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Investigation of the Oscillatory Properties of Fourth-Order Delay Differential Equations Using a Comparison Approach with First- and Second-Order Equations

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Abstract: This paper investigates the oscillatory behavior of solutions to fourth-order functional differential equations (FDEs) with multiple delays and a middle term. By employing a different comparison method approach with lower-order equations, the study introduces enhanced oscillation criteria. A key strength of the proposed method is its ability to reduce the complexity of the fourth-order equation by converting it into first- and second-order forms, allowing for the application of well-established oscillation theories. This approach not only extends existing criteria to higher-order FDEs but also offers more efficient and broadly applicable results. Detailed comparisons with previous research confirm the method's effectiveness and broader relevance while demonstrating the feasibility and significance of our results as an expansion and improvement of previous results.

Keywords: functional differential equations; oscillatory behavior; differential equations; middle term; comparison theorem

MSC: 34C10; 34K11



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1. Introduction

Differential equations (DEs) are defined as a mathematical relationship linking dependent independent variables (as functions of the independent variable, which is often time) and their time rate of change (as derivatives). This type of equation is constantly used to describe and analyze many life and scientific phenomena in various fields. As a result of the great scientific discoveries witnessed in the twentieth century and the rapid developments that occurred in most branches of science, such as genetic engineering, biology, population dynamics, chemistry, medicine, social sciences, and economics. Many mathematical models of different types of differential equations, such as ordinary, fractional, and functional equations, have been relied upon to describe different branches of science. This development and the need for such models led to the need to invent new methods for finding solutions to DEs, as traditional methods often fail. From here, what is known as the theory of oscillation of solutions to DEs came to light, which is one of the branches of qualitative theory that is concerned with describing and analyzing solutions of DEs, regardless of finding the formulas of these solutions.

Delay differential equations (DDEs) are a type of functional differential equation (FDE) in which the derivative of a function that is unknown at a given time is given in terms of the values of the function at previous times. Past dependence naturally appears in many applications in different sciences. It is easy to notice that various phenomena and problems often have temporal memory and need to be described mathematically, taking into account their past behavior or future behavior. Hence, interest in DDEs continues to grow in all scientific fields, especially biology, economics, control engineering, and others [1–3]. For a comprehensive overview of fractional and partial differential equations, please consult references [4–6] and their associated citations. Additionally, for an in-depth discussion on solving PDEs using advanced methods such as meshless techniques and deep learning, refer to references [7,8] and related works.

Over the past ten years, the oscillation theory of differential equations has advanced extremely quickly as a component of the qualitative theory of differential equations. Its area of interest was studying the characteristics of solutions that are oscillatory and those that are not. Fite published a seminal work [9] in the first quarter of the 20th century that laid the foundation for the theory of oscillation of differential equations with oblique arguments. Subsequently, a great deal of study has been conducted on the oscillation solutions of many differential and functional equation classes. Numerous monographs attest to the interest in this subject, for example, Agarwal et al. [10–12], Gyori and Ladas [13], and Erbe et al. [14].

1.1. Damped Delay Differential Equation

In this work, we investigate the oscillatory behavior of solutions of the fourth-order half-linear DDE with a middle term

$$\left[r(s)(x'''(s))^\kappa \right]' + \alpha(s)(x'''(s))^\kappa + \sum_{i=1}^m \beta_i(s)x^\kappa(g_i(s)) = 0, \quad s \in [s_0, \infty), \quad (1)$$

which is a type of damped DDE. In our results, we assume that the following assumptions hold eventually:

(H1) $\kappa \in \mathbb{Q}_{\text{odd}}^+$ is a result of dividing two odd positive integers;

(H2) $r \in \mathbf{C}^1([s_0, \infty), \mathbb{R}^+)$ and $r'(s) \geq 0$, with

$$\int_{s_0}^{\infty} \left(\frac{1}{r(\zeta)} \exp \left[- \int_{s_0}^{\zeta} \frac{\alpha(u)}{r(u)} du \right] \right)^{1/\kappa} d\zeta = \infty;$$

(H3) $\alpha, \beta_i \in \mathbf{C}([s_0, \infty), \mathbb{R}^+)$, and $\alpha(s) > 0$, for $i = 1, 2, \dots, m, m \in \mathbb{Z}^+$;

(H4) $g_i \in \mathbf{C}([s_0, \infty), \mathbb{R})$, $g_i(s) \leq s$, $\lim_{s \rightarrow \infty} g_i(s) = \infty$, and $\widehat{g}'(s) \geq 0$, where

$$\widehat{g}(s) = \min \{ g_i(s) : i = 1, 2, \dots, m \}.$$

The function $x \in \mathbf{C}^3([s_*, \infty), \mathbb{R})$, $s_* \in [s_0, \infty)$, is said to be a proper solution of (1) if $r(x''')^\kappa \in \mathbf{C}^1([s_*, \infty), \mathbb{R})$, x satisfies (1), and $\sup \{ |x(s)| : s \geq s_* \} > 0$ for $s_* \geq s_0$. If x is neither the positive nor negative solution of (1), eventually, then x is called oscillatory, and otherwise, it is called nonoscillatory. If all solutions of (1) oscillate, then the equation itself is called an oscillatory DE. Otherwise, it is called nonoscillatory.

