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Two-Dimensional Time Fractional River-Pollution Model and Its Remediation by Unsteady Aeration

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Abstract: This study contains a mathematical model for river pollution and its remediation for an unsteady state and investigates the effect of aeration on the degradation of pollutants. The governing equation is a pair of nonlinear time-fractional two-dimensional advection-diffusion equations for pollutant and dissolved oxygen (DO) concentration. The coupling of these equations arises due to the chemical interactions between oxygen and pollutants, forming harmless chemicals. The Fractional Reduced Differential Transform Method (FRDTM) is applied to provide approximate solutions for the given model. Also, the convergence of solutions is checked for efficacy and accuracy. The effect of longitudinal and transverse diffusion coefficients of pollutant and DO on the concentration of pollutant and DO is analyzed numerically and graphically. Also, we checked the effect of change in the river's longitudinal and transverse seepage velocity on pollutant and DO concentration numerically and graphically. We analyzed the comparison of change in the value of half-saturated oxygen demand concentration for pollutant decay on pollutant and DO concentration numerically and graphically. Also, numerical and graphical analysis examined the effect of fractional parameters on pollution levels.

Keywords: fractional reduced differential transform method; river-pollution model; coupled non-linear PDE; unsteady aeration

MSC: 35R11; 35A22; 35C10; 35G25



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1. Introduction

Several factors must be considered when evaluating river water quality, including the concentration of dissolved oxygen, the presence of nitrates, chlorides, and phosphates, the amount of suspended solids, the occurrence of environmental hormones, the chemical oxygen demand, heavy metals, and the presence of bacteria. Pollutants originating from agricultural activities can be considered a substantial factor in the degradation of surface and groundwater quality. The advection-dispersion equations are relevant in several fields, such as water pollution, groundwater hydrology, chemical engineering, biosciences, environmental sciences, and petroleum engineering [1]. Huang et al. built a mathematical model to study the movement of solutes in porous materials with varying properties, considering the effect of scale-dependent dispersion. They hypothesized that the dispersion coefficient is directly proportional to the velocity of the pore water, having a linear relationship [2]. Pimpunchat et al. developed a fundamental mathematical model to study river pollution and examined the effects of aeration on reducing pollutants [3].

Hussain et al. examined a mathematical model that enables estimating river-pollutant concentration levels for the steady-state in a one-dimensional region without dispersion. The concentrations of pollutants and DO are determined by a set of coupled equations that include the processes of reaction, diffusion, and advection [4]. Ibrahim et al. analyzed a

study on river-pollution cleanup. They examined the use of unstable aeration with arbitrary initial and boundary conditions [5]. On a finite domain, Liu et al. solved a space-time fractional advection–dispersion equation using an implicit difference method (IDM) and an explicit difference method (EDM). They also verified the convergence and stability of the methods [6]. Attia et al. used a numerical approach to find an approximate solution of the time-fractional advection–diffusion equation (FADE) under the Atangana–Baleanu derivative in Caputo sense (ABC) with Mittag–Leffler kernel by using reproducing kernel Hilbert space method (RKHSM) [7].

Mardani et al. used a very effective and precise meshless technique to solve the time fractional advection–diffusion equation with variable coefficients. This approach is based on the moving least square (MLS) approximation [8]. Maisuria et al. [9] applied the fractional reduced differential transform method (FRDTM) to derive a solution for the time-fractional two-dimensional advection–dispersion equation. The equation considered variable dispersion coefficients, velocities, decay constant of the first order, production rate coefficient for the solute at the zero-order level, and retardation factor. Fractional-order models provide more versatility in describing complex dynamic phenomena because of their continuous nature which can more accurately represent real-world dynamics. Nonlinear fractional partial differential equations are often more challenging to solve than linear ones. Furthermore, no technique provides a precise solution for the fractional-order differential equation. Therefore, the presence of fractional-order nonlinear PDEs improves the significance of the study. They can provide more precise analyses of systems that possess memory and hereditary features. Fractional-order models explicitly include non-local memory effects, while integer-order models assume just local memory. Nevertheless, fractional-order models have some disadvantages. They may possess more mathematical complexity and provide a more demanding analysis compared to models based on integer order. Solving fractional-order differential equations needs the use of specific methods.

In this work, we have used the latest method available, the FRDTM—an effective and reliable approach to solving the coupled non-linear time-fractional two-dimensional advection–diffusion equations that characterize the concentration of dissolved oxygen (DO) and the amount of pollution. For nonlinear time-fractional differential equations, the FRDTM produces very accurate numerical answers without needing spatial discretization, linearization, transformation, or perturbation. The originality of our work is that the FRDTM can be used to generate a wide range of coupled fractional-order nonlinear equations with different physical structures that arise in science.

The structure of the manuscript is as follows. Section 2 presents the mathematical formulation of the problem. In Section 3, the core concepts of the fractional reduced differential transform method and the associated convergence analysis are introduced. Section 4 provides a detailed discussion of the numerical results, including three-dimensional graphical comparisons and tabulated data. Finally, Section 5 offers a summary of the conclusions.

2. Mathematical Formulation of the Problem

In this study, a two-dimensional mathematical model is discussed to represent turbulent river flow. The advection–diffusion equation can be the best fit to approximate river pollution. The cross-sectional area of river is assumed to be constant. The two-dimensional nature of the unstable flow in the river defined by distances $x^*(m)$ and $y^*(m)$ measured from the origin with $x^* = 0$ and $y^* = 0$ have been considered for the present study. The governing equations have been described as follow [1,10,11]:

$$\frac{\partial(A^*p^*)}{\partial t^*} = D_{p_x}^* \frac{\partial^2(A^*p^*)}{\partial x^{*2}} + D_{p_y}^* \frac{\partial^2(A^*p^*)}{\partial y^{*2}} - \frac{\partial(u_x^*A^*p^*)}{\partial x^*} - \frac{\partial(u_y^*A^*p^*)}{\partial y^*} - k_1^* \frac{q^*}{q^* + k^*} A^*p^* + f^* \quad (1)$$

$$\frac{\partial(A^*q^*)}{\partial t^*} = D_{q_x}^* \frac{\partial^2(A^*q^*)}{\partial x^{*2}} + D_{q_y}^* \frac{\partial^2(A^*q^*)}{\partial y^{*2}} - \frac{\partial(u_x^*A^*q^*)}{\partial x^*} - \frac{\partial(u_y^*A^*q^*)}{\partial y^*} - k_2^* \frac{q^*}{q^* + k^*} A^*p^* + \beta^*(K^* - q^*) \quad (2)$$

where $0 \leq x^* \leq L^*, 0 \leq y^* \leq W^*, t^* \geq 0$

Let us consider the following non-dimensional variables and parameters [1,12]

$$x = \frac{x^*}{L^*}, y = \frac{y^*}{L^*}, t = k_2^* t^*, p = \frac{p^*}{p_0^*}, q = \frac{q^*}{q_0^*}, W = \frac{W^*}{L^*}, D_{p_x} = \frac{D_{p_x}^*}{L^{*2} k_1^*}, D_{p_y} = \frac{D_{p_y}^*}{L^{*2} k_1^*}, u_x = \frac{u_x^*}{L^* k_1^*}, u_y = \frac{u_y^*}{L^* k_1^*}, k = \frac{k^*}{q_0^*}, f = \frac{f^*}{A^* k_1^* p_0^*},$$

$$D_{q_x} = \frac{D_{q_x}^*}{L^{*2} k_1^*}, D_{q_y} = \frac{D_{q_y}^*}{L^{*2} k_1^*}, k_3 = \frac{k_3^*}{k_1^*}, \beta = \frac{\beta^*}{A^* k_1^*}, K = \frac{K^*}{q_0^*}$$

Hence, governing Equations (1) and (2) have been converted in the given form:

$$\frac{\partial p}{\partial t} = D_{p_x} \frac{\partial^2 p}{\partial x^2} + D_{p_y} \frac{\partial^2 p}{\partial y^2} - \frac{\partial(u_x p)}{\partial x} - \frac{\partial(u_y p)}{\partial y} - \frac{q}{q+k} p + f \quad (3)$$

$$\frac{\partial q}{\partial t} = D_{q_x} \frac{\partial^2 q}{\partial x^2} + D_{q_y} \frac{\partial^2 q}{\partial y^2} - \frac{\partial(u_x q)}{\partial x} - \frac{\partial(u_y q)}{\partial y} - k_3 \frac{q}{q+k} p + \beta(K - q) \quad (4)$$

where $0 \leq x \leq 1, 0 \leq y \leq W, t \geq 0$

The initial conditions of Equations (3) and (4) are [12,13]:

$$p(x, y, 0) = C_1 e^{-\left(x+y\sqrt{\frac{D_{p_y}}{D_{p_x}}}\right)} \quad (5)$$

$$q(x, y, 0) = C_2 e^{-\left(x+y\sqrt{\frac{D_{q_y}}{D_{q_x}}}\right)} \quad (6)$$

3. Fractional Reduced Differential Transform Method (FRDTM)

This section contains an explanation of several fundamental concepts and properties of fractional calculus theory [14,15].

Definition 1. The fractional derivative of the function $m \in \mathbb{C}_\mu$ of order $\alpha \geq 0$, in the sense of Caputo, is denoted by $D^\alpha m(\tau)$ and is defined as

$$D^\alpha m(\tau) = \frac{1}{\Gamma(k-\alpha)} \int_0^1 (\tau-t) m^k(t) dt, \quad (7)$$

for $k-1 < \alpha \leq k, k \in \mathbb{N}, \tau > 0, m \in \mathbb{C}_\mu, \mu \geq -1$

Let $\psi(\varphi, \varkappa, \tau)$ be a function of three variables, which can be expressed as $\psi(\varphi, \varkappa, \tau) = a(\varphi)b(\varkappa)c(\tau)$. The differential transform's characteristics can be used to determine the representation of the function $\psi(\varphi, \varkappa, \tau)$.

$$\psi(\varphi, \varkappa, \tau) = \sum_{k=0}^{\infty} \Psi_k(\varphi, \varkappa) \tau^{\alpha k} \quad (8)$$

where $\Psi_k(\varphi, \varkappa)$ is a t -dimensional spectrum function. The transformed function of $\psi(\varphi, \varkappa, \tau)$ is $\Psi_k(\varphi, \varkappa)$. The fundamental concepts and operations of FRDTM are as outlined below:

The operator R_D represents the reduced differential transform, whereas R_D^{-1} represents the inverse reduced differential transform.

