0\} \subset D_1, 0 \in \overline{D_2} \cap \overline{D_1}$ and there exists $0 < a < 1$ such that $B(y, a|y|) \subset D_1$ for every $y \in D_2$. Let $D \subset \mathbb{R}^n$ be open such that $0 \in \partial D, D_1 \setminus \{0\} \subset D$ and let $f : D \rightarrow \mathbb{R}^n$ be a mapping of finite distortion satisfying relation (4) such that $\int_{D \cap B(0, r)} L(z, f)^n dz < \infty$ and suppose that there exists a path $\gamma : [0, 1) \rightarrow D_1$ such that $\lim_{t \rightarrow 1} \gamma(t) = 0$ and $\lim_{t \rightarrow 1} f(\gamma(t)) = c \in \mathbb{R}^n$. Then, $\lim_{\substack{y \rightarrow 0 \\ y \in D_2}} f(y) = c$.

We remind that Miklyukov showed in [48] that if $f : B^n \rightarrow \mathbb{R}^n$ is a bounded quasiregular mapping and $\int_{B^n} |f'(x)|^n dx < \infty$, then f has a.e. nontangential limits. Theorem 4 and Theorem 6 generalized this result in the class of mappings of finite distortion satisfying relation (4).

Corollary 1. Let $n \geq 2$, let X be a Riemannian n -manifold, let $G \subset X$ be open and $x \in \partial G$ be an isolated point of ∂G . Let $f : G \rightarrow Y$ be continuous and quasimonotone, satisfying condition (N^{-1}) such that $\mu(B_f) = 0, \nu(f(B_f)) = 0$ and $\int_G L(z, f)^n d\mu < \infty$. Suppose that there exists a path $\gamma : [0, 1) \rightarrow G$ such that $\lim_{t \rightarrow 1} \gamma(t) = x$ and $\lim_{t \rightarrow 1} f(\gamma(t)) = c$. Then, $\lim_{y \rightarrow x} f(y) = c$.

Remark 6. Usually in Theorem 5, we take X a Riemannian n -manifold and the mapping $\Phi : \overline{B}(0, r_x) \rightarrow \overline{U}_r$ a C^1 diffeomorphism. Also, in Theorem 3, we can take X a Riemannian n -manifold and $\Phi : \overline{B}^n \rightarrow \overline{D}$ a C^1 diffeomorphism. If $D \subset \mathbb{R}^n$ is a domain and $\Phi : \overline{B}^n \rightarrow \overline{D}$ is a homeomorphism such that there exist $m, M > 0$ such that $l(z, \Phi) > m, L(z, \Phi) < M$ for every $z \in \overline{B}^n$ and $f : D \rightarrow \mathbb{R}^n$ is a mapping of biconformal energy, then there exists $\lim_{t \rightarrow 1} f(\Phi(t\Phi^{-1}(y)))$ for $\mathcal{H}^{n-1}$ a.e. $y \in \partial D$.

Theorem 7. Let $n \geq 2$, let X, Y be Riemannian n -manifolds, let $D \subset X$ be a domain, let $\omega : D \rightarrow [0, \infty]$ be μ measurable and finite a.e. and let $f : D \rightarrow D_f$ be a homeomorphism such that $M_n(f(\Gamma)) \leq M_\omega^n(\Gamma)$ for every $\Gamma \in A(D)$. Let $x \in D$ be such that $\omega_x < \infty$. Then, there exist constants C_1, C_2, C_3, C_4 and a function $\eta_x : (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow 0} \eta_x(t) = 0, \eta_x$ is increasing on each interval $(0, 1)$ and $(1, \infty)$,

$$\eta_x(t) = \begin{cases} (C_1 / \mathcal{H}_n^{-1}(\frac{C_2 \omega_x}{(\ln \ln(\frac{t}{t})^n})), & \text{if } 0 < t < 1 \\ C_4 \exp(C_3 \omega_x(2 + t)^n), & \text{if } t \geq 1. \end{cases}$$

and there exists $r_x > 0$ such that

$$\frac{d(f(z), f(x))}{d(f(y), f(x))} \leq \eta_x\left(\frac{d(x, z)}{d(x, y)}\right) \text{ if } y, z \in B(x, r_x), x \neq y, x \neq z. \quad (16)$$

