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A Two-Dimensional Nonlocal Fractional Parabolic Initial Boundary Value Problem

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Abstract: In this paper, we investigate a two-dimensional singular fractional-order parabolic partial differential equation in the Caputo sense. The partial differential equation is supplemented with Dirichlet and weighted integral boundary conditions. By employing a functional analysis method based on operator theory techniques, we prove the existence and uniqueness of the solution to the posed nonlocal initial boundary value problem. More precisely, we establish an a priori bound for the solution from which we deduce the uniqueness of the solution. For proof of its existence, we use various density arguments.

Keywords: caputo fractional derivative; well-posedness; strong solution; fractional Sobolev space; nonlocal constraint

MSC: 35B45; 35D30; 35D35; 35R11

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1. Introduction

Fractional partial differential equations, especially those involving Caputo derivatives, have been extensively studied for their applications in various fields such as physics, engineering, and biology. The complexity introduced by nonlocal constraints and singular behaviors in such equations significantly enriches their dynamics. The investigation into the existence and uniqueness of solutions for fractional parabolic equations subject to initial and nonlocal boundary conditions has produced significant contributions, reflecting the complex interplay between fractional differentiation, parabolic dynamics, and boundary conditions. In this regard we cite [1], where the authors explored the existence and uniqueness of solutions for nonlocal boundary value problems with discontinuous matching conditions for loaded equations involving the Caputo fractional derivative, employing methods of integral energy and integral equations for in the proof. In the work [2], the authors showed the existence of solutions for anisotropic fractional-type degenerate parabolic equations under mixed-boundary conditions, addressing the challenge posed by nonlocal anisotropic diffusion effects. In [3–9], the authors established the existence of solutions for singular and non-singular fractional initial boundary value problems with local and nonlocal conditions using variational, energy methods, and fixed-point theorems. The contributions in [10–13], deal with the study of various initial boundary value problems for differential equations with fractional-order and nonlocal conditions. For numerical methods and simulations, the reader could, for example, refer to [14–19], where the authors try to develop efficient numerical methods for solving various fractional initial boundary value problems with local and nonlocal conditions, including finite difference methods, spectral methods, and the differential quadrature approach. For some papers dealing with the existence and uniqueness of solutions for a coupled system of Caputo fractional differential equations supplemented with nonlocal integral boundary conditions, we cite [20–23]. In this work, our main objective is to show the well-posedness of a given initial boundary

value problem with a nonlocal constraint for a singular two-dimensional Caputo fractional-order differential equation combined with a Dirichlet and weighted integral boundary conditions. We first establish an a priori estimate from which the uniqueness of the solution follows, then for the solvability of the posed problem, we use various density arguments. Briefly, we employ the theory operator techniques to provide various proofs. The structure of this paper is outlined as follows: In Section 2, we define the problem, present the fractional framework, and outline the abstract formulation of the problem. Section 3 covers the essential lemmas required for various proofs in subsequent sections. Section 4 is dedicated to presenting the primary outcome concerning the uniqueness and continuous dependence of the solution to the stated problem. Finally, Section 5 delves into the main result regarding the problem's solvability.

2. Statement of the Problem

In this paper, we are concerned with the following fractional initial boundary value problem, which deals with the solution to a time fractional heat conduction problem for a finite medium in which the temperature $\theta = \theta(r, y, t)$, and it is subjected to a heat source $f(r, y, t)$.

In the domain $\Lambda = \Omega \times [0, T]$, where $0 < T < \infty$, $\Omega = (0, a) \times (0, b)$, and $a, b > 0$, we consider the following two-dimensional singular fractional partial differential equation:

$$\mathcal{L}\theta = \partial_t^\beta \theta - A(t)\left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy}\right) = f(r, y, t), \tag{1}$$

where $\partial_t^\sigma \theta$ indicates the right Caputo fractional derivative of order β, $0 < \beta \leq 1$ [24], defined as follows:

$$\partial_t^\beta \theta = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\theta_\tau(\tau)}{(t-\tau)^\beta} d\tau, \quad \forall t \in [0, T].$$

Equation (1) is supplemented by the initial condition

$$\ell\theta = \theta(r, y, 0) = \phi(r, y), \ (r, y) \in \Omega, \tag{2}$$

the Dirichlet boundary conditions

$$\theta(a, y, t) = 0, \ \theta(r, 0, t) = 0, \ \theta(r, b, t) = 0, \tag{3}$$

and the nonlocal boundary condition

$$\int_0^a r\theta(r, y, t) dr = 0, \quad t \in [0, T], \tag{4}$$

with the compatibility conditions

$$\phi(a, y) = 0 \ , \ \phi(r, b) = 0, \ \phi(r, 0) = 0, \ \int_0^a r\phi dr = 0. \tag{5}$$

For every $t \in [0, T]$, we have the following conditions on the $C^2[0, T]$ function A:

$$c_0 \leqslant A \leqslant c_1, \ \ A' \leqslant c_2, \tag{6}$$

$$c_3 \leqslant A'' \leqslant c_4, \tag{7}$$

where c_0, c_1, c_2, c_3, and c_4 are positive constants, and $f \in L^2(0, T; L^2(\Omega))$ and $\phi \in H_\sigma^{1,r}(\Omega)$ are given functions.

To investigate the well-posedness of the problem defined in (1)–(4), we introduce the Hilbert space $L^2(\Lambda)$ of square integrable functions with inner products:

$$(F,G)_{L^2(\Lambda)} = \int_\Lambda FGdrdydt.$$

The weighted Hilbert spaces $L^2_\rho(\Lambda)$ $(\rho = r^2)$, $L^2_\gamma(\Lambda)$ $(\gamma = r)$, and $L^2_\sigma(\Lambda)$ $(\sigma = r^3)$ have the following inner products, respectively:

$$(F,G)_{L^2_\rho(\Lambda)} = (r^2F,G)_{L^2(\Lambda)},\ (F,G)_{L^2_\gamma(\Lambda)} = (rF,G)_{L^2(\Lambda)},\ (F,G)_{L^2_\sigma(\Lambda)} = (r^3F,G)_{L^2(\Lambda)}.$$