Many life phenomena in various branches of science use fourth-order DEs to model and analyze them. This is due to the ability of this type of DE to describe complex physical systems and capture higher-order behaviors. We find it employed to model material elasticity and deformation in materials science and engineering, see [15]. In biomechanics, fourth-order DEs are used to describe joint movement and muscle mechanics in living organisms; see [16,17]. For more applications, you can review the research [18–20].

Therefore, in this manuscript, we are interested in studying the behavior of the solutions of fourth-order DEs, specifically studying the monotonic properties of positive solutions and improving them and then relying on the famous comparison theorem to deduce oscillation criteria that are more effective and applicable than the previous ones in

the canonical case of (1). This paper is divided into five main sections. Section 1 contains the basic definitions of this study and an overview of the history of the study of (1) and its most important previous influential works. Section 2 is concerned with studying the monotonic properties of positive solutions to (1) and improving them, in addition to the basic part of the study, which is the classification of these solutions. Section 3, entitled Comparison with First-Order Equations, in which the theorem of comparison and oscillating first-order DEs, is used to deduce criteria that guarantee the oscillation of our fourth-order equation. In Section 4, we use the same technique but with second-order oscillatory equations. This section also contains some examples and comparisons with previous works. Section 5 contains the conclusion and we summarize what our results add and their impact on the theory of the oscillation of DEs, in addition to some future recommendations.

1.2. Review of Relevant Literature

Here, we review some earlier results that helped build the theory of oscillation for fourth-order DDEs, in addition to some important previous work that we will use to compare with our results to confirm their originality and importance. But first, let us provide a simple overview of the oscillatory equations of the first and second order, which we will use as a basis for comparison and to conclude our results.

It is self-evident to know that the beginnings of the study of the oscillation of differential equations were largely directed at first-order equations, as they received great attention from researchers due to their ease and the lack of classifications of their positive solutions. In the classic paper [21], the Kneser theorem is introduced for the existence of nonoscillatory solutions under specific conditions. Work continued to deduce oscillation criteria using different techniques. Please see [14,22–24].

In 1987, Ladde et al. [25] studied the first-order DDE

$$x'(s) + \beta(s)x(g_1(s)) = 0, \quad (2)$$

and provided the following oscillation theorem:

Theorem 1. *If*

$$\liminf_{s \rightarrow \infty} \int_{g_1(s)}^s \beta(u) du > \frac{1}{e},$$

then every solution of (2) oscillates.

On the other hand, for the second-order DDE, this type has received greater attention and the study of more general types. To be more specific to the topic of our research, you can read [11,26,27], works that are concerned with studying second-order DEs in the linear case; works [10,28,29] for types of sublinear, superlinear, and nonlinear equations; the works of Essam et al. [30] and Santra et al. [31], which focus on adding several delays to the studied equations; the works of [32,33], which studied the noncanonical case; and [34,35], which concern the neutral DDEs (that is, the higher derivative appears with and without delay).

In 2020, the work of Jadlovská and Džurina [36], concerned the second order linear DDE

$$\left(r(s)(x'(s))^\kappa \right)' + \beta(s)x^\kappa(g_1(s)) = 0, \quad (3)$$

and established the following oscillation theorem:

Theorem 2. *For*

$$R(s) := \int_{s_0}^s r^{-1/\kappa}(u) du.$$

Let $\kappa \geq 1$ and

$$\rho := \liminf_{s \rightarrow \infty} \frac{R(s)}{R(g_1(s))} < \infty.$$

If

$$\liminf_{s \rightarrow \infty} \left[r^{1/\kappa}(s) R(s) R^\kappa(g_1(s)) \beta(s) \right] > \hat{\rho},$$

where

$$\hat{\rho} = \kappa \max_{\epsilon \in (0,1)} \left\{ \frac{\epsilon(1-\epsilon)^\kappa}{\rho^{\kappa\epsilon}} \right\}.$$

then every solution of (3) oscillates.

Returning to the fourth-order equations, for the canonical case, Zhang et al. [37] relied on the Riccati technique to develop oscillation criteria for the equation

$$\left(r(s) (x'''(s))^\kappa \right)' + \beta(s) x^\kappa(g_1(s)) = 0.$$

Then, Grace et al. [38] considered the nonlinear DDE

$$\left(r(s) (x'''(s))^\kappa \right)' + \beta(s) f(x(g_1(s))) = 0.$$

Bazighifan et al. [39] ensured the oscillation of the fourth-order DDE with a middle term

$$\left[r |x''''|^{\kappa-2} x'''' \right]'(s) + \alpha(s) |x'''(s)|^{\kappa-2} x'''(s) + \beta(s) |x'''(g_1(s))|^{\kappa-2} x'''(g_1(s)) = 0,$$

by using the comparison theorem, under the condition

$$\int_{s_0}^{\infty} \left(\frac{1}{r(\zeta)} \exp \left[- \int_{s_0}^{\zeta} \frac{\alpha(u)}{r(u)} du \right] \right)^{1/\kappa-1} d\zeta = \infty$$

and $\kappa > 0$.

For the noncanonical case, in 2019, Grace et al. [40] were interested in the oscillation of the DDE

$$\left(r_3(s) \left(r_2(s) \left(r_1(s) (x'(s))' \right)' \right)' \right)' + \beta(s) x(g_1(s)) = 0,$$

for $r_i \in C^1([s_0, \infty), \mathbb{R}^+)$, $i = 1, 2, 3$ and

$$\int_{s_0}^s \frac{1}{r_i(u)} du < \infty.$$

In the same year, Chatzarakis et al. [41] generalized the previous study by considering the neutral DDE

$$\left(r(s) \left([x(s) + a(s)x(h(s))]''' \right)^\kappa \right)' + \int_a^b \beta(s, u) f(x(g_1(s, u))) du = 0.$$

They also obtained some improved oscillation criteria by using the Riccati technique. Please see [42–46] for further details.