Proof of Theorem 7. Let $r_x > 0$ and $\Phi : \overline{B}(0, r_x) \rightarrow \overline{B}(x, r_x)$ be a C^1 diffeomorphism such that $\Phi(0) = x$ and $\Phi(S(0, t)) = S(x, t)$ for every $0 < t < r_x$ and let $V_x \in \mathcal{V}(f(x))$ be such that there exists $\psi : B(0, 1) \rightarrow V_x$ a C^1 diffeomorphism such that $\psi(0) = f(x)$ and $f(\overline{B}(x, r_x)) \subset V_x$. Let $y, z \in B(x, r_x)$ be such that $0 < d(x, y) < d(x, z) < \frac{r_x}{e}$, and let $a, b \in$

$B(0, r_x)$ be such that $\Phi(a) = y$ and $\Phi(b) = z$. Let $F : B(0, 1) \rightarrow B(0, 1)$, $F = \psi^{-1} \circ f \circ \Phi$. Then, $F : B(0, 1) \rightarrow F(B(0, 1))$ is a homeomorphism between open sets from $\mathbb{R}^n$, $F(0) = 0$ and let $\Gamma = \Gamma_{0, |a|, |b|}$. Then, $F(\Gamma)$ is a ring $R(C_0, C_1)$ and $F(0), F(a) \in C_0$, $F(b), \infty \in C_1$. Let L be such that $\frac{1}{L}d(\alpha, \beta) \leq d(\psi^{-1}(\beta), \psi^{-1}(\alpha)) \leq Ld(\alpha, \beta)$ for every $\alpha, \beta \in V_x$. Using Lemma 1 and Lemma 7, we have the following:

$$\begin{aligned} \mathcal{H}_n(L^2 \frac{d(f(z), f(x))}{d(f(y), f(x))}) &= \mathcal{H}_n(\frac{L^2 d(f \circ \Phi(b), f \circ \Phi(0))}{d(f \circ \Phi(a), f \circ \Phi(0))}) \leq \mathcal{H}_n(\frac{d(F(b), F(0))}{d(F(a), F(0))}) \leq \\ &\leq M_n(F(\Gamma)) \leq CM_n(f \circ \Phi(\Gamma)) \leq CM_\omega^n(\Phi(\Gamma)) = CM_\omega^n(\Gamma_{x, d(x, y), d(x, z)}) \leq \frac{C\omega_x}{(\ln \ln(\frac{ed(x, z)}{d(x, y)}))^n} \end{aligned}$$

if $0 < d(x, y) < d(x, z) < \frac{r_x}{e}$. We proved that

$$\frac{d(f(z), f(x))}{d(f(y), f(x))} \geq \frac{1}{L^2} \mathcal{H}_n^{-1}(\frac{C\omega_x}{(\ln \ln(\frac{ed(x, z)}{d(x, y)}))^n})$$

if $0 < d(x, y) < d(x, z) < \frac{r_x}{e}$.

Suppose now that $\frac{1}{L_1}d(\alpha, \beta) \leq d(\Phi(\alpha), \Phi(\beta)) \leq L_1d(\alpha, \beta)$ if $\alpha, \beta \in \bar{B}(0, r_x)$. Let $y, z \in B(x, r_x)$ be such that $0 < d(x, y) \leq d(x, z) \leq 2d(x, y) + d(x, z) < r_x$ and let $a, b \in B(0, r_x)$ be such that $\Phi(a) = y$, $\Phi(b) = z$, and as before, let $F = \psi^{-1} \circ f \circ \Phi$. Let $w \in S(0, |a| + |b|)$ be such that $L(0, F, |a| + |b|) = d(F(0), F(w))$ and let $B \in S(0, 2|a| + |b|)$ be such that $L(0, F, 2|a| + |b|) = d(F(0), F(B))$. Let P be the plane determined by the point $0, a, w$ and let C be the circle $P \cap S(a, |a|)$. Let E be the tangency point of C of a ray emerging from 0 to w and let γ be the shortest arc in the circle C joining 0 with E and let $E_1 = [E, w] \cup |\gamma|$. Let d be the line perpendicular on the plane P in the point a and let $M = d \cap S(0, 2|a| + |b|)$. Let $E_2 = [a, M] \cup S(0, 2|a| + |b|)$. Then, $d(E_1, E_2) \geq |a|$ and $d(F(0), F(a)) \leq L(0, F, |a|) \leq L(0, F, |a| + |b|) = d(F(0), F(w)) \leq L(0, F, 2|a| + |b|) = d(F(0), F(B))$. Let $\Delta = \Delta(E_1, E_2, B(0, 2|a| + |b|))$. Then, $\frac{1}{|a|}\chi_{B(0, 2|a| + |b|)} \in F(\Delta)$. Let $\Gamma' = \Delta(F(E_1), F(E_2), C_{0, d(F(0), F(a)), d(F(0), F(w))})$.

