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Multidimensional Fractional Calculus: Theory and Applications

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Abstract: In this paper, we introduce several new types of partial fractional derivatives in the continuous setting and the discrete setting. We analyze some classes of the abstract fractional differential equations and the abstract fractional difference equations depending on several variables, providing a great number of structural results, useful remarks and illustrative examples. Concerning some specific applications, we would like to mention here our investigation of the fractional partial differential inclusions with Riemann–Liouville and Caputo derivatives. We also establish the complex characterization theorem for the multidimensional vector-valued Laplace transform and provide certain applications.

Keywords: multidimensional fractional calculus; multidimensional discrete fractional calculus; multidimensional generalized Weyl fractional calculus; abstract partial fractional differential equations; abstract partial fractional difference equations; multidimensional vector-valued Laplace transform; multivalued linear operators

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1. Introduction and Preliminaries

Fractional calculus is an important field of theoretical and applied mathematics which generalizes the classical differential and integral calculus with the operations of integration and differentiation of noninteger order. Fractional calculus and fractional differential equations have earned considerable popularity and importance in the past few decades in various fields of applied science; for further information in this direction, see [1–8] and the references quoted therein. We will only mention here that fractional differential equations are invaluable and important in modeling of various phenomena appearing in mathematical physics, viscoelasticity, optics, acoustics, rheology, bioengineering, control theory, electrical and mechanical engineering and so on.

Discrete fractional calculus is also a rapidly developing branch of mathematics. The first serious study of discrete fractional differences can be attributed to F. Atici and P. Eloe ([9], 2009); for more details about this topic, we refer the reader to the research monograph [10] by C. Goodrich and A. C. Peterson, and the references quoted therein. Fractional difference equations are extremely useful in modeling discrete phenomena in different fields such as economics, physics, engineering and biology, and undoubtedly, there is a vast literature on them; for example, T. Zhang and Y. Li [11] recently analyzed the global exponential stability of discrete-time almost automorphic Caputo–Fabrizio BAM fuzzy neural networks (it would be very difficult to summarize and quote here all relevant references concerning discrete fractional calculus and its applications). The stability, boundedness, periodicity and asymptotic behavior of solutions for various classes of the fractional difference equations are very well explored by now. Concerning the existence and uniqueness of almost periodic solutions to the abstract fractional difference equations and the abstract Volterra difference equations, we refer the reader to the research monograph [12] and the list of references quoted therein.

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The partial fractional derivatives of functions have not attracted as much attention of the authors working in the field of fractional calculus to date. With the exception of the structural theory developed in Chapter 5 of the fundamental research monograph [7] by S. G. Samko, A. A. Kilbas and O. I. Marichev and some structural results about the partial fractional differential equations given in Chapter 7 in the fundamental research monograph [4] by A. A. Kilbas, M. Srivastava and J. J. Trujillo, we can freely say that almost all established results about partial fractional derivatives of functions and partial fractional differential equations given to date are rather fragmentary and concern some very special kinds of functions and partial fractional differential equations. In this research study, we tried to overcome the shortcomings of existing research by investigating many interesting topics that have not attracted the attention of authors working in the field of fractional calculus yet; for example, we initiated the study of the abstract partial fractional differential-difference inclusions with multivalued linear operators here (the multidimensional Laplace transform of functions with values in complex Banach spaces is also a very unexplored chapter of the theory of integral transforms).

For example, H. M. Srivastava, R. C. Singh Chandel and P. K. Vishwakarma analyzed, in [13], the partial fractional derivatives of certain generalized hypergeometric functions of several variables (see also [14]); the partial fractional differential equations with Riesz space fractional derivatives of positive real order (see [7] (Section 25, p. 357) for the notion and more details) were analyzed by H. Jiang et al. in [15] (see also [16]). It is also worth mentioning the recent research article [17] by V. Pilipauskaitė and D. Surgailis, where the authors analyzed certain fractional operators and fractionally integrated random fields on $\mathbb{Z}^n$. Further on, M. O. Mamchuev [18] and A. V. Pshku [19] considered the systems of multidimensional fractional partial differential equations containing the terms of form $D^{\alpha_j}_{x_j} u(x_1, ..., x_n)$ with just one index $j \in \mathbb{N}_n$ and not the general forms of partial fractional derivatives introduced in this paper. More precisely, A. V. Pshku considered, in [19], the well-posedness of the following multidimensional fractional partial differential equation:

$$\sum_{k=1}^{n} a_k \frac{\partial^{\sigma_k}}{\partial x_k^{\sigma_k}} u(x) + \lambda u(x) = f(x), \quad x \in [0, \infty)^n, \tag{1}$$

where $(\partial^{\sigma_k} / \partial x_k^{\sigma_k} u)$ denotes the fractional partial derivative of order σ_k with respect to the variable x_k with origin $x_k = 0$ (in the sense of the Riemann–Liouville, Caputo or Dzhrbashyan–Nersesyan approach); here $a_k > 0$ for $1 \leq k \leq n$, $\lambda \in \mathbb{R}$ and $f(\cdot)$ is a locally integrable function. We also refer the reader to the works mentioned in [7] (pp. 623–624) and some recent results about nonlinear fractional partial differential equations obtained in [20–25].

The structure and main ideas of this paper can be briefly summarized as follows. First of all, we explain the notation and terminology used throughout the paper and recall the basic facts about the generalized Hilfer fractional derivatives and differences (cf. Section 1.1). Section 2 examines the multidimensional generalized Hilfer fractional derivatives and differences. We first introduce the notion of a multidimensional generalized Hilfer fractional derivative $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u$ for a class of locally integrable functions $u : [0, \infty)^n \to X$; here and hereafter, $(X, \|\cdot\|)$ denotes a complex Banach space. After that, we introduce the multidimensional generalized Hilfer fractional discrete derivative $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u$, for any sequence $u : \mathbb{N}_0^n \to X$. It seems that the notion introduced in this section is not considered elsewhere in the existing literature, even for the Riemann–Liouville or Caputo fractional derivatives.

Section 3, which is broken down into two subsections, examines the multidimensional generalized Weyl fractional derivatives and differences. The first subsection investigates the generalized Weyl fractional derivatives and differences in the one-dimensional setting. In Definition 3, we introduce the notion of a generalized Weyl fractional derivative $D_W^{\alpha,a} u$ of function $u(\cdot)$. After that, we examine the basic structural properties of the introduced fractional derivatives. If $a : \mathbb{N}_0 \to \mathbb{C}$ and $f : \mathbb{Z} \to X$ are given sequences, then we define the Weyl fractional difference operator $\Delta_{W,a,m} f$. We show that the approach of R. Hilfer [3] is meaningless for the definitions of Weyl fractional derivatives and differences.

The second subsection investigates the generalized Weyl fractional derivatives and differences in the multidimensional setting (concerning some predecessors of this work, we would like to mention here the research articles [26] by V. B. L. Chaurasia and R. S. Dubey, [27] by S. P. Goyal and Trilok Mathur, [28] by B. B. Jaimini and H. Nagar and [29] by R. K. Raina; see also the lists of references quoted therein). We first introduce the notion of a generalized Weyl $(\alpha, \mathbf{a})$-fractional derivative $\mathbb{D}_W^{\alpha,\mathbf{a}} u$; a very special case of the partial fractional derivative $\mathbb{D}_W^{\alpha,\mathbf{a}} u$ is the generalized Weyl $(\alpha, \mathbf{a})$-fractional derivative $\mathbb{D}_W^{\alpha} u$. After that, if the sequences $a_j : \mathbb{N}_0 \to \mathbb{C}$ and $u : \mathbb{Z}^n \to X$ are given and $m_j \in \mathbb{N}$ is a given integer $(1 \leq j \leq n)$, then we introduce the multidimensional Weyl fractional difference operator $\mathbb{D}_{W,\mathbf{a},\mathbf{m}} u$. We investigate the law of exponents for generalized Weyl derivatives and integrals and provide an interesting open problem about the generation of C-regularized solution operator families by the Weyl fractional differential operators with constant coefficients. Furthermore, we reconsider the well-known Clairaut's theorem on equality of mixed partial derivatives (sometimes also called Schwartz's theorem or Young's theorem) in the fractional setting and prove that it is not valid for the Riemann–Liouville and Caputo fractional derivatives (see [7], p. 342) for the first results established in this direction) as well as that it is valid for the Weyl fractional derivatives under certain reasonable assumptions.