Definition 2. Assuming that $\psi(\varphi, \varkappa, \tau)$ is analytic and continuous differentiable with respect to $\varphi, \varkappa$, and τ inside the domain of interest. The reduced differential transformations for the function $\psi(\varphi, \varkappa, \tau)$ is defined as follows [16,17]:

$$R_D[\psi(\varphi, \varkappa, \tau)] \approx \Psi_k(\varphi, \varkappa) = \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial \tau^{\alpha k}} \psi(\varphi, \varkappa, \tau) \right]_{\tau=0} \quad (9)$$

The time-fractional order derivative is represented by the symbol α .

Definition 3. The inverse differential transformations of the function $\Psi_k(\varphi, \varkappa)$ is presented as:

$$R_D^{-1}[\Psi_k(\varphi, \varkappa)] \approx \psi(\varphi, \varkappa, \tau) = \sum_{k=0}^{\infty} \Psi_k(\varphi, \varkappa) \tau^{\alpha k} \quad (10)$$

From Equations (9) and (10), we obtain

$$\psi(\varphi, \varkappa, \tau) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial \tau^{\alpha k}} \psi(\varphi, \varkappa, \tau) \right]_{\tau=0} \tau^{\alpha k} \quad (11)$$

We examine the general fractional non-linear non-homogeneous partial differential equation in order to understand the operation of the FRDTM [18].

$$M\psi(\varphi, \varkappa, \tau) + A\psi(\varphi, \varkappa, \tau) + N\psi(\varphi, \varkappa, \tau) = g(\varphi, \varkappa, \tau) \quad (12)$$

with initial condition

$$\psi(\varphi, \varkappa, 0) = \Xi(\varphi, \varkappa) \quad (13)$$

where $M = \frac{\partial^{\alpha}}{\partial \tau^{\alpha}}$, the linear operator A has partial derivatives, whereas the operator $N\psi(\varphi, \varkappa, \tau)$ is non-linear, and $g(\varphi, \varkappa, \tau)$ represents an inhomogeneous term.

We can derive the following iteration formula from the concepts of the FRDTM:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k + 1)} \Psi_{k+1}(\varphi, \varkappa) = G_k(\varphi, \varkappa) - A\Psi_k(\varphi, \varkappa) - N\Psi_k(\varphi, \varkappa) \quad (14)$$

The functions $\Psi_k(\varphi, \varkappa)$, $A\Psi_k(\varphi, \varkappa)$, $N\Psi_k(\varphi, \varkappa)$, and $G_k(\varphi, \varkappa)$ represent the transformations functions of the $\psi(\varphi, \varkappa, \tau)$, $A\psi(\varphi, \varkappa, \tau)$, $N\psi(\varphi, \varkappa, \tau)$, and $g(\varphi, \varkappa, \tau)$ functions, respectively.

From the initial condition (13), we obtain

$$\Psi_0(\varphi, \varkappa) = \Xi(\varphi, \varkappa) \quad (15)$$

By substituting Equation (15) into Equation (14) and carrying out a basic iterative computation, we obtain the values of $\Psi_k(\varphi, \varkappa)$ as follows. Subsequently, the inverse translation of the set of values $\{\Psi_k(\varphi, \varkappa)\}_{k=0}^n$ yields the n -terms approximation solution in the following manner.

$$\psi_n(\varphi, \varkappa, \tau) = \sum_{k=0}^n \Psi_k(\varphi, \varkappa) \tau^{\alpha k} \quad (16)$$

Therefore, the exact answer is provided by

$$\psi(\varphi, \varkappa, \tau) = \lim_{n \rightarrow \infty} \psi_n(\varphi, \varkappa, \tau) \quad (17)$$

Theorem 1. If $\psi(\varphi, \varkappa, \tau) = \sum_{k=0}^{\infty} \Psi_k(\varphi, \varkappa) \tau^{\alpha k}$ is a given series [19–21];

1. If $\exists 0 < \lambda < 1$ such that $\frac{\|\Psi_{k+1}\|}{\|\Psi_k\|} \leq \lambda$ then the given series solution is convergent.
2. If $\exists \lambda > 1$ such that $\frac{\|\Psi_{k+1}\|}{\|\Psi_k\|} \geq \lambda$ then the given series solution is divergent.

The proof of Theorem 1, which is a particular case of Banach's fixed point theorem, can be found in the literature.

Corollary 1. The series solution $\sum_{k=0}^{\infty} \Psi_k(\varphi, \varkappa, \tau)$ exhibits convergence towards the exact solution $\psi(\varphi, \varkappa, \tau)$ with the condition that $0 \leq \chi_n < 1$, where n is a whole number and for every n [19–21]:

$$\chi_n = \begin{cases} \frac{\|\Psi_{n+1}\|}{\|\Psi_n\|}, & \|\Psi_n\| \neq 0 \\ 0, & \|\Psi_n\| = 0 \end{cases}$$

The fundamental mathematical operations of the FRDTM are shown in Table 1.

Table 1. Transform table [22–25].

Function	Transformation
$\psi(\varphi, \varkappa, \tau)$	$\Psi_i(\varphi, \varkappa) = \frac{1}{\Gamma(\alpha i + 1)} \left[\frac{\partial^{\alpha i}}{\partial \tau^{\alpha i}} \psi(\varphi, \varkappa, \tau) \right]_{\tau=0}$
$co(\varphi, \varkappa, \tau) \pm dp(\varphi, \varkappa, \tau)$	$cO_i(\varphi, \varkappa) \pm dP_i(\varphi, \varkappa)$, c and d are constant.
$o(\varphi, \varkappa, \tau)p(\varphi, \varkappa, \tau)$	$\sum_{k=0}^i O_k(\varphi, \varkappa) P_{i-k}(\varphi, \varkappa)$
$\frac{\partial^{\alpha k}}{\partial \tau^{\alpha k}} \psi(\varphi, \varkappa, \tau)$	$\frac{\Gamma(i\alpha + k\alpha + 1)}{\Gamma(i\alpha + 1)} \Psi_{i+k}(\varphi, \varkappa)$
$\frac{\partial^k}{\partial \varphi^k} \psi(\varphi, \varkappa, \tau)$	$\frac{\partial^k}{\partial \varphi^k} \Psi_i(\varphi, \varkappa)$.
$\frac{\partial^k}{\partial \varkappa^k} \psi(\varphi, \varkappa, \tau)$	$\frac{\partial^k}{\partial \varkappa^k} \Psi_i(\varphi, \varkappa)$.
$\varphi^m \varkappa^j \tau^k$	$\varphi^m \varkappa^j \delta(i\alpha - k)$ where $\delta(i\alpha - k) = \begin{cases} 1 & i\alpha = k \\ 0, & i\alpha \neq k \end{cases}$.

4. Results and Discussion

This section shows the effect of various parameters on concentration functions as well as the numerical and graphical outcomes of the obtained solutions.

We fractionalized Equations (3) and (4) into time-fractional partial differential equations in order to comprehend the anomalous behavior of the coupled system of equations. These equations are provided by

$$\frac{\partial^\lambda p}{\partial t^\lambda} = D_{p_x} \frac{\partial^2 p}{\partial x^2} + D_{p_y} \frac{\partial^2 p}{\partial y^2} - \frac{\partial(u_x p)}{\partial x} - \frac{\partial(u_y p)}{\partial y} - \frac{q}{q+k} p + f \quad (18)$$

$$\frac{\partial^\lambda q}{\partial t^\lambda} = D_{q_x} \frac{\partial^2 q}{\partial x^2} + D_{q_y} \frac{\partial^2 q}{\partial y^2} - \frac{\partial(u_x q)}{\partial x} - \frac{\partial(u_y q)}{\partial y} - k_3 \frac{q}{q+k} p + \beta(K - q) \quad (19)$$

where $0 \leq \lambda \leq 1$.

Applying FRDTM to both sides of Equations (18) and (19), we obtain

$$\frac{\Gamma(\lambda((k+1)+1))}{\Gamma(\lambda k + 1)} P_{k+1} = D_{p_x} \frac{\partial^2 P_k}{\partial x^2} + D_{p_y} \frac{\partial^2 P_k}{\partial y^2} - u_x \frac{\partial P_k}{\partial x} - u_y \frac{\partial P_k}{\partial y} - \sum_{r=0}^k G_r P_{k-r} + f \delta(k\lambda) \quad (20)$$

$$\frac{\Gamma(\lambda((k+1)+1))}{\Gamma(\lambda k + 1)} Q_{k+1} = D_{q_x} \frac{\partial^2 Q_k}{\partial x^2} + D_{q_y} \frac{\partial^2 Q_k}{\partial y^2} - u_x \frac{\partial Q_k}{\partial x} - u_y \frac{\partial Q_k}{\partial y} - k_3 \sum_{r=0}^k G_r P_{k-r} + \beta(K \delta(k\lambda) - Q_k) \quad (21)$$

$$P_{k+1} = \frac{\Gamma(\lambda k + 1)}{\Gamma(\lambda((k+1)+1))} \left(D_{p_x} \frac{\partial^2 P_k}{\partial x^2} + D_{p_y} \frac{\partial^2 P_k}{\partial y^2} - u_x \frac{\partial P_k}{\partial x} - u_y \frac{\partial P_k}{\partial y} - \sum_{r=0}^k G_r P_{k-r} + f \delta(k\lambda) \right) \quad (22)$$

$$Q_{k+1} = \frac{\Gamma(\lambda k + 1)}{\Gamma(\lambda((k+1)+1))} \left(D_{q_x} \frac{\partial^2 Q_k}{\partial x^2} + D_{q_y} \frac{\partial^2 Q_k}{\partial y^2} - u_x \frac{\partial Q_k}{\partial x} - u_y \frac{\partial Q_k}{\partial y} - k_3 \sum_{r=0}^k G_r P_{k-r} + \beta(K \delta(k\lambda) - Q_k) \right) \quad (23)$$

where G_r is the transformation function of $g = \frac{q}{q+k}$. From the initial conditions (5) and (6), we write

$$P_0(x, y) = C_1 e^{-\left(x+y\sqrt{\frac{D_{p_y}}{D_{p_x}}}\right)} \quad (24)$$

$$Q_0(x, y) = C_2 e^{-\left(x+y\sqrt{\frac{D_{q_y}}{D_{q_x}}}\right)} \quad (25)$$