As for equations of the fourth order, the following theorems guarantee the oscillation of the equation:

$$(x(s) + a(s)x(h(s)))^{(4)} + \beta(s)x(g_1(s)) = 0, \quad (4)$$

for $a(s) \in C([s_0, \infty), \mathbb{R}^+)$ and $0 \leq a(s) < 1$.

Theorem 3 (Zafer, 1998 [47]). *If*

$$\liminf_{s \rightarrow \infty} \int_{g_1(s)}^s g_1^3(u)(1 - a(g_1(u)))\beta(u)du > \frac{192}{e},$$

then every solution of (4) oscillates.

Theorem 4 (Zhang and Yan, 2006 [48]). *Assume that there exists a nonnegative integer j . If*

$$\liminf_{s \rightarrow \infty} \int_{g_1(s)}^s \eta_j(u)du > \frac{1}{e^j},$$

where

$$\eta_0(s) = \frac{g_1^3(s)(1 - a(g_1(s)))\beta(s)}{6},$$

$$\eta_1(s) = \int_{\rho_0(s)}^s \eta_0(u)du,$$

for $s \in [\rho_{-1}(s), \infty)$, $\rho_{-1}(s) = \sup\{\rho_0(\varepsilon) = s : \zeta \geq s_1\}$, and $\rho_0(s) = \max\{g_1(\zeta), s_1 \leq \zeta \leq s\}$, and

$$\eta_{j+1}(s) = \int_{\rho_0(s)}^s \Psi_0(v)\Psi_j(s)dv$$

for $s \in [\rho_{-(j+1)}(s), \infty)$ and $\rho_{-(j+1)}(s) = \rho_{-1}(\rho_{-j}(s))$, then every solution of (4) oscillates.

Theorem 5 (Karpuz et al., 2010 [49]). *If*

$$\liminf_{s \rightarrow \infty} \int_{g_1(s)}^s g_1^3(u)(1 - a(g_1(u)))\beta(u)du > \frac{6}{e},$$

then every solution of (4) oscillates.

Theorem 6 (Agarwal et al. [50]). *Assume that there exists a positive function $\pi_j \in C^1([s_0, \infty), \mathbb{R}^+)$ for $j = 1, 2$. If*

$$\int_{s_1}^{\infty} \pi_1(u)\beta(u)(1 - a(g_1(u)))\frac{g_1^3(u)}{u^3} - \frac{1}{2\epsilon} \frac{(\pi_1'(u))^2}{u^2\pi_1(u)}du = \infty$$

and

$$\int_{s_1}^{\infty} \pi_2(u) \left(\int_u^{\infty} (\varsigma - u)\beta(\varsigma)(1 - a(g_1(\varsigma)))\frac{g_1(\varsigma)}{\varsigma}d\varsigma \right) - \frac{(\pi_2'(u))^2}{4\pi_2(u)}du = \infty.$$

then every solution of (4) oscillates.

Inspired by all of these previous works and based on the theorem of comparison with lower orders. This paper provides several key contributions to the study of oscillatory behavior in fourth-order functional differential equations (FDEs) with multiple delays and a middle term by:

1. Applying to all possible cases of κ and considering multiple delayed arguments $(g_i(s), i \in \mathbb{N})$;
2. Adopting a developed approach by comparing fourth-order equations (1) with multiple lower-order equations (first and second order), leading to improve oscillation criteria;
3. Introducing a novel method to reduce the fourth-order equation to first- and second-order forms, enabling the application of well-established oscillation results from these lower-order equations;

4. Deriving new and more effective oscillation criteria for fourth-order functional differential equations;
5. Improving the works of Zafer [47], Zhang and Yan [48], Karpuz et al. [49], and Agarwal et al. [50];
6. Broadening the applicability and impact of our findings, this in turn represents a significant advancement in the field.

Subsequently, we presume that all functional inequalities hold in due course or that they are satisfied for all sufficiently large values of t .

2. Auxiliary Lemmas

In this part, we introduce some auxiliary lemmas to facilitate the study of the properties of positive solutions of DDE (1). Throughout the results, we denote the class of all eventually positive solutions of DDE (1) with the symbol $\mathbb{X}^+$. Moreover, we define the following notations:

$$\begin{aligned}\hat{\alpha}(s, s_1) &:= \exp \left[\int_{s_1}^s \frac{\alpha(u)}{r(u)} du \right], \\ \mathbb{A}(s, s_1) &:= \int_{s_1}^s \left(\frac{1}{\hat{\alpha}(u, s_1)r(u)} \right)^{1/\kappa} du, \\ \overline{\mathbb{A}}(s, s_1) &:= \int_{s_1}^s \mathbb{A}(u, s_1) du,\end{aligned}$$

and

$$\overline{\overline{\mathbb{A}}}(s, s_1) := \int_{s_1}^s \overline{\mathbb{A}}(u, s_1) du.$$

Lemma 1. Assume that $x \in \mathbb{X}^+$. Then, x satisfies eventually

$$x'(s) > 0, \quad x'''(s) > 0, \quad \left[\hat{\alpha}(s, s_1)r(s)(x'''(s))^\kappa \right]' \leq 0,$$

and $x''(s)$ is of one sign.