We see that $\Gamma' > F(\Delta)$, that $f(E_1)$ is a path joining $F(0)$ with $F(w)$ and that $F(E_2)$ contains a subpath joining $F(a)$ with $F(B)$, and since $d(F(0), F(w)) \leq d(F(0), F(B))$, we see from (13) that $M_n(\Gamma') \geq C(n) \ln(\frac{d(F(0), F(w))}{d(F(0), F(a))})$. We see that $d(\Phi(E_1), \Phi(E_2)) \geq \frac{d(E_1, E_2)}{L_1} \geq \frac{d(x, y)}{L_1}$ and hence $\rho = \frac{L_1}{d(x, y)} \chi_{B(x, 2d(x, y) + d(x, z))} \in F(\Phi(\Delta))$. We have the following:

$$\begin{aligned} C(n) \ln(\frac{d(f(z), f(x))}{L^2 d(f(y), f(x))}) &= C(n) \ln(\frac{d(f(\Phi(b))), d(f(\Phi(0)))}{L^2 d(f(\Phi(a))), f(\Phi(0))}) \leq \\ &C(n) \ln(\frac{d(F(b), F(0))}{d(F(a), F(0))}) \leq C(n) \ln(\frac{d(F(w), F(0))}{d(F(a), F(0))}) \leq M_n(\Gamma') \leq \\ &\leq M_n(F(\Delta)) = M_n(\Psi^{-1} \circ f \circ \Phi(\Delta)) \leq CM_n(f \circ \Phi(\Delta)) \leq \\ &CM_\omega^n(\Phi(\Delta)) \leq C \int_X \rho^n(z) \omega(z) d\mu \leq \frac{L_1^n C}{d(x, y)^n} \int_{B(x, 2d(x, y) + d(x, z))} \omega(u) du \leq \\ &\leq (\text{if } r_x \text{ is small enough}) \leq \frac{CL_1^n (2d(x, y) + d(x, z))^n}{d(x, y)^n} \int_{B(x, 2d(x, y) + d(x, z))} \omega(u) du \leq \\ &\leq CL_1^n \omega_x (2 + \frac{d(x, z)}{d(x, y)})^n \text{ if } 0 < d(x, y) \leq d(x, z) < 2d(x, y) + d(x, z) < r_x. \end{aligned}$$

We proved that

$$\frac{d(f(z), f(x))}{d(f(y), f(x))} \leq L^2 \exp\left(\frac{CL_1^n \omega_x}{C(n)} \left(2 + \frac{d(x, z)}{d(x, y)}\right)\right)^n$$

if $0 < d(x, y) \leq d(x, z) < 2d(x, y) + 2d(x, z) < r_x$. $\square$

Corollary 2. Let $f : D \rightarrow D_f$ be a homeomorphism satisfying condition (N) and such that $\int_Y L(y, g)^n dv < \infty$, where $g : D_f \rightarrow D$ is the inverse of f . Then, relation (16) is satisfied for a.e. $x \in D$ and $H(x, f) < \infty$ a.e.

Proof of Corollary 2. We see from Theorem 1 that there exists $\omega \in L^1(X)$ such that $M_n(\Gamma') \leq M_\omega^n(g(\Gamma))$ for every $\Gamma' \in A(D_f)$ and this implies that $M_n(f(\Gamma)) \leq M_\omega^n(\Gamma)$ for every $\Gamma \in A(D)$. Since $\omega \in L^1(X)$, we see that $\omega_x < \infty$ μ a.e. in D and we apply Theorem 7. Also, the proof of Corollary 3 follows immediately from Corollary 2. $\square$

Corollary 3. Let $f : D \rightarrow D_f$ be a homeomorphism between two domains in $\mathbb{R}^n$ such that $f \in W_{loc}^{1,n}(D, D_f)$ and $f^{-1} \in W_{loc}^{1,n}(D_f, D)$. Then, relation (16) holds for a.e. $x \in D$.