We also introduce the function space $H^{1,r}_\rho(\Lambda)$ of elements $\theta \in L^2_\sigma(\Lambda)$ having generalized first-order derivatives with respect to r, summable on Λ. This space is provided with the inner product

$$(F,G)_{H^{1,r}_\sigma(\Lambda)} = (F,G)_{L^2_\sigma(\Lambda)} + (F_r,G_r)_{L^2_\sigma(\Lambda)},$$

and the space of functions $L^2(0,t;H^\beta_\sigma(\Omega))$ of abstract strongly measurable functions h on the interval $[0,T]$ into the space $H^\beta_\sigma(\Omega)$ having the following norm:

$$\|h\|^2_{L^2(0,T;H^\beta_\sigma(\Omega))} = \left\|\partial^\beta_t h\right\|^2_{L^2(0,T;L^2_\sigma(\Omega))} + \|h\|^2_{L^2(0,T;L^2_\sigma(\Omega))}.$$

Abstract Formulation. The given problem (1)–(4) can be presented in the operator form $\mathcal{N}\theta = \mathcal{M}$, where $\mathcal{N} = (\mathcal{L},\ell)$, and $\mathcal{M} = (f,\phi)$. The operator $\mathcal{N}: S \to H$ is an unbounded operator with the following domain of definition:

$$\mathcal{D}(\mathcal{N}) = \begin{cases} \theta \in L^2(\Lambda) : \theta_t, \theta_r, \theta_{rr}, \theta_y, \theta_{yy}, \theta_{rt}, \theta_{yt}, \partial^\beta_t\theta \in L^2(\Lambda), \\ \theta(a,y,t) = 0,\ \theta(r,0,t) = 0,\ \theta(r,b,t) = 0,\ \int\limits_0^a r\theta(r,y,t)dr = 0 \end{cases}$$

The operator acts on the Banach space S into the Hilbert space H, where S is the set of functions $\theta \in L^2_\sigma(\Lambda)$ verifying conditions (3) and (4) and having the following finite norm:

$$\|\theta\|^2_S = \|\theta\|^2_{L^2(0.T;H^\beta_\sigma(\Omega))} + \sup_{t\in[0,T]}\left(D^{\beta-1}\|\theta_r\|^2_{L^2_\sigma(\Omega)}\right),$$

and H is the Hilbert space $L^2(0,T;L^2(\Omega)) \times H^{1,r}_\sigma(\Omega)$ with the following associated norm:

$$\|\mathcal{N}\theta\|^2_H = \|\mathcal{M}\|^2_H = \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{H^{1,r}_\sigma(\Omega)}.$$

On the basis of an a priori bound and relying on the density of the set of values of the operator generated by the considered problem, we prove the existence and uniqueness of a strong solution to the problem defined in (1)–(4). Therefore, we need to introduce the notion of a strong solution to the problem posed in (1)–(4). Let $\overline{\mathcal{N}}$ be the closure of the operator $\mathcal{N}$, and define the strong solution to our problem as the solution to the operator equation $\mathcal{N}\theta = (f,\phi)$, where $(f,\phi) \in H = L^2(0,T;L^2(\Omega)) \times H^{1,r}_\sigma(\Omega)$.

3. Preliminaries

Lemma 1 ([25]). *Let $Q(s)$ be nonnegative and absolutely continuous on $[0,T]$, and for almost all $s \in [0,T]$, it satisfies the following differential inequality:*

$$\frac{dQ}{ds} \le A_1(s)Q(t) + B_1(s),$$

where the functions $A_1(s)$ and $B_1(s)$ are summable and nonnegative on $[0,T]$. Then,

$$Q(s) \leq \exp\left(\int_0^s A_1(t)dt\right)\left(Q(0)+\int_0^s B_1(t)dt\right).$$

Lemma 2. *(A weighted Poincare inequality) For any function $\theta \in L^2_\sigma((0,a)\times(0,b))$, the following inequality holds:*

$$\int_0^b\int_0^a r^3\theta^2 drdy \leq a^2\int_0^b\int_0^a r^3\theta_r^2 drdy.$$

Proof. We use the notation $\Im_r(\xi\theta)$ for $\int_0^r \xi\theta(\xi,y,t)d\xi$. Then, by using the boundary condition (4) and Young's inequality, we obtain the following:

$$\begin{aligned}
\int_0^b\int_0^a r^3\theta^2 drdy &= \int_0^b r^2\theta\Im_r(\xi\theta)\Big|_{r=0}^{r=a}dy - 2\int_0^b\int_0^a r\theta\Im_r(\xi\theta)drdy\\
&\quad -\int_0^b\int_0^a r^2\theta_r\Im_r(\xi\theta)drdy\\
&= -\int_0^b(\Im_r(\xi\theta))^2\Big|_{r=0}^{r=a}dy - \int_0^b\int_0^a r^2\theta_r\Im_r(\xi\theta)drdy\\
&= -\int_0^b\int_0^a r^2\theta_r\Im_r(\xi\theta)drdy\\
&\leq \frac{\varepsilon_1}{2}\int_0^b\int_0^a r^3\theta_r^2 drdy + \frac{1}{2\varepsilon_1}\int_0^b\int_0^a r(\Im_r(\xi\theta))^2 drdy. \qquad (8)
\end{aligned}$$

Now, we estimate the term $\int_0^b\int_0^a r(\Im_r(\xi\theta))^2 drdy$ as follows:

$$\begin{aligned}
\int_0^b\int_0^a r(\Im_r(\xi\theta))^2 drdy &= \int_0^b \frac{r^2}{2}(\Im_r(\xi\theta))^2\Big|_{r=0}^{r=a}dy - \int_0^b\int_0^a r^3\theta\Im_r(\xi\theta)drdy\\
&\leq \frac{\varepsilon_2}{2}\int_0^b\int_0^a r^3\theta^2 drdy + \frac{1}{2\varepsilon_2}\int_0^b\int_0^a r^3(\Im_r(\xi\theta))^2 drdy\\
&\leq \frac{\varepsilon_2}{2}\int_0^b\int_0^a r^3\theta^2 drdy\\
&\quad + \frac{a^2}{2\varepsilon_2}\int_0^b\int_0^a r(\Im_r(\xi\theta))^2 drdy. \qquad (9)
\end{aligned}$$

By choosing $\varepsilon_2 = a^2$, the inequality (9) yields the following:

$$\int_0^b\int_0^a r(\Im_r(\xi\theta))^2 drdy \leq a^2\int_0^b\int_0^a r^3\theta^2 drdy. \qquad (10)$$

In light of (10), and by taking $\varepsilon_1 = a^2$, we infer from (8) the following:

$$\int_0^b \int_0^a r^3\theta^2 dr dy \leq a^2 \int_0^b \int_0^a r^3\theta_r^2 dr dy.$$

□

Lemma 3 ([26]). *For any absolutely continuous function $J(t)$ on the interval $[0, T]$, the following fractional differential inequality holds:*

$$J(t)\, \partial_t^\sigma J(t) \geq \frac{1}{2}\, \partial_t^\sigma J^2(t), \qquad 0 < \sigma < 1.$$

Lemma 4 ([26]). *Let a nonnegative absolutely continuous function $M(t)$ satisfy the inequality*

$$\partial_t^\sigma M(t) \leq b_1 M(t) + b_2(t), \quad 0 < \sigma < 1,$$

for almost all $t \in [0, T]$, where b_1 is a positive constant and b_2 is an integrable nonnegative function on $[0, T]$. Then,

$$M(t) \leq M(0)E_\sigma(b_1 t^\sigma) + \Gamma(\sigma)E_{\sigma,\sigma}(b_1 t^\sigma)D_t^{-\sigma} b_2(t),$$

where

$$E_\sigma(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\sigma n + 1)} \text{ and } E_{\sigma,\mu}(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\sigma n + \mu)},$$

are the Mittag–Leffler functions.

