

Article

The Singularity of the K_4 Homeomorphic Graph

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Abstract: Let G be a finite simple graph and let $A(G)$ be its adjacency matrix. Then, G is singular if $A(G)$ is singular. The singularity of graphs is of certain interest in graph theory and algebraic combinatorics. For positive integers $a_i \geq 2$, $i = 1, 2, \dots, 6$. Insert $a_1 - 2$, $a_2 - 2$, $a_3 - 2$, $a_4 - 2$, $a_5 - 2$ and $a_6 - 2$ vertices in the six edges of the complete graph K_4 , respectively, then the resulting graph is called the K_4 homeomorphic graph, denoted by $K(a_1, a_2, a_3, a_4, a_5, a_6)$. In this paper, we give the necessary and sufficient condition for the singularity of $K(a_1, a_2, a_3, a_4, a_5, a_6)$, and we also show that the probability of a K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$ being a singular graph is equal to $\frac{193}{512}$.

Keywords: singular graphs; homeomorphic; probability; nullity

MSC: 05C50

1. Introduction

Suppose that G is a finite undirected simple graph, and its vertex set is denoted as $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix of G is defined as the $n \times n$ matrix $A(G) = (a_{ij})$,

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

where $v_i \sim v_j$ indicates that the vertices v_i and v_j are neighboring to each other. Evidently, $A(G)$ is a real symmetric matrix whose elements are either 0 or 1, and its characteristic roots are real numbers. The characteristic roots of $A(G)$ are also referred to as the characteristic roots of G . Moreover, the characteristic roots of G form the spectrum of G . The quantity of non-zero characteristic roots and zero characteristic roots within the spectrum of G are, respectively, named the rank and the nullity of G , and they are denoted by $r(G)$ and $\eta(G)$. It is clear that $r(G) + \eta(G) = n$.

In chemistry, when we represent the atoms in a molecule as vertices and the bonds between atoms as edges, we will obtain a graph. This graph is called the molecular graph of this molecule. Molecular graphs have many applications [1–6]. For example, according to Hückel's molecular orbital theory, the π -electron energy of a molecule is approximately equal to the sum of the absolute values of all the characteristic roots of the molecular graph it represents. For another example, It is a necessary condition for the stability of the chemical properties of the molecule represented by a graph that the nullity of the graph is equal to 0 [3]. In 1957, Collatz and Sinogowitz [4] put forward the problem of characterizing all graphs whose nullity is greater than zero. We say that a graph is singular if it has a zero characteristic roots. In other words, its nullity is greater than 0. Thus, the problem raised by Collatz and Sinogowitz is equivalent to the task of depicting all singular graphs, which



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is a highly difficult problem indeed. For the purpose of delving into this issue, plenty of people examine the structural traits of singular graphs, and many results have been derived concerning the correlation between a graph's nullity and its structure (see [7–14]).

Singular graphs possess numerous equivalent definitions. For instance, a graph G is singular precisely when the kernel space of $A(G)$ is not a null space; A graph G is singular precisely when the homogeneous system of linear equations $AX = 0$ with $A(G)$ as the coefficient matrix has a non-zero solution. From this perspective, the authors [15–20] studied the singularity of graphs and provided many necessary or sufficient conditions for a graph to be singular. A graph is singular precisely when there exists a nonzero vector $X = (x_1, x_2, \dots, x_n)$ such that using this vector to weight the vertices of the graph, each vertex $u \in V(G)$ satisfies

$$\sum_{v \sim u} x_v = 0, \quad (1)$$

Figure 1 gives the vertex weighting of the grid graph $P_7 \times P_7$ and the wheel graph W_8 satisfying Equation (1), which implies that the grid graph $P_7 \times P_7$ and the wheel graph W_8 are singular.

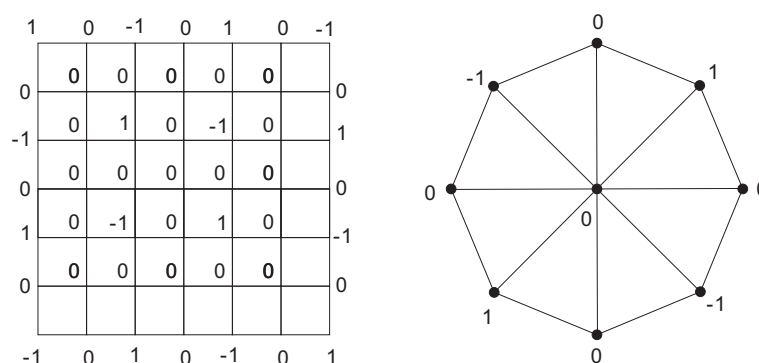


Figure 1. A weighting on the grid graph $P_7 \times P_7$ and the wheel graph W_8 that satisfies Equation (1).

A graph G is singular if and only if $A(G)$ turns out to be a singular matrix, which means that the determinant of $A(G)$ equals zero. From this perspective, the authors [21–23] provided the necessary and sufficient conditions for the singularity of several categories of tricyclic graphs as well as the probability of singular graphs occurring in these graph classes. The latest investigations have also demonstrated that singular graphs are associated with other branches of mathematics, like the representation theory of finite groups, combinatorial mathematics, algebraic geometry and the like (refer to [24–26]).