Theorem 8. Let $n \geq 2$, $D \subset \mathbb{R}^n$ a domain, $a, b \in D$, $a \neq b$, $n - 1 < q \leq n$ and $r > 0$ be such that $H_D^q \neq \emptyset$. Then, there exists an injective minimizer $f \in H_D^q$ for the total energy $E_t^q(D)$ such that $E_{t,D}^q(f) = E_t^q(D)$. Also, if $H_{D,M}^q \neq \emptyset$, there exist injective minimizers $f_1, f_2 \in H_{D,M}^q$ such that $E_M^q(D) = E_D(f_1)$ and $E_{t,M}^q(D) = E_{t,D}^q(f_2)$ and f_1 and f_2 may be different.

Proof of Theorem 8. Let $I = E_t^q(D) < \infty$ and let $f_j \in H_D^q$ be such that $I < E_{t,D}^q(f_j) < I + \frac{1}{j}$ for every $j \in \mathbb{N}$. We see from Corollary 7.5.1 in [6] that the family $(f_j)_{j \in \mathbb{N}}$ is equicontinuous and, eventually taking a subsequence, we find $f : D \rightarrow \mathbb{R}^n$ such that $f_j \rightarrow f$ uniformly on the compact subsets D . We see from Lemma 4 that $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$ and that $\int_D |f'(x)|^n dx \leq \liminf_{j \rightarrow \infty} \int_D |f_j'(x)|^n dx = I$. Let $g_j : D_{f_j} \rightarrow D$ be the inverse of $f_j : D \rightarrow D_{f_j}$ for every $j \in \mathbb{N}$. Then, g_j is a.e. differentiable, satisfies condition (N^{-1}) and $\int_{D_{f_j}} |g_j'(y)|^q dy = \int_D K_{I,q}(x, f_j) \leq$

$E_{t,D}^q(f_j) < \infty$ for every $j \in \mathbb{N}$. We see from Theorem 1 that there exists $\omega_j \in L^1(\mathbb{R}^n)$ such that $M_q(\Gamma') \leq M_{\omega_j}^q(g_j(\Gamma'))$ for every $\Gamma' \in A(D_{f_j})$ and every $j \in \mathbb{N}$ and hence $M_q(f_j(\Gamma)) \leq M_{\omega_j}^q(\Gamma)$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$. We see that

$$\begin{aligned} \int_D \omega_j(z) dz &= \int_{g_j(f_j(D))} \omega_j(z) dz \leq \int_{f_j(D)} L(y, g_j)^q dy = \int_{f_j(D)} |g_j'(y)|^q dy = \\ &= \int_D K_{I,q}(x, f_j) dx \leq E_{t,D}^q(f_j) < I + 1 \end{aligned}$$

for every $j \in \mathbb{N}$.

We see from Lemma 3 that either f is constant on D or there exists a domain $G \subset \mathbb{R}^n$ such that $f : D \rightarrow G$ is a homeomorphism and $f_j \rightarrow f$ uniformly on the compact subsets of D . Since $q(f_j(a), f_j(b)) \geq r$ for every $j \in \mathbb{N}$, we see that $q(f(a), f(b)) \geq r > 0$ and hence there exists a domain $G \subset \mathbb{R}^n$ such that $f : D \rightarrow G$ is a homeomorphism. Using Lemma 1 in [13], we find that G is a component of $\text{Ker}_{j \rightarrow \infty} D_{f_j}$ and let $g : G \rightarrow D$ be the inverse of $f : D \rightarrow G$. Let now K and U be domains such that $K \subset \subset G$ and $U \subset \subset D$ and $K \subset f(U)$. Then, $K \subset f(U) \subset \text{Ker}_{j \rightarrow \infty} f_j(U)$. Let us show that there exists $j_0 \in \mathbb{N}$ such that $g_j(K) \subset \bar{U}$ for $j \geq j_0$. Indeed, otherwise we find infinitely many