Putting (24) and (25) into (22) and (23), we obtain the approximate analytical solution as follows:

$$p(x, y, t) = P_0(x, y) + P_1(x, y)t^\lambda + P_2(x, y)t^{2\lambda} + P_3(x, y)t^{3\lambda} + \dots = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \dots \quad (26)$$

$$q(x, y, t) = Q_0(x, y) + Q_1(x, y)t^\lambda + Q_2(x, y)t^{2\lambda} + Q_3(x, y)t^{3\lambda} + \dots = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \dots \quad (27)$$

where

$$P_1(x, y) = \frac{\left(f + D_{p_x} C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + u_x C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + u_y C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \right. \\ \left. + \frac{D_{p_y}^2 C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} + C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right) \right)}{\Gamma(\lambda+1)}$$

$$P_2(x, y) = \frac{\left(D_{p_x} p e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-2x-2y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \right. \\ \left. + u_x C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + \frac{D_{p_y}^2 C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + \frac{D_{p_y} u_y C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} \right. \\ \left. + C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right) \right. \\ \left. + C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right) \right)}{\Gamma(2\lambda+1)}$$

$$+ \frac{\left(D_{p_y} C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + 2 C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-2x-2y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \right. \\ \left. + \frac{D_{p_y}^3 C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}^2} + \frac{D_{p_y} u_x C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} + \frac{3 D_{q_y} C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-2x-2y} \sqrt{\frac{D_{q_y}}{D_{q_x}}}}{D_{q_x}} \right. \\ \left. + \frac{D_{p_y} u_y C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} + \frac{D_{p_y} C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right)}{D_{p_x}} \right. \\ \left. + \frac{D_{q_y} C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right)}{D_{q_x}} \right. \\ \left. + 2 C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right) \right)}{\Gamma(2\lambda+1)}$$

$$+ \frac{\left(D_{p_x} C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + u_x C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + 5 C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-2x-2y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \right. \\ \left. + u_y C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} \sqrt{\frac{D_{p_y}}{D_{p_x}}} + \frac{D_{p_y}^2 C_1 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}}}{D_{p_x}} \right. \\ \left. + 4 C_1 C_2 e^{-x-y} \sqrt{\frac{D_{p_y}}{D_{p_x}}} e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} \left(k + C_2 e^{-x-y} \sqrt{\frac{D_{q_y}}{D_{q_x}}} - 2 \right) \right)}{\Gamma(2\lambda+1)}$$

$$\begin{aligned}
& + \frac{u_x \left(u_x C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} + C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} \sqrt{\frac{D_{py}}{D_{px}}} \right.}{\Gamma(2\lambda+1)} \\
& \left. + \frac{D_{py}^2 C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}}}{D_{px}} + 2 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right. \\
& \left. + D_{px} C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} \right) \\
& + \frac{C_1 \Gamma(\lambda+1) e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} \left(\left(C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \left(\beta \left(K - C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right. \right.}{\Gamma(\lambda+1)} \\
& \left. \left. + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \right. \right. \\
& \left. \left. + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right) \right) \\
& + \frac{C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(\beta \left(K - C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right.}{\Gamma(\lambda+1)} \\
& \left. + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \right. \\
& \left. + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right) \right) \\
& + \frac{C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \left(f + D_{px} C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} + u_x C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} \right.}{\Gamma(2\lambda+1)} \\
& \left. + u_y C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} \sqrt{\frac{D_{py}}{D_{px}}} + \frac{D_{py}^2 C_1 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}}}{D_{px}} \right. \\
& \left. + C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right) \\
& + \frac{\beta \left(K - C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{\Gamma(\lambda+1)} \\
& \left. + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \right. \\
& \left. + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right) \\
& Q_1(x, y) = \frac{\beta \left(K - C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{\Gamma(\lambda+1)} \\
& \left. + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \right. \\
& \left. + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
Q_2(x, y) = & \frac{u_x \left(u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - \beta C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \right. \\
& + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + k_3 C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}} \\
& \left. + 2k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right)}{\Gamma(2\lambda+1)} \\
& + \frac{u_y \left(u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} - \beta C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \right. \\
& + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}}}{D_{qx}} + \frac{u_y D_{qy} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + k_3 C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \\
& + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{py}}{D_{px}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \\
& \left. + k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right)}{\Gamma(2\lambda+1)} \\
& + \frac{D_{qx} \left(u_x C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + D_{qx} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - \beta C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \right. \\
& + \frac{D_{qy}^2 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + 5k_3 C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}} \\
& \left. + 4k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \right)}{\Gamma(2\lambda+1)} \\
& + \frac{C_2 \left(D_{qy} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} + \frac{D_{qy}^3 C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}^2} + \frac{u_x D_{qy} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} - \frac{D_{qy} \beta C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \right. \\
& + \frac{u_y D_{qy} C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}}}{D_{qx}} + 2k_3 C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{py}}{D_{px}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \\
& + 2k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{py}}{D_{px}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \\
& + \frac{D_{py} k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right)}{D_{px}} + \frac{3 D_{qy} k_3 C_1 C_2^2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-2x-2y \sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} \\
& \left. + \frac{D_{qy} k_3 C_1 C_2 e^{-x-y \sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y \sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right)}{D_{qx}} \right)}{\Gamma(2\lambda+1)}
\end{aligned}$$

$$\begin{aligned}
& \beta \left(K - C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + D_{qx} C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \\
& + u_y C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \\
& + \frac{D_{qy}^2 C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + k_3 C_1 C_2 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(k + C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} - 2 \right) \Bigg) \\
& \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)} \\
& + \left(C_1 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} \left(\left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \left(\beta \left(K - C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \right. \right. \right. \\
& \left. \left. + D_{qx} C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \right. \right. \\
& \left. \left. + \frac{D_{qy}^2 C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + k_3 C_1 C_2 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \right) \right) \\
& \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1)} \\
& + \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(\beta \left(K - C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \right) + u_x C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \right. \right. \\
& \left. \left. + D_{qx} C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + u_y C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \sqrt{\frac{D_{qy}}{D_{qx}}} \right. \right. \\
& \left. \left. + \frac{D_{qy}^2 C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}}}{D_{qx}} + k_3 C_1 C_2 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \right) \right) \\
& \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1)} \\
& + \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \left(f + D_{px} C_1 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} + u_x C_1 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} \right. \right. \\
& \left. \left. + u_y C_1 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} \sqrt{\frac{D_{py}}{D_{px}}} + \frac{D_{py}^2 C_1 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}}}{D_{px}} \right. \right. \\
& \left. \left. + C_1 C_2 e^{-x-y\sqrt{\frac{D_{py}}{D_{px}}}} e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} \left(C_2 e^{-x-y\sqrt{\frac{D_{qy}}{D_{qx}}}} + k - 2 \right) \right) \right) \\
& \frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}
\end{aligned}$$

In addition, to analyze the convergence of an individual series solution, we compute the terms ζ_k using Corollary 1. For the series solution of $p(x, y, t)$, we obtain $\zeta_0 = \frac{\|\chi_1\|}{\|\chi_0\|} = 0.91992 < 1, \zeta_1 = \frac{\|\chi_2\|}{\|\chi_1\|} = 0.75091 < 1$. And for the series solution of $q(x, y, t)$, we obtain $\zeta_0 = \frac{\|\psi_1\|}{\|\psi_0\|} = 0.23513 < 1, \zeta_1 = \frac{\|\psi_2\|}{\|\psi_1\|} = 0.05883 < 1$. This observation verifies that the FRDTM produces a series solution for p and q , demonstrating convergence towards the exact answer.

Now, we have obtained the following tables of numerical values and graphs of the concentration functions using MATLAB 2023 for fixed parameters $D_{px} = 1, D_{py} = 0.1, u_x = 1, u_y = 0.1, f = 1, D_{qx} = 1, D_{qy} = 0.1, k_3 = 0.5, \beta = 2, K = 2, k = 1$.

The numerical values of pollutant concentration for fixed $t = 0.1, 0.5$ are in Tables 2 and 3. The numerical values of DO concentration for fixed $t = 0.1, 0.5$, respectively, are given in Tables 4 and 5. The impact of change in time on pollutant and DO concentration can be seen graphically in Figures 1 and 2, respectively. The numerical values of p and q for fixed $x = 0.4, 0.8$ are given in Tables 6–9. Figures 3 and 4 give the three-dimensional graphical comparison of pollutant and DO concentration function for fixed x , respectively. The numerical values of p and q for fixed $y = 0.01, 0.05$ are given in Tables 10–13. The impact of change in y on pollutant and DO concentration functions can be seen graphically in Figures 5 and 6, respectively. It has been seen that the amount of pollution and DO concentration rises as t rises. And the levels of p and q decrease when x and y rise.

Table 2. $p(x, y, 0.1)$.

$x \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	0.413736	0.411974	0.410227	0.408494	0.406775	0.405069	0.403378	0.4017	0.400035	0.398383
0.2	0.364269	0.362893	0.361528	0.360172	0.358827	0.357491	0.356166	0.35485	0.353544	0.352248
0.3	0.325284	0.324189	0.323101	0.322021	0.320949	0.319883	0.318825	0.317774	0.31673	0.315694
0.4	0.293999	0.293112	0.29223	0.291355	0.290485	0.28962	0.288761	0.287907	0.287059	0.286216
0.5	0.268475	0.267745	0.26702	0.266299	0.265583	0.26487	0.264162	0.263458	0.262758	0.262062
0.6	0.247342	0.246734	0.246129	0.245527	0.244929	0.244334	0.243742	0.243153	0.242568	0.241985
0.7	0.229613	0.2291	0.228589	0.22808	0.227575	0.227071	0.226571	0.226073	0.225577	0.225084
0.8	0.214567	0.214129	0.213692	0.213258	0.212826	0.212396	0.211968	0.211543	0.211119	0.210697
0.9	0.201667	0.20129	0.200914	0.20054	0.200167	0.199796	0.199427	0.19906	0.198694	0.19833
1	0.190511	0.190183	0.189856	0.189531	0.189207	0.188885	0.188564	0.188245	0.187926	0.18761

Table 3. $p(x, y, 0.5)$.

$x \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	5.732374	5.67944	5.627131	5.575438	5.524354	5.473869	5.423977	5.374668	5.325936	5.277773
0.2	4.324727	4.288016	4.251715	4.21582	4.180325	4.145226	4.110516	4.076192	4.042247	4.008678
0.3	3.339795	3.313837	3.288154	3.262742	3.237599	3.212721	3.188105	3.163747	3.139645	3.115795
0.4	2.637396	2.618698	2.600187	2.581862	2.56372	2.54576	2.527978	2.510373	2.492942	2.475685
0.5	2.12733	2.113624	2.100048	2.0866	2.07328	2.060086	2.047016	2.034069	2.021245	2.00854
0.6	1.750571	1.740357	1.730236	1.720205	1.710265	1.700414	1.690651	1.680975	1.671385	1.661881
0.7	1.467825	1.460098	1.452437	1.444841	1.43731	1.429843	1.422439	1.415098	1.407819	1.400602
0.8	1.25249	1.246561	1.24068	1.234847	1.22906	1.223321	1.217628	1.211981	1.206379	1.200823
0.9	1.086249	1.08164	1.077066	1.072528	1.068025	1.063556	1.059122	1.054722	1.050355	1.046022
1	0.956285	0.952658	0.949059	0.945485	0.941938	0.938417	0.934922	0.931452	0.928008	0.924589

Table 4. $q(x, y, 0.1)$.

