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Necessary and Sufficient Criteria for a Four-Weight Weak-Type Maximal Inequality in the Orlicz Class

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Abstract: Let Φ_i $(i = 1,2)$ be two N-functions, f be a μ-measurable function, and ω_i $(i = 1,2,3,4)$ be four weight functions. This study presents necessary and sufficient conditions for weight functions $(\omega_1, \omega_2, \omega_3, \omega_4)$ such that the inequality $\int_{\{x:Mf(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \leq c_1 \int_X \Phi_2(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x)$ holds, which extends several established results.

Keywords: weight; weak-type inequality; N-function; Orlicz class

MSC: 42B25; 46E30

1. Introduction

In the field of weighted theory, the significance of weight inequalities cannot be overstated, as they represent the core of ongoing research. Gogatishvili and Kokilashvili [1] have elucidated both necessary and sufficient conditions for weighted inequalities within Orlicz classes applicable to maximal functions on spaces of homogeneous type. Similarly, criteria that are both necessary and sufficient for weighted integral inequalities linked with Doob's maximal operator in the martingale Orlicz frameworks have been delineated by Chen and Liu [2]. Berkovits [3] has ventured into the parabolic analog realm by proposing a variant of Muckenhoupt's $A(p)$ class and has successfully formulated a John–Nirenberg inequality applicable to a BMO class. Furthermore, Ren [4] explored a four-weight weak-type maximal inequality in martingale contexts. An essential and sufficient criterion for a three-weight weak-type one-sided Hardy–Littlewood maximal inequality on $\mathbb{R}^2$ has been identified by Zhang and Ren [5], which also integrates previous two-weight weak and extra-weak inequalities into a unified three-weight weak-type framework. Additional studies that extend these discussions are documented in references [6–12].

In particular, in a seminal work from 1972, Muckenhoupt [13] put forth a landmark study delineating the behavior of weak-type inequalities associated with the Hardy–Littlewood maximal function. This function is characterized as follows:

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum encompasses all Q cubes that contain the point x within $\mathbb{R}^n$. Within this framework, Muckenhoupt established that when considering the Hardy–Littlewood maximal function, a noteworthy finding arises with respect to weighted weak-type inequalities. Specifically, he demonstrated that under the influence of two distinct weight functions, u and v, the inequality maintains its form, i.e., the inequality

$$u(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{c}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad 1 < p < \infty \tag{1}$$

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holds if and only if $(u, v) \in A_p$. There is a positive constant c such that we have the following:

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}dx\right)^{p/p'} \leq c,$$

where $p' = \frac{p}{p-1}$.

The A_p condition proposed by Muckenhoupt has significantly influenced the weighted theory, prompting a series of scholarly efforts to expand these concepts into various function spaces. Within the realm of Orlicz spaces, the application and adaptation of inequality (1) were examined by researchers, including Gallardo [14], Bagby [15], Bloom and Kerman [16], and Gogatishvili and Kokilashvili [17]. During this period, the primary challenge in the study of weighted weak-type inequalities for maximal functions in Orlicz spaces involved overcoming the $\triangle_2$ condition. Notably, Pick [18] advanced this field significantly by introducing a Φ-extension of (1) along with its weighted formulation and obtained its weighted equivalent representation, i.e., the weighted inequality, as follows:

$$\int_{\{x\in X: Mf(x)>\lambda\}} \Phi_1(\lambda)u(x)d\mu(x) \leq c\int_X \Phi_2(c|f(x)|)v(x)d\mu(x) \tag{2}$$

holds, if and only if $(u, v) \in A_{\Phi_1,\Phi_2}$, namely

$$\Phi_1\left(\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2\left(\frac{\lambda}{v(x)}\right)v(x)d\mu(x)\right)u(B) \leq c\int_B \Psi_2\left(\frac{\lambda}{v(x)}\right)v(x)d\mu(x),$$

where Ψ_2 is a complement function of Φ_2. Recently, in 2022, extensive research by Ren and Ding [19] elaborated on the essential and sufficient criteria for the inequality (2), further extending the insights previously established in [18]. Building upon these foundational studies on [18,19], the current paper delves into the four-weight generalization of inequality (2), as follows:

$$\int_{\{x:Mf(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \leq c_1\int_X \Phi_2(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x) \tag{3}$$

and establishes necessary and sufficient criteria for the four-weight weak-type maximal inequality (3), which provides a relatively complete four-weight characterization.