Let K_n , P_n and C_n denote the complete graph, path and cycle with n vertices, respectively. For positive integers $a_i \geq 2$, $i = 1, 2, \dots, 6$. Insert $a_1 - 2$, $a_2 - 2$, $a_3 - 2$, $a_4 - 2$, $a_5 - 2$ and $a_6 - 2$ vertices in the six edges of the complete graph K_4 , respectively, then the resulting graph is called the K_4 homeomorphic graph, denoted by $K(a_1, a_2, a_3, a_4, a_5, a_6)$, see Figure 2. In this paper, it is always agreed that the paths marked a_1 and a_4 have no common vertices, the paths marked a_2 and a_5 have no common vertices, and the paths marked a_3 and a_6 have no common vertices. So the K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$ defined above (see Figure 2) has 4 vertices of degree 3, marked as A, B, C and D ; there are 6 paths between them. We denote the graph formed by three 3-degree vertices and the path between them by a triangle, for instance, the triangle ABC , denoted by $\triangle ABC$. The graph composed of the vertices A, B, C, D and the four paths connected in sequence is called a quadrilateral (excluding the two paths on it that do not have common vertices), for instance, the quadrilateral $A - a_1 - B - a_3 - C - a_4 - D - a_6 - A$, denoted by $\square ABCD$. If two triangles have a common path, we say that the two triangles are interdependent. Denote O as the set of odd numbers in $a_1, a_2, a_3, a_4, a_5, a_6$.

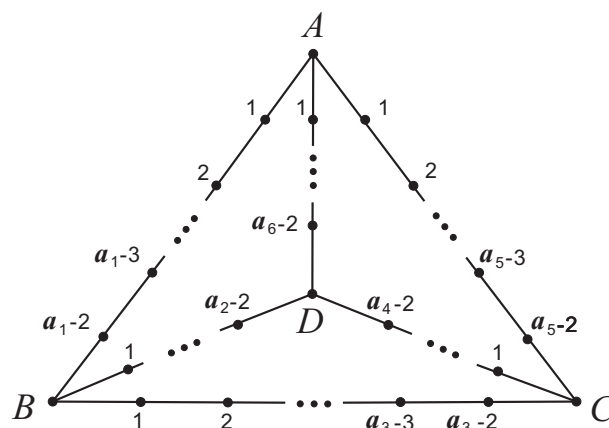


Figure 2. The K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$.

In the context of graph theory, when considering two graphs G and H , the symbol $G \cup H$ signifies the disjoint union of G and H . Meanwhile, mG represents the disjoint union of m copies of G . Notably, mK_2 designates the disjoint union of the m isolated edges. For any notations and terminologies that have not been expounded upon above, one should consult [1] for further clarification.

We know that the singularity of a graph G can be ascertained from the subsequent perspectives: (1) whether 0 constitutes a characteristic root of $A(G)$; (2) whether the kernel space of $A(G)$ is a null space; (3) whether there exists a particular weighting scheme for the vertices of this graph that complies with Equation (1); (4) whether the value of the determinant of $A(G)$ equals zero. A great number of preceding research findings predominantly focused on ascertaining whether a graph was singular by considering the 0 characteristic root of the graph. Merely a small number of academic papers depicted singular graphs from the standpoint of the determinant. Moreover, only extremely restricted literature investigated the necessary and sufficient conditions for the singularity of a particular class of graphs. In this paper, from the perspective of determinant, we give the necessary and sufficient condition for the singularity of $K(a_1, a_2, a_3, a_4, a_5, a_6)$, and we also show that the probability of a K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$ being a singular graph is equal to $\frac{193}{512}$. We will prove the following results in Section 3.

Theorem 1. The K_4 homeomorphic graph $G = K(a_1, a_2, a_3, a_4, a_5, a_6)$ is singular if and only if one of the following holds.

- (i) $|O| = 1$, the length of at least one triangle in G is a multiple of 4.
- (ii) $|O| = 2$, the paths marked by two numbers in O have no common vertex in G , and G has at least one pair of interdependent triangles with odd paths as a common side, and the lengths of these triangles are all multiples of 4.
- (iii) $|O| = 3$, the paths marked by three numbers in O form a triangle in G .
- (iv) $|O| = 4$. (1) The paths marked by three numbers in O form a triangle in G , and the length of this triangle is a multiple of 4; (2) the paths marked by four numbers in O form a quadrilateral in G , and the length of this quadrilateral is a multiple of 4.
- (v) $|O| \geq 5$.

Theorem 2. The probability that any given K_4 homeomorphic graph $G = K(a_1, a_2, a_3, a_4, a_5, a_6)$ is a singular graph is equal to $\frac{193}{512}$.

2. Lemma

A subgraph H of G is designated as an elementary subgraph (Sachs subgraph) provided that each connected component of H is either a cycle or a copy of K_2 within G . An

elementary subgraph that encompasses all the vertices of G is referred to as a spanning elementary subgraph (spanning Sachs subgraph) of G . A matching is defined as a collection of edges where no vertices are shared among them, and a perfect matching is a matching that ensures every vertex of G is saturated. It is quite evident that the quantity of vertices of a graph G which possesses a perfect matching must be an even number.

Lemma 1 ([1]). Let G be a graph of order n , $A(G)$ be the adjacency matrix of G , then

$$\det(A(G)) = (-1)^n \sum_{H \in \mathcal{H}} (-1)^{p(H)} 2^{c(H)},$$

where $\mathcal{H}$ is the set of spanning Sachs subgraphs of G , and $p(H)$ and $c(H)$ are the number of connected components and cycles in H , respectively.

To understand the concept of spanning Sachs subgraphs and Lemma 1, we first give two examples.

Example 1. As shown in Figure 3, there exists one spanning Sachs subgraph of graph X with two cycles: $C_5 \cup C_6 \cup K_2$ (see (1)). There exist two spanning Sachs subgraphs of graph X with cycle C_5 ; they are $C_5 \cup 4K_2$ (see (2) and (3)). There is no spanning Sachs subgraph of graph X with cycle C_6 because the number of vertices remaining after removing C_6 from X is odd. There is no perfect matching in X because the number of vertices of X is odd. By Lemma 1

$$\det(A(X)) = (-1)^{13} [(-1)^3 \times 2^2 + (-1)^5 \times 2 \times 2] = 8.$$

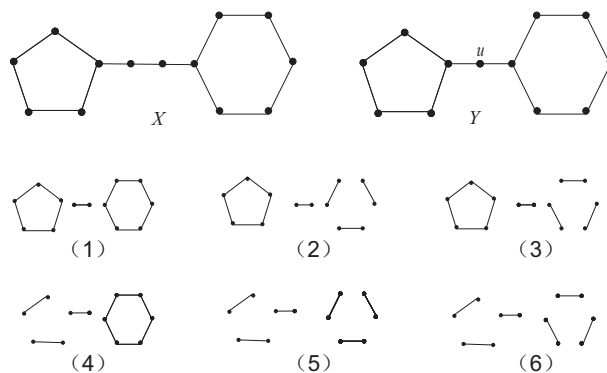


Figure 3. The graph X and Y , and the spanning Sachs subgraphs of these two graphs.