$x \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	2.126643	2.120715	2.114811	2.108929	2.103069	2.097232	2.091417	2.085624	2.079854	2.074105
0.2	1.949676	1.944415	1.939172	1.933949	1.928745	1.923561	1.918395	1.913248	1.90812	1.903011
0.3	1.792241	1.78755	1.782875	1.778217	1.773575	1.76895	1.764342	1.75975	1.755174	1.750614
0.4	1.65164	1.647444	1.643261	1.639094	1.634941	1.630802	1.626678	1.622568	1.618472	1.61439
0.5	1.52571	1.521946	1.518196	1.514458	1.510732	1.50702	1.50332	1.499633	1.495958	1.492296
0.6	1.41267	1.409289	1.405919	1.40256	1.399212	1.395876	1.392551	1.389237	1.385934	1.382642
0.7	1.311031	1.307989	1.304956	1.301934	1.298921	1.295918	1.292926	1.289943	1.28697	1.284007
0.8	1.219524	1.216783	1.214051	1.211328	1.208614	1.205909	1.203212	1.200525	1.197846	1.195176
0.9	1.137054	1.134582	1.132119	1.129664	1.127217	1.124777	1.122346	1.119922	1.117507	1.115099
1	1.06267	1.06044	1.058217	1.056002	1.053793	1.051592	1.049398	1.047211	1.045031	1.042859

Table 5. $q(x, y, 0.5)$.

$x \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	5.166659	5.138586	5.110808	5.083321	5.05612	5.029204	5.002566	4.976204	4.950115	4.924294
0.2	4.404801	4.384408	4.364212	4.344208	4.324395	4.30477	4.285331	4.266076	4.247003	4.228108
0.3	3.84405	3.82881	3.813702	3.798727	3.783881	3.769163	3.754572	3.740107	3.725765	3.711546
0.4	3.419727	3.408031	3.396426	3.384914	3.373493	3.362161	3.350918	3.339763	3.328694	3.317712
0.5	3.09033	3.081134	3.072003	3.062939	3.05394	3.045005	3.036133	3.027325	3.018579	3.009895
0.6	2.8287	2.821313	2.813975	2.806686	2.799444	2.79225	2.785102	2.778001	2.770946	2.763937
0.7	2.616693	2.610649	2.604643	2.598672	2.592738	2.586839	2.580976	2.575147	2.569354	2.563595
0.8	2.44192	2.436898	2.431903	2.426937	2.421998	2.417087	2.412203	2.407346	2.402516	2.397712
0.9	2.295735	2.291505	2.287297	2.283111	2.278948	2.274805	2.270684	2.266585	2.262506	2.258449
1	2.171961	2.168359	2.164775	2.161209	2.157659	2.154128	2.150613	2.147116	2.143635	2.140172

Table 6. $p(0.4, y, t)$.

$t \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	0.293999	0.293112	0.29223	0.291355	0.290485	0.28962	0.288761	0.287907	0.287059	0.286216
0.2	0.556634	0.554305	0.551997	0.549707	0.547438	0.545188	0.542957	0.540744	0.538551	0.536376
0.3	0.989466	0.984101	0.978785	0.973521	0.968305	0.963139	0.958021	0.952951	0.947929	0.942953
0.4	1.660414	1.649798	1.639286	1.628878	1.618571	1.608365	1.59826	1.588252	1.578342	1.568528
0.5	2.637396	2.618698	2.600187	2.581862	2.56372	2.54576	2.527978	2.510373	2.492942	2.475685
0.6	3.988331	3.958101	3.928178	3.898558	3.869237	3.840213	3.811481	3.783038	3.75488	3.727005
0.7	5.781137	5.735308	5.689947	5.645048	5.600607	5.556618	5.513074	5.469972	5.427305	5.38507
0.8	8.083734	8.017619	7.952183	7.887417	7.823314	7.759865	7.697064	7.634901	7.573369	7.512462
0.9	10.96404	10.87233	10.78157	10.69175	10.60284	10.51485	10.42775	10.34155	10.25622	10.17176
1	14.48997	14.36676	14.24481	14.12412	14.00468	13.88646	13.76945	13.65364	13.53901	13.42556

Table 7. $q(0.4, y, t)$.

$t \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	1.65164	1.647444	1.643261	1.639094	1.634941	1.630802	1.626678	1.622568	1.618472	1.61439
0.2	2.012667	2.007864	2.003082	1.998321	1.99358	1.98886	1.98416	1.97948	1.97482	1.97018
0.3	2.39768	2.391597	2.385547	2.379532	2.37355	2.367601	2.361685	2.355802	2.349951	2.344132
0.4	2.851696	2.8434	2.835161	2.826979	2.818853	2.810782	2.802767	2.794805	2.786898	2.779043
0.5	3.419727	3.408031	3.396426	3.384914	3.373493	3.362161	3.350918	3.339763	3.328694	3.317712
0.6	4.146789	4.130246	4.113847	4.097589	4.081472	4.065493	4.049652	4.033946	4.018375	4.002935
0.7	5.077895	5.054804	5.031925	5.009256	4.986794	4.964537	4.942483	4.920629	4.898973	4.877512
0.8	6.25806	6.226461	6.195164	6.164165	6.133461	6.103049	6.072924	6.043083	6.013523	5.98424
0.9	7.732297	7.689975	7.648068	7.60657	7.565478	7.524785	7.484487	7.44458	7.405058	7.365918
1	9.545622	9.490104	9.435139	9.380722	9.326846	9.273503	9.220688	9.168393	9.116613	9.065342

Table 8. $p(0.8, y, t)$.

$t \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	0.214567	0.214129	0.213692	0.213258	0.212826	0.212396	0.211968	0.211543	0.211119	0.210697
0.2	0.370469	0.369579	0.368695	0.367816	0.366944	0.366077	0.365216	0.36436	0.363511	0.362666
0.3	0.580169	0.578341	0.576526	0.574724	0.572937	0.571162	0.569401	0.567654	0.565919	0.564197
0.4	0.866549	0.863104	0.859687	0.856296	0.852933	0.849596	0.846285	0.843001	0.839742	0.836509
0.5	1.25249	1.246561	1.24068	1.234847	1.22906	1.223321	1.217628	1.211981	1.206379	1.200823
0.6	1.760874	1.751401	1.742007	1.732689	1.723448	1.714282	1.70519	1.696173	1.687229	1.678357
0.7	2.41458	2.400316	2.386169	2.372138	2.358223	2.344421	2.330733	2.317156	2.30369	2.290334
0.8	3.236492	3.215995	3.195668	3.175508	3.155513	3.135683	3.116015	3.096508	3.077161	3.057972
0.9	4.24949	4.221131	4.193005	4.165111	4.137447	4.11001	4.082799	4.05581	4.029041	4.002491
1	5.476456	5.438412	5.400683	5.363264	5.326153	5.289347	5.252843	5.216638	5.180729	5.145112

Table 9. $q(0.8, y, t)$.

$t \setminus y$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.1	1.219524	1.216783	1.214051	1.211328	1.208614	1.205909	1.203212	1.200525	1.197846	1.195176
0.2	1.536579	1.533657	1.530747	1.527847	1.524958	1.522079	1.51921	1.516352	1.513504	1.510666
0.3	1.828475	1.825168	1.821874	1.818594	1.815327	1.812075	1.808836	1.80561	1.802398	1.799199
0.4	2.121495	2.117514	2.113552	2.10961	2.105686	2.101782	2.097897	2.094031	2.090184	2.086355
0.5	2.44192	2.436898	2.431903	2.426937	2.421998	2.417087	2.412203	2.407346	2.402516	2.397712
0.6	2.816031	2.80952	2.803049	2.796617	2.790226	2.783873	2.77756	2.771285	2.765049	2.75885
0.7	3.27011	3.261581	3.253109	3.244693	3.236332	3.228027	3.219777	3.21158	3.203438	3.195349
0.8	3.830438	3.819283	3.808205	3.797205	3.786281	3.775434	3.764661	3.753963	3.743338	3.732786
0.9	4.523296	4.508825	4.494459	4.480196	4.466037	4.451978	4.438021	4.424163	4.410404	4.396743
1	5.374966	5.35641	5.337991	5.319709	5.301561	5.283546	5.265664	5.247912	5.23029	5.212797

Table 10. $p(x, 0.01, t)$.

$x \setminus t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.413736	0.911818	1.848428	3.397352	5.732374	9.027281	13.45586	19.19189	26.40916	35.28146
0.2	0.364269	0.756092	1.463546	2.611475	4.324727	6.728149	9.946587	14.10489	19.3279	25.74046
0.3	0.325284	0.641936	1.189003	2.057839	3.339795	5.126225	7.50848	10.57791	14.42588	19.14372
0.4	0.293999	0.556634	0.989466	1.660414	2.637396	3.988331	5.781137	8.083734	10.96404	14.48997
0.5	0.268475	0.491729	0.841848	1.369994	2.12733	3.165018	4.534219	6.286097	8.471814	11.14253
0.6	0.247342	0.441496	0.730803	1.154186	1.750571	2.558881	3.618042	4.966978	6.644613	8.689872
0.7	0.229613	0.401995	0.645954	0.991296	1.467825	2.105348	2.93367	3.982595	5.28193	6.86148
0.8	0.214567	0.370469	0.580169	0.866549	1.25249	1.760874	2.41458	3.236492	4.24949	5.476456
0.9	0.201667	0.344959	0.528465	0.769718	1.086249	1.495592	2.015278	2.66284	3.45581	4.411722
1	0.190511	0.324054	0.487307	0.693606	0.956285	1.288678	1.704121	2.215948	2.837494	3.582093

Table 11. $q(x, 0.01, t)$.