Proof. This result can be derived directly from Lemma 4 in [51]. $\square$

Lemma 2. Assume that $x \in \mathbb{X}^+$. Then, DDE (1) can be written in the form

$$\left[\hat{\alpha}(s, s_1)r(s)(x'''(s))^\kappa \right]' + \hat{\alpha}(s, s_1) \sum_{i=1}^m \beta_i(s)x^\kappa(g_i(s)) = 0, \quad (5)$$

for $s \geq s_1 \geq s_0$. Moreover, if $x''(s) > 0$, eventually, then

(i) the following functions are nonincreasing:

$$\frac{x''(s)}{\mathbb{A}(s, s_1)}, \quad \frac{x'(s)}{\overline{\mathbb{A}}(s, s_1)}, \quad \text{and} \quad \frac{x(s)}{\overline{\overline{\mathbb{A}}}(s, s_1)};$$

(ii) $\mathbb{A}(s, s_1)x(s) \geq \overline{\overline{\mathbb{A}}}(s, s_1)x''(s)$;

(iii) $x(s) \geq [\hat{\alpha}(s, s_1)r(s)]^{1/\kappa} \overline{\overline{\mathbb{A}}}(s, s_1)x'''(s)$.

Proof. Suppose that $x \in \mathbb{X}^+$, for $s \geq s_0$. Then, there is a $s_1 \geq s_0$ such that $x(g_i(s)) > 0$ for $s \geq s_1$ and $i = 1, 2, \dots, m$. It follows from (1) that

$$\begin{aligned}
0 &\geq -\sum_{i=1}^m \beta_i(s) x^\kappa(g_i(s)) \\
&= \left[r(s) (x'''(s))^\kappa \right]' + \alpha(s) (x'''(s))^\kappa \\
&= \exp \left[-\int_{s_1}^s \frac{\alpha(u)}{r(u)} du \right] \left(\exp \left[\int_{s_1}^s \frac{\alpha(u)}{r(u)} du \right] r(s) (x'''(s))^\kappa \right)' \\
&= \frac{1}{\widehat{\alpha}(s, s_1)} \left[\widehat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa \right]'.
\end{aligned}$$

Next, we assume $x''(s) > 0$ for $s \geq s_1$. Hence, we have

$$\begin{aligned}
x''(s) &\geq \int_{s_1}^s \left(\frac{1}{\widehat{\alpha}(u, s_1) r(u)} \right)^{1/\kappa} \left[(\widehat{\alpha}(u, s_1) r(u))^{1/\kappa} x'''(u) \right] du \\
&\geq [\widehat{\alpha}(s, s_1) r(s)]^{1/\kappa} x'''(s) \int_{s_1}^s \left(\frac{1}{\widehat{\alpha}(u, s_1) r(u)} \right)^{1/\kappa} du \\
&= [\widehat{\alpha}(s, s_1) r(s)]^{1/\kappa} x'''(s) \mathbb{A}(s, s_1).
\end{aligned} \tag{6}$$

This brings us to

$$\begin{aligned}
\left[\frac{x''(s)}{\mathbb{A}(s, s_1)} \right]' &= \frac{\mathbb{A}(s, s_1) x'''(s) - [\widehat{\alpha}(s, s_1) r(s)]^{-1/\kappa} x''(s)}{\mathbb{A}^2(s, s_1)} \\
&\leq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x'(s) &\geq \int_{s_1}^s \mathbb{A}(u, s_1) \frac{x''(u)}{\mathbb{A}(u, s_1)} du \\
&\geq \frac{x''(s)}{\mathbb{A}(s, s_1)} \int_{s_1}^s \mathbb{A}(u, s_1) du \\
&= \frac{\overline{\mathbb{A}}(s, s_1)}{\mathbb{A}(s, s_1)} x''(s),
\end{aligned} \tag{7}$$

and so

$$\left[\frac{x'(s)}{\overline{\mathbb{A}}(s, s_1)} \right]' = \frac{\overline{\mathbb{A}}(s, s_1) x''(s) - \mathbb{A}(s, s_1) x'(s)}{\overline{\mathbb{A}}^2(s, s_1)} \leq 0. \tag{8}$$

Likewise, we obtain

$$\left[\frac{x(s)}{\overline{\overline{\mathbb{A}}}(s, s_1)} \right]' \leq 0.$$

Now, from (6) and (8), we find

$$x(s) \geq \frac{\overline{\overline{\mathbb{A}}}(s, s_1)}{\mathbb{A}(s, s_1)} x'(s) \geq \frac{\overline{\overline{\mathbb{A}}}(s, s_1)}{\mathbb{A}(s, s_1)} \frac{\overline{\mathbb{A}}(s, s_1)}{\mathbb{A}(s, s_1)} x''(s) = \frac{\overline{\overline{\mathbb{A}}}(s, s_1)}{\mathbb{A}(s, s_1)} x''(s),$$

and

$$x(s) \geq [\widehat{\alpha}(s, s_1) r(s)]^{1/\kappa} \overline{\overline{\mathbb{A}}}(s, s_1) x'''(s).$$

The proof is complete. $\square$

3. Comparison with First-Order Equations

In this section, we present some criteria to verify the oscillation of solutions of DDE (1) by comparing it with two first-order DEs.