Lemma 5. *For a function $\theta \in L^2_\sigma(\Omega)$, the following estimate holds:*

$$\|\Im_r(\xi\theta)\|^2_{L^2_\sigma(\Omega)} \leq 16a^4 \|\theta\|^2_{L^2_\sigma(\Omega)}. \tag{11}$$

Proof.

$$\begin{aligned} \|\Im_r(\xi\theta)\|^2_{L^2_\sigma(\Omega)} &= \int_0^b \int_0^a r^3 (\Im_r(\xi\theta))^2 dr dy = \int_0^b \int_0^a r^{\frac{7}{2}} \frac{(\Im_r(\xi\theta))^2}{r^{\frac{1}{2}}} dr dy \\ &\leq a^{\frac{7}{2}} \int_0^b \int_0^a r^{\frac{-1}{2}} (\Im_r(\xi\theta))^2 dr dy. \end{aligned} \tag{12}$$

We now estimate the integral on the right side of the inequality (12) as follows:

$$\begin{aligned} \int_0^b \int_0^a r^{\frac{-1}{2}} (\Im_r(\xi\theta))^2 dr dy &= \int_0^b \left(2r^{\frac{1}{2}} (\Im_r(\xi\theta))^2\right)_0^a dy - 4 \int_0^b \int_0^a r^{\frac{3}{2}} \Im_r(\xi\theta)\theta dr dy \\ &= -4 \int_0^b \int_0^a r^{\frac{7}{4}} \theta \frac{\Im_r(\xi\theta)}{r^{\frac{1}{4}}} dr dy \\ &\leq 2\varepsilon \int_0^b \int_0^a r^{\frac{-1}{2}} (\Im_r(\xi\theta))^2 dr dy + \frac{2}{\varepsilon} \int_0^b \int_0^a r^{\frac{7}{2}} \theta^2 dr dy \\ &\leq 2\varepsilon \int_0^b \int_0^a r^{\frac{-1}{2}} (\Im_r(\xi\theta))^2 dr dy + \frac{2}{\varepsilon} a^{\frac{1}{2}} \int_0^b \int_0^a r^3 \theta^2 dr dy. \end{aligned}$$

Let $\varepsilon = 1/4$. Then, we have the following:

$$\int_0^b \int_0^a r^{\frac{-1}{2}} (\Im_r(\xi\theta))^2 dr dy \leq 8a^{\frac{1}{2}} \int_0^b \int_0^a r^3 \theta^2 dr dy. \tag{13}$$

A combination of (12) and (13) gives the desired inequality (11). □

4. Main Result for the Uniqueness and Continuous Dependence of the Solution

In this section, we show that the solution to the problem defined in (1)–(4) satisfies an ideal energy inequality.

Theorem 1. *Assume that function A satisfies conditions (6) for every $t \in [0, T]$. Then, for any $\theta \in \mathcal{D}(\mathcal{N})$, there exists a positive constant C^* such that the following a priori estimate is satisfied:*

$$\begin{aligned} &\|\theta\|^2_{L^2(0.T;H^\beta_\sigma(\Omega))} + \sup_{t\in[0,T]} \left(D^{\beta-1} \|\theta_r\|^2_{L^2_\sigma(\Omega)} \right) \\ \leq\ & C^* \left(\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{H^{1,r}_\sigma(\Omega)} \right), \end{aligned} \tag{14}$$

where

$$C^* = \max\left[\frac{K(1+\varkappa T^\beta)}{\beta\Gamma(\beta)}, \frac{\varkappa T^\beta K(1-\beta) + T^{\beta-1}}{(1-\beta)\Gamma(1-\beta)} \right],$$

with $\varkappa$ and K given by (29) and (33).

Proof. For the differential operators

$$\begin{aligned} \mathcal{M}_1\theta &= r^3 \partial_t^\beta \theta, \mathcal{M}_2\theta = -r^3 \Im_r(\xi\theta) = -r^3 \int_0^r \xi\theta(\xi, y, t) d\xi \\ \mathcal{M}_3\theta &= -2r^2 A(t)\theta_r, \end{aligned}$$

we consider the following expressions:

$$\begin{aligned} &\left(\mathcal{M}_1\theta, \partial_t^\beta \theta - A(t) \left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy} \right) \right)_{L^2(\Omega)} \\ =\ & \left(\partial_t^\beta \theta, \partial_t^\beta \theta \right)_{L^2_\sigma(\Omega)} - \left(\partial_t^\beta \theta, \frac{A(t)}{r}(r\theta_r)_r \right)_{L^2_\sigma(\Omega)} - \left(\partial_t^\beta \theta, \frac{A(t)}{r^2}\theta_{yy} \right)_{L^2_\sigma(\Omega)}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\left(\mathcal{M}_2\theta, \partial_t^\beta \theta - A(t) \left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy} \right) \right)_{L^2(\Omega)} \\ =\ & -\left(\Im_r(\xi\theta), \partial_t^\beta \theta \right)_{L^2_\sigma(\Omega)} + \left(\Im_r(\xi\theta), \frac{A(t)}{r}(r\theta_r)_r \right)_{L^2_\sigma(\Omega)} \\ & + \left(\Im_r(\xi\theta), \frac{A(t)}{r^2}\theta_{yy} \right)_{L^2_\sigma(\Omega)}, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left(\mathcal{M}_3\theta, \partial_t^\beta \theta - A(t) \left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy} \right) \right)_{L^2(\Omega)} \\ =\ & -\left(2r^2 A(t)\theta_r, \partial_t^\beta \theta \right)_{L^2(\Omega)} + \left(2r^2 A(t)\theta_r, \frac{A(t)}{r}(r\theta_r)_r \right)_{L^2(\Omega)} \\ & + \left(2r^2 A(t)\theta_r, \frac{A(t)}{r^2}\theta_{yy} \right)_{L^2(\Omega)}. \end{aligned} \tag{17}$$