The subsequent sections of this manuscript are organized in the following manner: Section 2 offers a preliminary discussion, delivering a succinct summary of the essential concepts and lemmas that underpin our study. Section 3 presents the central findings of this research, accompanied by a detailed and rigorous substantiation. Finally, Section 4 articulates the conclusions drawn from this work and delineates various prospects for future inquiry.

2. Preliminaries

This section presents a summary of the crucial elements related to the complete measure space, its Besicovitch property, and N-functions relevant to our analysis. For detailed discussions, the reader is referred to references[9,18,20].

Consider (X, d, μ) as a complete measure space, where d serves as a quasimetric. It is postulated that μ is a doubling measure relative to d and the space (X, d, μ) exhibits the Besicovitch property, as noted in [20].

We examine a function f, which is both locally integrable and measurable in relation to μ. We define the following:

$$(f)_B = \frac{1}{\mu B}\int_B f(x)d\mu.$$

We present the definition of the maximum function of f as follows:

$$Mf(x) = \sup_{x\in B}\frac{1}{\mu B}\int_B |f(y)|d\mu(y), \quad x \in X$$

where the supremum is taken over all balls B with $x \in B$.

An almost everywhere positive μ-measurable locally integrable function is referred to as a weight function. If ω represents a weight, we denote the following:

$$\omega(B) = \int_B \omega(x)d\mu(x).$$

A mapping $\Phi : \mathbb{R} \to \mathbb{R}^+$ is considered an N-function if it is even and convex, satisfies $\Phi(x) = 0$ only when $x = 0$, and has $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$. If Φ is an N-function, then the complementary function of Φ given by $\Psi(t) = \sup\{st - \Phi(s)\}$ can also be regarded as an N-function. This pair of complementary N-functions (Φ, Ψ) satisfies Young's inequality, as follows:

$$st \leq \Phi(s) + \Psi(t),$$

see [20].

Lemma 1 ([18]). *Let (Φ, Ψ) be two complementary N-functions, then $\frac{\Phi(x)}{x}$ and $\frac{\Psi(x)}{x}$ increase on $(0, \infty)$. Additionally, they meet the following inequalities:*

$$\Psi(\frac{\Phi(x)}{x}) \leq \Phi(x) \leq \Psi(2\frac{\Phi(x)}{x}) \tag{4}$$

and

$$\Phi(\frac{\Psi(x)}{x}) \leq \Psi(x) \leq \Phi(2\frac{\Psi(x)}{x}) \tag{5}$$

for all $x > 0$.

Throughout this paper, $c_i > 0$ and $c > 0$ are constants that can vary across different instances.

3. Main Result

In 2022, Ren and Ding [19] presented some necessary and sufficient conditions for the two-weight weak-type maximal inequality (i.e., Theorem 1).

Theorem 1. *Let (Φ_i, Ψ_i) $(i = 1, 2)$ be two pairs of complementary N-functions, f be a μ-measurable function, and ϱ and σ be weight functions. Then the subsequent statements are equivalent:*

(i) The following inequality holds for any f and $\lambda > 0$, with a constant $c_1 \geq 1$:

$$\int_{\{x:Mf(x)>\lambda\}} \Phi_1(\lambda)\varrho(x)d\mu(x) \leq c_1 \int_X \Phi_2(c_1|f(x)|)\sigma(x)d\mu(x); \tag{6}$$

(ii) The following inequality holds for any f and $\lambda > 0$, with a constant $c_2 \geq 1$:

$$\int_{\{x:Mf(x)>\lambda\}} \Psi_2(\frac{\lambda}{\sigma(x)})\sigma(x)d\mu(x) \leq c_2 \int_X \Psi_1(c_2\frac{|f(x)|}{\varrho(x)})\varrho(x)d\mu(x); \tag{7}$$