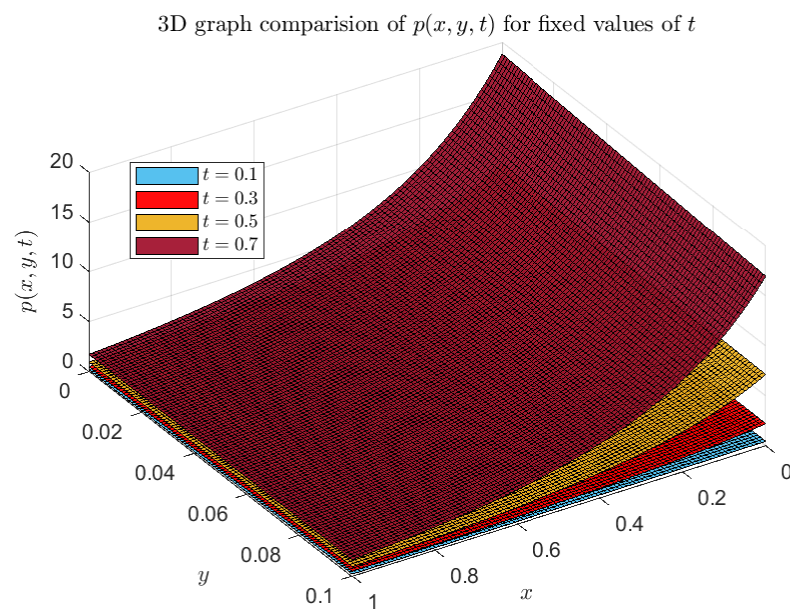
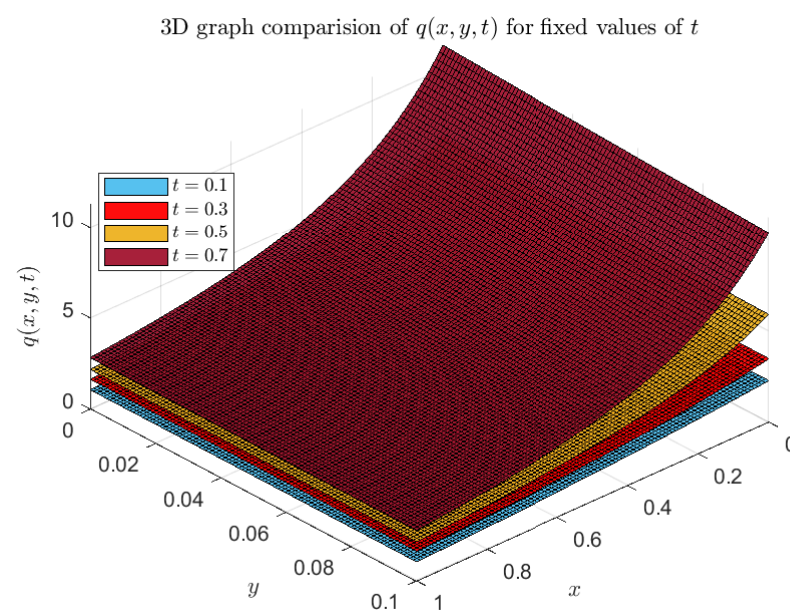
$x \setminus t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	2.126643	2.588718	3.189644	4.019073	5.166659	6.722056	8.774918	11.4149	14.73165	18.81483
0.2	1.949676	2.365732	2.8677	3.524438	4.404801	5.577646	7.111831	9.07621	11.53964	14.57098
0.3	1.792241	2.176119	2.609525	3.147242	3.84405	4.754731	5.934066	7.436837	9.317824	11.63181
0.4	1.65164	2.012667	2.39768	2.851696	3.419727	4.146789	5.077895	6.25806	7.732297	9.545622
0.5	1.52571	1.870274	2.220525	2.614523	3.09033	3.686007	4.439616	5.389219	6.572876	8.028648
0.6	1.41267	1.745221	2.070085	2.42025	2.8287	3.32842	3.952395	4.73361	5.705051	6.899702
0.7	1.311031	1.634709	1.940753	2.258352	2.616693	3.044963	3.57235	4.228043	5.041227	6.041092
0.8	1.219524	1.536579	1.828475	2.121495	2.44192	2.816031	3.27011	3.830438	4.523296	5.374966
0.9	1.137054	1.449117	1.730245	2.004447	2.295735	2.628119	3.025611	3.512221	4.111961	4.84884
1	1.06267	1.370938	1.643773	1.90338	2.171961	2.471721	2.824864	3.253594	3.780116	4.426632

Table 12. $p(x, 0.05, t)$.

$x \setminus t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.406775	0.889236	1.791998	3.281542	5.524354	8.686916	12.93571	18.43723	25.35795	33.86436
0.2	0.358827	0.739673	1.423609	2.530528	4.180325	6.492893	9.588126	13.58592	18.60616	24.76875
0.3	0.320949	0.629761	1.160195	2.000168	3.237599	4.960406	7.256507	10.21382	13.92026	18.46376
0.4	0.290485	0.547438	0.968305	1.618571	2.56372	3.869237	5.600607	7.823314	10.60284	14.00468
0.5	0.265583	0.484661	0.826036	1.339109	2.07328	3.077947	4.402511	6.096371	8.208927	10.78958
0.6	0.244929	0.435974	0.718797	1.13102	1.710265	2.494153	3.520306	4.826346	6.449893	8.42857
0.7	0.227575	0.397614	0.636701	0.973658	1.43731	2.056479	2.85999	3.876666	5.13533	6.664806
0.8	0.212826	0.366944	0.572937	0.852933	1.22906	1.723448	2.358223	3.155513	4.137447	5.326153
0.9	0.200167	0.342085	0.522738	0.759069	1.068025	1.46655	1.971589	2.600087	3.368989	4.29524
1	0.189207	0.321682	0.482716	0.685178	0.941938	1.265867	1.669834	2.166708	2.769361	3.490661

Table 13. $q(x, 0.05, t)$.

$x \setminus t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	2.103069	2.558347	3.144592	3.948338	5.05612	6.554475	8.529936	11.06904	14.25832	18.18431
0.2	1.928745	2.34008	2.831955	3.471138	4.324395	5.458493	6.940198	8.836278	11.2135	14.13863
0.3	1.773575	2.154121	2.580456	3.105927	3.783881	4.667662	5.810616	7.266091	9.087431	11.32798
0.4	1.634941	1.99358	2.37355	2.818853	3.373493	4.081472	4.986794	6.133461	7.565478	9.326846
0.5	1.510732	1.853564	2.200155	2.587837	3.05394	3.635795	4.370732	5.296083	6.449178	7.867347
0.6	1.399212	1.730489	2.052657	2.398161	2.799444	3.288951	3.899125	4.66241	5.61125	6.778089
0.7	1.298921	1.621652	1.92568	2.239783	2.592738	3.013322	3.530312	4.172485	4.968618	5.947489
0.8	1.208614	1.524958	1.815327	2.105686	2.421998	2.790226	3.236332	3.786281	4.466037	5.301561
0.9	1.127217	1.438741	1.718698	1.990847	2.278948	2.606759	2.998041	3.476553	4.066054	4.790305
1	1.053793	1.36165	1.633579	1.891581	2.157659	2.453816	2.802052	3.22437	3.742772	4.379261


Figure 1. Comparison of pollutant concentration for fixed t .

Figure 2. Comparison of DO concentration for fixed t .

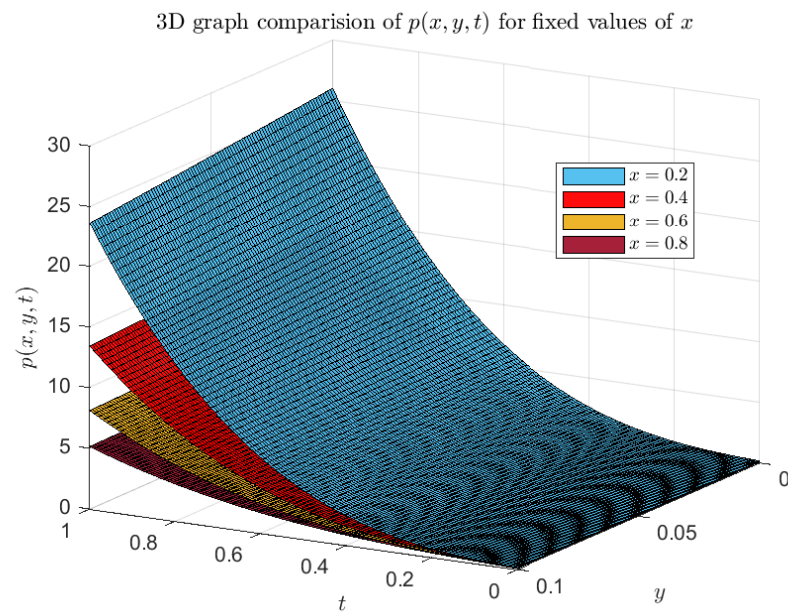


Figure 3. Comparison of pollutant concentration for fixed x .

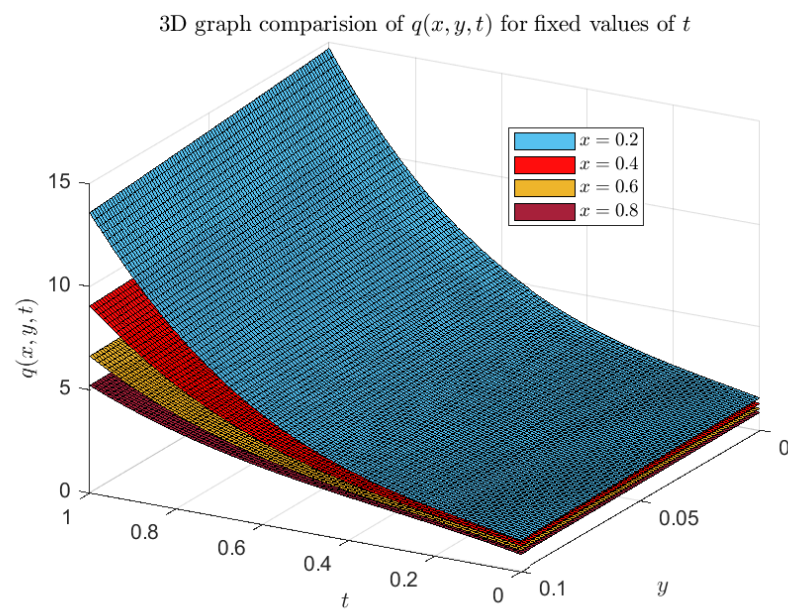


Figure 4. Comparison of DO concentration for fixed x .

Tables 14 and 15 provide the numerical values of pollutant and DO concentration for various values of λ ($\lambda = 0.1, 0.3, 0.5, 0.7, 1$). The graphical representation of Figures 7 and 8 illustrates the effect of variations in λ on the functions of pollutant and DO concentration. The numerical tables and figures provided show that when the value of λ has increased from 0 to 1, then the values of p and q are reduced.