Theorem 7. Assume that the first-order DDEs

$$\psi'(s) + \hat{\alpha}(s, s_1) \overline{\hat{\alpha}}^\kappa(\hat{g}(s), s_1) \psi(\hat{g}(s)) \sum_{i=1}^m \beta_i(s) = 0 \quad (9)$$

and

$$w'(s) + \lambda \hat{g}(s) w(\hat{g}(s)) \int_s^\infty \left[\frac{1}{\hat{\alpha}(\zeta, s_1) r(\zeta)} \int_\zeta^\infty \hat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta = 0 \quad (10)$$

are oscillatory for some any $\lambda \in (0, 1)$. Then, every solution of DDE (1) is oscillatory.

Proof. Assume the contrary of the results of this theorem, that x be a nonoscillatory solution of (1). Without loss of generality, we assume that $x \in \mathbb{X}^+$. From Lemma 1, we have two cases for x , which are

$$(C_1) \quad x'(s) > 0, x''(s) > 0, x'''(s) > 0, [\hat{\alpha}(s, s_2) r(s) (x'''(s))^\kappa]' \leq 0,$$

$$(C_2) \quad x'(s) > 0, x''(s) < 0, x'''(s) > 0, [\hat{\alpha}(s, s_2) r(s) (x'''(s))^\kappa]' \leq 0.$$

Assume that case (C_1) holds. From Lemma 2, we obtain that (i)–(iii) hold. It follows from the increasing monotonicity of z , the definition of $\hat{g}$, and Equation (5) that

$$[\hat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa]' + \hat{\alpha}(s, s_1) x^\kappa(\hat{g}(s)) \sum_{i=1}^m \beta_i(s) \leq 0. \quad (11)$$

Combining (iii) with (11), we arrive at

$$0 \geq [\hat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa]' + \hat{\alpha}(s, s_1) \hat{\alpha}(\hat{g}(s), s_1) r(\hat{g}(s)) \left(\overline{\hat{\alpha}}(\hat{g}(s), s_1) x'''(\hat{g}(s)) \right)^\kappa \sum_{i=1}^m \beta_i(s).$$

Assume that $\psi(s) := \hat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa > 0$; then,

$$\psi'(s) + \hat{\alpha}(s, s_1) \overline{\hat{\alpha}}^\kappa(\hat{g}(s), s_1) \psi(\hat{g}(s)) \sum_{i=1}^m \beta_i(s) \leq 0.$$

This inequality has a positive solution according to [52], which contradicts the oscillatory behavior of (9). And this completes the proof of this part. Assume that case (C_2) holds. Integrating (5) from s to ∞ , we arrive at

$$\hat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa \geq \int_s^\infty \hat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) x^\kappa(g_i(u)) du.$$

Since $x'(s) > 0$, we obtain

$$\begin{aligned} \hat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa &\geq \int_s^\infty \hat{\alpha}(u, s_1) x^\kappa(\hat{g}(u)) \sum_{i=1}^m \beta_i(u) du \\ &\geq x^\kappa(\hat{g}(s)) \int_s^\infty \hat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du, \end{aligned}$$

i.e.,

$$x'''(s) \geq x(\widehat{g}(s)) \left[\frac{1}{\widehat{\alpha}(s, s_1)r(s)} \int_s^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa}.$$

Integrating once more, we have

$$\begin{aligned} x''(s) &\leq - \int_s^\infty x(\widehat{g}(\zeta)) \left[\frac{1}{\widehat{\alpha}(\zeta, s_1)r(\zeta)} \int_\zeta^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta \\ &\leq -x(\widehat{g}(s)) \int_s^\infty \left[\frac{1}{\widehat{\alpha}(\zeta, s_1)r(\zeta)} \int_\zeta^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta. \end{aligned} \quad (12)$$

From the decreasing monotonicity of x'' , we obtain that

$$x(s) \geq x(s) - x(s_1) = \int_{s_1}^s x'(u) du \geq (s - s_1)x'(s) \geq \lambda s x'(s), \quad (13)$$

for all $s \geq s_1$ and $\lambda \in (0, 1)$. Combining (12) with (13), one obtains

$$x''(s) + \lambda \widehat{g}(s)x'(\widehat{g}(s)) \int_s^\infty \left[\frac{1}{\widehat{\alpha}(\zeta, s_1)r(\zeta)} \int_\zeta^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta \leq 0.$$

Assume that $w(s) := x'(s) > 0$. Then,

$$w'(s) + \lambda \widehat{g}(s)w(\widehat{g}(s)) \int_s^\infty \left[\frac{1}{\widehat{\alpha}(\zeta, s_1)r(\zeta)} \int_\zeta^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta \leq 0.$$

This inequality has a positive solution according to [52], which contradicts the oscillatory behavior of (10). This completes the proof. $\square$

Corollary 1. Assume that

$$\liminf_{s \rightarrow \infty} \int_{\widehat{g}(s)}^s \widehat{\alpha}(u, s_1) \overline{\mathbb{A}}^\kappa(\widehat{g}(u), s_1) \sum_{i=1}^m \beta_i(u) du > \frac{1}{e} \quad (14)$$

and

$$\liminf_{s \rightarrow \infty} \int_{\widehat{g}(s)}^s \widehat{g}(v) \int_v^\infty \left[\frac{1}{\widehat{\alpha}(\zeta, s_1)r(\zeta)} \int_\zeta^\infty \widehat{\alpha}(u, s_1) \sum_{i=1}^m \beta_i(u) du \right]^{1/\kappa} d\zeta dv > \frac{1}{\lambda e}. \quad (15)$$

Then, every solution of DDE (1) is oscillatory.