points $a_j \in K$ such that $\alpha_j = g_j(a_j) \notin \bar{U}$. Then, $f_j(\alpha_j) = a_j \in K \subset f(U) \subset \text{Ker}_{j \rightarrow \infty} f_j(U)$; and for j great enough, we see that $f_j(\alpha_j) \subset f_j(U)$ and hence we find $\beta_j \in U$ such that $f_j(\alpha_j) = f_j(\beta_j)$; and since $\alpha_j \neq \beta_j$, this contradicts the injectivity of the mapping f_j . We proved that there exists $j_0 \in \mathbb{N}$ such that $g_j(K) \subset \bar{U}$ for every $j \geq j_0$ and we can suppose that $K \subset f(U) \subset D_{f_j}$ for every $j \geq j_0$. We can take $j_0 = 1$ and then the mappings $g_j|_K : K \rightarrow D$ are well defined for every $j \in \mathbb{N}$. The mappings g_j satisfy condition (N^{-1}) and $\int_K |g'_j(y)|^q dy \leq \int_{D_{f_j}} |g'_j(y)|^q dy = \int_D K_{L,q}(x, f_j) dx \leq E_{t,D}^q(f_j) < I + 1$ for every $j \in \mathbb{N}$ and we see from Theorem 2 that the family $W = (g_j|_K)_{j \in \mathbb{N}}$ is equicontinuous. In fact, the family $W_1 = (g_j|_{K_1})_{j \in \mathbb{N}}$ is equicontinuous even if $K \subset \subset K_1$ and $K_1 \subset f(U)$. We can suppose that there exists $h : \bar{K} \rightarrow \mathbb{R}^n$ such that $g_j \rightarrow h$ uniformly on $\bar{K}$. We see that $h(\bar{K}) \subset \bar{U} \subset D$ and let us show that $g_j \rightarrow g$ uniformly on $\bar{K}$. Indeed, otherwise we can find $\epsilon > 0$ and $z_j, z \in \bar{K}$ such that $z_j \rightarrow z$ and $|g_j(z_j) - g(z_j)| > \epsilon$ for every $j \in \mathbb{N}$. Now, $w_j = g_j(z_j) \rightarrow h(z) = w \in \bar{U} \subset D$ and let $x_j = g(z_j)$ for every $j \in \mathbb{N}$. We can suppose that $x_j \rightarrow x \in g(\bar{K}) \subset D$, and since the family $(f_j)_{j \in \mathbb{N}}$ is equicontinuous at w , we find that $f_j(w_j) \rightarrow f(w)$. Then, $f_j(w_j) = f_j(g_j(z_j)) = z_j = f(x_j) \rightarrow f(x)$; and letting $j \rightarrow \infty$, we find that $f(w) = f(x)$; and since f is injective, it results that $x = w$. On the other side, $|x_j - w_j| = |g(z_j) - g_j(z_j)| > \epsilon$ for every $j \in \mathbb{N}$ and then $|x - w| \geq \epsilon$ and we reached a contradiction.

We proved that $g_j \rightarrow g$ uniformly on K and hence $g_j \rightarrow g$ in $W_{loc}^{1,q}(K, \mathbb{R}^n)$. As before, using Lemma 4, we have

$$\int_K |g'(y)|^q dy \leq \liminf_{j \rightarrow \infty} \int_K |g'_j(y)|^q dy \leq \liminf_{j \rightarrow \infty} \int_{D_{f_j}} |g'_j(y)|^q dy$$

and letting $K \nearrow G$, we find that $g \in W_{loc}^{1,q}(G, \mathbb{R}^n)$ and

$$\int_G |g'(y)|^q dy \leq \liminf_{j \rightarrow \infty} \int_{D_{f_j}} |g'_j(y)|^q dy.$$

We have

$$\begin{aligned} E_{t,D}^q(f) &= \int_D (|f'(x)|^n + K_{L,q}(x, f)) dx = \int_D |f'(x)|^n dx + \int_G |g'(y)|^q dy \leq \\ &\leq \liminf_{j \rightarrow \infty} \int_D |f'_j(x)|^n dx + \liminf_{j \rightarrow \infty} \int_{D_{f_j}} |g'_j(y)|^q dy = \\ &= \liminf_{j \rightarrow \infty} \int_D |f'_j(x)|^n dx + \liminf_{j \rightarrow \infty} \int_D K_{L,q}(x, f_j) dx \leq \\ &\leq \liminf_{j \rightarrow \infty} \int_D (|f'_j(x)|^n + K_{L,q}(x, f_j)) dx = \liminf_{j \rightarrow \infty} E_{t,D}^q(f_j) = I. \end{aligned}$$

We proved that $f \in H_D^q$ and that $E_{t,D}^q(f) = E_t^q(D)$.