Using boundary conditions (3)–(4), Equations (15), (16), and (17) become the following, respectively:

$$\begin{aligned} & \left\|\partial_t^\beta \theta\right\|_{L^2_{\tilde{\sigma}}(\Omega)} + \left(A(t)\partial_t^\beta \theta_r, \theta_r\right)_{L^2_{\tilde{\sigma}}(\Omega)} + 2\left(r^2 A(t)\partial_t^\beta \theta, \theta_r\right)_{L^2(\Omega)} \\ & + \left(A(t)\partial_t^\beta \theta_y, \theta_y\right)_{L^2_{\tilde{\rho}}(\Omega)} \\ = \; & \left(\partial_t^\beta \theta, f\right)_{L^2_{\tilde{\sigma}}(\Omega)}, \end{aligned} \tag{18}$$

$$4\left\|\sqrt{A(t)}\theta\right\|^2_{L^2_{\tilde{\sigma}}(\Omega)} = \left(\Im_r(\xi\theta), \partial_t^\beta \theta\right)_{L^2_{\tilde{\sigma}}(\Omega)} - \left(\Im_r(\xi\theta), f\right)_{L^2_{\tilde{\sigma}}(\Omega)}, \tag{19}$$

$$\begin{aligned} & -2\left(r^2 A(t)\partial_t^\beta \theta, \theta_r\right)_{L^2(\Omega)} + a^2 \|A(t)\theta_r(a,y,t)\|^2_{L^2((0,t)\times(0,b))} \\ & + \|A(t)\theta_y(0,y,t)\|^2_{L^2((0,t)\times(0,b))} \\ = \; & -\left(2r^2 A(t)\theta_r, f\right)_{L^2(\Omega)}. \end{aligned} \tag{20}$$

By summing side-to-side equalities (18)–(20), we obtain the following:

$$\begin{aligned} & \left\|\partial_t^\beta \theta\right\|_{L^2_{\tilde{\sigma}}(\Omega)} + \left(A(t)\partial_t^\beta \theta_r, \theta_r\right)_{L^2_{\tilde{\sigma}}(\Omega)} + \left(A(t)\partial_t^\beta \theta_y, \theta_y\right)_{L^2_{\tilde{\rho}}(\Omega)} \\ & + 4\left\|\sqrt{A(t)}\theta\right\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + a^2 \|A(t)\theta_r(a,y,t)\|^2_{L^2((0,t)\times(0,b))} \\ & + \|A(t)\theta_y(0,y,t)\|^2_{L^2((0,t)\times(0,b))} \\ = \; & -\left(\Im_r(\xi\theta), f\right)_{L^2_{\tilde{\sigma}}(\Omega)} + \left(\partial_t^\beta \theta, f\right)_{L^2_{\tilde{\sigma}}(\Omega)} + \left(\Im_r(\xi\theta), \partial_t^\beta \theta\right)_{L^2_{\tilde{\sigma}}(\Omega)} \\ & - \left(2r^2 A(t)\theta_r, f\right)_{L^2(\Omega)}. \end{aligned} \tag{21}$$

By taking into account inequality (11), Young's inequality, and (6), we observe that the left-hand side of (21) can be estimated as follows:

$$\begin{aligned} \left(\Im_r(\xi\theta), \partial_t^\beta \theta\right)_{L^2_{\tilde{\sigma}}(\Omega)} & \leq \frac{\nu_1}{2}\left\|\partial_t^\beta \theta\right\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{1}{2\nu_1}\|\Im_r(\xi\theta)\|^2_{L^2_{\tilde{\sigma}}(\Omega)} \\ & \leq \frac{\nu_1}{2}\left\|\partial_t^\beta \theta\right\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{8a^4}{\nu_1}\|\theta\|^2_{L^2_{\tilde{\sigma}}(\Omega)}, \end{aligned} \tag{22}$$

$$\left(\partial_t^\beta \theta, f\right)_{L^2_{\tilde{\sigma}}(\Omega)} \leq \frac{\nu_2}{2}\left\|\partial_t^\beta \theta\right\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{a^3}{2\nu_2}\|f\|^2_{L^2(\Omega)}, \tag{23}$$

$$\begin{aligned} -\left(\Im_r(\xi\theta), f\right)_{L^2_{\tilde{\sigma}}(\Omega)} & \leq \frac{\nu_3}{2}\|\Im_r(\xi\theta)\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{a^3}{2\nu_3}\|f\|^2_{L^2(\Omega)} \\ & \leq 8a^4\nu_3\|\theta\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{a^3}{2\nu_3}\|f\|^2_{L^2(\Omega)}, \end{aligned} \tag{24}$$

$$-\left(2r^2 A(t)\theta_r, f\right)_{L^2(\Omega)} \leq c_1\nu_4\|\theta_r\|^2_{L^2_{\tilde{\sigma}}(\Omega)} + \frac{a^3 c_1}{2\nu_4}\|f\|^2_{L^2(\Omega)}. \tag{25}$$

In the light of Lemma 3, condition (6), and inequalities (22)–(25), the equality (21) reduces to the following:

$$\begin{aligned}
&\left\|\partial_t^\beta\theta\right\|^2_{L^2_\sigma(\Omega)}+\frac{c_0}{2}\partial_t^\beta\|\theta_r\|^2_{L^2_\sigma(\Omega)}+\frac{c_0}{2}\partial_t^\beta\|\theta_y\|^2_{L^2_\rho(\Omega)}+4c_0\|\theta\|^2_{L^2_\sigma(\Omega)}\\
&+a^2c_0^2\|\theta_r(a,y,t)\|^2_{L^2((0,t)\times(0,b))}+c_0^2\left\|\theta_y(0,y,t)\right\|^2_{L^2((0,t)\times(0,b))}\\
\leq\ &\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)\left\|\partial_t^\beta\theta\right\|^2_{L^2_\sigma(\Omega)}+\left(\frac{8a^4}{\nu_1}+8a^4\nu_3\right)\|\theta\|^2_{L^2_\sigma(\Omega)}\\
&+\left(\frac{a^3}{2\nu_3}+\frac{a^3c_1}{2\nu_4}+\frac{a^3}{2\nu_2}\right)\|f\|^2_{L^2(\Omega)}+c_1\nu_4\|\theta_r\|^2_{L^2_\sigma(\Omega)}
\end{aligned}\tag{26}$$