Proof. Using the standard oscillation criterion in [25] for first-order equations, we find that conditions (14) and (15) are sufficient to guarantee the oscillation of the solutions of Equations (9) and (10), respectively. $\square$

4. Comparison with Second-Order Equations

This section present some oscillation criteria of solutions of DDE (1) by comparing it with two second-order DDEs.

Theorem 8. Assume that $\kappa \geq 1$ and the second order DDEs

$$\left[(\widehat{\alpha}(s, s_1)r(s))^{\frac{1}{\kappa}} y'(s) \right]' + \frac{1}{\kappa} \widehat{\alpha}(s, s_1) \frac{\overline{\mathbb{A}}^\kappa(\widehat{g}(s), s_1)}{\mathbb{A}(\widehat{g}(s), s_1)} y(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s) = 0 \quad (16)$$

and

$$x''(s) + x(\widehat{g}(s)) \int_s^\infty (\widehat{\alpha}(\varsigma, s_1) r(\varsigma))^{-\frac{1}{\kappa}} \left(\int_s^\infty \widehat{\alpha}(\nu, s_1) \sum_{i=1}^m \beta_i(\nu) d\nu \right)^{\frac{1}{\kappa}} d\varsigma = 0 \quad (17)$$

are oscillatory. Then, every solution of DDE (1) is oscillatory.

Proof. Assume on the contrary that x be a nonoscillatory solution of (1). Without loss of generality, we assume that $x \in \mathbb{X}^+$. From Lemma 1, we have two cases for x , which are (C_1) and (C_2) as we discussed previously in the proof of Theorem 7.

Firstly, assume that case (C_1) holds. From (11), we obtain that

$$\begin{aligned} \left[(\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} (x'''(s)) \right]' &= \left[(\widehat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa)^{\frac{1}{\kappa}} \right]' \\ &= \frac{1}{\kappa} (\widehat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa)^{\frac{1}{\kappa}-1} (\widehat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa)' \\ &= \frac{1}{\kappa} (\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} x'''(s)^{1-\kappa} (\widehat{\alpha}(s, s_1) r(s) (x'''(s))^\kappa)' \\ &\leq -\frac{1}{\kappa} (\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} x'''(s)^{1-\kappa} \widehat{\alpha}(s, s_1) x^\kappa(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s). \end{aligned} \quad (18)$$

But, the nonincreasing monotonicity of $x(s)/\overline{\mathbb{A}}(s, s_1)$ and (iii)-part of Lemma 2 denotes that

$$\frac{x(\widehat{g}(s))}{\overline{\mathbb{A}}(\widehat{g}(s), s_1)} \geq \frac{x(s)}{\overline{\mathbb{A}}(s, s_1)} \geq [\widehat{\alpha}(s, s_1) r(s)]^{1/\kappa} x'''(s).$$

Now, take power $1 - \kappa$ for both sides of the last inequality, then

$$\left(\frac{x(\widehat{g}(s))}{\overline{\mathbb{A}}(\widehat{g}(s), s_1)} \right)^{1-\kappa} \leq (\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} x'''(s)^{1-\kappa}.$$

Substituting from the previous inequality into (18), yields

$$\begin{aligned} \left[(\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} (x'''(s)) \right]' &\leq -\frac{1}{\kappa} \left(\frac{x(\widehat{g}(s))}{\overline{\mathbb{A}}(\widehat{g}(s), s_1)} \right)^{1-\kappa} \widehat{\alpha}(s, s_1) x^\kappa(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s) \\ &= -\frac{1}{\kappa} (\overline{\mathbb{A}}(\widehat{g}(s), s_1))^{\kappa-1} \widehat{\alpha}(s, s_1) x(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s). \end{aligned} \quad (19)$$

Again, from (ii)-part of Lemma 2, we have

$$x(\widehat{g}(s)) \geq \frac{\overline{\mathbb{A}}(\widehat{g}(s), s_1)}{\mathbb{A}(\widehat{g}(s), s_1)} x''(\widehat{g}(s)),$$

then (19) becomes

$$\left[(\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} (x'''(s)) \right]' \leq -\frac{1}{\kappa} \widehat{\alpha}(s, s_1) \frac{\overline{\mathbb{A}}^\kappa(\widehat{g}(s), s_1)}{\mathbb{A}(\widehat{g}(s), s_1)} x''(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s),$$

i.e.,

$$\left[(\widehat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} (x'''(s)) \right]' + \frac{1}{\kappa} \widehat{\alpha}(s, s_1) \frac{\overline{\mathbb{A}}^\kappa(\widehat{g}(s), s_1)}{\mathbb{A}(\widehat{g}(s), s_1)} x''(\widehat{g}(s)) \sum_{i=1}^m \beta_i(s) \leq 0.$$

For simplification, let us assume the positive function $y(s) = x''(s)$, and so, the last inequality becomes

$$\left[(\hat{\alpha}(s, s_1) r(s))^{\frac{1}{\kappa}} y'(s) \right]' + \frac{1}{\kappa} \hat{\alpha}(s, s_1) \frac{\overline{\mathbb{A}}(\hat{g}(s), s_1)}{\mathbb{A}(\hat{g}(s), s_1)} y(\hat{g}(s)) \sum_{i=1}^m \beta_i(s) \leq 0.$$

This inequality has a positive solution according to [53], which contradicts the oscillatory behavior of (16). And this completes the proof of this part.