In Section 4, we introduce and analyze the partial fractional derivatives of functions defined on some special regions in $\mathbb{R}^n$ and the partial fractional differences of sequences defined on some special subsets of $\mathbb{Z}^n$ (we tried to furnish an illustrative example for each partial fractional derivative introduced in this paper; unfortunately, in the present situation, we cannot precisely explain the physical meaning for each partial fractional derivative introduced here). Further on, the investigation of two-dimensional scalar-valued Laplace transform starts probably with the works of D. L. Bernstein [30,31] and J. C. Jaeger [32] (1939–1941); for more details about the multidimensional scalar-valued Laplace transform and its applications to (fractional) partial integro-differential equations, we refer the reader to the research articles [33–38] and the doctoral dissertations [39–41]. For the purpose of our investigations of the partial fractional integro-differential inclusions, we provide the basic details and results about the multidimensional vector-valued Laplace transform in Section 5 (we will systematically analyze multidimensional vector-valued Laplace transform elsewhere). Our main structural result established in this section is Theorem 1, where we clarify the complex inversion theorem for the multidimensional vector-valued Laplace transform.

The fractional partial differential inclusions with Riemann–Liouville and Caputo derivatives are investigated in Section 6.1, whose main results are Theorems 2 and 3 (cf. also Remarks 3 and 4); Section 6.2, whose main result is Theorem 4, investigates the abstract multiterm fractional partial differential equations with Riemann–Liouville and Caputo derivatives, while Section 6.3 investigates the fractional partial difference equations with generalized Weyl derivatives. Many other types of fractional partial differential-difference equations will be considered in [12].

We introduce many new types of partial fractional derivatives in this paper. Before fixing the notation and explaining some preliminaries, we would like to emphasize that it is our duty to say that the motivation behind our innovations is still not sufficiently explained as well as that future research studies should shed a light on these new concepts.

Notation and terminology. In the sequel, we will always assume that $m,\ n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a complex Banach space, $L(X)$ is the Banach space of all bounded linear operators on X and $C \in L(X)$; $\mathbb{N}_n := \{1, ..., n\}$, $\mathbb{N}_0^n := \{0, 1, ..., n\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(n \in \mathbb{N};\ s \in \mathbb{R})$. If A is a closed linear operator on X, then $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm. The finite convolution product $*_0$ of the Lebesgue measurable functions $a(\cdot)$ and $b(\cdot)$ defined on $[0, \infty)$ is given by $(a *_0 b)(t) := \int_0^t a(t-s)b(s)\,ds$, $t \geq 0$; if the sequences $(a_k)_{k\in\mathbb{N}_0}$ and $(b_k)_{k\in\mathbb{N}_0}$ are given, then we define $(a *_0 b)(\cdot)$ by $(a *_0 b)(k) := \sum_{j=0}^k a_{k-j} b_j$, $k \in \mathbb{N}_0$. If A and B are non-empty sets, then we define $B^A := \{f \,|\, f : A \to B\}$. By $\Gamma(\cdot)$ we denote the Euler Gamma function; we set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$ and $g_0(t) := \delta(t)$, the Dirac δ-distribution. If $\alpha \in (0, \pi]$, then

we define $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$; further on, if $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set, then $L^1_{loc}(\Omega)$ denotes the space of all locally integrable complex-valued functions defined on Ω. For more details about the multivalued linear operators, we refer the reader to [5]; we will use the same terminology as in this monograph.

If $u \in L^1_{loc}([0,\infty)^n)$, $j \in \mathbb{N}_n$ and $\alpha_j > 0$, then we define

$$J^{\alpha_j}_{t_j} u(x_1,...,x_{j-1},x_j,x_{j+1},...,x_n) := \int_0^{x_j} g_{\alpha_j}(x_j - s)\, u(x_1,...,x_{j-1},s,x_{j+1},...,x_n)\, ds,$$
$$\mathbf{x} = (x_1,...,x_{j-1},x_j,x_{j+1},...,x_n) \in [0,\infty)^n.$$

If $\alpha > 0$, then the Cesàro sequence $(k^\alpha(v))_{v\in\mathbb{N}_0}$ is defined by

$$k^\alpha(v) := \frac{\Gamma(v+\alpha)}{\Gamma(\alpha)v!}.$$

It is well-known that for every $\alpha > 0$ and $\beta > 0$, we have $k^\alpha *_0 k^\beta \equiv k^{\alpha+\beta}$. Define $k^0(0) := 1$ and $k^0(v) := 0$, $v \in \mathbb{N}$; then $k^\alpha *_0 k^\beta \equiv k^{\alpha+\beta}$ for all $\alpha,\ \beta \geq 0$.

If (u_k) is a one-dimensional sequence in X, then the Euler forward difference operator Δ is defined by $\Delta u_k := u_{k+1} - u_k$. The operator Δ^m is defined inductively; then, for every integer $m \geq 1$, we have

$$\Delta^m u_k = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} u_{k+j}.$$

If A and B are non-empty sets, then we define $B^A := \{f \mid f : A \to B\}$.