Suppose now that $H_{D,M}^q \neq \emptyset$. Then, $I = E_M^q(D) \leq E_{t,M}^q(D) < M$ and let $f_j \in H_{D,M}^q$ be such that $I < E_D(f_j) < I + \frac{1}{j}$ for every $j \in \mathbb{N}$. As before, taking eventually a subsequence, we can suppose that there exists $f \in W_{loc}^{1,n}(D, \mathbb{R}^n)$ such that $f_j \rightarrow f$ uniformly on the compact subsets of D and $\int_D |f'(x)|^n dx \leq \liminf_{j \rightarrow \infty} \int_D |f'_j(x)|^n dx$. Let $g_j : D_{f_j} \rightarrow D$ be the inverse of $f_j : D \rightarrow D_{f_j}$ for every $j \in \mathbb{N}$. Then, g_j satisfies condition (N^{-1}) ,

is a.e. differentiable and $\int_{D_{f_j}} |g'_j(y)|^q dy = \int_D K_{I,q}(x, f_j) dx < M$ for every $j \in \mathbb{N}$. Using

Theorem 1, we find $\omega_j \in L^1(\mathbb{R}^n)$ such that $M_q(\Gamma') \leq M_{\omega_j}^q(g_j(\Gamma'))$ for every $\Gamma' \in A(D_{f_j})$ and every $j \in \mathbb{N}$ and hence $M_q(f_j(\Gamma)) \leq M_{\omega_j}^q(\Gamma)$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$. Then, $\int_D \omega_j(z) dz = \int_{g_j(f_j(D))} \omega_j(z) dz \leq \int_{D_{f_j}} L(z, g_j)^q dz = \int_{D_{f_j}} |g'_j(y)|^q dy = \int_D K_{I,q}(x, f_j) dx \leq E_{t,D}^q(f_j) < M$ for every $j \in \mathbb{N}$.

Using Lemma 3, we find a domain $G \subset \mathbb{R}^n$ such that $f : D \rightarrow G$ is a homeomorphism, and if $g : G \rightarrow D$ is the inverse of $f : D \rightarrow G$, we prove as before that $\int_G |g'(y)|^q dy \leq \liminf_{j \rightarrow \infty} \int_{D_{f_j}} |g'_j(y)|^q dy$ and hence that $E_{t,D}^q(f) \leq \lim_{j \rightarrow \infty} E_{t,D}^q(f_j) < M$. We proved that $f \in H_{D,M}^q$ and that $E_M^q(D) = E_D(f)$. $\square$

5. Conclusions

Vuorinen (see [4,43–45]) was the first who remarked that the mappings of a bounded Dirichlet integral satisfy a modular inequality. He used this method to prove some geometric properties of such mappings. We develop his method and, by showing that such mappings satisfy an inverse Poletsky modular inequality, we can obtain strong generalizations of some classical theorems from complex analysis. We open the systematic study of geometric properties of some classes of mappings of finite distortion using the modulus method. The conditions for finding injective minimizers may be useful both in theoretical and applied mathematics.

Funding: This research received no external funding.

Data Availability Statement: The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Reshetnyak, Y.G. *Space Mappings of Bounded Distortion, Translations of Mathematical Monographs*; AMS: Providence, RI, USA, 1989; Volume 73.
2. Rickman, S. *Quasiregular Mappings*; Springer: Berlin/Heidelberg, Germany, 1993; Volume 26.
3. Väisälä, J. *Lectures on n-Dimensional Quasiconformal Mappings, Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 1971; Volume 229.
4. Vuorinen, M. *Conformal Geometry and Quasiregular Mappings, Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 1988; Volume 1319.
5. Hencl, S.; Koskela, P. *Lectures on Mappings of Finite Distortion, Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 2014; Volume 2096.
6. Iwaniec, T.; Martin, G. *Geometric Function Theory and Non-Linear Analysis, Oxford Math. Monographs*; The Clarendon Press: Oxford, UK; Oxford University Press: New York, NY, USA, 2001.
7. Cristea, M. Open, discrete mappings having local ACL^n inverses. *Complex Var. Elliptic Equ.* **2010**, *55*, 61–90. [[CrossRef](#)]
8. Cristea, M. Cluster sets theorems on metric spaces. *Anal. Math. Phys.* **2022**, *12*, 139. [[CrossRef](#)]
9. Koskela, P.; Onninen, J. Mappings of finite distortion: Capacity and modulus inequalities. *J. Reine Angew. Math.* **2005**, *599*, 1–26. [[CrossRef](#)]
10. Afanas'eva, E. Boundary extension of mappings with integrally bounded moduli in Finsler manifolds. *Complex Anal. Oper. Theory* **2021**, *13*, 2839–2851. [[CrossRef](#)]
11. Afanas'eva, E.; Golberg, A. Topological mappings of finite area distortion. *Anal. Math. Phys.* **2022**, *12*, 1–29. [[CrossRef](#)]
12. Cristea, M. Boundary behaviour of the mappings satisfying generalized inverse modular inequalities. *Complex Var. Elliptic Equ.* **2015**, *60*, 437–469. [[CrossRef](#)]
13. Cristea, M. The limit mapping of generalized ring homeomorphisms. *Complex Var. Elliptic Equ.* **2016**, *61*, 608–622. [[CrossRef](#)]