There exist no spanning Sachs subgraphs of graph Y with two cycles $C_5 \cup C_6$; otherwise, vertex u does not match. There is no spanning Sachs subgraph of graph Y with cycle C_5 because the number of vertices remaining after removing C_5 from Y is odd. There exists one spanning Sachs subgraph of graph Y with cycle C_6 ; they are $C_6 \cup 3K_2$ (see (4)). There exist two perfect matches in Y (see (5) and (6)). By Lemma 1

$$\det(A(Y)) = (-1)^{12} [(-1)^4 \times 2 + (-1)^6 \times 2] = 4.$$

Lemma 2. Let G_1 be a graph consisting of paths of length $a_1 - 1, a_2 - 1, a_3 - 1, a_5 - 1, a_6 - 1$ as shown in Figure 4; a_1 is an odd number, and a_2, a_3, a_5, a_6 are all even numbers; then

$$(-1)^{\frac{(a_1-1)+a_2+a_3}{2}} + (-1)^{\frac{(a_1-1)+a_5+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} = 0, \quad (2)$$

if and only if the length of at least one of triangle $\triangle ABD$ and triangle $\triangle ABC$ have a multiple of 4.

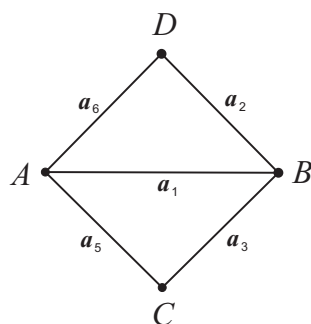


Figure 4. The graph G_1 .

Proof. We prove this lemma in the following two cases.

(i) $a_1 \equiv 1 \pmod{4}$. In this case, Equation (2) becomes:

$$(-1)^{\frac{a_2+a_3}{2}} + (-1)^{\frac{a_5+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} = 0, \quad (3)$$

Equation (3) is equivalent to

$$[(-1)^{\frac{a_2}{2}} + (-1)^{\frac{a_6}{2}}][(-1)^{\frac{a_3}{2}} + (-1)^{\frac{a_5}{2}}] = 0.$$

Then, Equation (3) holds if and only if $a_2 \not\equiv a_6 \pmod{4}$ or $a_3 \not\equiv a_5 \pmod{4}$ holds. That is, $a_2 + a_6 \equiv 2 \pmod{4}$ or $a_3 + a_5 \equiv 2 \pmod{4}$ holds. When and only when $a_1 + a_2 + a_6 - 3 \equiv 0 \pmod{4}$ or $a_1 + a_3 + a_5 - 3 \equiv 0 \pmod{4}$ holds. That is, the length of $\triangle ABD$ is a multiple of 4 or the length of $\triangle ABC$ is a multiple of 4.

(ii) $a_1 \equiv 3 \pmod{4}$. In this case, Equation (2) becomes:

$$-(-1)^{\frac{a_2+a_3}{2}} - (-1)^{\frac{a_5+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} = 0, \quad (4)$$

Equation (4) is equivalent to

$$[(-1)^{\frac{a_6}{2}} - (-1)^{\frac{a_2}{2}}][(-1)^{\frac{a_3}{2}} - (-1)^{\frac{a_5}{2}}] = 0.$$

Then, Equation (4) holds if and only if $a_2 \equiv a_6 \pmod{4}$ or $a_3 \equiv a_5 \pmod{4}$ holds. That is $a_1 + a_2 + a_6 - 3 \equiv 0 \pmod{4}$ or $a_1 + a_3 + a_5 - 3 \equiv 0 \pmod{4}$ holds. This is the case if and only if the length of $\triangle ABD$ is a multiple of 4 or the length of $\triangle ABC$ is a multiple of 4.

Hence, we have finished the proof of the lemma. $\square$

For convenience, suppose that M represents the set of all even numbers; M_0 and M_2 represent the set of even numbers with a remainder of 0 and 2 divided by 4, respectively. Then, $M_0 \cup M_2 = M$, $\overline{M_0} = M_2$, $\overline{M_2} = M_0$. The following lemma is obvious.

Lemma 3. If $a_2, a_3, a_5, a_6 \in M$ and $a_2 + a_3 + a_5 + a_6 \equiv 0 \pmod{4}$, then one of the following holds.

- (i) $a_2, a_3, a_5, a_6 \in M_i$ ($i = 0$ or 2 .)
- (ii) $a_2, a_3 \in M_i; a_5, a_6 \in \overline{M_i}$ ($i = 0$ or 2 .)
- (iii) $a_2, a_5 \in M_i; a_3, a_6 \in \overline{M_i}$ ($i = 0$ or 2 .)
- (iv) $a_2, a_6 \in M_i; a_3, a_5 \in \overline{M_i}$ ($i = 0$ or 2 .)