Now, let case (C₂) holds. By integrating (11) from s to infinity, yields

$$\hat{\alpha}(s, s_1) r(s) (x'''(s))^{\kappa} \geq \int_s^{\infty} \hat{\alpha}(v, s_1) x^{\kappa}(\hat{g}(v)) \sum_{i=1}^m \beta_i(v) dv.$$

Since x is a positive increasing function, we have

$$\hat{\alpha}(s, s_1) r(s) (x'''(s))^{\kappa} \geq x^{\kappa}(\hat{g}(s)) \int_s^{\infty} \hat{\alpha}(v, s_1) \sum_{i=1}^m \beta_i(v) dv,$$

i.e.,

$$x'''(s) \geq x(\hat{g}(s)) (\hat{\alpha}(s, s_1) r(s))^{\frac{-1}{\kappa}} \left(\int_s^{\infty} \hat{\alpha}(v, s_1) \sum_{i=1}^m \beta_i(v) dv \right)^{\frac{1}{\kappa}}.$$

One more time, integrate the last inequality from s to infinity, which implies that

$$\begin{aligned} -x''(s) &\geq \int_s^{\infty} x(\hat{g}(\zeta)) (\hat{\alpha}(\zeta, s_1) r(\zeta))^{\frac{-1}{\kappa}} \left(\int_{\zeta}^{\infty} \hat{\alpha}(v, s_1) \sum_{i=1}^m \beta_i(v) dv \right)^{\frac{1}{\kappa}} d\zeta \\ &\geq x(\hat{g}(s)) \int_s^{\infty} (\hat{\alpha}(\zeta, s_1) r(\zeta))^{\frac{-1}{\kappa}} \left(\int_{\zeta}^{\infty} \hat{\alpha}(v, s_1) \sum_{i=1}^m \beta_i(v) dv \right)^{\frac{1}{\kappa}} d\zeta. \end{aligned}$$

Similarly, as in the first part, the last inequality has a positive solution based on [53], which contradicts the oscillation of (17). The proof is complete. $\square$

Below, we will use the results of this theorem to deduce criteria that guarantee the oscillation of (1) through comparison with the criteria for the oscillation of the second-order equation found in [36]. But first, for convenience, let us define the following positive constants:

$$\begin{aligned} \mu_* &= \liminf_{s \rightarrow \infty} \frac{s}{\hat{g}(s)}, \\ \lambda_* &= \liminf_{s \rightarrow \infty} \frac{\mathbb{A}(s, s_0)}{\mathbb{A}(\hat{g}(s), s_0)} \end{aligned}$$

with

$$\begin{aligned} \delta_{\mu_*} &= \max_{\epsilon \in (0,1)} \left\{ \frac{\epsilon(1-\epsilon)^{\kappa}}{\mu_*^{\epsilon\kappa}} \right\}, \\ \delta_{\lambda_*} &= \max_{\epsilon \in (0,1)} \left\{ \frac{\epsilon(1-\epsilon)^{\kappa}}{\lambda_*^{\epsilon\kappa}} \right\}. \end{aligned}$$

Corollary 2. Let $\kappa \geq 1$ and their exist bounded constants $\mu_*, \lambda_* < \infty$. If

$$\liminf_{s \rightarrow \infty} \left[r^{1/\kappa}(s) \mathbb{A}(s, s_1) \overline{\mathbb{A}}(\hat{g}(s), s_1) \hat{\alpha}^{(1+\kappa)/\kappa}(s, s_1) \sum_{i=1}^m \beta_i(s) \right] > \kappa \delta_{\mu_*} \quad (20)$$

and

$$\liminf_{s \rightarrow \infty} \left[s \hat{g}(s) \int_s^\infty (\hat{\alpha}(\zeta, s_1) r(\zeta))^{-1/\kappa} \left(\int_\zeta^\infty \hat{\alpha}(\nu, s_1) \sum_{i=1}^m \beta_i(\nu) d\nu \right)^{1/\kappa} d\zeta \right] > \delta_{\lambda_*} \quad (21)$$

satisfies for $m \in \mathbb{Z}^+$, then every solution of (1) oscillates.

Proof. Assume on the contrary that (1) has a nonoscillatory solution x on $s \geq s_0$. Then, exactly as we prove in Lemma 1 and Theorem 7, x belongs to the only two possible cases (C_1) and (C_2) . By following the same steps as in proving Theorem 8, we can conclude that Equations (16) and (17) have a positive solution, which contradicts conditions (20) and (21), which guarantee their oscillation, as mentioned in [36]. And so, the proof is complete. $\square$

By taking $\kappa \equiv 1$ and $r(s) \equiv 1$, we can deduce the following corollary which denotes the oscillation criteria of the linear form of (1)

$$x^{(4)}(s) + \alpha(s)x^{(3)}(s) + \sum_{i=1}^m \beta_i(s)x(g_i(s)) = 0. \quad (22)$$

Corollary 3. Let $\kappa = 1$ and there exists a bounded constant $\mu_* < \infty$. If

$$\liminf_{s \rightarrow \infty} \mathbb{A}(s, s_1) \overline{\mathbb{A}}(\hat{g}(s), s_1) \hat{\alpha}^2(s, s_1) \sum_{i=1}^m \beta_i(s) > \delta_{\mu_*} \quad (23)$$

and

$$\liminf_{s \rightarrow \infty} \hat{g}(s) \int_s^\infty \frac{1}{\hat{\alpha}(\zeta, s_1)} \left(\int_\zeta^\infty \hat{\alpha}(\nu, s_1) \sum_{i=1}^m \beta_i(\nu) d\nu \right) d\zeta > \delta_{\mu_*} \quad (24)$$

satisfies for $m \in \mathbb{Z}^+$, then, every solution of the linear Equation (22) oscillates.

