

Co-Secure Domination Number in Some Graphs

Jiatong Cui ¹, Tianhao Li ¹, Jiayuan Zhang ¹, Xiaodong Chen ^{1,*} and Liming Xiong ^{2,*} 

¹ School of Mathematics, Liaoning Normal University, Dalian 116029, China; cuijiatong1220@126.com (J.C.); litianhao0323@163.com (T.L.); zhangjiayuan202402@163.com (J.Z.)

² School of Mathematics and Statistics, Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, China

* Correspondence: xiaodongchen74@126.com (X.C.); lmxiong@bit.edu.cn (L.X.)

Abstract: Let G be a graph and $S \subseteq V(G)$. If S is a dominating set of G , and for each vertex $u \in V(G) - S$, there is a neighbor of u in S , denoted by v , such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G , then S is a secure dominating set (SDS) of G . Conversely, S is a co-secure dominating set (CSDS) of G if S is a dominating set of G and for each vertex v in S , $V(G) - S$ contains a neighbor of v , denoted by u , such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G . The minimum cardinality of a CSDS (resp. SDS) of G is the co-secure (resp. secure) domination number of G . We use $\gamma_{cs}(G)$ and $\gamma_s(G)$ to denote the co-secure domination number and secure domination number of G , respectively. Arumugam et al. proposed two questions: (1) Characterize a graph G with $\gamma_{cs}(G) = \alpha(G)$, where $\alpha(G)$ is the independence number of G ; (2) Characterize a graph G with $\gamma_{cs}(G) = \gamma_s(G)$. In this paper, we characterize some forbidden induced subgraphs for a graph G with $\gamma_{cs}(G) = \alpha(G)$; moreover, we obtain that $\gamma_{cs}(G) = \gamma_s(G)$ for each $\{K_3, C_5, P_5\}$ -free graph G with $\delta(G) \geq 2$. Our conclusions can generalize some known results.

Keywords: co-secure dominating set; secure dominating set; co-secure domination number

MSC: 05C69



Academic Editor: Roman Dmytryshyn

Received: 18 November 2024

Revised: 24 December 2024

Accepted: 25 December 2024

Published: 27 December 2024

Citation: Cui, J.; Li, T.; Zhang, J.; Chen, X.; Xiong, L. Co-Secure Domination Number in Some Graphs. *Axioms* **2025**, *14*, 10. <https://doi.org/10.3390/axioms14010010>

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

We consider only simple graphs in this paper. For notation and terminology not defined here, readers are referred to [1]. Let G be a graph and $S \subseteq V(G)$. We use $G[S]$ and $E(G[S])$ to denote the subgraph of G induced by S , and the edge set of $G[S]$, respectively. Let $N_S(v) = \{u \in S : uv \in E(G)\}$ and $d_S(v) = |N_S(v)|$; and if there is no confusion, we set $N(v) = N_G(v)$ and $d(v) = d_G(v)$ for brevity. Let $N[v] = N(v) \cup \{v\}$. For an induced subgraph H_1 of G and a graph H_2 , let $H_1 = H_2$ if H_1 is isomorphic to H_2 . A complete graph of order n is denoted by K_n , and a complete bipartite graph with two color classes A, B is denoted by $K_{p,q}$, where $|A| = p$ and $|B| = q$. G is H -free if G contains no induced subgraph isomorphic to a graph H . Figure 1 shows some of the forbidden induced subgraphs of G in this paper. A claw is a graph isomorphic to $K_{1,3}$. We use $G[v, v_1, v_2, v_3]$ to denote an induced subgraph H of G with $H = K_{1,3}$ such that the degree sequence of H is $d_H(v) = 3, d_H(v_i) = 1$ for each $i \in \{1, 2, 3\}$. Let K_4^- denote the graph obtained from K_4 by deleting an edge. We use C_4, C_5 , and P_5 to denote a 4-cycle, 5-cycle, and a path of order 5, respectively. Let $P = v_1 v_2 \cdots v_k$ denote a path P of order $k \geq 2$ from v_1 to v_k in G . The bull graph, denoted by $B_{1,1}$, is a graph obtained by joining two disjoint vertices to two distinct vertices of a triangle. If an induced subgraph H of G consisting of $v_i, 1 \leq i \leq 5$, and isomorphic to $B_{1,1}$, then let $G[\{v_1, v_2, v_3, v_4, v_5\}] = H$ such that $d_H(v_1) = 2, d_H(v_2) = d_H(v_3) = 3, d_H(v_4) = d_H(v_5) = 1$, and $v_4 v_2, v_3 v_5 \in E(H)$.

If an induced subgraph H of G consisting of $w_i, 1 \leq i \leq 5$, and isomorphic to Z_2 , then let $G[\{w_1, w_2, w_3, w_4, w_5\}] = H$ such that $d_H(w_1) = 3, d_H(w_2) = d_H(w_3) = d_H(w_4) = 2, d_H(w_5) = 1$, and $w_1 w_4 w_5$ is an induced path of order 3 in H . For two graphs H_1 and H_2 , let $G_1 = H_1 \vee H_2$ denote the join of H_1 and H_2 such that $V(G_1) = V(H_1) \cup V(H_2)$ and $E(G_1) = E(H_1) \cup E(H_2) \cup \{uv \in E(G_1) : u \in V(H_1), v \in V(H_2)\}$. The diameter of G , denoted by $diam(G)$, is the maximum distance of a pair of vertices in G . Let $\alpha(G)$ denote the independence number of G .

If each vertex of G is in S or adjacent to some vertex in S , then S is a *dominating set* (DS) of G . If S is a dominating set of G , and for each vertex $u \in V(G) - S$, there is a neighbor of u in S , denoted by v , such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G , then S is a *secure dominating set* (SDS) of G . Conversely, S is a *co-secure dominating set* (CSDS) of G if S is a dominating set of G and for each vertex v in S , $V(G) - S$ contains a neighbor of v , denoted by u , such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G . The minimum cardinality of a CSDS (resp. SDS) of G is the *co-secure* (resp. *secure*) *domination number* of G . Let $\gamma_{cs}(G)$ and $\gamma_s(G)$ denote the co-secure domination number and secure domination number of G , respectively. Let $G = K_m \vee (K_2 \cup K_1), m \geq 3$. It is easy to verify that $\gamma_{cs}(G) = 1$ and $\gamma_s(G) = 2$, which implies there is no implication between SDS and CSDS.

The secure domination and co-secure domination problems have practical applications, such as the assignment of guards for locations. We can obtain a graph G in the following way: the given locations consist of the vertex set of G , and if two locations are connected by a street, then there is an edge between them in G . We assume that if a location u has a guard, then each neighbor of u is protected by the guard. The secure domination problem is that we assign a minimum number of guards for a subset of locations, denoted by S , such that for each location u in $V(G) - S$, S contains a neighbor v of u such that the guard in v can move to u , and then the guard team can also protect all the locations. The problem with the co-secure domination number is that we assign a minimum number of guards for a subset S of locations such that for each location u in S , its guard can move to a neighbor of u , and then the guard team can also protect all the locations.

Cockayne et al. introduced the concept of a secure dominating set and secure domination number. The exact values of secure domination number for paths, cycles, complete multipartite graphs and some other graph classes were given in [2], and the bounds of secure domination number for different graph classes were given in [3–9]. The concept of co-secure domination was introduced by Arumugan et al. [10]. They also determined the co-secure domination number of some families of graphs, such as paths and cycles, and gave sharp upper and lower bounds of graphs; moreover, they proved that the decision problem for $\gamma_{cs}(G)$ of a graph G is NP-complete even if G is a bipartite, chordal or planar graph. Joseph and Sangeetha [11] gave some bounds for the co-secure domination number of some classes of graphs.

Arumugan et al. [10] proved that for a connected graph G , the independence number is an upper bound of $\gamma_{cs}(G)$ as follows.

Theorem 1 ([10]). *Every maximum independent set is a co-secure dominating set of G for each non-trivial graph G , i.e., $\gamma_{cs}(G) \leq \alpha(G)$.*

Moreover, in [10], they gave the upper bound $\gamma_s(G)$ for $\gamma_{cs}(G)$ of a graph G as follows.