(iii) The following inequality holds for any f and ball B, with a constant $c_3 \geq 1$:

$$\int_B \Psi_2(\frac{|f|_B}{\sigma(x)})\sigma(x)d\mu(x) \leq c_3 \int_B \Psi_1(c_3\frac{|f(x)|}{\varrho(x)})\varrho(x)d\mu(x); \tag{8}$$

(iv) The following inequality holds for any $\lambda > 0$ and ball B, with constants $c_4 \geq 1$ and $\varepsilon > 0$:

$$\int_B \Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda)\varrho(x)d\mu(x)}{\lambda\mu B\sigma(x)})\sigma(x)d\mu(x) \leq c_4 \int_B \Phi_1(\lambda)\varrho(x)d\mu(x). \tag{9}$$

Subsequently, this paper expands the initial two-weight weak-type maximal inequality into a more complex four-weight weak-type maximal inequality. To present the core Theorem 2, it is essential to incorporate and discuss Lemmas 2 and 3, as follows:

Lemma 2. *Let* (Φ_i, Ψ_i) $(i = 1, 2)$ *denote two pairs of complementary N-functions,* f *denote an* μ*-measurable function, and* ω_i $(i = 1, 2, 3, 4)$ *denote weight functions. Then the following assertions are equivalent:*

(i) The following inequality holds for any f *and* $\lambda > 0$*, with a constant* $c_1 \geq 1$*:*

$$\int_{\{x:Mf(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \leq c_1 \int_X \Phi_2(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x); \quad (10)$$

(ii) The following inequality holds for any f *and ball* B*, with a constant* $c_2 \geq 1$*:*

$$\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) \leq c_2 \int_B \Phi_2(c_2|f(x)|\omega_3(x))\omega_4(x)d\mu(x); \quad (11)$$

(iii) The following inequality holds for any $\lambda > 0$ *and ball* B*, with constants* $c_3 \geq 1$ *and* $\varepsilon > 0$*:*

$$\int_B \Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq c_3 \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x). \quad (12)$$

Proof. We establish the proof by demonstrating the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Since B is a subset of $\{x \in X : M(2f\chi_B)(x) > |f|_B\}$, we can directly obtain the inequality (11) from (10).

$(ii) \Rightarrow (iii)$. Let B denote a given ball and k denote a natural number; we set the following:

$$B_k = \{x \in B : \frac{1}{k} < \omega_3(x)\omega_4(x) < k\}$$

and we set the following:

$$g(x) = (\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})^{-1}\Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})\chi_{B_k},$$

with ε to be specified later.

Then, we have the following:

$$\begin{aligned} I &= \int_{B_k} \Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \\ &= \frac{1}{\lambda\mu B}\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x)\int_B \frac{g(x)}{\omega_3(x)}d\mu(x). \end{aligned}$$

If for the aforementioned ball B and λ, it holds that $\frac{1}{\mu B}\int_B \frac{g(x)}{\omega_3(x)}d\mu(x) \leq \lambda$, then we have the following:

$$I \leq \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x). \quad (13)$$

If for the aforementioned ball B and λ, it holds that $\frac{1}{\mu B}\int_B \frac{g(x)}{\omega_3(x)}d\mu(x) > \lambda$, in (11), we set $f(x) = \frac{g(x)}{\omega_3(x)}$, then $|f|_B = \frac{1}{\mu B}\int_B |f(x)|d\mu(x)$. By noting that $\frac{\Phi(t)}{t}$ is increasing on $(0, \infty)$, we have the following:

$$\begin{aligned} I &= \frac{1}{\lambda}\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x)|f|_B \\ &= \int_B \frac{\Phi_1(\lambda\omega_1(x))}{\lambda\omega_1(x)}\omega_1(x)\omega_2(x)d\mu(x)|f|_B \\ &\leq \int_B \frac{\Phi_1(|f|_B\omega_1(x))}{|f|_B\omega_1(x)}\omega_1(x)\omega_2(x)d\mu(x)|f|_B \\ &= \int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x), \end{aligned}$$

from (11), we have the following:

$$I \leq \int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) \leq c_2\int_B \Phi_2(c_2|g(x)|)\omega_4(x)d\mu(x). \tag{14}$$