Proof. Four even numbers a_2, a_3, a_5, a_6 such that $a_2 + a_3 + a_5 + a_6 \equiv 0 \pmod{4}$ holds forces all four of them to belong to M_0 (or M_2), or two of them to belong to M_0 and the other two to belong to M_2 . $\square$

Lemma 4. Let G_2 be a K_4 homeomorphic graph consisting of paths of length $a_1 - 1, a_2 - 1, a_3 - 1, a_4 - 1, a_5 - 1, a_6 - 1$ as shown in Figure 5, and a_1, a_4 are two odd numbers, and a_2, a_3, a_5, a_6 are all even numbers. Then,

$$\begin{aligned} & (-1)^{\frac{(a_1-1)+a_2+a_3}{2}} + (-1)^{\frac{a_3+(a_4-1)+a_5}{2}} + (-1)^{\frac{(a_1-1)+a_5+a_6}{2}} + (-1)^{\frac{a_2+(a_4-1)+a_6}{2}} \\ & + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{(a_1-1)+a_2+a_3+(a_4-1)+a_5+a_6}{2}} \times 2 = 0, \end{aligned} \quad (5)$$

if and only if the two pairs of interdependent triangles in G_2 with odd paths have common sides; the length of at least one pair of triangles is a multiple of 4 (Figure 5).

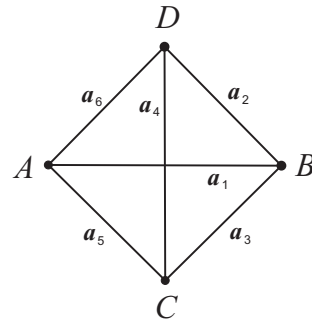


Figure 5. The graph G_2 .

Proof. We prove this lemma in the following four cases.

Case 1. $a_1 \equiv 1 \pmod{4}, a_4 \equiv 1 \pmod{4}$. In this case, Equation (5) becomes:

$$\begin{aligned} & (-1)^{\frac{a_2+a_3}{2}} + (-1)^{\frac{a_3+a_5}{2}} + (-1)^{\frac{a_5+a_6}{2}} + (-1)^{\frac{a_2+a_6}{2}} \\ & + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{a_2+a_3+a_5+a_6}{2}} \times 2 = 0, \end{aligned} \quad (6)$$

Multiplying both sides of Equation (6) by $(-1)^{\frac{a_2+a_3+a_5+a_6}{2}}$, we obtain

$$(-1)^{\frac{a_5+a_6}{2}} + (-1)^{\frac{a_2+a_6}{2}} + (-1)^{\frac{a_2+a_3}{2}} + (-1)^{\frac{a_3+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + 2 = 0, \quad (7)$$

Subtracting Equation (7) from Equation (6), we obtain

$$(-1)^{\frac{a_2+a_3+a_5+a_6}{2}} = 1, \quad (8)$$

that is

$$a_2 + a_3 + a_5 + a_6 \equiv 0 \pmod{4}. \quad (9)$$

It follows from (9) and Lemma 3 that a_2, a_3, a_5, a_6 are congruent $\pmod{4}$, or they are congruent in pairs $\pmod{4}$. We construct Table 1 below to check the cases where Equation (7) holds.

In Table 1, the elements below M_i indicate that these elements belong to M_i . $\emptyset$ indicates that none of the four elements belong to $\overline{M_i}$, $\checkmark$ indicates that Equation (7) holds, $\times$ indicates that Equation (7) does not hold.

It is known from Table 1 that Equation (7) holds if and only if a_2, a_3, a_5, a_6 cannot all belong to M_0 (or M_2).

(ii) or (iii) of Table 1 hold if and only if a_2 and a_6 do not belong to the same set M_i , a_3 and a_5 do not belong to the same set, if and only if $a_1 + a_2 + a_6 - 3 \equiv 0 \pmod{4}$, $a_1 + a_3 + a_5 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles above $\triangle ABD$ and below $\triangle ABC$ are multiples of 4.

(iii) or (iv) of Table 1 holds if and only if a_2 and a_3 do not belong to the same set M_i , a_5 and a_6 do not belong to the same set, if and only if $a_4 + a_5 + a_6 - 3 \equiv 0 \pmod{4}$, $a_2 + a_3 + a_4 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles left $\triangle ADC$ and right $\triangle BDC$ are multiples of 4.

Table 1. The various situations in which Equation (7) holds.

Case	M_i ($i = 0$ or 2)	$\overline{M_i}$	Equation (7)
(i)	a_2, a_3, a_5, a_6	$\emptyset$	$\times$
(ii)	a_2, a_3	a_5, a_6	$\checkmark$
(iii)	a_2, a_5	a_3, a_6	$\checkmark$
(iv)	a_2, a_6	a_3, a_5	$\checkmark$

Case 2. $a_1 \equiv 1 \pmod{4}$, $a_4 \equiv 3 \pmod{4}$. In this case, Equation (5) becomes:

$$\begin{aligned} & (-1)^{\frac{a_2+a_3}{2}} - (-1)^{\frac{a_3+a_5}{2}} + (-1)^{\frac{a_5+a_6}{2}} - (-1)^{\frac{a_2+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} \\ & - (-1)^{\frac{a_2+a_3+a_5+a_6}{2}} \times 2 = 0. \end{aligned} \quad (10)$$

Multiplying both sides of Equation (10) by $(-1)^{\frac{a_2+a_3+a_5+a_6}{2}}$, we obtain

$$(-1)^{\frac{a_5+a_6}{2}} - (-1)^{\frac{a_2+a_6}{2}} + (-1)^{\frac{a_2+a_3}{2}} - (-1)^{\frac{a_3+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} - 2 = 0, \quad (11)$$

Subtracting Equation (11) from Equation (10), we obtain

$$(-1)^{\frac{a_2+a_3+a_5+a_6}{2}} = 1, \quad (12)$$

that is

$$a_2 + a_3 + a_5 + a_6 \equiv 0 \pmod{4}. \quad (13)$$

It follows from (13) and Lemma 3 that a_2, a_3, a_5, a_6 are congruent $\pmod{4}$, or they are congruent in pairs $\pmod{4}$. We construct Table 2 below to check the cases where Equation (11) holds.