Theorem 2 ([10]). *$\gamma_{cs}(G) \leq \gamma_s(G)$ for each graph G with $\delta(G) \geq 2$.*

Based on Theorems 1 and 2, Arumugan et al. [10] asked the following two problems.

Problem 1 ([10]). *Characterize a graph G with $\gamma_{cs}(G) = \alpha(G)$.*

Problem 2 ([10]). Characterize a graph G with $\gamma_{cs}(G) = \gamma_s(G)$.

Inspired by Theorems 1 and 2 and Problems 1 and 2, in the Section 2, we give sufficient conditions for a graph G that satisfies $\gamma_{cs}(G) = \alpha(G)$ and $\gamma_{cs}(G) = \gamma_s(G)$, respectively.

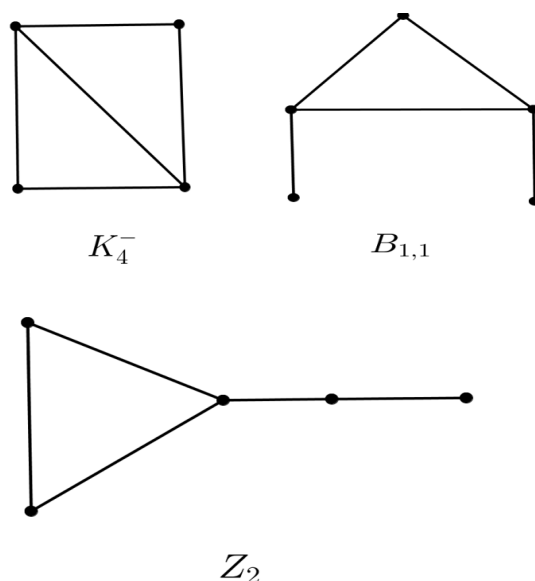


Figure 1. The forbidden induced subgraphs in this paper K_4^- , $B_{1,1}$ and Z_2 .

2. Main Results and Their Proofs

Suppose that S is a dominating set of a graph G with $v \in S, u \in N_{G-S}(v)$. If $N_S(u) = \{v\}$, then u is an *external private neighbor* of v with respect to S . If $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G , then u is *S -defended by v* , and we can say u is *S -defended* for brevity. If for each vertex $u' \in S$, u is not S -defended by u' , then u is *S -undefended*. Let $epn(v, S) = \{u \in V(G) - S : u \text{ is an } S\text{-external private neighbor of } v\}$ and $rpn(v, S) = \{w \in V(G) - S : w \text{ is } S\text{-defended by } v\}$. It is easy to verify that $u \in rpn(v, S)$ if and only if $epn(v, S) \subseteq N[u]$. By the definitions of secure dominating set and co-secure dominating set, if S is a secure dominating set of G , it is easy to verify that for each $u \in G - S$, $u \in rpn(v, S)$ and $epn(v, S) \subseteq N[u]$ for some vertex $v \in S$; if S is a co-secure dominating set of G , then for each vertex $v \in S$, $rpn(v, S) \neq \emptyset$ and $epn(v, S) \subseteq N[u]$ for each vertex $u \in rpn(v, S)$. We mainly applied the previous definitions and properties to the proof of our main results in this section.

According to Problem 1, we obtain Theorems 3 and 4, which gives an infinite family $\mathcal{F}$ of graphs with $\gamma_{cs}(G) = \alpha(G)$ if $G \in \mathcal{F}$.

Theorem 3. Every non-trivial $\{K_4^-, C_4, P_5, B_{1,1}\}$ -free graph contains a minimum co-secure dominating set which is a maximum independent set.

Proof. Let G be a non-trivial $\{K_4^-, C_4, P_5\}$ -free graph. Suppose that G consists of k components $G_1, \dots, G_k$ with $k \geq 1$. Clearly, $\sum_{i=1}^k \gamma_{cs}(G_i) = \gamma_{cs}(G)$. Thus, without loss of generality, we assume that G is connected, i.e., $k = 1$. We begin with the following result.

Claim 1. For a vertex x in a CSDS S of G , if $epn(x, S) \neq \emptyset$, then $G[epn(x, S)]$ is a clique.

Proof. Suppose that S is a CSDS of G and $x \in S$ with $epn(x, S) \neq \emptyset$. If $|epn(x, S)| = 1$, then we are done. Let $|epn(x, S)| \geq 2$. Suppose on the contrary that $x_1, x_2 \in epn(x, S)$ and $x_1x_2 \notin E(G)$. By the definition of a CSDS, we have $rpn(x, S) \neq \emptyset$ and $x_1, x_2 \notin$

$rpn(x, S)$. Assume that $y \in rpn(x, S)$. Then, $y \notin \{x_1, x_2\}$ and $x_1y, x_2y \in E(G)$, which implies $G[\{x, x_1, x_2, y\}] = K_4^-$, a contradiction. Thus, $G[epn(x, S)]$ is a clique. $\square$

For a CSDS S of G , since $u \in rpn(v, S)$ if and only if $epn(v, S) \subseteq N[u]$ for a vertex $v \in S$ and a vertex $u \in N_{G-S}(v)$, we can obtain the following claim by Claim 1.

Claim 2. For a vertex x in a CSDS S of G , $epn(x, S) \subseteq rpn(x, S)$.

By the definition of a CSDS and Claims 1 and 2, we can obtain the following result.

Claim 3. Let S be a dominating set such that $d_{G-S}(x) \geq 1$ for each vertex $x \in S$. If S is not a CSDS of G , then S contains a vertex x such that $rpn(x, S) = \emptyset$, and $epn(x, S)$ contains two non-adjacent vertices.

Claim 4. If S is a minimum CSDS of G with $E(G[S]) \neq \emptyset$, then there is a vertex $x \in S$ with $d_S(x) \geq 1$ such that $epn(x, S) \neq \emptyset$.

Proof. Suppose on the contrary that $epn(x, S) = \emptyset$ for each vertex $x \in S$ with $d_S(x) \geq 1$. Let $u, v \in S$ such that $uv \in E(G)$. Then, by the hypothesis, we have $epn(u, S) = epn(v, S) = \emptyset$. Let $S' = S \setminus \{u\}$. Since S is a CSDS of G , we have $rpn(x, S) \neq \emptyset$ and $N_{G-S}(x) \neq \emptyset$ for each vertex $x \in S$. Moreover, since $epn(u, S) = \emptyset$ and $uv \in E(G)$, we have $d_{S'}(x) \geq 1$ for each vertex $x \in V(G) - S'$. Thus, S' is a dominating set of G . Since S is a minimum CSDS of G and $|S'| < |S|$, S' is not a CSDS of G . Thus, by Claim 3, there exists a vertex y in S' with $epn(y, S') \neq \emptyset$ and $rpn(y, S') = \emptyset$. Moreover, let $y_1, y_2 \in epn(y, S')$ with $y_1y_2 \notin E(G)$. By $y \in S'$, we have $y \neq u$. Suppose that $y_1, y_2 \notin epn(y, S)$. Then, $N_S(y_i) = \{u, y\}$ for each $i \in \{1, 2\}$. If $uy \notin E(G)$, then $G[\{y_1, y_2, y, u\}] = y_1uy_2yy_1$ is a 4-cycle by $y_1y_2 \notin E(G)$, a contradiction. Thus, $uy \in E(G)$, which implies $G[\{y_1, y_2, y, u\}] = K_4^-$, a contradiction. Thus, $epn(y, S)$ contains at least one vertex in $\{y_1, y_2\}$. Moreover, by Claim 1 and $y_1y_2 \notin E(G)$, we have $|\{y_1, y_2\} \cap epn(y, S)| = 1$.

Without loss of generality, assume that $y_1 \in epn(y, S)$ and $y_2 \notin epn(y, S)$. Then, $N_S(y_2) = \{u, y\}$ by $y_2 \in epn(y, S')$. Since $y_1 \in epn(y, S)$, we have $d_S(y) = 0$ by $epn(x, S) = \emptyset$ for each vertex $x \in S$ with $d_S(x) \geq 1$. Thus, $y \neq v$ and $yu, yv \notin E(G)$. By $y \neq v$ and $N_S(y_2) = \{u, y\}$, we have $y_2v \notin E(G)$. We have $y_1u, y_1v \notin E(G)$ by $y_1 \in epn(y, S)$ and $y \notin \{u, v\}$. It follows that $G[\{v, u, y, y_1, y_2\}] = vuy_2yy_1 = P_5$, a contradiction. Thus, $epn(x, S) \neq \emptyset$ for some vertex $x \in S$ with $d_S(x) \geq 1$. $\square$

Claim 5. G contains a minimum CSDS S such that S is an independent set.