Combining (13) and (14), we obtain the following:

$$I \leq \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) + c_2\int_B \Phi_2(c_2|g(x)|)\omega_4(x)d\mu(x).$$

Now, we select ε to be sufficiently small such that $c_2\varepsilon < 1$ and $c_2^2\varepsilon < 1$, and substitute $g(x)$ into the aforementioned inequality, yielding the following:

$$I \leq \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) + c_2^2\varepsilon I,$$

from which we obtain the following:

$$\int_{B_k} \Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq \frac{1}{1-c_2^2\varepsilon}\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x),$$

letting $k \to \infty$ yields (12).

$(iii) \Rightarrow (i)$. $\forall n \in \mathbb{N}$, we have the following:

$$M^n f(x) = \sup \frac{1}{\mu B}\int_B |f(y)|d\mu(y),$$

where the supremum is taken over all balls B satisfying $x \in B$ and $r \leq n$. $\forall B \in X$, we have the following:

$$\lambda \leq \frac{1}{\mu B}\int_B |f(x)|d\mu(x),$$

using Young's inequality and (12), we have the following:

$$\begin{aligned} \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) &\leq \int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x)\frac{1}{\lambda\mu B}\int_B |f(x)|d\mu(x) \\ &= \frac{1}{2c_3}\int_B \frac{2c_3}{\varepsilon}|f(x)|\omega_3(x)\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)}\omega_4(x)d\mu(x) \\ &\leq \frac{1}{2c_3}\int_B \Phi_2(\frac{2c_3}{\varepsilon}|f(x)|\omega_3(x))\omega_4(x)d\mu(x) \\ &\quad + \frac{1}{2c_3}\int_B \Psi_2(\varepsilon\frac{\int_B \Phi_1(\lambda\omega_1(y))\omega_2(y)d\mu(y)}{\lambda\mu B\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \\ &\leq \frac{1}{2}c_4\int_B \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x) \\ &\quad + \frac{1}{2}\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x), \end{aligned}$$

where $c_4 = \max\{\frac{1}{c_3}, \frac{2c_3}{\varepsilon}\}$. We then obtain the following:

$$\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \le c_4 \int_B \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x). \tag{15}$$

For arbitrary $x \in \{x : M^n f(x) > \lambda\}$, there exists a ball B', such that $x \in B'$ and $0 < r_{B'} \le n$, satisfying the following:

$$\frac{1}{\mu(B')}\int_{B'} |f(x)|d\mu(x) > \lambda.$$

By utilizing the Besicovitch property, we select an—at most—countable number of balls from the ball family $\{B'\}$ to satisfy the following:

$$\{x \in X : M^n f(x) > \lambda\} \subset \bigcup_i B_i, \quad \sum_i \chi_{B_i} \le c'. \tag{16}$$

Then, by combining (15) and (16), we deduce the following:

$$\begin{aligned}
\int_{\{x\in X:M^n f(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) &\le \sum_i \int_{B_i} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \\
&\le c_4 \sum_i \int_{B_i} \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x) \\
&= c_4 \sum_i \int_X \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)\chi_{B_i}d\mu(x) \\
&\le c'c_4 \int_X \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x),
\end{aligned}$$

letting $n \to \infty$, we obtain the following:

$$\int_{\{x\in X:Mf(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \le c_1 \int_X \Phi_2(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x),$$

where $c_1 = \max\{c'c_4, c_4\}$. □

Lemma 3. *Let (Φ_i, Ψ_i) $(i = 1, 2)$ be two pairs of complementary N-functions, f be a μ-measurable function, and ω_i $(i = 1, 2, 3, 4)$ be weight functions. Then, the following assertions are equivalent:*