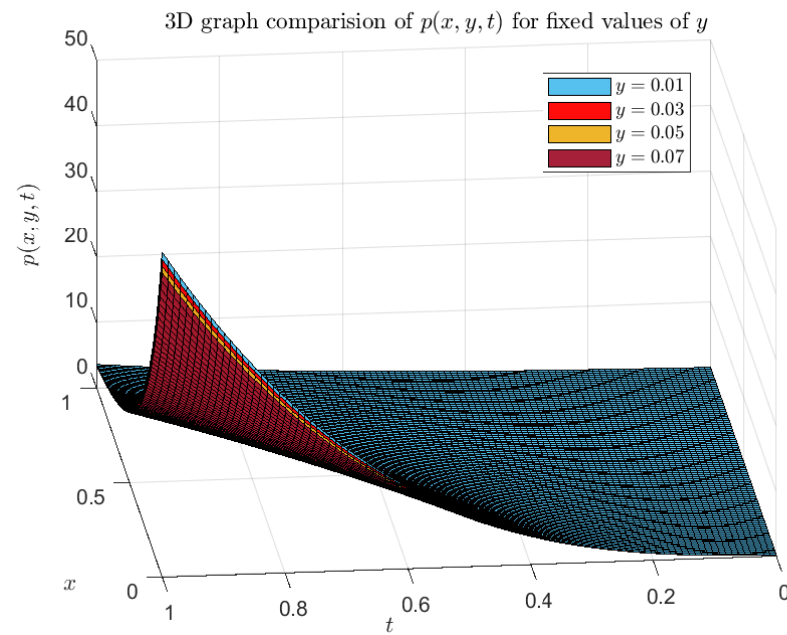


Figure 5. Comparison of pollutant concentration for fixed y .

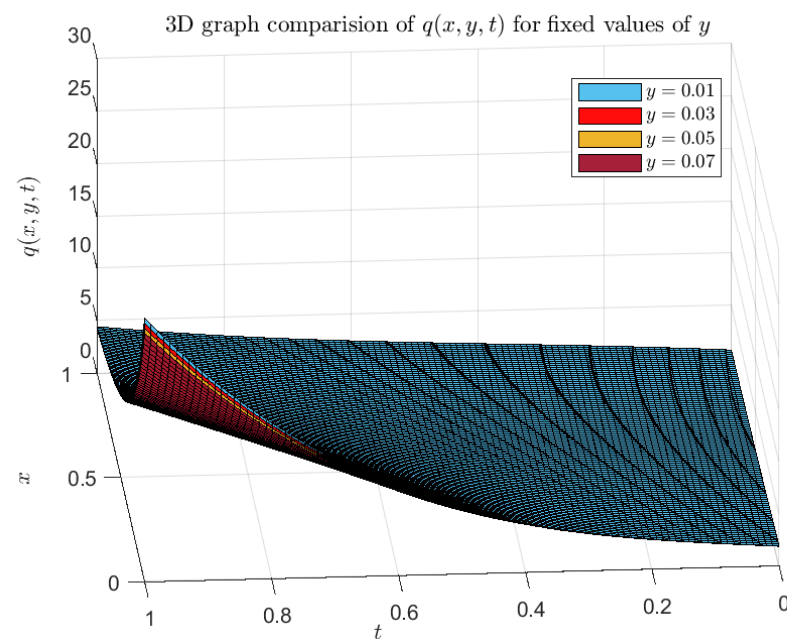


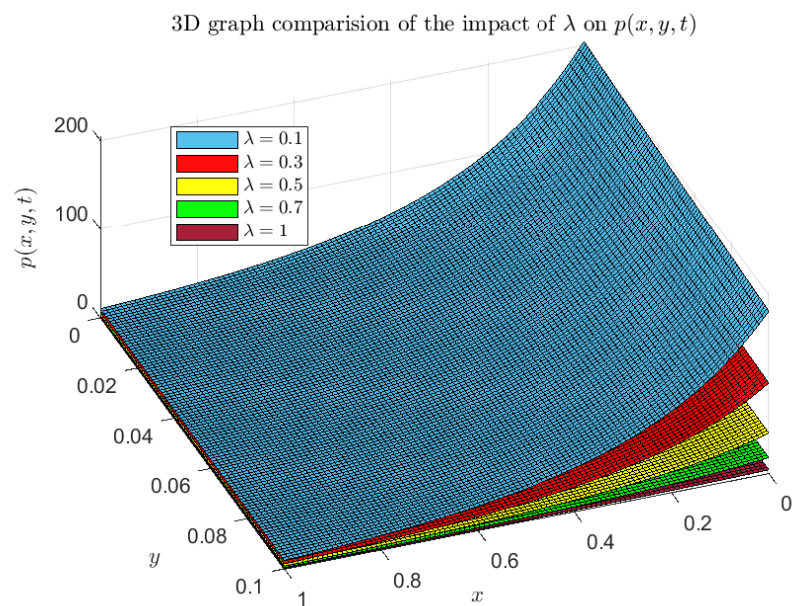
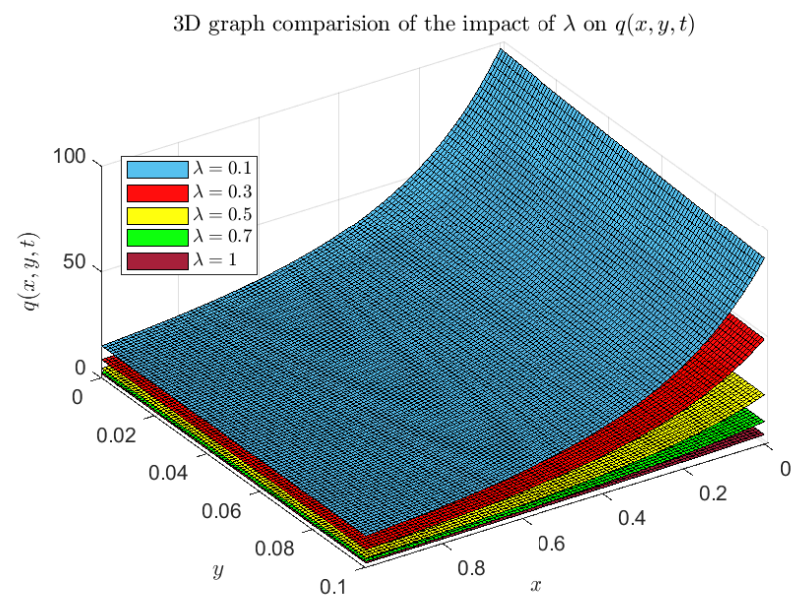
Figure 6. Comparison of DO concentration for fixed y .

Table 14. $p(x, 0.07, 0.4)$ for different values of λ .

x	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 1$
0.1	131.948316	73.8004016	33.1716726	13.2108399	3.225622782
0.2	94.13683331	52.8718564	23.96775981	9.699857389	2.491370321
0.3	68.29978316	38.5362045	17.6351535	7.266718406	1.972220367
0.4	50.30194048	28.5254298	13.19330039	5.547936266	1.59825953
0.5	37.52960574	21.4036321	10.01941793	4.31135652	1.32409346
0.6	28.304331	16.2470678	7.711488848	3.406233193	1.119740513
0.7	21.53037552	12.4515558	6.005632159	2.733013382	0.96505829
0.8	16.48021325	9.61516849	4.725662367	2.224819191	0.846285232
0.9	12.66246199	7.46590809	3.751914664	1.835952131	0.753864312
1	9.73954369	5.81655292	3.001728331	1.534661489	0.681053708

Table 15. $q(x, 0.07, 0.4)$ for different values of λ .

x	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 1$
0.1	66.05390856	37.4465357	17.84592924	8.485860191	3.913953704
0.2	50.03237409	28.4863522	13.80185495	6.839155968	3.445155514
0.3	39.19137003	22.3902622	11.01742358	5.676722457	3.085736626
0.4	31.68921312	18.146462	9.053560949	4.834434204	2.8027665
0.5	26.37960345	15.1237465	7.635089859	4.208517426	2.574740807
0.6	22.53836044	12.922309	6.586790145	3.732217372	2.387303087
0.7	19.70063084	11.2847595	5.795182815	3.361797688	2.230643742
0.8	17.56290471	10.0424983	5.185440381	3.068042559	2.097897281
0.9	15.92342254	9.08305228	4.707281924	2.831038014	1.984140789
1	14.64553736	8.32995883	4.326257377	2.636927524	1.885758787


Figure 7. Comparison of pollutant concentration for different values of λ .

Figure 8. Comparison of DO concentration for different values of λ .

The numerical values of the pollutant and DO concentration for various values of $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$ ($D_{p_x} = D_{q_x} = 1, 3, 5$ & $D_{p_y} = D_{q_y} = 0.1, 0.3, 0.5$) are provided in Tables 16 and 17. The three-dimensional graphical comparison of p and q for various values of $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$ is shown in Figures 9 and 10. If the values of $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$ are increased, then the values of p and q are raised.

Table 16. $p(x, 0.07, 0.4)$ for different values of $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$.

x	$D_{p_x} = D_{q_x} = 1 \text{ \& } D_{p_y} = D_{q_y} = 0.1$	$D_{p_x} = D_{q_x} = 3 \text{ \& } D_{p_y} = D_{q_y} = 0.3$	$D_{p_x} = D_{q_x} = 5 \text{ \& } D_{p_y} = D_{q_y} = 0.5$
0.1	3.225622782	10.35100308	22.15170425
0.2	2.491370321	7.686782206	16.33898209
0.3	1.972220367	5.784754721	12.15687743
0.4	1.59825953	4.412625224	9.126022393
0.5	1.32409346	3.413322842	6.915223144
0.6	1.119740513	2.679198634	5.29313882
0.7	0.96505829	2.135536472	4.096574258
0.8	0.846285232	1.729871087	3.209395814
0.9	0.753864312	1.424971585	2.548326985
1	0.681053708	1.194168312	2.053258012

Table 17. $q(x, 0.07, 0.4)$ for different values of $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$.

x	$D_{p_x} = D_{q_x} = 1 \text{ \& } D_{p_y} = D_{q_y} = 0.1$	$D_{p_x} = D_{q_x} = 3 \text{ \& } D_{p_y} = D_{q_y} = 0.3$	$D_{p_x} = D_{q_x} = 5 \text{ \& } D_{p_y} = D_{q_y} = 0.5$
0.1	3.913953704	9.187685447	18.65057115
0.2	3.445155514	7.622698858	15.20393681
0.3	3.085736626	6.444496397	12.59892502
0.4	2.8027665	5.540903037	10.60017544
0.5	2.574740807	4.835723633	9.044135138
0.6	2.387303087	4.276354163	7.815633855
0.7	2.230643742	3.825895594	6.832548434
0.8	2.097897281	3.45806434	6.035625304
0.9	1.984140789	3.153860602	5.381634143
1	1.885758787	2.899355674	4.838701197

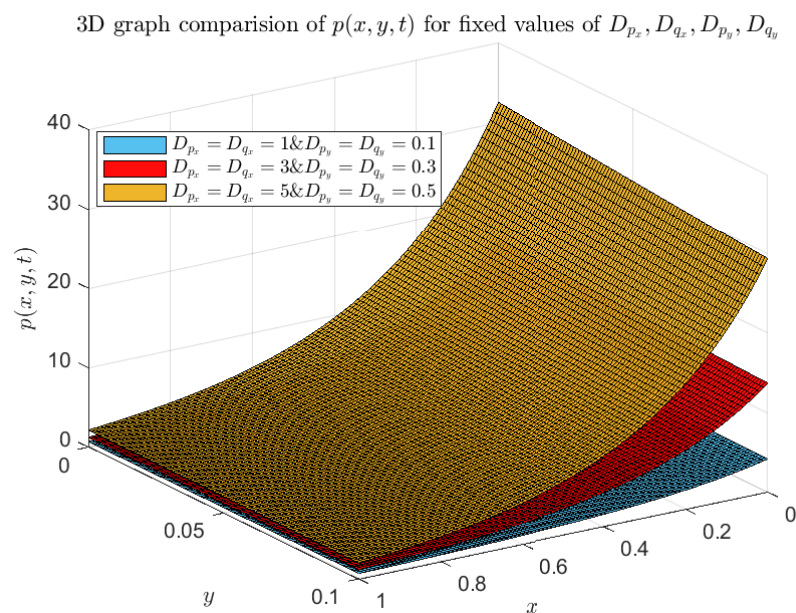


Figure 9. Comparison of pollutant concentration for fixed $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$.