Referring to Lemma 2, inequality (26) yields the following:

$$\begin{aligned}
&\left\|\partial_t^\beta\theta\right\|^2_{L^2_\sigma(\Omega)}+\frac{c_0}{2}\partial_t^\beta\|\theta_r\|^2_{L^2_\sigma(\Omega)}+4c_0\|\theta\|^2_{L^2_\sigma(\Omega)}+\frac{c_0}{2}\partial_t^\beta\|\theta_y\|^2_{L^2_\rho(\Omega)}\\
&+a^2c_0^2\|\theta_r(a,y,t)\|^2_{L^2((0,t)\times(0,b))}+c_0^2\left\|\theta_y(0,y,t)\right\|^2_{L^2((0,t)\times(0,b))}\\
\leq\ &\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)\left\|\partial_t^\beta\theta\right\|^2_{L^2_\sigma(\Omega)}+\left(\frac{8a^6}{\nu_1}+8a^6\nu_3+c_1\nu_4\right)\|\theta_r\|^2_{L^2_\sigma(\Omega)}\\
&+\left(\frac{a^3}{2\nu_3}+\frac{a^3c_1}{2\nu_4}+\frac{a^3}{2\nu_2}\right)\|f\|^2_{L^2(\Omega)}.
\end{aligned}\tag{27}$$

Given the choice $\nu_1=\nu_2=1l2$, $\nu_3=\nu_4=4$, and neglecting the last three terms on the left-hand side of (27), it becomes the following:

$$\|\theta\|^2_{H^\beta_\sigma(\Omega)}+\partial_t^\beta\|\theta_r\|^2_{L^2_\sigma(\Omega)}\leq K\Big(\|\theta_r\|^2_{L^2_\sigma(\Omega)}+\|f\|^2_{L^2(\Omega)}\Big),\tag{28}$$

where

$$K=\frac{\max\left[48a^6+c_1,3a^3+c_1a^4\right]}{\min[1,c_0]}.\tag{29}$$

Integration of (28) with respect to time yields the following estimate:

$$\begin{aligned}
&\int_0^t\|\theta\|^2_{H^\beta_\sigma(\Omega)}ds+D^{\beta-1}\|\theta_r\|^2_{L^2_\sigma(\Omega)}\\
\leq\ &K\left(\int_0^t\|\theta_r\|^2_{L^2_\sigma(\Omega)}ds+\int_0^t\|f\|^2_{L^2(\Omega)}ds\right)\\
&+\frac{T^{\beta-1}}{(1-\beta)\Gamma(1-\beta)}\|\phi_r\|^2_{L^2_\sigma(\Omega)}.
\end{aligned}\tag{30}$$

We are now in a position to apply Lemma 4 to (30) by discarding the first term and letting the following:

$$M(t)=\int_0^t\|\theta_r\|^2_{L^2_\sigma(\Omega)}ds,\ \ M(0)=0,\ \partial_t^\sigma M(t)=D^{\beta-1}\|\theta_r\|^2_{L^2_\sigma(\Omega)},\tag{31}$$

We then have the following:

$$\int_0^t\|\theta_r\|^2_{L^2_\sigma(\Omega)}ds\leq\varkappa\left(D_t^{-\beta}\int_0^t\|f\|^2_{L^2(\Omega)}ds+D_t^{-\beta}\|\phi_r\|^2_{L^2_\sigma(\Omega)}\right),\tag{32}$$

where

$$\varkappa = \Gamma(\beta)E_{\beta,\beta}(KT^{\beta})\max\left(K, \frac{T^{\beta-1}}{(1-\beta)\Gamma(1-\beta)}\right). \tag{33}$$

In the light of (32), inequality (30) becomes the following:

$$\begin{aligned} & \int_0^t \|\theta\|^2_{H^{\beta}_{\sigma}(\Omega)} ds + D^{\beta-1}\|\theta_r\|^2_{L^2_{\sigma}(\Omega)} \leq C^*\left(\int_0^t \|f\|^2_{L^2(\Omega)} ds + \|\phi_r\|^2_{L^2_{\sigma}(\Omega)}\right) \\ \leq \; & C^*\left(\int_0^t \|f\|^2_{L^2(\Omega)} ds + \|\phi\|^2_{H^{1,r}_{\sigma}(\Omega)}\right) \\ \leq \; & C^*\left(\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{H^{1,r}_{\sigma}(\Omega)}\right), \end{aligned} \tag{34}$$

where

$$C^* = \max\left[\frac{K(1+\varkappa T^{\beta})}{\beta\Gamma(\beta)}, \frac{\varkappa T^{\beta}K(1-\beta)+T^{\beta-1}}{(1-\beta)\Gamma(1-\beta)}\right]. \tag{35}$$

The independence on t of the right-hand side of (34) gives the following:

$$\|\theta\|^2_{L^2(0.T;H^{\beta}_{\sigma}(\Omega))} + \sup_{t\in[0,T]}\left(D^{\beta-1}\|\theta_r\|^2_{L^2_{\sigma}(\Omega)}\right) \leq C^*\left(\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{H^{1,r}_{\sigma}(\Omega)}\right). \tag{36}$$

It can be easily shown that the operator $\mathcal{N} : S \to H$ admits a closure; that is, the closure of the graph $G(\mathcal{N}) \subset S \times H$ is a graph $G(\overline{\mathcal{N}})$ of $\overline{\mathcal{N}}$. □

Proposition 1. *The operator $\mathcal{N} : S \to H$ is closable.*

The a priori estimate (14) can be extended to the solutions $\theta \in D(\overline{\mathcal{N}})$ by taking the limit to obtain the following:

$$\|\theta\|^2_S \leq C^*\|\mathcal{N}\theta\|^2_H, \text{ for all } \theta \in D(\overline{\mathcal{N}}). \tag{37}$$

We conclude from (37) that the problem defined in (1)–(4) admits a unique strong solution that depends continuously on $f \in L^2(\Lambda)$, and $\phi \in H^{1,r}_{\sigma}(\Omega)$. We also see that the range of the operator $\overline{\mathcal{N}}$ is closed in space H and coincides with the closure of the range of the operator $\mathcal{N}$.