(i) The following inequality holds for any f and $\lambda > 0$, with a constant $c_1 \ge 1$:

$$\int_{\{x:Mf(x)>\lambda\}} \Psi_2\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)d\mu(x) \le c_1 \int_X \Psi_1\Big(c_1\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\Big)\omega_2(x)d\mu(x); \tag{17}$$

(ii) The following inequality holds for any f and ball B, with a constant $c_2 \ge 1$:

$$\int_B \Psi_2\Big(\frac{|f|_B}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)d\mu(x) \le c_2 \int_B \Psi_1\Big(c_2\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\Big)\omega_2(x)d\mu(x); \tag{18}$$

(iii) The following inequality holds for any f and ball B, with a constant $c_3 \ge 1$:

$$\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) \le c_3 \int_B \Phi_2(c_3|f(x)|\omega_3(x))\omega_4(x)d\mu(x); \tag{19}$$

(iv) The following inequality holds for any $\lambda > 0$ and ball B, with constants $c_4 \ge 1$ and $\varepsilon > 0$:

$$\int_B \Phi_1\Big(\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)d\mu(x)\omega_1(x)\Big)\omega_2(x)d\mu(x) \le c_4 \int_B \Psi_2\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)d\mu(x). \tag{20}$$

Proof. We establish the proof by demonstrating the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Since B is a subset of $\{x \in X : M(2f\chi_B)(x) > |f|_B\}$, we can directly obtain inequality (18) from (17).

$(ii) \Rightarrow (iii)$. Without loss of generality, setting $|f|_B > 0$. By applying Young's inequality and using (18), we have the following:

$$\begin{aligned}\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) &= \frac{1}{|f|_B\mu B}\int_B |f(y)|[\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)]d\mu(y)\\ &= \frac{1}{2c_2}\int_B 2c_2^2|f(y)|\omega_3(y)\cdot\frac{1}{\omega_3(y)\omega_4(y)}\\ &\quad\times\frac{1}{\mu B}[\frac{1}{c_2|f|_B}\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)]\omega_4(y)d\mu(y)\\ &\leq \frac{1}{2c_2}\int_B \Psi_2\{\frac{1}{\omega_3(x)\omega_4(x)}\frac{1}{\mu B}[\frac{1}{c_2|f|_B}\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)]\}\\ &\quad\cdot\omega_4(x)d\mu(x)+\frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|\omega_3(x))\omega_4(x)d\mu(x),\end{aligned}$$

we set $f(x)=\frac{1}{c_2|f|_B}\Phi_1(|f|_B\omega_1(x))\omega_2(x)$, then $|f|_B=\frac{1}{\mu B}\int_B|f(x)|d\mu(x)$, by (18) and (4), we have the following:

$$\begin{aligned}\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) &\leq \frac{1}{2c_2}\int_B \Psi_2\{\frac{|f|_B}{\omega_3(x)\omega_4(x)}\}\omega_4(x)d\mu(x)\\ &\quad+\frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|\omega_3(x))\omega_4(x)d\mu(x)\\ &\leq \frac{1}{2}\int_B \Psi_1(c_2\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x)\\ &\quad+\frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|\omega_3(x))\omega_4(x)d\mu(x)\\ &= \frac{1}{2}\int_B \Psi_1(\frac{\Phi_1(|f|_B\omega_1(x))\omega_2(x)}{|f|_B\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x)\\ &\quad+\frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|\omega_3(x))\omega_4(x)d\mu(x)\\ &\leq \frac{1}{2}\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)\\ &\quad+\frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|\omega_3(x))\omega_4(x)d\mu(x).\end{aligned}$$

So, we have the following:

$$\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)\leq c_3\int_B \Phi_2(c_3|f(x)|\omega_3(x))\omega_4(x)d\mu(x),$$

where $c_3=\max\{\frac{1}{c_2},2c_2^2\}$.