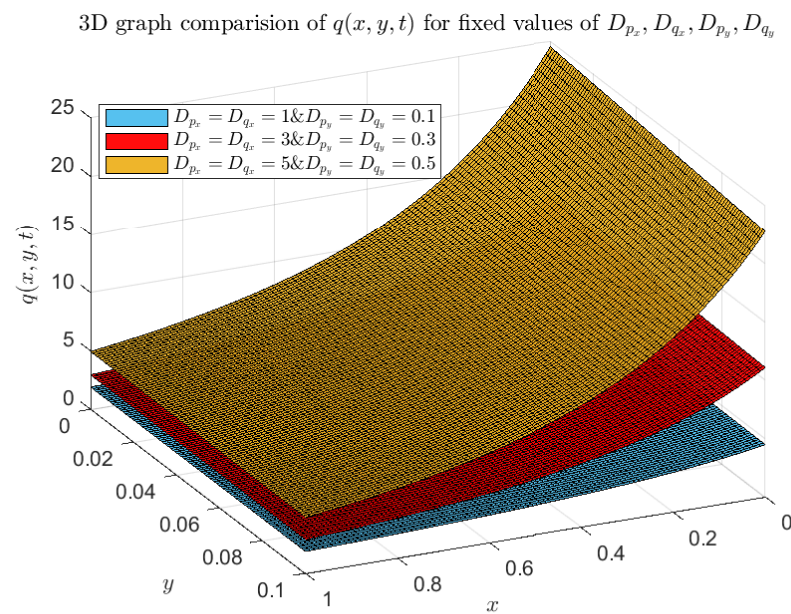


Figure 10. Comparison of DO concentration for fixed $D_{p_x}, D_{q_x}, D_{p_y}, D_{q_y}$.

The numerical values of the pollutant and DO concentrations for the different u_x, u_y values ($u_x = 1, 3, 5$ and $u_y = 0.1, 0.3, 0.5$) are given in Tables 18 and 19. The impact of variations in u_x, u_y on the functions of pollutant and DO concentration is shown graphically in Figures 11 and 12. We observe that when the values of u_x, u_y rise, the values of p and q also rise.

Table 18. $p(x, 0.07, 0.4)$ for distinct values of u_x, u_y .

x	$u_x = 1, u_y = 0.1$	$u_x = 3, u_y = 0.3$	$u_x = 5, u_y = 0.5$
0.1	3.225622782	6.184395	10.36456421
0.2	2.491370321	4.69464747	7.829728301
0.3	1.972220367	3.627437831	5.99953251
0.4	1.59825953	2.852467117	4.663582708
0.5	1.32409346	2.282575405	3.678457678
0.6	1.119740513	1.858551313	2.945060114
0.7	0.96505829	1.539573101	2.394058674
0.8	0.846285232	1.297105722	1.976393285
0.9	0.753864312	1.110946437	1.656992113
1	0.681053708	0.966624183	1.410554156

Table 19. $q(x, 0.07, 0.4)$ for distinct values of u_x, u_y .

x	$u_x = 1, u_y = 0.1$	$u_x = 3, u_y = 0.3$	$u_x = 5, u_y = 0.5$
0.1	3.913953704	7.26423765	13.17560781
0.2	3.445155514	6.252528358	11.29058683
0.3	3.085736626	5.466032573	9.801609533
0.4	2.8027665	4.841455011	8.603480282
0.5	2.574740807	4.335990014	7.623323371
0.6	2.387303087	3.920094326	6.809704515
0.7	2.230643742	3.572959316	6.125655747
0.8	2.097897281	3.279632101	5.544134335
0.9	1.984140789	3.029156722	5.045020123
1	1.885758787	2.813354725	4.613098884

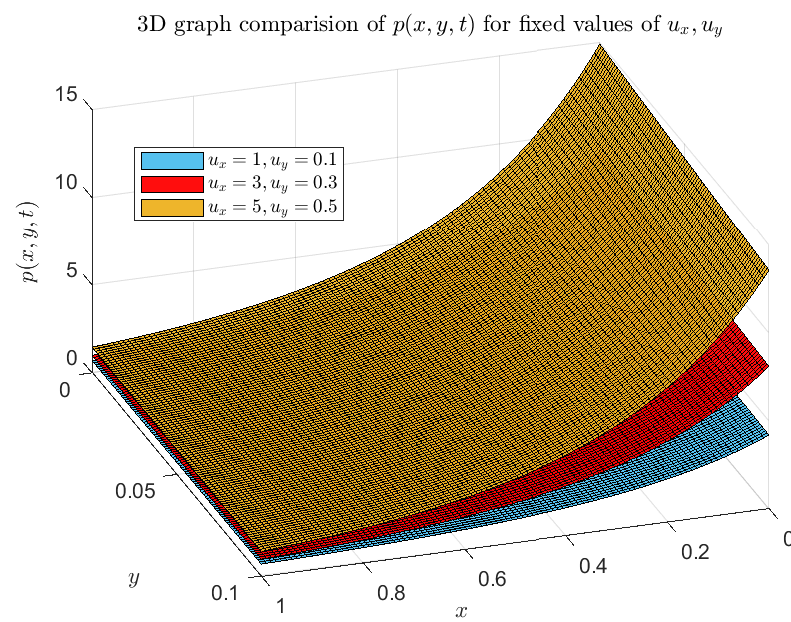


Figure 11. Comparison of pollutant concentration for fixed u_x, u_y .

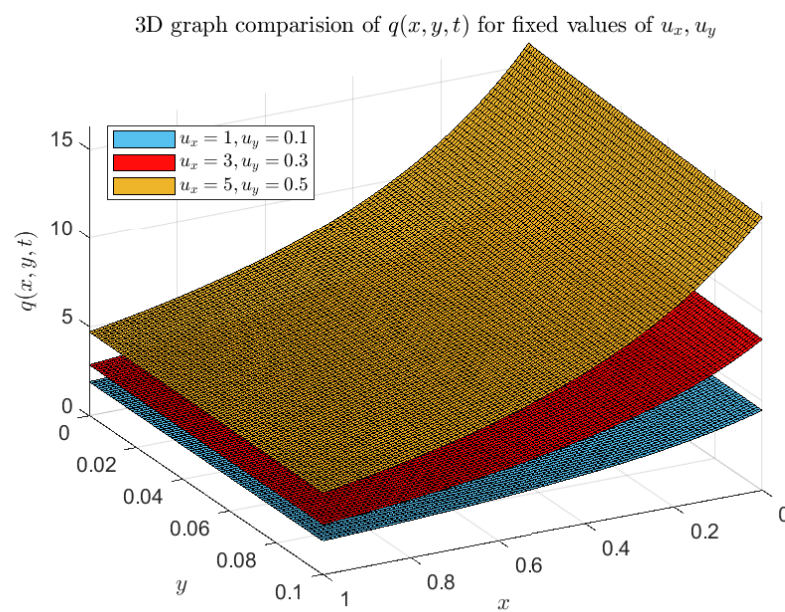


Figure 12. Comparison of DO concentration for fixed u_x, u_y .

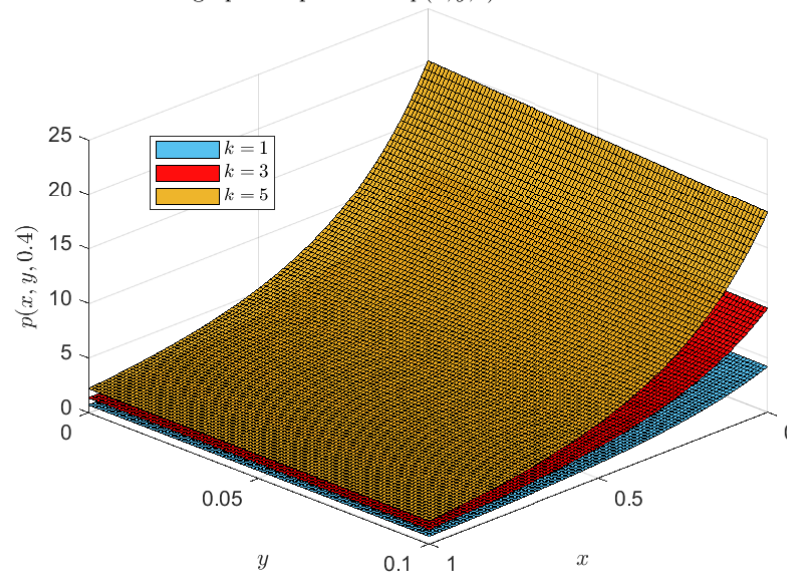
Tables 20 and 21 provide the numerical values of the pollutant and DO concentrations for the various k values ($k = 1, 3, 5$). In Figures 13 and 14, the effects of changes in k on the functions of pollutant and DO concentration are graphically displayed. We can observe that when the value of k increases, the values of p and q rise.

Table 20. $p(x, 0.07, 0.4)$ for distinct values of k .

x	$k = 1$	$k = 3$	$k = 5$
0.1	3.225623	7.292483	13.93789
0.2	2.49137	5.561109	10.50596
0.3	1.97222	4.33521	8.074585
0.4	1.59826	3.452954	6.32677
0.5	1.324093	2.807882	5.052338
0.6	1.119741	2.328956	4.11013
0.7	0.965058	1.968107	3.404136
0.8	0.846285	1.692361	2.868241
0.9	0.753864	1.478794	2.456363
1	0.681054	1.311259	2.136

Table 21. $q(x, 0.07, 0.4)$ for distinct values of k .

x	$k = 1$	$k = 3$	$k = 5$
0.1	3.913954	5.732002	8.794366
0.2	3.445156	4.805046	7.069187
0.3	3.085737	4.123853	5.825489
0.4	2.802767	3.611715	4.911947
0.5	2.574741	3.218179	4.228409
0.6	2.387303	2.90952	3.707685
0.7	2.230644	2.662828	3.304071
0.8	2.097897	2.462277	2.986054
0.9	1.984141	2.29675	2.731601
1	1.885759	2.158305	2.525098

3D graph comparison of $p(x, y, t)$ for fixed values of k

Figure 13. Comparison of pollutant concentration for fixed k .