5. Solvability and Existence of the Solution

Theorem 2. *Under the conditions (6) and (7), and for arbitrary functions $f \in L^2(\Lambda)$ and $\phi \in H^{1,r}_{\sigma}(\Omega)$, there exists a function $\theta \in L^2(0.T; H^{\beta}_{\sigma}(\Omega))$ which solves the problem defined in (1)–(4). In other words, for all elements $Z = (f, \phi) \in H$, there exists a unique strong solution $\theta = \overline{\mathcal{N}}^{-1}Z = \overline{\mathcal{N}^{-1}}Z$ which solves the problem defined in (1)–(4).*

From the previous section, we see that $\overline{\mathcal{N}}$ is one-to-one (injective). Thus, in order to prove the existence of the solution to the problem defined in (1)–(4), we need to show that the operator $\overline{\mathcal{N}}$ is surjective. That is, we have to prove that $R(\mathcal{N})$ (the range of operator $\mathcal{N}$) is dense everywhere in space H. More precisely, $R(\mathcal{N})$ is a subset of the Hilbert space H, so it is a Hilbert space under the same inner product. The orthogonal complement of the set $R(\mathcal{N})$ is denoted by $R(\mathcal{N})^{\perp}$ and defined as follows:

$$R(\mathcal{N})^{\perp} = \{\mathcal{W} = (\Psi, \omega) \in H, (\mathcal{N}\theta, \mathcal{W})_H = 0 \text{ for all } \theta \in \mathcal{D}(\mathcal{N})\}.$$

To prove that $\overline{R(\mathcal{N})} = H$, we must show that $R(\mathcal{N})^{\perp} = \{0\}$; that is, $\Psi = 0$ and $\omega = 0$.

We will prove the density in two cases. We first prove the following result.

Theorem 3. *Under the assumptions of Theorem 2, if*

$$(\mathcal{L}\theta, \Psi)_{L^2(0,T.,L^2\Omega)} = 0, \tag{38}$$

for arbitrary $\theta \in \mathcal{D}_0(\mathcal{N}) = \{\theta \in \mathcal{D}(\mathcal{N}), \ell\theta = \theta(r, y, 0) = 0\}$, *and for some function* $\Psi \in L^2(0, T., L^2\Omega)$, *then* Ψ *is zero almost everywhere in* Λ.

Assume for the moment that Theorem 3 has been demonstrated, and continue the proof of Theorem 2.

Let

$$(\mathcal{N}\theta, \mathcal{W})_H = (\mathcal{L}\theta, \Psi)_{L^2(0,T;L^2(\Omega))} + (\ell\theta, \rightarrow)_{H^{1,r}_{\sigma}(\Omega)} = 0, \text{ for all } \theta \in \mathcal{D}(\mathcal{N}). \tag{39}$$

If we take $\theta \in \mathcal{D}_0(\mathcal{N})$, then according to Theorem 9, we conclude that $\Psi \equiv 0$. Consequently,

$$(\ell\theta, \rightarrow)_{H^{1,r}_{\sigma}(\Omega)} = 0, \text{ for all } \theta \in \mathcal{D}(\mathcal{N}). \tag{40}$$

Owing to the fact that the range of the trace operator ℓ is dense in the space $H^{1,r}_{\sigma}(\Omega)$, (40) implies that $\rightarrow = 0$. Hence, $\overline{R(\mathcal{N})} = H$.

To prove Theorem 2, we need to prove Theorem 3.

Proof of Theorem 3. Equation (38) can be read as follows:

$$\int_0^T \left(\partial_t^{\beta}\theta - A(t)\left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy}\right), \Psi \right)_{L^2\Omega} dt = 0. \tag{41}$$

Consider a function $\mathcal{K}(x, y, t)$ that verifies conditions (2)–(4) such that $\mathcal{K}$, $\mathcal{K}_r$, $\mathcal{K}_y$, $\Im_t\mathcal{K}$, $\Im_t\mathcal{K}_r$, $\Im_t\mathcal{K}_y$, $\Im_r(\xi\Im_t\mathcal{K})$, and $\partial_t^{\beta}(\Im_t\mathcal{K})$ belong to $L^2(0, T; L^2(\Omega))$. Now, we set

$$\theta = \Im_t\mathcal{K} = \int_0^t \mathcal{K}(r, y, s)ds, \tag{42}$$

and let the function Ψ be defined as

$$\Psi(r, y, t) = r^3\partial_t^{\beta}(\Im_t\mathcal{K}) - r^3\Im_r(\xi\Im_t\mathcal{K}) - 2r^2A(t)\Im_t\mathcal{K}_r. \tag{43}$$

Then, Equation (41) takes the following form:

$$\begin{aligned} &\int_0^T \left(\partial_t^{\beta}\Im_t\mathcal{K}, r^3\partial_t^{\beta}(\Im_t\mathcal{K}) - r^3\Im_r(\xi\Im_t\mathcal{K}) - 2r^2A(t)\Im_t\mathcal{K}_r\right)_{L^2\Omega} dt \\ &- \int_0^T \left(A(t)\frac{1}{r}(r\Im_t\mathcal{K}_r)_r, r^3\partial_t^{\beta}(\Im_t\mathcal{K}) - r^3\Im_r(\xi\Im_t\mathcal{K}) - 2r^2A(t)\Im_t\mathcal{K}_r\right)_{L^2\Omega} dt \\ &- \int_0^T \left(A(t)\frac{1}{r^2}\Im_t\mathcal{K}_{yy}, r^3\partial_t^{\beta}(\Im_t\mathcal{K}) - r^3\Im_r(\xi\Im_t\mathcal{K}) - 2r^2A(t)\Im_t\mathcal{K}_r\right)_{L^2\Omega} dt \\ = \; &0. \end{aligned} \tag{44}$$

According to conditions (2)–(4) on function $\mathcal{K}$, we evaluate the different inner products as follows:

$$\left(\partial_t^{\beta}\Im_t\mathcal{K}, r^3\partial_t^{\beta}(\Im_t\mathcal{K})\right)_{L^2\Omega} = \left\|\partial_t^{\beta}\Im_t\mathcal{K}\right\|^2_{L^2_{\sigma}\Omega}, \tag{45}$$

$$-\left(\partial_t^\beta \Im_t \mathcal{K}, r^3 \Im_r(\xi \Im_t \mathcal{K})\right)_{L^2\Omega} = -\left(\partial_t^\beta \Im_t \mathcal{K}, \Im_r(\xi \Im_t \mathcal{K})\right)_{L^2_\sigma\Omega}, \tag{46}$$

$$-2\left(\partial_t^\beta \Im_t \mathcal{K}, r^2 A(t) \Im_t \mathcal{K}_r\right)_{L^2\Omega} = -2\left(\partial_t^\beta \Im_t \mathcal{K}, A(t) \Im_t \mathcal{K}_r\right)_{L^2_\rho\Omega}, \tag{47}$$