If $\mathrm{j} = (j_1,...,j_n) \in \mathbb{N}_0^n$ and $\mathrm{k} = (k_1,...,k_n) \in \mathbb{N}_0^n$, then we write $\mathrm{j} \leq \mathrm{k}$ if and only if $j_m \leq k_m$ for all $1 \leq m \leq n$. If the sequences $(a_\mathrm{k})_{\mathrm{k}\in\mathbb{N}_0^n}$ and $(b_\mathrm{k})_{\mathrm{k}\in\mathbb{N}_0^n}$ are given, then we define $(a *_0 b)(\cdot)$ by

$$(a *_0 b)(\mathrm{k}) := \sum_{j\in\mathbb{N}_0^n; j\leq \mathrm{k}} a_{\mathrm{k}-j} b_j,\ \mathrm{k} \in \mathbb{N}_0^n;$$

and it can be simply proved that the convolution product $*_0$ is commutative and associative. If the sequences $(a_\mathrm{k})_{\mathrm{k}\in\mathbb{N}_0^n}$ and $(b_\mathrm{k})_{\mathrm{k}\in\mathbb{Z}^n}$ are given, then we define the Weyl convolution product $(a \circ b)(\cdot)$ by

$$(a \circ b)(\mathrm{v}) := \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}} a(\mathrm{v}-l)b(l), \quad v \in \mathbb{Z}^n,$$

whenever the last series is absolutely convergent.

Finally, if $a(\cdot)$ is a given sequence in X which depends on the variables $v_1,...,v_n$, then we define

$$\Delta_{v_i} a(v_1,...,v_i,...,v_n) := a(v_1,...,v_i+1,...,v_n) - a(v_1,...,v_i,...,v_n).$$

After that, we set $\Delta^2_{v_iv_j} a := \Delta_{v_i}\Delta_{v_j} a$ and $\Delta^2_{v_iv_i} a := \Delta_{v_i}\Delta_{v_i} a$; the terms

$$\Delta^m_{v_{i_1}...v_{i_m}} a \text{ and } \Delta^{|\alpha|}_{v_1^{\alpha_1}\cdot....\cdot v_n^{\alpha_n}} a$$

are defined recursively, as for the partial derivatives of functions ($\alpha_i \in \mathbb{N}_0$; $|\alpha| = \alpha_1 + ... + \alpha_n$). It is worth noting that for every permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, we have

$$\Delta^{|\alpha|}_{v_1^{\alpha_1}\cdot....\cdot v_n^{\alpha_n}} a = \Delta^{|\alpha|}_{v_{\sigma(1)}^{\alpha_{\sigma(1)}}\cdot....\cdot v_{\sigma(n)}^{\alpha_{\sigma(n)}}} a, \tag{2}$$

as easily proved. Many other important results of mathematical analysis, like Green's formula in the plane and the Grönwall inequality, have analogues for the difference operators; see [42] (pp. 23–25, 43–44) for more details in this direction.

1.1. Generalized Hilfer Fractional Derivatives and Differences

If $\delta(t)$ denotes the Dirac delta distribution, then we accept the formal convention $\int_0^t \delta(t-s)f(s)\,ds \equiv f(t)$. Suppose now that $u : [0,\infty) \to X$ is locally integrable, $\alpha > 0$, $m = \lceil \alpha \rceil$, $a \in L^1_{loc}([0,\infty))$ or $a(t) = \delta(t)$, and $b \in L^1_{loc}([0,\infty))$ or $b(t) = \delta(t)$. Set

$$v_a(t) := \int_0^t a(t-s)u(s)\,ds, \quad t \geq 0.$$

The following extension of the usual Hilfer fractional derivative $D_t^{\alpha,\beta}u(t)$, when $a(t) = g_{(1-\beta)(m-\alpha)}(t)$ and $b(t) = g_{\beta(m-\alpha)}(t)$ for some $\beta \in [0,1]$, was recently introduced in [43] (for $\beta = 0$, resp. $\beta = 1$, we obtain the usual Riemann–Liouville fractional derivative $D_R^\alpha u$ of order α, resp., the Caputo fractional derivative $\mathbf{D}_C^\alpha u$ of order α).