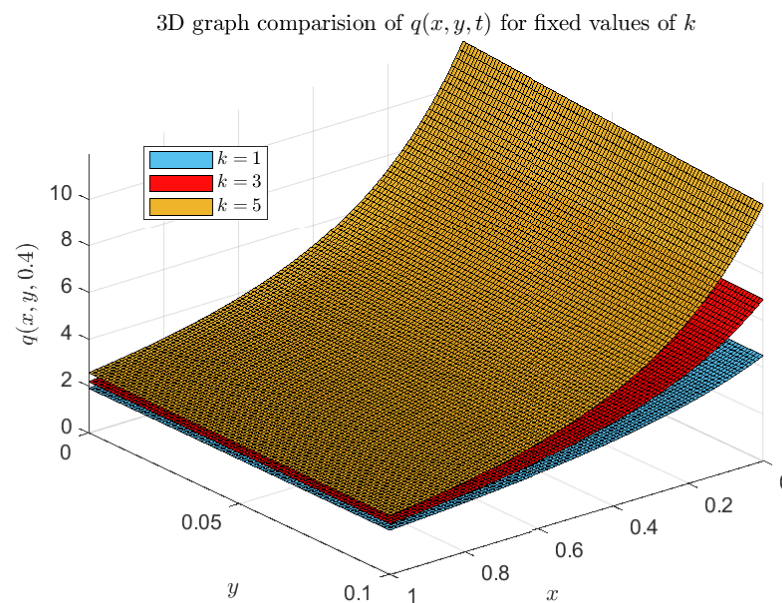


Figure 14. Comparison of DO concentration for fixed k .

5. Conclusions

In the present study, we have presented the two-dimensional time fractional mathematical model for river pollution and its remediation for an unsteady state and investigated the effect of aeration. We successfully applied the fractional reduced differential transform method to solve the coupled nonlinear two-dimensional time fractional advection-diffusion equations. The research area is supposed to be initially polluted in the exponentially decayed form. We have observed that the concentration of DO and the value of pollutants decrease as the length and width of the river increase. The concentration value of DO and pollutants increases as time increases. When fractional order values rise from 0 to 1, the concentration of pollutant and DO decreases. We also checked the effect of different parameters on concentrations. We observed that the concentration of pollutant and DO increases as the longitudinal and transverse diffusion coefficients of the pollutant and DO increase. Also, the concentration of contaminants and DO will rise as the river's longitudinal and transverse seepage velocity increases, and it will increase if the half-saturated oxygen demand concentration for pollutant degradation increases.

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Nomenclature

β^*	The mass transfer of oxygen from air to water $\left(\frac{\text{m}^2}{\text{day}}\right)$
A^*	River's cross section area (m^2)
$D_{p_x}^*$	Longitudinal diffusion coefficient of pollutant $\left(\frac{\text{m}^2}{\text{day}}\right)$
$D_{p_y}^*$	Transverse diffusion coefficient of pollutant $\left(\frac{\text{m}^2}{\text{day}}\right)$
$D_{q_x}^*$	Longitudinal diffusion coefficient of DO $\left(\frac{\text{m}^2}{\text{day}}\right)$
$D_{q_y}^*$	Transverse diffusion coefficient of DO $\left(\frac{\text{m}^2}{\text{day}}\right)$
f^*	The added pollutant rate along the river $\left(\frac{\text{kg}}{\text{m day}}\right)$
K^*	Saturated oxygen concentration $\left(\frac{\text{kg}}{\text{m}^3}\right)$
k^*	Half-saturated oxygen demand concentration for pollutant decay $\left(\frac{\text{kg}}{\text{m}^3}\right)$
k_1^*	Degradation rate coefficient of pollutant $\left(\frac{1}{\text{day}}\right)$
k_2^*	Degradation rate coefficient of DO $\left(\frac{1}{\text{day}}\right)$
p^*	Pollutant concentration at position (x^*, y^*) and time t^* $\left(\frac{\text{kg}}{\text{m}^3}\right)$
q^*	DO concentration at position (x^*, y^*) and time t^* $\left(\frac{\text{kg}}{\text{m}^3}\right)$
t^*	Time (days)
u_x^*	Longitudinal seepage velocity of the river $\left(\frac{\text{m}}{\text{day}}\right)$
u_y^*	Transverse seepage velocity of the river $\left(\frac{\text{m}}{\text{day}}\right)$
x^*	Length of the river from origin (m)
y^*	Width of the river (m)

References

- Hadouda, M.K.A.; Wahid, T.Z.A. On the Numerical Solution for Pollution in a River and Its Remediation by Two-Dimensional Unsteady Aeration. *Int. J. Appl. Eng. Res.* **2017**, *12*, 24.
- Kangle, H.; van Genuchten, M.T.; Renduo, Z. Exact solutions for one-dimensional transport with asymptotic scale-dependent dispersion. *Appl. Math. Model.* **1996**, *20*, 298–308. [\[CrossRef\]](#)
- Pimpunchat, B.; Sweatman, W.L.; Triampo, W.; Wake, G.C.; Parshotam, A. Modelling river pollution and removal by aeration. In Proceedings of the MODSIM 2007: International Congress on Modelling and Simulation, Land, Water & Environmental Management: Integrated Systems for Sustainability, Modelling and Simulation Society of Australia and New Zealand, Christchurch, New Zealand, 10–13 December 2007; pp. 2431–2437.
- Hussain, S.A.; Atshan, W.G.; Najam, Z.M. Mathematical model for the concentration of pollution and dissolved oxygen in the diwaniya river (Iraq). *Am. J. Sci. Res.* **2012**, *78*, 33–37.
- Ibrahim, F.N.; Dimian, M.F.; Wadi, A.S. Remediation of pollution in a river by unsteady aeration with arbitrary initial and boundary conditions. *J. Hydrol.* **2015**, *525*, 793–797. [\[CrossRef\]](#)
- Liu, F.; Zhuang, P.; Anh, V.; Turner, I.; Burrage, K. Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation. *Appl. Math. Comput.* **2007**, *191*, 12–20. [\[CrossRef\]](#)
- Attia, N.; Akgul, A.; Seba, D.; Nour, A. On solutions of time-fractional advection–diffusion equation. *Numer. Methods Partial. Differ. Equ.* **2023**, *39*, 4489–4516. [\[CrossRef\]](#)
- Mardani, A.; Hooshmandasl, M.R.; Heydari, M.H.; Cattani, C. A meshless method for solving the time fractional advection–diffusion equation with variable coefficients. *Comput. Math. Appl.* **2018**, *75*, 122–133. [\[CrossRef\]](#)
- Maisuria, M.A.; Tandel, P.V.; Patel, T. Solution of two-dimensional solute transport model for heterogeneous porous medium using fractional reduced differential transform method. *Axioms* **2023**, *12*, 1039. [\[CrossRef\]](#)
- Pimpunchat, B.; Sweatman, W.L.; Wake, G.C.; Triampo, W.; Parshotam, A. A mathematical model for pollution in a river and its remediation by aeration. *Appl. Math. Lett.* **2009**, *22*, 304–308. [\[CrossRef\]](#)
- Manitharoen, N.; Pimpunchat, B. Analytical and numerical solutions of pollution concentration with uniformly and exponentially increasing forms of sources. *J. Appl. Math.* **2020**, *2020*, 9504835. [\[CrossRef\]](#)
- Hadhouda, M.K.; Hassan, Z.S. Mathematical model for unsteady remediation of river pollution by aeration. *Inf. Sci. Lett.* **2022**, *11*, 323–329.
- Saleh, A.; Ibrahim, F.N.; Hadhouda, M.K. Remediation of pollution in a river by releasing clean water. *Inf. Sci. Lett.* **2022**, *11*, 127–133.
- Patel, H.; Patel, T.; Pandit, D. An efficient technique for solving fractional-order diffusion equations arising in oil pollution. *J. Ocean Eng. Sci.* **2023**, *8*, 217–225. [\[CrossRef\]](#)
- Alshammari, S.; Abuasad, S. Exact solutions of the 3D fractional helmholtz equation by fractional differential transform method. *J. Funct. Spaces* **2022**, *2022*, 7374751. [\[CrossRef\]](#)

16. Patel, T.; Patel, H. An analytical approach to solve the fractional-order $(2 + 1)$ -dimensional Wu–Zhang equation. *Math. Methods Appl. Sci.* **2023**, *46*, 479–489. [[CrossRef](#)]
17. Osman, M.; Xia, Y.; Omer, O.A.; Hamoud, A. On the fuzzy solution of linear-nonlinear partial differential equations. *Mathematics* **2022**, *10*, 2295. [[CrossRef](#)]
18. Al-Amr, M.O. New applications of reduced differential transform method. *Alex. Eng. J.* **2014**, *53*, 243–247. [[CrossRef](#)]
19. Moosavi Noori, S.R.; Taghizadeh, N. Study of convergence of reduced differential transform method for different classes of differential equations. *Int. J. Differ. Equ.* **2021**, *2021*, 6696414. [[CrossRef](#)]
20. Noori, S.R.M.; Taghizadeh, N. Study of convergence of reduced differential transform method for different classes of nonlinear differential equations. *Authorea Prepr.* **2020**.
21. Varsoliwala, A.C.; Singh, T.R. Mathematical modeling of atmospheric internal waves phenomenon and its solution by Elzaki Adomian decomposition method. *J. Ocean Eng. Sci.* **2022**, *7*, 203–212. [[CrossRef](#)]
22. Tandel, P.; Patel, H.; Patel, T. Tsunami wave propagation model: A fractional approach. *J. Ocean Eng. Sci.* **2022**, *7*, 509–520. [[CrossRef](#)]
23. Singh, B.K.; Kumar, P. FRDTM for numerical simulation of multi-dimensional, time-fractional model of Navier–Stokes equation. *Ain Shams Eng. J.* **2018**, *9*, 827–834. [[CrossRef](#)]
24. Abuasad, S.; Moaddy, K.; Hashim, I. Analytical treatment of two-dimensional fractional Helmholtz equations. *J. King Saud Univ.-Sci.* **2019**, *31*, 659–666. [[CrossRef](#)]
25. Al-rabtah, A.; Abuasad, S. Effective Modified Fractional Reduced Differential Transform Method for Solving Multi-Term Time-Fractional Wave-Diffusion Equations. *Symmetry* **2023**, *15*, 1721. [[CrossRef](#)]

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