$$\begin{aligned} -\left(A(t)(r\Im_t \mathcal{K}_r)_r, r^2 \partial_t^\beta(\Im_t \mathcal{K})\right)_{L^2\Omega} &= -\int_0^b r^3 A(t) \Im_t \mathcal{K}_r \partial_t^\beta(\Im_t \mathcal{K})\Big|_{r=0}^{r=a} dy \\ &\quad + \left(A(t)\Im_t \mathcal{K}_r, \partial_t^\beta(\Im_t \mathcal{K}_r)\right)_{L^2_\sigma\Omega} \\ &\quad + 2\left(A(t)\Im_t \mathcal{K}_r, \partial_t^\beta(\Im_t \mathcal{K})\right)_{L^2_\rho\Omega}, \end{aligned} \tag{48}$$

$$\begin{aligned} \left(A(t)\frac{1}{r}(r\Im_t \mathcal{K}_r)_r, r^3 \Im_r(\xi \Im_t \mathcal{K})\right)_{L^2\Omega} &= \int_0^b r^3 A(t) \Im_r(\xi \Im_t \mathcal{K})(\Im_t \mathcal{K}_r)\Big|_{r=0}^{r=a} dy \\ &\quad - \left(r^4 A(t) \Im_t \mathcal{K}_r, \Im_t \mathcal{K})\right)_{L^2\Omega} \\ &\quad - 2(A(t)\Im_t \mathcal{K}_r, \Im_r(\xi \Im_t \mathcal{K})_{L^2_\rho\Omega} \\ &= -\int_0^b r^4 A(t)(\Im_t \mathcal{K})^2\Big|_{r=0}^{r=a} dy + 2\left\|\sqrt{A(t)}\Im_t \mathcal{K}\right\|^2_{L^2_\sigma\Omega} \\ &\quad - 2(A(t)\Im_t \mathcal{K}_r, \Im_r(\xi \Im_t \mathcal{K})_{L^2_\rho\Omega}, \end{aligned} \tag{49}$$

$$\begin{aligned} \left(A(t)\frac{1}{r}(r\Im_t \mathcal{K}_r)_r, 2r^2 A(t)\Im_t \mathcal{K}_r\right)_{L^2\Omega} &= \int_0^b r^2 A^2(t)(\Im_t \mathcal{K}_r)^2\Big|_{r=0}^{r=a} dy \\ &= a^2 \int_0^b A^2(t)\left(\Im_t \mathcal{K}_r^2(a, y, t)\right) dy, \end{aligned} \tag{50}$$

$$\begin{aligned} -\left(A(t)\frac{1}{r^2}\Im_t \mathcal{K}_{yy}, r^3 \partial_t^\beta(\Im_t \mathcal{K})\right)_{L^2\Omega} &= -\int_0^a rA(t)\partial_t^\beta(\Im_t \mathcal{K})(\Im_t \mathcal{K}_y)^2\Big|_{y=0}^{y=b} dx \\ &\quad + \left(A(t)\Im_t \mathcal{K}_y, \partial_t^\beta(\Im_t \mathcal{K}_y)\right)_{L^2_\gamma\Omega}, \end{aligned} \tag{51}$$

$$\begin{aligned} \left(A(t)\frac{1}{r^2}\Im_t \mathcal{K}_{yy}, r^3 \Im_r(\xi \Im_t \mathcal{K})\right)_{L^2\Omega} &= \int_0^a rA(t)\Im_t \mathcal{K}_y \Im_r(\xi \Im_t \mathcal{K})\Big|_{y=0}^{y=b} dx \\ &\quad - \left(A(t)r\Im_t \mathcal{K}_y, \Im_r(\xi \Im_t \mathcal{K}_y)\right)_{L^2\Omega} \\ &= -\frac{1}{2}\int_0^a r^2 A^2(t)\Im_r(\xi \Im_t \mathcal{K}_y)\Big|_{r=0}^{r=a} dy \\ &= 0, \end{aligned} \tag{52}$$

$$
\begin{aligned}
2\left(A(t)\frac{1}{r^2}\Im_t\mathcal{K}_{yy}, r^2A(t)\Im_t\mathcal{K}_r\right)_{L^2\Omega} &= \int_0^a A^2(t)\Im_t\mathcal{K}_r\Im_t\mathcal{K}_y\Big|_{r=0}^{r=a}dy \\
&\quad -2(A^2(t)\Im_t\mathcal{K}_{yr}, \Im_t\mathcal{K}_y)_{L^2\Omega} \\
&= -\int_0^b A^2(t)(\Im_t\mathcal{K}_y)^2\Big|_{r=0}^{r=a}dy \\
&= \int_0^b A^2(t)(\Im_t\mathcal{K}_y^2(0,y,t)dy. \qquad (53)
\end{aligned}
$$

Combining equalities (45)–(53) yields the following:

$$
\begin{aligned}
&\left\|\partial_t^\beta\Im_t\mathcal{K}\right\|^2_{L^2_\sigma\Omega} + \left(A(t)\partial_t^\beta(\Im_t\mathcal{K}_r), \Im_t\mathcal{K}_r\right)_{L^2_\sigma\Omega} + 2\left\|\sqrt{A(t)}\Im_t\mathcal{K}\right\|^2_{L^2_\sigma\Omega} \\
&+\left(A(t)\partial_t^\beta(\Im_t\mathcal{K}_y), \Im_t\mathcal{K}_y\right)_{L^2_\gamma\Omega} + a^2\int_0^b A^2(t)\left(\Im_t\mathcal{K}_r^2(a,y,t\right)dy \\
&+\int_0^b A^2(t)(\Im_t\mathcal{K}_y^2(0,y,t)dy \\
= &\left(\partial_t^\beta\Im_t\mathcal{K}, \Im_r(\xi\Im_t\mathcal{K})\right)_{L^2_\sigma\Omega} + 2(A(t)\Im_t\mathcal{K}_r, \Im_r(\xi\Im_t\mathcal{K})_{L^2_\rho\Omega}. \qquad (54)
\end{aligned}
$$

Based on condition (6) and Young's inequality, the terms on the right-hand side can be bounded above as follows:

$$
\left(\partial_t^\beta\Im_t\mathcal{K}, \Im_r(\xi\Im_t\mathcal{K})\right)_{L^2_\sigma\Omega} \le \frac{d_1}{2}\left\|\partial_t^\beta\Im_t\mathcal{K}\right\|^2_{L^2_\sigma\Omega} + \frac{1}{2d_1}\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\sigma\Omega}, \qquad (55)
$$

$$
2(A(t)\Im_t\mathcal{K}_r, \Im_r(\xi\Im_t\mathcal{K})_{L^2_\rho\Omega} \le c_1^2d_2\|\Im_t\mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \frac{1}{d_2}\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\gamma\Omega}. \qquad (56)
$$