Table 2. The various situations in which Equation (11) holds.

Case	M_i ($i = 0$ or 2)	$\overline{M_i}$	Equation (11)
(i)	a_2, a_3, a_5, a_6	$\emptyset$	$\checkmark$
(ii)	a_2, a_3	a_5, a_6	$\checkmark$
(iii)	a_2, a_5	a_3, a_6	$\checkmark$
(iv)	a_2, a_6	a_3, a_5	$\times$

It is known from Table 2 that Equation (11) holds if and only if (i), (ii), or (iii) holds.

(i) or (ii) holds if and only if $a_2 \equiv a_3 \pmod{4}$, $a_5 \equiv a_6 \pmod{4}$ (that is, a_2 and a_3 belong to the same set M_i , a_5 and a_6 belong to same set), if and only if $a_4 + a_5 + a_6 - 3 \equiv 0 \pmod{4}$, $a_2 + a_3 + a_4 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles left $\triangle ADC$ and right $\triangle BDC$ are multiples of 4.

(ii) or (iii) holds if and only if $a_2 \not\equiv a_6 \pmod{4}$, $a_3 \not\equiv a_5 \pmod{4}$ (that is, a_2 and a_6 do not belong to the same set M_i , a_3 and a_5 do not belong to the same set, and each set M_i has exactly two elements), if and only if $a_1 + a_2 + a_6 - 3 \equiv 0 \pmod{4}$, $a_1 + a_3 + a_5 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles above $\triangle ABD$ and below $\triangle ABC$ are multiples of 4.

Case 3. $a_1 \equiv 3 \pmod{4}$, $a_4 \equiv 1 \pmod{4}$. Similar to case 3, this is omitted.

Case 4. $a_1 \equiv 3 \pmod{4}$, $a_4 \equiv 3 \pmod{4}$. In this case, Equation (5) becomes:

$$-(-1)^{\frac{a_2+a_3}{2}} - (-1)^{\frac{a_3+a_5}{2}} - (-1)^{\frac{a_5+a_6}{2}} - (-1)^{\frac{a_2+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{a_2+a_3+a_5+a_6}{2}} \times 2 = 0. \quad (14)$$

Multiplying both sides of Equation (14) by $(-1)^{\frac{a_2+a_3+a_5+a_6}{2}}$, we obtain

$$-(-1)^{\frac{a_5+a_6}{2}} - (-1)^{\frac{a_2+a_6}{2}} - (-1)^{\frac{a_2+a_3}{2}} - (-1)^{\frac{a_3+a_5}{2}} + (-1)^{\frac{a_3+a_6}{2}} + (-1)^{\frac{a_2+a_5}{2}} + 2 = 0, \quad (15)$$

Subtracting Equation (15) from Equation (14), we obtain

$$(-1)^{\frac{a_2+a_3+a_5+a_6}{2}} = 1, \quad (16)$$

that is

$$a_2 + a_3 + a_5 + a_6 \equiv 0 \pmod{4}. \quad (17)$$

It follows from (17) and Lemma 3 that a_2, a_3, a_5, a_6 are congruent $\pmod{4}$, or they are congruent in pairs $\pmod{4}$. We construct Table 3 below to check the cases where Equation (15) holds.

Table 3. The various situations in which Equation (15) holds.

Case	M_i ($i = 0$ or 2)	$\overline{M_i}$	Equation (15)
(i)	a_2, a_3, a_5, a_6	$\emptyset$	$\checkmark$
(ii)	a_2, a_3	a_5, a_6	$\checkmark$
(iii)	a_2, a_5	a_3, a_6	$\times$
(iv)	a_2, a_6	a_3, a_5	$\checkmark$

It is known from Table 3 that Equation (15) holds if and only if (i), (ii), or (iv) holds.

(i) or (ii) holds if and only if $a_2 \equiv a_3 \pmod{4}$, $a_5 \equiv a_6 \pmod{4}$, if and only if $a_4 + a_5 + a_6 - 3 \equiv 0 \pmod{4}$, $a_2 + a_3 + a_4 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles left $\triangle ADC$ and right $\triangle BDC$ are multiples of 4.

(ii) or (iv) holds if and only if $a_2 \not\equiv a_5 \pmod{4}$, $a_3 \not\equiv a_6 \pmod{4}$ (that is, a_2 and a_5 do not belong to the same set M_i , a_3 and a_6 do not belong to the same set, and each set M_i has exactly two elements), when and only when $a_2 \equiv a_3 \not\equiv a_5 \equiv a_6 \pmod{4}$ or $a_2 \equiv a_6 \not\equiv a_3 \equiv a_5 \pmod{4}$, when and only when $a_2 + a_3 + a_4 - 3 \equiv 0 \pmod{4}$, $a_4 + a_5 + a_6 - 3 \equiv 0 \pmod{4}$, or $a_1 + a_2 + a_6 - 3 \equiv 0 \pmod{4}$, $a_1 + a_3 + a_5 - 3 \equiv 0 \pmod{4}$, if and only if the lengths of both triangles left $\triangle ADC$ and right $\triangle BDC$ are multiples of 4, or the lengths of both triangles above $\triangle ABD$ and below $\triangle ABC$ are multiples of 4.

Hence, we have finished the proof of the lemma. $\square$

Let A, B be two random events, and let $p(A)$ and $p(A|B)$ represent the probability of event A occurring and the conditional probability of event A occurring if event B has occurred, respectively.

Lemma 5. Let the K_4 homomorphic graph have exactly two odd paths and four even paths. Then, the conditional probability of event A occurring if event $O = 2$ has occurred is equal to $\frac{3}{40}$.

A : these two odd paths have no common vertices, and the two pairs of interdependent triangles in a K_4 homomorphic graph with odd paths have common sides; the length of at least one pair of triangles is a multiple of 4.