Proof. Suppose to the contrary that each minimum CSDS of G is not an independent set. We choose a minimum CSDS S of G such that $|E(G[S])|$ is minimum. Since S is not an independent set, we have $|E(G[S])| \geq 1$, which implies $\gamma_{cs}(G) = |S| \geq 2$. By Claim 4, without loss of generality, assume that $d_S(u) \geq 1$ and $u' \in epn(u, S)$. Let $S_1 = (S \setminus \{u\}) \cup \{u'\}$. By Claim 1, $epn(u, S) \subseteq N[u']$. Moreover, $rpn(x, S) \neq \emptyset$ for each vertex $x \in S_1 \setminus \{u'\}$. Thus, S_1 is a dominating set of G with $d_{S_1}(x) \geq 1$ for each vertex $x \in S_1$. We have $d_{S_1}(u') = 0$ by $u' \in epn(u, S)$, which implies $|E(G[S_1])| < |E(G[S])|$. Thus, by the choice of S , S_1 is not a CSDS of G . By Claim 3, there is a vertex $w \in S_1$ and $z_1, z_2 \in epn(w, S_1)$ such that $rpn(w, S_1) = \emptyset$ and $z_1z_2 \notin E(G)$. By $u \notin S_1$, we have $w \neq u$. By Claim 1 and $z_1z_2 \notin E(G)$, at least one vertex in $\{z_1, z_2\}$ is not in $epn(w, S)$. In the following, we claim that there is a contradiction if $\{z_1, z_2\} \cap epn(w, S) = \emptyset$ or $|\{z_1, z_2\} \cap epn(w, S)| = 1$.

Suppose that $\{z_1, z_2\} \cap \text{epn}(w, S) = \emptyset$. Then, $N_S(z_i) = \{u, w\}$ for each $i \in \{1, 2\}$. If $wu \in E(G)$, then $G[\{z_1, u, z_2, w\}] = K_4^-$, a contradiction. If $wu \notin E(G)$, then $G[\{z_1, u, z_2, w\}] = z_1wz_2uz_1$ is a 4-cycle, a contradiction. Thus, $|\{z_1, z_2\} \cap \text{epn}(w, S)| = 1$. Without loss of generality, assume that $z_1 \notin \text{epn}(w, S)$ and $z_2 \in \text{epn}(w, S)$. Then, $N_S(z_1) = \{u, w\}$. By $u' \in \text{epn}(u, S)$ and $z_2 \in \text{epn}(w, S)$, we have $u'w, z_2u \notin E(G)$. Suppose that $uw \in E(G)$. We have $u'z_2 \in E(G)$ by $G[\{u', u, z_1, w, z_2\}] \neq B_{1,1}$, a contradiction to $z_2 \in \text{epn}(w, S_1)$. Suppose that $uw \notin E(G)$. Then, we have $u'z_2 \in E(G)$ by $u'uz_1wz_2 \neq P_5$, a contradiction to $z_2 \in \text{epn}(w, S_1)$. Thus, the claim is true. $\square$

Let us resume the proof of Theorem 3. If G contains a minimum CSDS which is a maximum independent set of G , then we are done. Thus, we assume that each minimum CSDS of G is not a maximum independent set. Then, $\gamma_{cs}(G) < \alpha(G)$ by Theorem 1, which implies $\alpha(G) \geq 2$. We have $\gamma_{cs}(G) \geq 2$. Otherwise, suppose $S = \{v\}$ is a minimum CSDS of G . Then, each vertex in $V(G) - \{v\}$ is adjacent to v , i.e., $\text{epn}(v, S) = V(G) \setminus \{v\}$. Then, by Claim 1, G is a complete graph, a contradiction to $\alpha(G) \geq 2$.

Let S' be a maximum independent of G . By Claim 5, we take a minimum CSDS S of G such that S is an independent set and $|S \cap S'|$ is maximum. Then, $|S| < |S'|$ by $\gamma_{cs}(G) < \alpha(G)$, and hence $S' - S \neq \emptyset$ and $|S' - S| > |S - S'|$. Since S is a dominating set of G , each vertex in $S' - S$ is adjacent to a vertex of $S - S'$. The rest proof boils down to two cases as follows.

Case 1. $d_S(x) \geq 2$ for each vertex $x \in S' - S$.

Since $|S' - S| > |S - S'|$ and S is a dominating set of G , there is a vertex $v \in S - S'$ such that $d_{S'-S}(v) \geq 2$. Assume that $v_1, v_2 \in N_{S'-S}(v)$. By the hypothesis of the case, $d_S(v_i) \geq 2$ for each $i \in \{1, 2\}$. Suppose that $|N_S(v_1) \cap N_S(v_2)| \geq 2$ and assume $w \in (N_S(v_1) \cap N_S(v_2)) \setminus \{v\}$. Since v_1, v_2 are in the independent set S' , $v_1v_2 \notin E(G)$. We have $vw \notin E(G)$ since S is an independent set. Thus, $G[\{v, v_1, w, v_2\}] = v_1vv_2wv_1$ is a 4-cycle, a contradiction. Thus, $N_S(v_1) \cap N_S(v_2) = \{v\}$. Suppose $w_i \in N_S(v_i) \setminus \{v\}$, $i \in \{1, 2\}$. By $N_S(v_1) \cap N_S(v_2) = \{v\}$, we have $w_1 \neq w_2$ and $v_1w_2, v_2w_1 \notin E(G)$. Moreover, since S is an independent set, $G[\{w_1, v_1, v, v_2, w_2\}] = w_1v_1vv_2w_2 = P_5$, a contradiction.

Case 2. There is a vertex $u \in S' - S$ and a vertex $v \in S$ such that $u \in \text{epn}(v, S)$.

Let $S_1 = (S \setminus \{v\}) \cup \{u\}$. By $u \in \text{epn}(v, S)$, $ux \notin E(G)$ for each vertex $x \in S \setminus \{v\}$, which implies that S_1 is an independent set with $|S_1| = |S|$. Since $\text{rpn}(x, S) \neq \emptyset$ for each vertex $x \in S$ and $u \in \text{epn}(v, S)$, S_1 is a dominating set of G with $d_{G-S_1}(x') \geq 1$ for each vertex $x' \in S_1$ by Claim 1. If S_1 is a CSDS of G , then $|S_1 \cap S'| > |S \cap S'|$, a contradiction to the choice of S' . Thus, S_1 is not a CSDS of G . By Claim 3, there exists a vertex $x \in S_1$ and $x_1, x_2 \in \text{epn}(x, S_1)$ such that $\text{rpn}(x, S_1) = \emptyset$, and $x_1x_2 \notin E(G)$. Since $v \notin S_1$, we have $x \neq v$. Moreover, it is easy to verify that $v \in \text{rpn}(u, S_1)$, and hence $u \neq x$ by $\text{rpn}(x, S_1) = \emptyset$. By Claim 1 and $x_1x_2 \notin E(G)$, we have $|\{x_1, x_2\} \cap \text{epn}(x, S)| \leq 1$.

Suppose that $x_1, x_2 \notin \text{epn}(x, S)$. Then, $N_S(x_i) = \{v, x\}$ for each $i \in \{1, 2\}$. Since S is an independent set and $v \neq x$, we have $xv \notin E(G)$, which implies that $G[\{x_1, v, x_2, x\}] = x_1vx_2xx_1$ is a 4-cycle, a contradiction. Thus, $|\{x_1, x_2\} \cap \text{epn}(x, S)| = 1$. Without loss of generality, assume that $x_1 \in \text{epn}(x, S)$ and $x_2 \notin \text{epn}(x, S)$. Then, $N_S(x_2) = \{v, x\}$. By $x_1 \in \text{epn}(x, S)$ and $u \in \text{epn}(v, S)$, we have $x_1v, ux \notin E(G)$. Since $uvx_2xx_1 \neq P_5$ and $vx, x_1v, ux_2, x_1x_2, ux \notin E(G)$, we have $x_1u \in E(G)$, which implies $d_{S_1}(x_1) \geq 2$, a contradiction to $x_1 \in \text{epn}(x, S_1)$.