$(iii)\Rightarrow(iv)$. Let us set $f(x)=\varepsilon\Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\cdot\frac{1}{\lambda}\omega_4(x)$, then $|f|_B=\frac{\varepsilon}{\mu B}\int_B\Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\cdot\frac{1}{\lambda}\omega_4(x)d\mu(x)$, by (19), we have the following:

$$\begin{aligned}&\int_B \Phi_1(\frac{\varepsilon}{\mu B}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\cdot\frac{1}{\lambda}\omega_4(x)d\mu(x)\omega_1(x))\omega_2(x)d\mu(x)\\ &\qquad=\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x)\\ &\qquad\leq c_3\int_B \Phi_2(c_3|f(x)|\omega_3(x))\omega_4(x)d\mu(x).\end{aligned}$$

Now, we put $f(x)$ into the above inequality and choose ε, such that $c_3\varepsilon=1$, and by (5), we have the following:

$$c_3\int_B \Phi_2(c_3\varepsilon\Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\cdot\frac{1}{\lambda}\omega_4(x)\omega_3(x))\omega_4(x)d\mu(x) \leq c_3\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x).$$

So, we have the following:

$$\int_B \Phi_1(\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\omega_1(x))\omega_2(x)d\mu(x) \le c_4\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x),$$

where $c_4 = c_3$.

$(iv) \Rightarrow (i)$. $\forall n \in \mathbb{N}$, we set the following:

$$M^n f(x) = \sup \frac{1}{\mu B}\int_B |f(y)|d\mu(y),$$

where the supremum is evaluated across all balls B, within which x is included and whose radii do not exceed n. For each ball B belonging to set X, the following is established:

$$\lambda \le \frac{1}{\mu B}\int_B |f(x)|d\mu(x),$$

by applying Young's inequality and using (20), we have the following:

$$\begin{aligned}
\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) &\le \frac{1}{\lambda\mu B}\int_B |f(y)|(\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x))d\mu(y)\\
&= \frac{1}{2c_4}\int_B \frac{2c_4}{\varepsilon}\frac{|f(y)|}{\omega_1(y)\omega_2(y)}\\
&\quad \times\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\omega_1(y)\omega_2(y)d\mu(y)\\
&\le \frac{1}{2c_4}\int_B \Phi_1(\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\omega_1(x))\\
&\quad \cdot\omega_2(x)d\mu(x) + \frac{1}{2c_4}\int_B \Psi_1(\frac{2c_4}{\varepsilon}\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x)\\
&\le \frac{1}{2}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\\
&\quad +\frac{1}{2}c_5\int_B \Psi_1(c_5\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x),
\end{aligned}$$

where $c_5 = \max\{\frac{1}{c_4}, \frac{2c_4}{\varepsilon}\}$. Subsequently, we obtain the following:

$$\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \le c_5\int_B \Psi_1(c_5\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x). \tag{21}$$

For arbitrary $x \in \{x : M^n f(x) > \lambda\}$, there exists a ball B' such that $x \in B', 0 < r_{B'} \le n$, satisfying the following:

$$\frac{1}{\mu(B')}\int_{B'} |f(x)|d\mu(x) > \lambda.$$

By utilizing the Besicovitch property, we select an—at most—countable number of balls from the ball family $\{B'\}$ to satisfy the following:

$$\{x \in X : M^n f(x) > \lambda\} \subset \bigcup_i B_i, \quad \sum_i \chi_{B_i} \le c'. \tag{22}$$

Then, by combining (21) and (22), we deduce the following:

$$\begin{aligned}\int_{\{x\in X:M^n f(x)>\lambda\}} \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) &\leq \sum_i \int_{B_i} \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\\ &\leq c_5\sum_i \int_{B_i} \Psi_1(c_5\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x)\\ &= c_5\sum_i \int_X \Psi_1(c_5\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)\chi_{B_i}d\mu(x)\\ &\leq c'c_5\int_X \Psi_1(c_5\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x),\end{aligned}$$

letting $n\to\infty$, we obtain the following:

$$\int_{\{x\in X:Mf(x)>\lambda\}} \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq c_1\int_X \Psi_1(c_1\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x),$$

where $c_1=\max\{c'c_5,c_5\}$. □

Remark 1. *Recently, a new equivalent characterization for the two-weight weak-type maximal inequality, labeled inequality (9), was established in [12]. Efforts were made to extend this inequality to a four-weight form and incorporate it into Lemmas 2 and 3. Despite these attempts, the results did not meet expectations. Nonetheless, the pursuit of uncovering and exploring new extensible two-weight inequalities continues.*