14. Cristea, M. Eliminability results for mappings satisfying generalized modular inequalities. *Complex Var. Elliptic Equ.* **2019**, *64*, 676–684. [\[CrossRef\]](#)
15. Cristea, M. Boundary behaviour of open, light mappings in metric measure spaces. *Ann. Fenn. Math.* **2021**, *46*, 1179–1201. [\[CrossRef\]](#)
16. Cristea, M. The local behaviour of open, light mappings satisfying generalized modular inequalities. *Complex Var. Elliptic Equ.* **2022**, *67*, 1598–1623. [\[CrossRef\]](#)
17. Cristea, M. On the radial limits of mappings on Riemannian manifolds. *Anal. Math. Phys.* **2023**, *13*, 60. [\[CrossRef\]](#)
18. Golberg, A.; Sevost'yanov, E. Absolute continuity on path of spatial discrete mappings. *Anal. Math. Phys.* **2018**, *81*, 25–35. [\[CrossRef\]](#)
19. Ilyutko, D.; Sevost'yanov, E. Boundary behaviour of open, discrete mappings on Riemannian manifolds. *Sb. Math.* **2018**, *209*, 605–651. [\[CrossRef\]](#)
20. Martio, O.; Ryazanov, V.; Srebro, U.; Yakubov, E. Mappings of finite length distortion. *J. Anal. Math.* **2004**, *93*, 215–236. [\[CrossRef\]](#)
21. Martio, O.; Ryazanov, V.; Srebro, U.; Yakubov, E. On Q -homeomorphisms. *Ann. Fenn. Math.* **2005**, *30*, 49–69.
22. Martio, O.; Ryazanov, V.; Srebro, U.; Yakubov, E. *Moduli in Modern Mapping Theory*; Springer: New York, NY, USA, 2009.
23. Ryazanov, V.; Salimov, R.; Sevost'yanov, E. On convergence analysis of space homeomorphisms. *Siberian Adv. Math.* **2013**, *23*, 263–293. [\[CrossRef\]](#)
24. Ryazanov, V.; Volkov, S. On the boundary behaviour of mappings in the class $W_{loc}^{1,1}$ on Riemann surfaces. *Complex Anal. Oper. Theory* **2017**, *11*, 1503–1520. [\[CrossRef\]](#)
25. Salimov, R.; Sevost'yanov, E. ACL and differentiability of the open, discrete mappings. *Complex Var. Elliptic Equ.* **2010**, *55*, 49–59. [\[CrossRef\]](#)
26. Salimov, R.; Sevost'yanov, E. The Poletskii and Väisälä inequalities for the mappings with (p, q) distortion. *Complex Var. Elliptic Equ.* **2014**, *59*, 217–234. [\[CrossRef\]](#)
27. Sevost'yanov, E. Local and boundary behaviour of maps in metric spaces. *St. Petersburg Math. J.* **2017**, *28*, 807–824. [\[CrossRef\]](#)
28. Sevost'yanov, E. *Mappings with Direct and Inverse Poletsky Inequalities*; Springer: Berlin/Heidelberg, Germany, 2023.
29. Cristea, M. Quasiregularity in metric spaces. *Rev. Roumaine Math. Pures Appl.* **2006**, *51*, 607–631.
30. Guo, C.Y.; Williams, M. The branch set of quasiregular mappings between metric manifolds. *C. R. Acad. Sci. Paris Ser. I* **2016**, *354*, 155–159. [\[CrossRef\]](#)
31. Guo, C.Y.; Williams, M. Geometric function theory: The art of pullback factorisation. *arXiv* **2016**, arXiv:1611.02478.
32. Heinonen, J. *Lectures on Analysis on Metric Spaces*; Springer: New York, NY, USA, 2001.
33. Heinonen, J.; Holopainen, I. Quasiregular Maps on Carnot Groups. *J. Geom. Anal.* **1997**, *7*, 109–148. [\[CrossRef\]](#)
34. Heinonen, J.; Koskela, P. Definitions of quasiconformality. *Invent. Math.* **1995**, *120*, 61–79. [\[CrossRef\]](#)
35. Heinonen, J.; Koskela, P. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* **1998**, *181*, 1–61. [\[CrossRef\]](#)