It follows that Theorem 3 is true. $\square$

By Theorems 1 and 3, we can obtain the following corollary, which characterizes a graph G with $\gamma_{cs}(G) = \alpha(G)$.

Corollary 1. Let G be a non-trivial $\{K_4^-, C_4, P_5, B_{1,1}\}$ -free graph. Then, $\gamma_{cs}(G) = \alpha(G)$.

By Corollary 1, it is obvious that G contains an induced subgraph isomorphic to K_4^-, C_4, P_5 or $B_{1,1}$ if $\gamma_{cs}(G) \neq \alpha(G)$ for a graph G . Clearly, a tree contains no induced subgraph in $\{K_4^-, B_{1,1}, C_4\}$. Thus, by Corollary 1, we have that if T is a tree with $\text{diam}(T) \leq 3$, then $\gamma_{cs}(T) = \alpha(T)$. Let T be a star or double star of order n . Clearly, $\alpha(T) = n - 2$ and $\text{diam}(T) \leq 3$. Thus, $\gamma_{cs}(T) = n - 2$ by Corollary 1, which is obtained by Arumugan in [10].

In the following we consider some other forbidden induced subgraphs in $\mathcal{F} = \{K_{1,3}, B_{1,1}, Z_2\}$ for a graph G , such that if G is $\mathcal{F}$ -free, i.e., G contains no any induced subgraph in $\mathcal{F}$, then $\gamma_{cs}(G) = \alpha(G)$.

Theorem 4. $\gamma_{cs}(G) = \alpha(G)$ for every $\{K_{1,3}, B_{1,1}, Z_2\}$ -free graph G with $\delta(G) \geq 3$ if $\alpha(G) \neq 2$.

Proof. We assume that G is connected and $\{K_{1,3}, B_{1,1}, Z_2\}$ -free graph with $\alpha(G) \neq 2$. As the proof of Theorem 3, it suffices to prove that there is a minimum CSDS of G , which is a maximum independent set of G . If G is a complete graph, i.e., $\alpha(G) = 1$, then $\gamma_{cs}(G) = \alpha(G) = 1$, and we are done. Suppose on the contrary that $\gamma_{cs}(G) < \alpha(G)$. Then, $\alpha(G) \geq 3$.

Claim 1. $\gamma_{cs}(G) \geq 2$.

Proof. Suppose on the contrary that $\gamma_{cs}(G) = 1$. Let $S = \{v\}$ be a CSDS of G . Then, each vertex in $G \setminus \{v\}$ is adjacent to v and each vertex in $G \setminus \{v\}$ is adjacent to v . Since $\delta(G) \geq 3$, $|V(G)| \geq 4$. Then, by $\alpha(G) \geq 3$, let $v_1, v_2, v_3 \in V(G) \setminus \{v\}$ such that $\{v_1, v_2, v_3\}$ is an independent set. Then, $G[v, v_1, v_2, v_3] = K_{1,3}$, a contradiction. Thus, $\alpha(G) \leq 2$, a contradiction. $\square$

Let S be a minimum CSDS of G such that $|E(G[S])|$ is minimum. By Claim 1, $|S| \geq 2$.

Claim 2. There exists a vertex $x \in S$ with $d_S(x) \geq 1$ such that $\text{epn}(x, S) \neq \emptyset$ if $E(G[S]) \neq \emptyset$.

Proof. Suppose to the contrary that $\text{epn}(x, S) = \emptyset$ for each vertex $x \in S$ with $d_S(x) \geq 1$. We assume that $uv \in E(G)$ with $u, v \in S$. Then, $\text{epn}(u, S) = \text{epn}(v, S) = \emptyset$. Let $S' = S \setminus \{u\}$. Since S is a CSDS of G , $N_{G-S}(x) \neq \emptyset$ for each vertex $x \in S$. Moreover, by $\text{epn}(u, S) = \emptyset$ and $uv \in E(G)$, we have S' is a dominating set of G such that $d_{G-S'}(x) \geq 1$ for each vertex $x \in S'$. By the minimality of $|S|$ and $|S'| < |S|$, S' is not a CSDS of G . Then, by Claim 3 of Theorem 3, let $x \in S'$ such that $\text{rpn}(x, S') = \emptyset$ and $x_1, x_2 \in \text{epn}(x, S')$ with $x_1 x_2 \notin E(G)$. We have $|\{x_1, x_2\} \cap \text{epn}(x, S)| < 2$ since $\text{rpn}(x, S) \neq \emptyset$. Clearly, $u \in N_S(x_i)$ if $x_i \notin \text{epn}(x, S)$, where $i \in \{1, 2\}$.

Suppose $\{x_1, x_2\} \cap \text{epn}(x, S) = \emptyset$. Then, $\{u, x\} = N_S(x_i)$ for each $i \in \{1, 2\}$. We have $x = v$; otherwise, $x_1, x_2 \notin N(v)$ by $N_{S'}(x_i) = \{x\}$, and so $G[u, x_1, x_2, v] = K_{1,3}$, a contradiction. Thus, $\text{epn}(v, S') \neq \emptyset$ by $x = v$ and $\{x_1, x_2\} \in \text{epn}(x, S')$. We have $\text{epn}(v, S') \subseteq N[u]$ by $\text{epn}(v, S) = \emptyset$. Thus, $u \in \text{rpn}(v, S')$, a contradiction to $\text{rpn}(v, S') = \emptyset$. It follows that $|\{x_1, x_2\} \cap \text{epn}(x, S)| = 1$ by $|\{x_1, x_2\} \cap \text{epn}(x, S)| < 2$.

Without loss of generality, assume $x_1 \notin \text{epn}(x, S)$ and $x_2 \in \text{epn}(x, S)$. Then, $x \neq v$ and $N_S(x) = \emptyset$ by the hypothesis that $\text{epn}(x', S) = \emptyset$ for each vertex $x' \in S$ with $d_S(x') \geq 1$. Note that $N_S(x_1) = \{u, x\}$. By $d(x) \geq 3$ and $N_S(x) = \emptyset$, assume $N_{G-S}(x) \setminus \{x_1, x_2\} \neq \emptyset$. We claim that $yx_2 \notin E(G)$, and $yx_1, yu \in E(G)$ for each vertex $y \in N_S(x) \setminus \{x_1, x_2\}$. Let $x_3 \in N_{G-S}(x) \setminus \{x_1, x_2\}$. We have $x_1 x_3 \in E(G)$ or $x_2 x_3 \in E(G)$ by $G[x, x_1, x_2, x_3] \neq K_{1,3}$ and $x_1 x_2 \notin E(G)$. Assume $x_2 x_3 \in E(G)$. Suppose $x_3 v \in E(G)$. Then, we have $x_1 x_3 \in E(G)$ by $G[x_2, x_3, x, v, x_1] \neq B_{1,1}$ and $vx, x_1 x_2, x_2 v, x_1 v \notin E(G)$, and hence $G[x_3, x_1, v, x_2] = K_{1,3}$, a contradiction. Thus, $x_3 v \notin E(G)$. If $x_3 u \in E(G)$, then $G[x_3, x_2, x, u, v] = Z_2$ by $x_2 u, x_2 v, x_3 v, x u, x v \notin E(G)$, a contradiction. Thus, $x_3 u \notin E(G)$, which implies $G[x, x_2, x_3,$

$x_1, u] = Z_2$, a contradiction. It follows that $x_2x_3 \notin E(G)$ and then $x_1x_3 \in E(G)$. By $G[x_3, x, x_1, x_2, u] \neq B_{1,1}$ and $x_3x_2, x_2x_1, x_2u \notin E(G)$, we have $x_3u \in E(G)$.