If we now apply Lemma 3, drop the last two positive terms on the left-hand side of (54), and combine them with (55) and (56), with $d_1 = d_2 = 1$, we obtain the following:

$$
\begin{aligned}
&\frac{1}{2}\left\|\partial_t^\beta\Im_t\mathcal{K}\right\|^2_{L^2_\sigma\Omega} + \frac{c_0}{2}\partial_t^\beta\|\Im_t\mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + \frac{c_0}{2}\partial_t^\beta\|\Im_t\mathcal{K}_y)\|^2_{L^2_\gamma\Omega} + 2c_0\|\Im_t\mathcal{K}\|^2_{L^2_\sigma\Omega} \\
\le\ & c_1^2\|\Im_t\mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \frac{1}{2}\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\sigma\Omega} + \|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\gamma\Omega}. \qquad (57)
\end{aligned}
$$

The last two terms on the right-hand side of (57) can be estimated as follows. Lemma 5 implies the following:

$$
\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\sigma\Omega} \le 16a^4\|\Im_t\mathcal{K})\|^2_{L^2_\sigma\Omega}. \qquad (58)
$$

Based on Lemma 2, inequality (58) becomes the following:

$$
\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\sigma\Omega} \le 16a^6\|\Im_t\mathcal{K}_r)\|^2_{L^2_\sigma\Omega}. \qquad (59)
$$

$$
\|\Im_r(\xi\Im_t\mathcal{K})\|^2_{L^2_\gamma\Omega} \le \frac{a^3b}{2}\|\Im_t\mathcal{K}_y)\|^2_{L^2_\gamma\Omega}. \qquad (60)
$$

Inequalities (57), (59) and (60) yield the following inequality:

$$
\begin{aligned}
&\partial_t^\beta\|\Im_t\mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + \partial_t^\beta\|\Im_t\mathcal{K}_y)\|^2_{L^2_\gamma\Omega} + \left\|\partial_t^\beta\Im_t\mathcal{K}\right\|^2_{L^2_\sigma\Omega} + \|\Im_t\mathcal{K}\|^2_{L^2_\sigma\Omega} \\
\le\ & B\left(\|\Im_t\mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \|\Im_t\mathcal{K}_y)\|^2_{L^2_\gamma\Omega}\right), \qquad (61)
\end{aligned}
$$

where

$$B = \frac{\max\left\{2c_1^2 + 16a^6, \frac{a^6 b}{2}\right\}}{\min\{1, c_0\}}. \tag{62}$$

We replace the time variable t with s in (61) and integrate over the interval $(0, t)$ to obtain the following:

$$\begin{aligned} & D^{\beta-1}\|\Im_t \mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + D^{\beta-1}\|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega} + \left\|\partial_t^\beta \Im_t \mathcal{K}\right\|^2_{L^2(0,t;L^2_\sigma\Omega)} \\ & + \|\Im_t \mathcal{K}\|^2_{L^2(0,t;L^2_\sigma\Omega)} \\ \leq \; & \frac{T^{1-\beta}B}{(1-\beta)\Gamma(1-\beta)}\left[\|\Im_t \mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega}\right]\Big|_{t=0} \\ & + B\int_0^t \left(\|\Im_t \mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega)}\right) ds. \end{aligned} \tag{63}$$

Since the term evaluated at $t = 0$ vanishes, inequality (63) reduces to the following:

$$\begin{aligned} & \partial_t^\beta \Im_t \mathcal{K}^2_{L^2(0,t;L^2_\sigma\Omega)} + \|\Im_t \mathcal{K}\|^2_{L^2(0,t;L^2_\sigma\Omega)} \\ & + D^{\beta-1}\left(\|\Im_t \mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega}\right) \\ \leq \; & B\int_0^t \left(\|\Im_t \mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega)}\right) ds. \end{aligned} \tag{64}$$

If we now discard the first two terms on the left-hand side of (64) and apply Lemma 4 by letting

$$\begin{aligned} M(t) &= \int_0^t \left(\|\Im_t \mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega}\right) ds, \\ \partial_t^\beta M(t) &= D^{\beta-1}\left(\|\Im_t \mathcal{K}_r)\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega}\right), \\ \frac{dM(t)}{dt} &= \|\Im_t \mathcal{K}_r\|^2_{L^2_\sigma\Omega} + \|\Im_t \mathcal{K}_y)\|^2_{L^2_\gamma\Omega}, \; M(0) = 0, \end{aligned} \tag{65}$$

we then have

$$M(t) \leq M(0)E_\beta(Bt^\beta) + \Gamma(\beta)E_{\beta,\beta}(Bt^\beta)D_t^{-\beta}(0) = 0 \;\; \forall t \in [0, T]. \tag{66}$$

It follows from (66) that $\Psi = 0$. □

Remark 1. *Our investigated problem can be solved in the nonlinear case*

$$\begin{cases} \partial_t^\beta \theta - A(t)\left(\frac{1}{r}(r\theta_r)_r + \frac{1}{r^2}\theta_{yy}\right) = f(r, y, t, \theta, \theta_r, \theta_y), \; \beta \in (0, 1] \\ \theta(r, y, 0) = \phi(r, y) \;\; , \; (r, y) \in \Omega, \\ \theta(a, y, t) = 0, \; \theta(r, 0, t) = 0, \; \theta(r, b, t) = 0, \\ \int_0^a r\theta(r, y, t)dr = 0, \quad t \in [0, T], \end{cases}$$

provided that the function f is a Lipschitzian function; that is, there exists a positive constant δ such that

$$\begin{aligned} & |f(r, y, t, U_1, V_1, W_1) - f(r, y, t, U_2, V_2, W_2)| \\ \leq \; & \delta(|U_1 - U_2| + |V_1 - V_2| + |W_1 - W_2|. \end{aligned}$$

6. Conclusions

The well-posedness of an initial nonlocal boundary value problem for a singular two-dimensional fractional-order partial differential equation in the Caputo sense is studied. The fractional partial differential equation is supplemented by Dirichlet and weighted integral boundary conditions. The uniqueness of the solution is established from an a priori estimate, and the solvability of the posed problem is demonstrated with the help of a density argument. We employed the theory operator techniques to provide various proofs.

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