36. Korany, A.; Reimann, H. Quasiconformal mappings on the Heisenberg group. *Invent. Math.* **1985**, *80*, 309–338. [\[CrossRef\]](#)
37. Markina, I. Singularities of quasiregular mappings on Carnot groups. *Sci. Ser. Math. Sci. Univ. Tehnică Federico Maria* **2005**, *11*, 69–81.
38. Onninen, J.; Rajala, K. Quasiregular maps to generalized manifolds. *J. Anal. Math.* **2009**, *109*, 33–79. [\[CrossRef\]](#)
39. Vodopyanov, S.; Uklov, D. Sobolev spaces of (p, q) -quasiconformal mappings on Carnot groups. *Sib. Mat. Zhurnal* **1998**, *39*, 776–795. [\[CrossRef\]](#)
40. Sevost'yanov, E.; Skvortshov, S. On mappings whose inverses satisfy the Poletsky inequality. *Ann. Acad. Sci. Fenn. Math.* **2020**, *45*, 259–277. [\[CrossRef\]](#)
41. Koskela, P.; Maly, J. Mappings of finite distortion: The zero set of the Jacobian. *J. Eur. Math. Soc.* **2003**, *5*, 95–105. [\[CrossRef\]](#)
42. Manfredi, J.; Villamor, E. An extension on Reshetnyak theorem. *Indiana Univ. Math. J.* **1998**, *47*, 1131–1145.
43. Vuorinen, M. Exceptional Sets and Boundary Behaviour of Quasiregular Mappings in n -Space. *Ann. Acad. Sci. Fenn.* **1976**.
44. Vuorinen, M. Capacity densities and angular limits of quasiregular mappings. *Trans. Am. Math. Soc.* **1981**, *263*, 343–354. [\[CrossRef\]](#)
45. Vuorinen, M. On functions with a finite or locally bounded Dirichlet integral. *Ann. Acad. Sci. Fenn. AI Math.* **1984**, *9*, 177–193. [\[CrossRef\]](#)
46. Manfredi, J.; Villamor, E. Traces of monotone Sobolev functions. *J. Geom. Anal.* **1996**, *6*, 433–444. [\[CrossRef\]](#)
47. Manfredi, J.; Villamor, E. Traces of monotone Sobolev functions in weighted Sobolev spaces III. *J. Math.* **2001**, *45*, 402–422.
48. Miklyukov, V. Boundary property of n -dimensional mappings with bounded distortion. *Mat. Zametki* **1972**, *11*, 159–164. [\[CrossRef\]](#)
49. Martio, O.; Rickman, S. Boundary behaviour of quasiregular mappings. *Ann. Acad. Sci. Fenn. Math. AI* **1972**, *507*, 1–17.
50. Äkkinen, T. Radial limits of mappings of bounded and finite distortion. *J. Geom. Anal.* **2014**, *24*, 1298–1322. [\[CrossRef\]](#)
51. Cristea, M. A generalization of Sard's lemma. A Jacobian condition for interiority. *Demonstr. Math.* **1988**, *21*, 399–406.

52. Iwaniec, T.; Onninen, J. Deformations of finite conformal energy: Boundary behaviour and limit theorems. *Trans. Am. Math. Soc.* **2011**, *363*, 5605–5648. [[CrossRef](#)]
53. Iwaniec, T.; Onninen, J.; Zhu, Z. Deformations of bi-conformal energy and a new characterization of quasiconformality. *Arch. Ration. Mech. Anal.* **2020**, *263*, 1709–1737. [[CrossRef](#)]
54. Iwaniec, T.; Onninen, J.; Pankka, P.; Radice, T. A neohookean model of plates. *SIAM J. Math. Anal.* **2021**, *53*, 509–529. [[CrossRef](#)]
55. Ball, J.M. Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. R. Soc. Eding. Sect. A* **1981**, *88*, 315–328. [[CrossRef](#)]
56. Caraman, P. Relations between p -capacity and p -module (I). *Rev. Roumaine Math. Pures Appl.* **1994**, *39*, 509–553.
57. Lee, J.M. *Riemannian Manifolds. An Introduction to Curvature, Graduate Texts in Math*; Springer: New York, NY, USA, 1997; Volume 176.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.