Proof. There are $\binom{6}{2}$ pairs of paths to choose from in the K_4 homeomorphic graph, but only 3 pairs of paths have no common vertices. Therefore, the probability that two odd paths have no common vertices is $\frac{3}{\binom{6}{2}}$.

Without loss of generality, let the two odd paths be labeled as a_1, a_4 , and the even paths be labeled a_2, a_3, a_5, a_6 (see Figure 5). Divide the set $\{a_2, a_3, a_5, a_6\}$ into two parts according to the remainder modulo 4; the part with fewer elements (in cases of equal number of elements, the part containing a_2) is denoted as X ; there are a total of 8 cases: $X = \phi, \{a_2\}, \{a_3\}, \{a_5\}, \{a_6\}, \{a_2, a_6\}, \{a_2, a_3\}, \{a_2, a_5\}$.

Let B represent the following random event: Under the premise that the two odd paths have no common vertices in this graph, the two pairs of interdependent triangles in the K_4 homomorphic graph with odd paths have common sides; the length of at least one pair of triangles is a multiple of 4. When $a_1 \equiv a_4 \equiv 1 \pmod{4}$, it follows from Table 1 that event B occurs if and only of $X = \{a_2, a_6\}, \{a_2, a_3\}, \{a_2, a_5\}$. That is,

$$p(B|a_1 \equiv a_4 \equiv 1) = \frac{3}{8}.$$

When $a_1 \equiv 1, a_4 \equiv 3 \pmod{4}$, it follows from Table 2 that event B occurs if and only of $X = \phi, \{a_2, a_3\}, \{a_2, a_5\}$. That is,

$$p(B|a_1 \equiv 1, a_4 \equiv 3) = \frac{3}{8}.$$

Similarly,

$$p(B|a_1 \equiv 3, a_4 \equiv 1) = \frac{3}{8}, \quad p(B|a_1 \equiv 3, a_4 \equiv 3) = \frac{3}{8}.$$

According to the total probability formula

$$\begin{aligned} p(B) &= p(a_1 \equiv 1, a_4 \equiv 1)p(B|a_1 \equiv 1, a_4 \equiv 1) + p(a_1 \equiv 1, a_4 \equiv 3)p(B|a_1 \equiv 1, a_4 \equiv 3) \\ &\quad + p(a_1 \equiv 3, a_4 \equiv 1)p(B|a_1 \equiv 3, a_4 \equiv 1) + p(a_1 \equiv 3, a_4 \equiv 3)p(B|a_1 \equiv 3, a_4 \equiv 3) \\ &= \frac{1}{4} \times \frac{3}{8} + \frac{1}{4} \times \frac{3}{8} + \frac{1}{4} \times \frac{3}{8} + \frac{1}{4} \times \frac{3}{8} = \frac{3}{8}. \end{aligned}$$

So

$$p(A|O = 2) = \frac{3}{\binom{6}{2}} \times \frac{3}{8} = \frac{3}{40}.$$

□

3. Proofs of Main Results

Proof of Theorem 1. Note that G is the K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, \text{ and } a_5, a_6)$, O is the set of odd numbers in $a_1, a_2, a_3, a_4, a_5, a_6$, then $|V(G)| = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - 8$. According to the value of $|O|$, we divided it into the following seven cases for the proof.

Case 1. $|O| = 0$. In this case, the spanning Sachs subgraph containing a triangular component does not exist in graph G (this is because, after deleting this triangle, the remaining vertices do not have a perfect matching). The spanning Sachs subgraphs containing a quadrilateral component in G are $C_{a_2+a_3+a_5+a_6-4} \cup \frac{a_1+a_4-4}{2}K_2$, $C_{a_1+a_3+a_4+a_6-4} \cup \frac{a_2+a_5-4}{2}K_2$, $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$. And G has three perfect matchings (because there are exactly three ways to match vertex A in Figure 2). By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_1+a_4-4}{2}+1} \times 2 + (-1)^{\frac{a_2+a_5-4}{2}+1} \times 2 + (-1)^{\frac{a_3+a_6-4}{2}+1} \times 2$$

$$+(-1)^{\frac{a_1+a_2+a_3+a_4+a_5+a_6-8}{2}} \times 3 = 0.$$

But this is impossible.

Case 2. $|O| = 1$. Without losing generality, we assume that a_1 is an odd number. In this case, the spanning Sachs subgraphs with a triangular component in G are $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$ and $C_{a_4+a_5+a_6-3} \cup \frac{a_1+a_2+a_3-5}{2}K_2$. (In fact, there are four triangles in graph G . However, a spanning Sachs subgraph with a triangular component $C_{a_1+a_2+a_6-3}$ does not exist. This is because after removing the cycle $C_{a_1+a_2+a_6-3}$ from G , the number of remaining vertices is odd and thus cannot form a perfect matching. Similarly, a spanning Sachs subgraph with a triangular component $C_{a_1+a_3+a_5-3}$ also does not exist.) The spanning Sachs subgraphs with a quadrilateral component in G are $C_{a_1+a_3+a_4+a_6-4} \cup \frac{a_2+a_5-4}{2}K_2$, $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$. (In fact, there are three quadrilaterals in graph G . However, a spanning Sachs subgraph with a quadrilateral component $C_{a_2+a_3+a_5+a_6-4}$ does not exist. This is because after removing the cycle $C_{a_2+a_3+a_5+a_6-4}$ from G , the number of remaining vertices is odd and thus cannot form a perfect matching.) G has no perfect matching (because the number of vertices in graph G is odd). By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_2+a_3-5}{2}+1} \times 2 + (-1)^{\frac{a_2+a_5-4}{2}+1} \times 2 + (-1)^{\frac{a_3+a_6-4}{2}+1} \times 2 = 0. \quad (18)$$

By Lemma 2, Equation (18) holds if and only if the length of at least one of the two triangles which are interdependent with odd paths as common sides is a multiples of 4.