We have $N_{G-S}(v) \cap N_{G-S}(x) = \emptyset$. Otherwise, without loss of generality, let $y \in N_{G-S}(v) \cap N_{G-S}(x)$. Then, $yx_1, yu \in E(G)$ and $yx_2 \notin E(G)$ by the above result. Thus, $G[y, u, v, x, x_2] = Z_2$, a contradiction. Moreover, we claim that $N_{G-S}(v) \cap N_{G-S}(x_1) = \emptyset$. Otherwise, let $v_1 \in N_{G-S}(v) \cap N_{G-S}(x_1)$. Then, $v_1x \notin E(G)$ by $N_{G-S}(v) \cap N_{G-S}(x) = \emptyset$. We have $v_1u \in E(G)$ by $G[x_1, u, x, v_1] \neq K_{1,3}$ and $ux, v_1x \notin E(G)$. Thus, $v_1x_2 \in E(G)$ by $G[x_1, u, v_1, x, x_2] \neq Z_2$ and $x_2u, x_2x_1, xv_1, xu \notin E(G)$. Then, we have $G[v_1, x_1, x_2, v] = K_{1,3}$, a contradiction. We have $N_{G-S}(v) \cap N_{G-S}(u) = \emptyset$. Otherwise, assume $v_2 \in N_{G-S}(v) \cap N_{G-S}(u)$. Then, $x_1v_2 \notin E(G)$ by $N_{G-S}(v) \cap N_{G-S}(x_1) = \emptyset$ and $v_2x \notin E(G)$ by $N_{G-S}(v) \cap N_{G-S}(x) = \emptyset$. Thus, $G[u, v_2, v, x_1, x] = Z_2$, a contradiction. Let $v' \in N_{G-S}(v)$ by $N_{G-S}(v) \neq \emptyset$. Then, $v'x, v'u, v'x_1 \notin E(G)$ by the previous proof, and hence $x_3v' \in E(G)$ by $G[u, x_1, x_3, v, v'] \neq Z_2$. It follows that $G[x_3, v', x, u] = K_{1,3}$, a contradiction. $\square$

Claim 3. S is an independent set.

Proof. Suppose on the contrary that S is not an independent set. By Claim 2, assume $u, v \in S$ with $uv \in E(G)$, and $u_1 \in \text{epn}(u, S)$. Clearly, $u_1v \notin E(G)$. Let $S_2 = (S \setminus \{u\}) \cup \{u_1\}$. For each vertex $u' \in N_{G-S}(u) \setminus \{u_1\}$, we have $u'u_1 \in E(G)$ or $u'v \in E(G)$ by $G[u, u_1, v, u'] \neq K_{1,3}$ and $u_1v \notin E(G)$. Thus, S_2 is a dominating set with $|S| = |S_2|$ and $|E(G[S_2])| < |E(G[S])|$, which implies S_2 is not a CSDS by the choice of S . Clearly, $u \in \text{rpn}(u_1, S_2)$. By Claim 3 of Theorem 3, let $x \in S_2 \setminus \{u_1\}$ such that $\text{rpn}(x, S_2) = \emptyset$, and let $x_1, x_2 \in \text{epn}(x, S_2)$ be two distinct non-adjacent vertices such that $|\{x_1, x_2\} \cap \text{epn}(x, S)| < 2$. By $x_1, x_2 \in \text{epn}(x, S_2)$, we have $x_1u_1, x_2u_1 \notin E(G)$. By $\{x_1, x_2\} \not\subseteq \text{epn}(x, S)$, without loss of generality, assume that $x_2 \notin \text{epn}(x, S)$. Then, $N_S(x_2) = \{u, x\}$. We have $x_2v \in E(G)$ by $G[u, x_2, v, u_1] \neq K_{1,3}$ and $x_2u_1, vu_1 \notin E(G)$, which implies $x = v$. We claim $x_1 \in \text{epn}(v, S)$. Otherwise, $N_S(x_1) = N_S(x_2) = \{u, v\}$. Then, $x_2x_1 \in E(G)$ by $G[u, x_2, x_1, u_1] \neq K_{1,3}$ and $x_1, x_2 \notin N(u_1)$, a contradiction. Thus, $N_S(x_1) = \{v\}$ and then $x_1u \notin E(G)$. By $G[x_2, u, v, u_1, x_1] \neq B_{1,1}$ and $u_1v, x_1u, u_1x_2, x_1x_2 \notin E(G)$, we have $u_1x_1 \in E(G)$, a contradiction to $x_1 \in \text{epn}(v, S_2)$. $\square$

Let S_1 be a maximum independent set of G . By Claim 3, we choose a minimum CSDS S of G such that S is an independent set and $|S \cap S_1|$ is maximum. Then, $|S_1 \setminus S| > |S \setminus S_1|$ by $\gamma_{cs}(G) < \alpha(G)$. Since S is a dominating set and $|S_1 \setminus S| > |S \setminus S_1|$, there are two vertices in $S_1 \setminus S$ having a common neighbor in $S \setminus S_1$. Assume $u, v \in S_1 \setminus S$ and $w \in S \setminus S_1$ such that $u, v \in N(w)$.

Claim 4. If $z \in \text{epn}(w, S)$, then $z \notin \text{rpn}(w, S)$, where $z \in \{u, v\}$.

Proof. Without loss of generality, let $u \in \text{epn}(w, S)$ and suppose on the contrary that $u \in \text{rpn}(w, S)$. Let $S' = (S \setminus \{w\}) \cup \{u\}$. Since $u \in \text{epn}(w, S)$, S' is an independent set. We have $\text{epn}(w, S) \subseteq N[u]$ by $u \in \text{rpn}(w, S)$. Thus, S' is a dominating set with $|S'| = |S|$. By $|S' \cap S_1| > |S \cap S_1|$ and the choice of S , S' is not a CSDS of G . Clearly, $w \in \text{rpn}(u, S')$. Thus, by Claim 3 of Theorem 3, let $x \in S' \setminus \{u\}$ such that $\text{rpn}(x, S') = \emptyset$ and $x_1, w_1 \in \text{epn}(x, S')$ such that $x_1w_1 \notin E(G)$ and $|\{x_1, w_1\} \cap \text{epn}(x, S)| < 2$. Since $x_1, w_1 \in \text{epn}(x, S')$ and $u \in S' \setminus \{x\}$, we have $x_1u, w_1u \notin E(G)$. By $|\{x_1, w_1\} \cap \text{epn}(x, S)| < 2$, without loss of generality, assume $w_1 \notin \text{epn}(x, S)$. Then, $N_S(w_1) = \{w, x\}$.

We claim that $|\{x_1, w_1\} \cap \text{epn}(x, S)| = 1$. Suppose $x_1 \notin \text{epn}(x, S)$. Then, $N_S(x_1) = N_S(w_1) = \{u, x\}$, and hence $G[w, u, x_1, w_1] = K_{1,3}$, a contradiction. Thus, $x_1 \in \text{epn}(x, S)$.

We claim that each vertex in $N_{G-S}(w) \setminus \{u, w_1\}$ is nonadjacent to u and adjacent to w_1 . Let $w_2 \in N_{G-S}(w) \setminus \{w_1, u\}$ since $d(w) \geq 3$ and S is an independent set. Then, $w_2w_1 \in E(G)$ or $w_2u \in E(G)$ by $G[w, w_1, w_2, u] \neq K_{1,3}$ and $uw_1 \notin E(G)$. Suppose $uw_2 \in E(G)$. Assume $w_2 \in epn(w, S)$. Then, $w_2x \notin E(G)$, which implies $w_2w_1 \in E(G)$ by $G[w, u, w_2, w_1, x] \neq Z_2$ and $wx, ux, uw_1 \notin E(G)$. Thus, $w_2x_1 \in E(G)$ by $G[w_1, w_2, w, x, x_1] \neq Z_2$ and $wx, x_1w, x_1w_1, w_2x \notin E(G)$. It follows that $G[w_2, u, w_1, x_1] = K_{1,3}$, a contradiction. Thus, $w_2 \notin epn(w, S)$, and moreover, we have $d_S(w_2) = 2$ since G is claw-free and S is an independent set. Let $\{y, w\} = N_S(w_2)$. Suppose $y \neq x$. Then, $w_2x \notin E(G)$, which implies $w_2w_1 \in E(G)$ by $G[w, u, w_2, w_1, x] \neq Z_2$. Since $u \in epn(w, S)$ and $N_S(w_1) = \{u, x\}$, we have $uy, w_1y \notin E(G)$, and hence $G[w_2, y, u, w_1] = K_{1,3}$, a contradiction. Thus, $y = x$, i.e., $N_S(w_2) = \{w, x\}$. By $G[x, w_1, w_2, x_1] \neq K_{1,3}$ and $w_1x_1 \notin E(G)$, we have $x_1w_2 \in E(G)$ or $w_1w_2 \in E(G)$. If $w_1w_2 \in E(G)$, then $w_2x_1 \in E(G)$ by $G[w_1, w_2, x, u, x_1] \neq B_{1,1}$ and $uw_1, ux_1, w_1x_1, ux \notin E(G)$, which implies $G[w_2, u, w_1, x_1] = K_{1,3}$, a contradiction. Thus, $w_2w_1 \notin E(G)$, and $w_2x_1 \in E(G)$, which implies $G[x_1, w_2, x, u, w_1] = B_{1,1}$, a contradiction. It follows that $w_2u \notin E(G)$, and $w_2w_1 \in E(G)$. In the following we use the result to obtain a contradiction.