Example 1. Consider the half-linear DDE

$$\left(s^{3\kappa+1} \left(x^{(3)}(s) \right)^\kappa \right)' + \beta_0 \sum_{i=1}^m x^\kappa(\theta_i s) = 0, \quad (25)$$

where $s, \beta_0 \in [0, \infty)$, $\theta_i \in (0, 1)$, $m \in \{1, 2, 3, \dots\}$, and $\kappa \geq 1$ is a quotient of two positive integers. Since, $r(s) = 1$, and so, $\mathbb{A}(s, 0) = s$, it clear that the assumptions (H1)–(H4) hold with $\hat{g}(s) = \min_{i \in [1, m]} \{\theta_i s\} = \theta s$. Moreover, we have $\overline{\mathbb{A}}(s, 0) = \frac{s^2}{2}$, $\overline{\mathbb{A}}(s, 0) = \frac{s^3}{6}$, and $\mu_* = \lambda_* = 1/\theta$.

By applying Theorem 8, we conclude that (25) is oscillatory if

$$\beta_0 > \kappa \max_{\epsilon \in (0, 1)} \left\{ \frac{\epsilon(1-\epsilon)^\kappa}{\theta^{\epsilon\kappa}} \right\} \left(\frac{6}{\theta^3} \right)^\kappa$$

and

$$\beta_0 > (3\kappa)^{1/\kappa} \max_{\epsilon \in (0, 1)} \left\{ \frac{\epsilon(1-\epsilon)^\kappa}{\theta^{\epsilon\kappa}} \right\} \frac{2}{\theta}$$

hold, i.e.,

$$\beta_0 > \max \left\{ \kappa \max_{\epsilon \in (0, 1)} \left\{ \frac{\epsilon(1-\epsilon)^\kappa}{\theta^{\epsilon\kappa}} \right\} \left(\frac{6}{\theta^3} \right)^\kappa, (3\kappa)^{1/\kappa} \max_{\epsilon \in (0, 1)} \left\{ \frac{\epsilon(1-\epsilon)^\kappa}{\theta^{\epsilon\kappa}} \right\} \frac{2}{\theta} \right\}. \quad (26)$$

Remark 1. By applying the preceding example to some significant works in the literature yields the criteria listed below:

I. Agarwal et al. [50]: for $\pi_1(s) = s^3$, $\pi_2(s) = s$

$$\beta_0 > \max \left\{ \frac{9}{2\theta^3}, \frac{3}{2\theta} \right\}. \quad (27)$$

II. Karpuz et al. [49]:

$$\beta_0 > \frac{6}{\theta^3 \ln \left(\frac{1}{\theta} \right) e}. \quad (28)$$

III. Zhang and Yan [48]: for $r \in \mathbb{Z}^+$

$$\beta_0 > \frac{6 \left(\frac{1}{e^r(1-\theta)} \right)^{1/r+1}}{\theta^3 \ln \left(\frac{1}{\theta} \right)}. \quad (29)$$

IV. Zafer [47]:

$$\beta_0 > \frac{192}{\theta^3 \ln \left(\frac{1}{\theta} \right) e}. \quad (30)$$

Remark 2. Assume the following special case of (25):

$$\left(s^4 x'''(s) \right)' + \beta_0 x \left(\frac{2s}{3} \right) = 0.$$

Table 1 compares the lowest bound values of coefficient β_0 for each criterion in Example 1 and Remark 1. From this comparison, it is evident that the results presented in (26) surpass previous findings (27)–(30) in several ways. Our results exhibit a broader applicability, addressing a wider range of cases and conditions compared to earlier studies. This expanded coverage confirms that our oscillation criteria are more comprehensive and effective. The enhanced criteria reflect a significant improvement over existing works in [47–50], underscoring their originality and effectiveness. This advancement not only demonstrates the superiority of our results but also highlights their contribution to refining and extending many aspects of prior research in the literature. Thus, our findings represent a notable advancement in the field, providing more robust and applicable oscillation criteria.

Table 1. The lowest bound values of β_0 for (26)–(30).

Oscillation Criteria	(26)	(27)	(28)	(29)	(30)
	6.2640	15.1875	18.3728	112.1786	587.93

5. Conclusions

This study has successfully contributed more effective and applicable oscillation criteria compared to earlier works. It facilitated the establishment of improved relationships between the positive solutions of the dependent variable in (1) and its time derivatives. The reduction of our equation's order from the fourth order to the two first-order Equations (9) and (10) in Section 3 allowed us to leverage the oscillation criteria of first-order equations, enhancing the applicability of our study. In Section 4, we further reduced our equation to the second order and conducted a comprehensive comparison with Equations (16) and (17) of the same order, resulting in the derivation of more efficient oscillation criteria akin to those observed in second-order equations. Through the development of Example 1 and the application of a special case in Remark 1 in this section, we conducted a thorough comparison with prior work, elucidating the effectiveness and originality of our findings in the realm of oscillations of fourth-order equations. Our results

showcased significantly broader applicability and coverage across a wider range, exhibiting a substantial numerical distinction from their predecessors.

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