Case 3. $|O| = 2$.

Subcase 3.1. The two paths marked by two numbers in O have a common vertex in G . Suppose a_1 and a_2 are odd numbers. In this subcase, the spanning Sachs subgraphs with a triangular component in G are $C_{a_1+a_3+a_5-3} \cup \frac{a_2+a_4+a_6-5}{2}K_2$, $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$. The spanning Sachs subgraph with a quadrilateral component in G is $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$. G has three perfect matchings (because there are exactly three ways to match vertex B in Figure 2). By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_2+a_4+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_3+a_6-4}{2}+1} \times 2 \\ + (-1)^{\frac{a_1+a_2+a_3+a_4+a_5+a_6-8}{2}} \times 3 = 0.$$

But this is impossible.

Subcase 3.2. The two paths marked by two numbers in O have no common vertices in G . Suppose a_1 and a_4 are odd numbers. In this subcase, the spanning Sachs subgraphs with a triangular component in G are $C_{a_4+a_5+a_6-3} \cup \frac{a_1+a_2+a_3-5}{2}K_2$, $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$, $C_{a_1+a_2+a_6-3} \cup \frac{a_3+a_4+a_5-5}{2}K_2$, $C_{a_1+a_3+a_5-3} \cup \frac{a_2+a_4+a_6-5}{2}K_2$. The spanning Sachs subgraphs with a quadrilateral component in G are $C_{a_1+a_3+a_4+a_6-4} \cup \frac{a_2+a_5-4}{2}K_2$, $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$; G has four perfect matchings (there are two perfect matchings where vertex A is saturated by an edge on path a_1 , and there are two perfect matchings where vertex A is not saturated by an edge on path a_1). By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_1+a_2+a_3-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_3+a_4+a_5-5}{2}+1} \times 2 \\ + (-1)^{\frac{a_2+a_4+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_2+a_5-4}{2}+1} \times 2 + (-1)^{\frac{a_3+a_6-4}{2}+1} \times 2 \\ + (-1)^{\frac{a_1+a_2+a_3+a_4+a_5+a_6-8}{2}} \times 4 = 0, \quad (19)$$

By Lemma 4, Equation (19) holds if and only if the lengths of at least one pair of triangles which are interdependent with odd paths as common sides are all multiples of 4.

Case 4. $|O| = 3$.

Subcase 4.1. The three paths marked by three numbers in O have a common vertex in G . Suppose a_1, a_2, a_3 are odd numbers. In this subcases, the spanning Sachs subgraphs with a triangular component in G are $C_{a_1+a_2+a_6-3} \cup \frac{a_3+a_4+a_5-5}{2}K_2$, $C_{a_1+a_3+a_5-3} \cup \frac{a_2+a_4+a_6-5}{2}K_2$, $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$. There is no Sachs subgraph with a quadrilateral component in G . There is no perfect matching in G . By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_3+a_4+a_5-5}{2}+1} \times 2 + (-1)^{\frac{a_2+a_4+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 = 0.$$

But this is impossible.

Subcase 4.2. The three paths marked by three numbers in O form a triangle in G . Suppose a_1, a_2, a_6 are odd numbers. In this subcase, G has no spanning Sachs subgraph, so G is singular.

Subcase 4.3. The three paths marked by three numbers in O form a path in G . Suppose a_1, a_2, a_4 are odd numbers. In this subcase, the spanning Sachs subgraphs with a triangular component in G are $C_{a_1+a_2+a_6-3} \cup \frac{a_3+a_4+a_5-5}{2}K_2$ and $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$. The Sachs subgraph with a quadrilateral component in G is $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$. There is no perfect matching in G . By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_3+a_4+a_5-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_3+a_6-4}{2}+1} \times 2 = 0.$$

But this is impossible.

Case 5. $|O| = 4$.

Subcase 5.1. The two paths marked by the two even numbers have a common vertex in G . Suppose a_1, a_2, a_3, a_4 are odd numbers. In this subcase, the spanning Sachs subgraph with a triangular component in G is $C_{a_2+a_3+a_4-3} \cup \frac{a_1+a_5+a_6-5}{2}K_2$. There is no spanning Sachs subgraph with a quadrilateral component in G . G has two perfect matchings (In this subcase, the vertex D cannot be saturated by an edge on the paths a_5 and a_6 (see Figure 2). Thus, there are only one possible matchings for vertex D , which in turn results in the graph G having only two perfect matchings. Without loss of generality, let $a_1 = a_2 = a_3 = a_4 = 3$ and $a_5 = a_6 = 4$. The thick lines in Figure 6(1,2) show two perfect matchings in $K(3,3,3,3,4,4)$). By Lemma 1, G is singular if and only if

$$(-1)^{\frac{a_1+a_5+a_6-5}{2}+1} \times 2 + (-1)^{\frac{a_1+a_2+a_3+a_4+a_5+a_6-8}{2}} \times 2 = 0, \quad (20)$$

Multiplying both sides of Equation (20) by $(-1)^{\frac{(a_1-1)+(a_2-1)+(a_3-1)+(a_4-1)+a_5+a_6-4}{2}}$, we obtain

$$(-1)^{\frac{(a_2-1)+(a_3-1)+(a_4-1)}{2}} = 1, \quad (21)$$

and Equation (21) holds if and only if the length of the triangle $\triangle BCD$ is a multiple of 4.