Since $u \in epn(w, S)$ and $d(u) \geq 3$, let $u_1, u_2 \in N_{G-S}(u)$. By the above result, we have $u_1, u_2 \notin N(w)$; otherwise, there is a vertex in $N(w) \setminus \{u, w_1\}$ adjacent to u , a contradiction. Thus, $u_1u_2 \in E(G)$ by $G[u, u_1, u_2, w] \neq K_{1,3}$. Moreover, we have $w_1u_i \in E(G)$ for some $i \in \{1, 2\}$ by $G[u, u_1, u_2, w, w_1] \neq Z_2$. Without loss of generality, assume $w_1u_1 \in E(G)$. Then, $u_1x \in E(G)$ by $G[w_1, u_1, w, x] \neq K_{1,3}$ and $u_1w, wx \notin E(G)$. We have $u_1x_1 \in E(G)$ by $G[w_1, u_1, x, u, x_1] \neq B_{1,1}$ and $ux, ux_1, uw_1, w_1x_1 \notin E(G)$. Thus, $G[u_1, w_1, u, x_1] = K_{1,3}$, a contradiction. $\square$

Claim 5. $\{u, v\} \cap epn(w, S) = \emptyset$.

Proof. Without loss of generality, suppose on the contrary that $u \in epn(w, S)$. Then, by Claim 4, $u \notin rp_n(w, S)$, and hence let $w_1 \in epn(w, S) \setminus \{u\}$ such that $uw_1 \notin E(G)$. Since S is a CSDS of G , $rp_n(w, S) \neq \emptyset$.

Let $w_2 \in rp_n(w, S)$. We have $w_2 \notin \{u, w_1\}$ by $u, w_1 \in epn(w, S)$ and $uw_1 \notin E(G)$. Moreover, $w_1w_2, w_2u \in E(G)$ by $w_2 \in rp_n(w, S)$ and $u, w_1 \in epn(w, S)$. We claim $rp_n(w, S) \subseteq epn(w, S)$. Suppose $w_2 \notin epn(w, S)$. Then, $d_S(w_2) \geq 2$. Let $x \in N_S(w_2) \setminus \{w\}$. We have $ux, w_1x \notin E(G)$ by $u, w_1 \in epn(w, S)$, and hence $G[w_2, w_1, u, x] = K_{1,3}$, a contradiction. Thus, $w_2 \in epn(w, S)$.

We claim that $N(w) = epn(w, S)$ by reduction to absurdity. Suppose $N(w) \setminus epn(w, S) \neq \emptyset$, and assume $w_3 \in N(w) \setminus epn(w, S)$. Then, $d_S(w_3) \geq 2$, and $w_3 \notin \{u, w_1, w_2\}$ by $\{u, w_1, w_2\} \subseteq epn(w, S)$. Let $y \in N_S(w_3) \setminus \{w\}$. Then, $w_3u \in E(G)$ or $w_3w_1 \in E(G)$ by $G[w, w_1, u, w_3] \neq K_{1,3}$ and $w_1u \notin E(G)$. Assume $w_3u \in E(G)$. Then, by $G[u, w, w_3, w_1, y] \neq B_{1,1}$ and $uw_1, uy, wy, w_1y \notin E(G)$, we have $w_1w_3 \in E(G)$, and hence $G[w_3, w_1, u, y] = K_{1,3}$, a contradiction. Thus, $w_3u \notin E(G)$ and similarly, $w_1w_3 \notin E(G)$, which implies a contradiction.

Since $|S| \geq 2$ and G is connected, assume that $x_1 \in N_{G-S}(w)$, $x_2 \in V(G) - S$ such that $x_1x_2 \in E(G)$ and $N_S(x_2) \setminus \{w\} \neq \emptyset$. Moreover, by $epn(w, S) = N(w)$, we have $x_2w \notin E(G)$. Let $z \in N_S(x_2)$. Then, $z \neq w$. Suppose $x_1 \notin rp_n(w, S)$. Then, without loss of generality, assume $x_1 = w_1$. By $x_2z \in E(G)$ and $N(w) = epn(w, S)$, we have $x_2w, w_1z, uz, w_2z \notin E(G)$. Thus, $w_2x_2 \in E(G)$ by $G[w_1, w_2, w, x_2, z] \neq Z_2$, which implies $ux_2 \in E(G)$ by $G[w_2, u, w, x_2, z] \neq Z_2$. It follows that $G[x_2, w_1, u, z] = K_{1,3}$, a contradiction. Thus, $x_1 \in rp_n(w, S)$, and then $x_2u \notin E(G)$. Without loss of generality, let $x_1 = w_2$. Then, $G[w_2, u(w_1), w, x_2, z] = Z_2$ by $x_2w_1, x_2u \notin E(G)$, a contradiction. $\square$

Claim 6. $epn(x, S) \neq \emptyset$ for each vertex x in $N_S(u) \cap N_S(v)$.

Proof. Without loss of generality, suppose on the contrary that $epn(w, S) = \emptyset$. Since $d(w) \geq 3$, let $w_1 \in N_{G-S}(w) \setminus \{u, v\}$. Then, $w_1u \in E(G)$ or $w_1v \in E(G)$ by $G[w, u, v, w_1] \neq K_{1,3}$ and $uv \notin E(G)$. Without loss of generality, assume that $w_1u \in E(G)$. Since $epn(w, S) = \emptyset$, $\min\{d_S(u), d_S(v)\} \geq 2$. Moreover, since G is claw-free and S is an independent set, $d_S(u) = d_S(v) = 2$. Assume $N_S(u) = \{w, x\}$ and $N_S(v) = \{w, y\}$.

We claim that $y = x$. Suppose $y \neq x$. By $G[w, w_1, u, v, y] \neq Z_2$ and $uy, uv, wy \notin E(G)$, we have $vw_1 \in E(G)$ or $w_1y \in E(G)$. Suppose that $yw_1 \in E(G)$. Then, $w_1x \notin E(G)$; otherwise, $G[w_1, w, y, x] = K_{1,3}$, a contradiction. Thus, $G[w, w_1, u, y, x] = B_{1,1}$, a contradiction. It follows that $yw_1 \notin E(G)$, and then $w_1v \in E(G)$. Since $epn(w, S) = \emptyset$, we assume that $z \in N_S(w_1) \setminus \{w\}$. By $w_1y \notin E(G)$, we have $z \neq y$. Moreover, we have $G[w, v, w_1, y, z] = B_{1,1}$ by $N_S(v) = \{y, w\}$, a contradiction. Thus, $y = x$.

Let $S' = (S \setminus \{w, x\}) \cup \{u, v\}$. Since $N_S(u) = N_S(v) = \{w, x\}$, S' is an independent set with $|S'| = |S|$. Moreover, each vertex in $N(w) \cup N(x)$ is adjacent to u or v , since G is claw-free. Clearly, $w, x \notin S_1$ and then $|S' \cap S_1| > |S \cap S_1|$. Thus, S' is not a CSDS of G by the choice of S . Then, let $z \in S'$ such that $rpn(z, S') = \emptyset$ and let x_1, x_2 be two non-adjacent vertices in $epn(z, S')$ such that $|\{x_1, x_2\} \cap epn(z, S)| < 2$. Without loss of generality, assume that $x_1 \notin epn(z, S)$. Then, $x_1 \cap \{w, x\} \neq \emptyset$, which implies $x_1u \in E(G)$ or $x_1v \in E(G)$. In the following, we claim that $z \in \{u, v\}$, and then obtain a forbidden subgraph to give a contradiction.