Our main conclusion is as follows:

Theorem 2. *Let (Φ_i,Ψ_i) $(i=1,2)$ denote two pairs of complementary N-functions, f denote a μ-measurable function, and ω_i $(i=1,2,3,4)$ denote weight functions. Then the subsequent statements are equivalent:*

(i) The following inequality holds for any f and $\lambda>0$, with a constant $c_1\geq 1$:

$$\int_{\{x:Mf(x)>\lambda\}} \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x) \leq c_1\int_X \Phi_2(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x); \quad (23)$$

(ii) The following inequality holds for any f and $\lambda>0$, with a constant $c_2\geq 1$:

$$\int_{\{x:Mf(x)>\lambda\}} \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq c_2\int_X \Psi_1(c_2\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x); \quad (24)$$

(iii) The following inequality holds for any f and ball B, with a constant $c_3\geq 1$:

$$\int_B \Psi_2(\frac{|f|_B}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq c_3\int_B \Psi_1(c_3\frac{|f(x)|}{\omega_1(x)\omega_2(x)})\omega_2(x)d\mu(x); \quad (25)$$

(iv) The following inequality holds for any f and ball B, with a constant $c_4\geq 1$:

$$\int_B \Phi_1(|f|_B\omega_1(x))\omega_2(x)d\mu(x) \leq c_4\int_B \Phi_2(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x); \quad (26)$$

(v) The following inequality holds for any $\lambda>0$, and ball B, with constants $c_5\geq 1$ and $\varepsilon_1>0$:

$$\int_B \Phi_1(\frac{\varepsilon_1}{\lambda\mu B}\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x)\omega_1(x))\omega_2(x)d\mu(x) \leq c_5\int_B \Psi_2(\frac{\lambda}{\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x); \quad (27)$$

(vi) The following inequality holds for any $\lambda>0$ and ball B, with constants $c_6\geq 1$ and $\varepsilon_2>0$:

$$\int_B \Psi_2(\varepsilon_2\frac{\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x)}{\lambda\mu B\omega_3(x)\omega_4(x)})\omega_4(x)d\mu(x) \leq c_6\int_B \Phi_1(\lambda\omega_1(x))\omega_2(x)d\mu(x). \quad (28)$$

Proof. It follows from Lemma 2 that $(i) \Leftrightarrow (iv) \Leftrightarrow (vi)$ and Lemma 3 that $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$. That is, we have the following equivalence relation graph (see Figure 1):

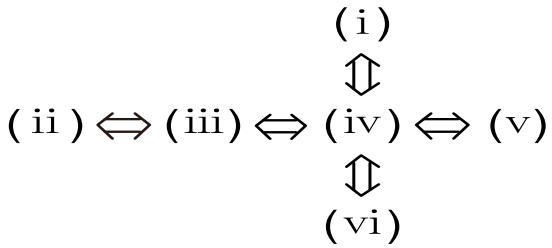

Figure 1. Equivalence relation graph.

So, we have that (i)–(vi) are equivalent. □

4. Conclusions

Upon analyzing Theorems 1 and 2, it is evident that inequalities (23) to (25) each represent the four-weight extension forms of inequalities (6) to (8), respectively. Moreover, inequality (28) extends inequality (9) in a similar manner. Additionally, inequalities (26) and (27) introduce novel four-weight extension forms that are not covered under Theorem 1. Consequently, Theorem 2 effectively encompasses a broader spectrum, offering a comprehensive and cohesive four-weight extension of Theorem 1, which elucidates a more complete characterization in four-weight terms. Future investigations will concentrate on developing new four-weight extension forms corresponding to inequality (23) and on formulating four-weight equivalent characterizations for inequalities on high dimensions.

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