Subcase 5.2. The two paths marked by the two even numbers have no common vertices in G . Suppose a_1, a_2, a_4, a_5 are odd numbers. In this subcase, there is no spanning Sachs subgraph with a triangular component in G . The spanning Sachs subgraph with a quadrilateral component in G is $C_{a_1+a_2+a_4+a_5-4} \cup \frac{a_3+a_6-4}{2}K_2$. G has two perfect matchings (In this subcase, the vertex D cannot be saturated by an edge on the path a_6 (see Figure 2). Thus, there are only two possible matchings for vertex D , which in turn results in the graph G having only two perfect matchings. Without loss of generality, let $a_1 = a_2 = a_4 = a_5 = 3$ and $a_3 = a_6 = 4$. The thick lines in Figure 6(3,4) show that perfect matching in $K(3,3,4,3,3,4)$). By Lemma 1

$$(-1)^{\frac{a_3+a_6-4}{2}+1} \times 2 + (-1)^{\frac{a_1+a_2+a_3+a_4+a_5+a_6-8}{2}} \times 2 = 0. \quad (22)$$

Multiplying both sides of Equation (22) by $(-1)^{\frac{(a_1-1)+(a_2-1)+a_3+(a_4-1)+(a_5-1)+a_6-4}{2}}$, we obtain

$$(-1)^{\frac{(a_1-1)+(a_2-1)+(a_4-1)+(a_5-1)}{2}} = 1, \quad (23)$$

and Equation (23) holds if and only if the length of the quadrilateral $\square ABDC$ is a multiple of 4.

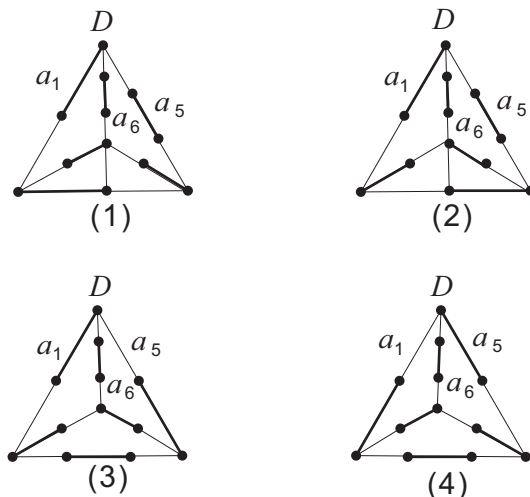


Figure 6. Two perfect matchings in Figure $K(3,3,3,3,4,4)$ (1,2). Two perfect matchings in Figure $K(3,3,4,3,3,4)$ (3,4).

Case 6. $|O| = 5$. In this case, G has no spanning Sachs subgraph, G is singular.

Case 7. $|O| = 6$. In this case, G has no spanning Sachs subgraph, G is singular.

To sum up, we have completed the proof of Theorem 1 $\square$

Now, we can prove Theorem 2

Proof of Theorem 2. Let A be an event: the graph $G = K(a_1, a_2, a_3, a_4, a_5, a_6)$ is a singular graph. By the full probability formula

$$\begin{aligned} p(A) &= p(|O| = 0)p(A||O| = 0) + p(|O| = 1)p(A||O| = 1) + p(|O| = 2)p(A||O| = 2) \\ &\quad + p(|O| = 3)p(A||O| = 3) + p(|O| = 4)p(A||O| = 4) + p(|O| = 5)p(A||O| = 5) \\ &\quad + p(|O| = 6)p(A||O| = 6), \end{aligned}$$

Since the random event $|O|$ follows a binomial distribution $B(6, \frac{1}{2})$, for $i = 0, 1, 2, 3, 4, 5, 6$, $p(|O| = i) = \binom{6}{i}(\frac{1}{2})^i(1 - \frac{1}{2})^{6-i} = \binom{6}{i}(\frac{1}{2})^6$.

(i) $|O| = 1$ (see Figure 4), the probability that at least one of the two triangles has a length that is a multiple of 4 is $\frac{3}{4}$, namely $p(A||O| = 1) = \frac{3}{4}$.

(ii) $|O| = 2$ (see Figure 5), by and Lemma 5, $p(A||O| = 2) = \frac{3}{40}$.

(iii) $|O| = 3$, the probability that three odd paths form a triangle is $\frac{4}{\binom{6}{3}}$.

(iv) $|O| = 4$, the probability that three odd-numbered paths form a triangle (two even-numbered paths have a common vertex) is $\frac{12}{\binom{6}{4}}$, and the probability that the length of this triangle is a multiple of 4 is $\frac{1}{2}$. The probability that four odd-numbered paths form a quadrilateral (two even-numbered paths have no common vertices) is $\frac{3}{\binom{6}{4}}$, and the probability that the length of this quadrilateral is a multiple of 4 is $\frac{1}{2}$. So $p(A||O| = 4) = \frac{12}{15} \times \frac{1}{2} + \frac{3}{15} \times \frac{1}{2} = \frac{1}{2}$.

To summarize,

$$p(A) = \binom{6}{0} \left(\frac{1}{2}\right)^6 \times 0 + \binom{6}{1} \left(\frac{1}{2}\right)^6 \times \frac{3}{4} + \binom{6}{2} \left(\frac{1}{2}\right)^6 \times \frac{3}{40} + \binom{6}{3} \left(\frac{1}{2}\right)^6 \times \frac{4}{20} \\ + \binom{6}{4} \left(\frac{1}{2}\right)^6 \times \frac{1}{2} + \binom{6}{5} \left(\frac{1}{2}\right)^6 \times 1 + \binom{6}{6} \left(\frac{1}{2}\right)^6 \times 1 = \frac{193}{512}.$$

Hence, we have finished the proof of Theorem 2. $\square$

4. Conclusions

The main contribution of this paper is the construction of the K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$, and the determination of the necessary and sufficient condition for the singularity of $K(a_1, a_2, a_3, a_4, a_5, a_6)$. We also show that the probability of a K_4 homeomorphic graph $K(a_1, a_2, a_3, a_4, a_5, a_6)$ being a singular graph is $\frac{193}{512}$. If the structure of the Sachs subgraphs of a graph can be determined, the singularity of this graph can be determined using the method employed in this paper. Finally, we claim that if the structure of the spanning Sachs subgraphs of a graph can be determined, then the singularity of this graph can be determined by the method used in this paper.

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