Suppose $z \notin \{u, v\}$. Then, $d_{S'}(x_1) \geq 2$ by $x_1 \in N(u) \cup N(v)$, a contradiction to $x_1 \in epn(z, S')$. Without loss of generality, assume that $z = u$. Then, $x_iv \notin E(G)$ by $x_i \in epn(u, S')$, and $N(x_i) \cap \{w, x\} \neq \emptyset$ for each $i \in \{1, 2\}$. Suppose that x_1 and x_2 has a common neighbor in $\{w, x\}$. Without loss of generality, assume that $w \in N(x_1) \cap N(x_2)$. Then, x_1v or $x_2v \in E(G)$ by $G[w, v, x_1, x_2] \neq K_{1,3}$ and $x_1x_2 \notin E(G)$, a contradiction. Thus, $N(x_1) \cap N(x_2) = \emptyset$. Without loss of generality, assume that $x_1w, x_2x \in E(G)$. Then, $x_1x, x_2w \notin E(G)$. It follows that $G[x_2, x, u, v, x_1] = B_{1,1}$, a contradiction. Thus, the claim is true. $\square$

Claim 7. If a vertex in $\{u, v\}$ is in $rpn(x, S)$, then $N_S(u) = N_S(v)$, where $x \in N_S(u) \cap N_S(v)$.

Proof. Without loss of generality, let $u \in rpn(w, S)$. We have $epn(w, S) \neq \emptyset$ by Claim 6. Then, each vertex in $epn(w, S)$ is adjacent to u . By Claim 5, $d_S(u) = d_S(v) = 2$ since G is claw-free. Assume $N_S(u) = \{w, x\}$ and $N_S(v) = \{w, y\}$. It suffices to prove $x = y$. Suppose $x \neq y$. We claim that $v \in rpn(w, S)$. Otherwise, let $w_1 \in epn(w, S)$ such that $w_1v \notin E(G)$. By $u \in rpn(w, S)$, we have $w_1u \in E(G)$. Moreover, $w_1y \notin E(G)$ by $w_1 \in epn(w, S)$. Thus, $G[w, u, w_1, v, y] = Z_2$, a contradiction, and then $v \in rpn(w, S)$.

Let $w_1 \in epn(w, S)$. Then, $w_1v, w_1u \in E(G)$ by $\{u, v\} \subseteq rpn(w, S)$. We claim that $v \in rpn(y, S)$. Otherwise, let $y_1 \in epn(y, S)$ such that $vy_1 \notin E(G)$. Clearly, $w_1y, y_1w \notin E(G)$ by $w_1 \in epn(w, S)$ and $y_1 \in epn(y, S)$. Thus, we have $w_1y_1 \in E(G)$ by $G[v, w_1, w, y, y_1] \neq Z_2$. Then, $uy_1 \in E(G)$ since $G[w_1, u, w, y_1, y] \neq Z_2$. Since $y_1 \in epn(y, S)$, we have $y_1x, y_1w \notin E(G)$, which implies $G[u, x, w, y_1] = K_{1,3}$, a contradiction. Similarly, $u \in rpn(x, S)$. Moreover, each vertex in $N(w) \setminus \{u, v\}$ is adjacent to u or v since G is claw-free. Let $S' = (S \setminus \{w, x, y\}) \cup \{u, v\}$. By the previous result, S' is a dominating set of G . Moreover, S' is an independent set.

We claim that $z_1u \in E(G)$ or $z_1v \in E(G)$ for each vertex in $z_1 \in N(z) \setminus (N(w) \cup epn(z, S))$ and each $z \in \{x, y\}$. Otherwise, without loss of generality, suppose that $z_1 \in N(x) \setminus (N(w) \cup epn(x, S))$ and $z_1u, z_1v \notin E(G)$. Then, we have $w_1z_1 \in E(G)$ by $G[u, w_1, w, x, z_1] \neq Z_2$, and hence $G[w_1, z_1, u, v] = K_{1,3}$, a contradiction. It follows that $z' \notin epn(z, S')$ if $N_S(z') \setminus \{w, x, y\} \neq \emptyset$ for each $z' \in N(z)$, where $z \in \{x, y, w\}$.

By $|S'| < |S|$, S' is not a CSDS of G . Then, let $z' \in S'$ such that $rpn(z', S') = \emptyset$ and let z_1, z_2 be two non-adjacent vertices in $epn(z', S')$ such that $|\{z_1, z_2\} \cap epn(z', S)| < 2$. Suppose

that $z' \notin \{u, v\}$. By $|\{z_1, z_2\} \cap \text{epn}(z', S)| < 2$, without loss of generality, assume that $z_1 \notin \text{epn}(z', S)$. Then, $N(z_1) \cap \{w, x, y\} \neq \emptyset$ by the above result. Since $N(z_1) \cap \{u, v\} \neq \emptyset$, we obtain a contradiction to $z_1 \in \text{epn}(z', S')$. Thus, $z' \in \{u, v\}$. Without loss of generality, assume that $z' = u$. Then, $z_1v, z_2v \notin E(G)$ by $z_1, z_2 \in \text{epn}(u, S')$. We have $z_1x \in E(G)$ or $z_2x \in E(G)$ by $G[u, z_1, z_2, x] \neq K_{1,3}$. Without loss of generality, assume that $z_1x \in E(G)$. Then, we have $z_1w \in E(G)$ by $G[u, z_1, x, w, v] \neq Z_2$, which implies $z_1y \in E(G)$ by $G[w, u, z_1, v, y] \neq Z_2$. Thus, $G[z_1, w, x, y] = K_{1,3}$, a contradiction. It follows that $x = y$. $\square$

Now we come to complete the theorem. We claim that $N_S(u) = N_S(v)$. By Claim 7, it suffices to prove $N_S(u) = N_S(v)$ if no vertex of $\{u, v\}$ is in $\text{rpn}(w', S)$ for each vertex $w' \in N_S(u) \cap N_S(v)$. Assume $\{u, v\} \cap \text{rpn}(w, S) = \emptyset$. Then, by Claim 6, let $w_1 \in \text{epn}(w, S)$ such that $w_1u \notin E(G)$. We have $w_1v \in E(G)$ by $G[w, u, v, w_1] \neq K_{1,3}$ and $uv \notin E(G)$. We have $d_S(u) = d_S(v) = 2$ since G is claw-free and by Claim 5. Assume that $N_S(u) = \{w, x\}$. Clearly, $w_1x \notin E(G)$ by $w_1 \in \text{epn}(w, S)$. Then, $vx \in E(G)$ by $G[w, v, w_1, u, x] \neq Z_2$. Thus, $N_S(u) = N_S(v) = \{w, x\}$.

Let $S' = (S \setminus \{w, x\}) \cup \{u, v\}$. Since G is claw-free and $uv \notin E(G)$, each vertex in $N(w) \cup N(x)$ is adjacent to u or v . Thus, S' is a dominating set, and moreover, S' is an independent set by $N_S(u) = N_S(v) = \{w, x\}$. Clearly, $x, w \notin S_1$ and then $|S' \cap S_1| > |S \cap S_1|$. Thus, S' is not a CSDS of G by the choice of S . Then, let $y \in S'$ such that $\text{rpn}(y, S') = \emptyset$, and let $y_1, y_2 \in \text{epn}(y, S')$ such that $y_1y_2 \notin E(G)$ and $|\{y_1, y_2\} \cap \text{epn}(y, S)| < 2$.

Without loss of generality, assume that $y_1 \notin \text{epn}(y, S)$. Then, $d_S(y_1) \geq 2$ and $N_S(y_1) \cap \{w, x\} \neq \emptyset$. Thus, $N(y_1) \cap \{u, v\} \neq \emptyset$. It follows that $d_{S'}(y_1) \geq 2$ if $y \notin \{u, v\}$, a contradiction to $y_1 \in \text{epn}(y, S')$. Thus, $y \in \{u, v\}$. Without loss of generality, assume that $y = u$. Then, $y_1v, y_2v \notin E(G)$ by $y_1, y_2 \in \text{epn}(u, S')$. We have $y_1w \in E(G)$ or $y_2w \in E(G)$ by $G[u, w, y_1, y_2] \neq K_{1,3}$. Without loss of generality, assume that $y_1w \in E(G)$. Then, $wy_2 \notin E(G)$. Otherwise, $G[w, y_1, y_2, v] = K_{1,3}$, a contradiction. It follows that $G[y_1, u, w, y_2, v] = B_{1,1}$, a contradiction. Thus, the theorem is true. $\square$

Let $G = K_m \vee (K_2 \cup K_1)$, $m \geq 3$. Then, G is a $\{K_{1,3}, B_{1,1}, Z_2\}$ -free graph with $\delta(G) \geq 3$ and $\alpha(G) = 2$. It is easy to verify that $\gamma_{cs}(G) = 1$. Thus, the condition that $\alpha(G) \neq 2$ in Theorem 4 is necessary.

By Theorem 4, if $\gamma_{cs}(G) < \alpha(G)$ for a graph G with $\delta(G) \geq 3$, then G contains an induced subgraph isomorphic to $K_{1,3}$, Z_2 , or $B_{1,1}$. Moreover, by Theorems 1 and 4, it is easy to obtain the following corollary.

Corollary 2. *Let G be a non-trivial every $\{K_{1,3}, B_{1,1}, Z_2\}$ -free graph G with $\delta(G) \geq 3$ and $\alpha(G) \neq 2$. Then, every maximum independent set of G is a minimum co-secured dominating set of G .*

According to Problem 2, we give the following result.

Theorem 5. $\gamma_{cs}(G) = \gamma_s(G)$ for every $\{K_3, C_5, P_5\}$ -free graph G with $\delta(G) \geq 2$.

Proof. Let G be a $\{K_3, C_5, P_5\}$ -free graph with $\delta(G) \geq 2$. By Theorem 2, it suffices to prove G contains a minimum CSDS S such that S is a SDS, i.e., $\gamma_{cs}(G) \geq \gamma_s(G)$. Suppose to the contrary that each minimum CSDS of G is not a SDS of G . Let S be a minimum CSDS of G . Then, there is an S -undefended vertex in $G - S$.

Firstly, we claim that for each vertex $u \in S$, $|\text{epn}(u, S)| \leq 1$. Suppose that there is a vertex $x \in S$ such that $|\text{epn}(x, S)| \geq 2$. Assume that $x_1, x_2 \in \text{epn}(x, S)$. We have $x_1x_2 \notin E(G)$, otherwise, $G[\{x, x_1, x_2\}] = K_3$, a contradiction. Thus, $x_i \notin \text{rpn}(x, S)$ for each $i \in \{1, 2\}$. By the definition of a CSDS, $\text{rpn}(x, S) \neq \emptyset$. Assume that $y \in \text{rpn}(x, S)$. Then, $y \notin \{x_1, x_2\}$ by $\{x_1, x_2\} \cap \text{rpn}(x, S) = \emptyset$, and hence $yx_i \in E(G)$ for each $i \in \{1, 2\}$. It

follows that $G[\{y, x_i, x\}] = K_3$ for each $i \in \{1, 2\}$, a contradiction. Thus, $|epn(x, S)| \leq 1$ and $rpn(x, S) = \{x'\}$ if $epn(x, S) = \{x'\}$. It follows that if a vertex $z \in V(G) - S$ is S -undefended, then $d_S(z) \geq 2$.

Suppose $z \in V(G) - S$ is S -undefended. Then, for each vertex w in $N_S(z)$, $epn(w, S) \neq \emptyset$ and the exactly one vertex in $epn(w, S)$ is non-adjacent to z . By $d_S(z) \geq 2$, we assume $w_1, w_2 \in N_S(z)$. Let $u_1 \in epn(w_1, S), u_2 \in epn(w_2, S)$. Then, $u_1w_2, u_2w_1 \notin E(G)$. By $G[\{z, w_1, w_2\}] \neq K_3$, we have $w_1w_2 \notin E(G)$. Then, $u_1u_2 \in E(G)$ by $G[\{u_1, w_1, z, w_2, u_2\}] \neq P_5$, which implies $G[\{u_1, w_1, z, w_2, u_2\}] = C_5$, a contradiction. Thus, each vertex in $G - S$ is S -defended, and S is a SDS of G . We have $\gamma_{cs}(G) \geq \gamma_s(G)$, and the theorem is true. $\square$

By Theorem 5, if $\gamma_{cs}(G) < \gamma_s(G)$ for a graph G with $\delta(G) \geq 2$, then G contains an induced subgraph isomorphic to K_3 , C_5 , or P_5 . Since a bipartite graph G contains no odd cycles, G is $\{K_3, C_5\}$ -free. Thus, by Theorem 5, we can obtain the following corollary.

Corollary 3. *If G is a bipartite graph with $\delta(G) \geq 2$ and contains no induced subgraph isomorphic to P_5 , then $\gamma_{cs}(G) = \gamma_s(G)$.*

By Corollary 3, we have $\gamma_{cs}(G) = \gamma_s(G)$ for a complete bipartite graph $G = K_{p,q}$ with $2 \leq p \leq q$. This generalizes Proposition 2.15 in [10].

3. Conclusions

Arumugam et al. [10] proposed two questions: (1) Characterize a graph G with $\gamma_{cs}(G) = \alpha(G)$; (2) Characterize a graph G with $\gamma_{cs}(G) = \gamma_s(G)$. In this paper, according to question (1), we prove that $\gamma_{cs}(G) = \alpha(G)$ for every non-trivial $\{K_4^-, C_4, B_{1,1}, P_5\}$ -free graph; moreover, we obtain that $\gamma_{cs}(G) = \alpha(G)$ for every non-trivial $\{K_{1,3}, B_{1,1}, Z_2\}$ -free graph G with $\delta(G) \geq 3$ and $\alpha(G) \neq 2$; according to question (2), we obtain that $\gamma_{cs}(G) = \gamma_s(G)$ for each $\{K_3, C_5, P_5\}$ -free graph G with $\delta(G) \geq 2$. Our conclusions can generalize some results on the bounds of co-secure domination number of some families of trees and bipartite graphs.

Author Contributions: Conceptualization, J.C. and T.L.; methodology, X.C. and T.L.; validation, X.C. and J.Z.; writing—original draft preparation, X.C. and T.L.; writing—review and editing, J.C.; project administration, L.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by National Natural Science Foundation of China (Grant No. 11901268, No. 12131013) and the Fundamental Research Funds for the Universities of Liaoning Province (2024).

Data Availability Statement: No data, models, or code were generated or used during the study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Bondy, J.; Murty, U. *Graph Theory with Applications*; American Elsevier: New York, NY, USA, 1976.
2. Cockayne, E.; Grobler, P.; Gründlingh, W.; Munganga, J.; Vuuren, J. Protection of a graph. *Util. Math.* **2005**, *67*, 19–32.
3. Araki, T.; Yumoto, I. On the secure domination numbers of maximal outerplanar graphs. *Discrete Appl. Math.* **2018**, *236*, 23–29. [[CrossRef](#)]
4. Cockayne, E.; Favaron, O.; Mynhardt, C. Secure domination, weak Roman domination and forbidden subgraphs. *Bull. Inst. Comb. Appl.* **2003**, *39*, 87–100.
5. Degawa, S.; Saito, A. A note on secure domination in C_5 -free graphs. *Discrete Appl. Math.* **2023**, *333*, 82–83. [[CrossRef](#)]
6. Haythorpe, M.; Newcombe, A. The secure domination number of Cartesian products of small graphs with paths and cycles. *Discrete Appl. Math.* **2022**, *309*, 32–45. [[CrossRef](#)]
7. Manjusha, P.; Chithra, M.R. Co-secure Domination in Mycielski Graph. *J. Comb. Math. Comb. Comput.* **2020**, *113*, 289–297.
8. Merouane, H.; Chellali, M. On secure domination in graphs. *Inform. Process. Lett.* **2015**, *115*, 786–790. [[CrossRef](#)]

9. Pothuvath, M.; Iyer, R.R.; Asiri, A.; Somasundaram, K. Co-Secure Domination in Jump Graphs for Enhanced Security. *Mathematics* **2024**, *12*, 3077. [[CrossRef](#)]
10. Arumugan, S.; Ebadi, K.; Manrique, M. Co-secure and secure domination in graphs. *Util. Math.* **2014**, *94*, 167–182.
11. Joseph, A.; Sangeetha, V. Bounds on co-secure domination in graphs. *Int. J. Math. Trends Technol.* **2018**, *55*, 158–164. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.