Definition 1. *The generalized Hilfer (a,b,α)-fractional derivative of function $u(\cdot)$, denoted shortly by $D_{a,b}^\alpha u$, is defined for any locally integrable function $u(\cdot)$ such that the function $v_a^{(m-1)}(t)$ is locally absolutely continuous for $t \geq 0$, by*

$$D_{a,b}^\alpha u(t) := \left(b *_0 v_a^{(m)}\right)(t) = \left(b *_0 \left(a *_0 u\right)^{(m)}\right)(t), \quad \text{a.e. } t \geq 0. \tag{3}$$

Suppose now that $u : \mathbb{N}_0 \to X$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $a : \mathbb{N}_0 \to \mathbb{C}$ and $b : \mathbb{N}_0 \to \mathbb{C}$. The following is a discrete version of the notion considered above (cf. [44] (Definition 3.1)).

Definition 2. *The generalized Hilfer (a,b,α)-fractional derivative of sequence $u(\cdot)$, denoted shortly by $D_{a,b}^\alpha u$, is defined by*

$$D_{a,b}^\alpha u(v) := \left(b *_0 \Delta^m \left(a *_0 u\right)\right)(v), \quad v \in \mathbb{N}_0.$$

If $0 \leq \beta \leq 1$, then the usual Hilfer fractional derivative $D^{\alpha,\beta}u$ of order α and type β is defined as the generalized Hilfer (a,b,α)-fractional derivative of $u(\cdot)$, with $a(v) = k^{(1-\beta)(m-\alpha)}(v)$ and $b(v) = k^{\beta(1-\alpha)}(v)$.

In both cases, the continuous one and the discrete one, we define

$$D_{a,b}^0 u := a *_0 b *_0 u.$$

2. Multidimensional Generalized Hilfer Fractional Derivatives and Differences

Suppose that $0 < T_j < +\infty$ and $I_j = [0,T_j)$, $I_j = [0,T_j]$ or $I_j = [0,+\infty)$ for $1 \leq j \leq n$. Set $I := I_1 \times I_2 \times \ldots \times I_n$. Suppose that $u : I \to X$ is a locally integrable function, and for every $j \in \mathbb{N}_n$, $a_j \in L^1_{loc}(I_j)$ or $a_j(t) = \delta(t)$, and $b_j \in L^1_{loc}(I_j)$ or $b_j(t) = \delta(t)$. Suppose further that $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$. Define $\alpha := (\alpha_1, \ldots, \alpha_n)$ and

$$\mathbb{D}_{\mathbf{a},\mathbf{b}}^\alpha u(x_1,\ldots,x_n) := \left[D_{a_1,b_1}^{\alpha_1}\left(D_{a_2,b_2}^{\alpha_2}\left(\ldots\left(D_{a_n,b_n}^{\alpha_n}u(\cdot,\ldots,\cdot)\right)\ldots\right)\right)\right](x_1,\ldots,x_n), \tag{4}$$

for a.e. $(x_1,\ldots,x_n) \in I$, provided that the right-hand side of (4) is well-defined. Here, we assume that the variables $x_1,\ x_2,\ldots,x_{n-1}$ are fixed in the computation of the term $D_{a_n,b_n}^{\alpha_n}u(x_1,\ldots,x_n),\ldots$, as well as that the variables $x_2,\ x_3,\ldots,x_n$ are fixed in the computation of the final term on the right-hand side of (4). We call $\mathbb{D}_{\mathbf{a},\mathbf{b}}^\alpha u$ the multidimensional generalized Hilfer $(\mathbf{a},\mathbf{b},\alpha)$-fractional derivative of the function $u(\cdot)$. If for each $j \in \mathbb{N}_n$, we have $D_{a_j,b_j}^{\alpha_j} = D_R^{\alpha_j}$, resp., for each $j \in \mathbb{N}_n$ we have $D_{a_j,b_j}^{\alpha_j} = \mathbf{D}_C^{\alpha_j}$, then the corresponding partial

fractional derivative $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}}$ is called the multidimensional Riemann–Liouville fractional operator (cf. also [7] (pp. 340–342)), resp., the multidimensional Caputo fractional operator, and it is denoted by $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} = \mathbb{D}^{\alpha}_{R}$, resp. $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} = \mathbb{D}^{\alpha}_{C}$.

In the discrete setting, we assume that $u : \mathbb{N}_0^n \to X$, $a_j : \mathbb{N}_0 \to \mathbb{C}$ and $b_j : \mathbb{N}_0 \to \mathbb{C}$ are given sequences $(1 \leq j \leq n)$. We define

$$\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u(v_1, ..., v_n) := \left[D^{\alpha_1}_{a_1,b_1} \left(D^{\alpha_2}_{a_2,b_2} \left(... \left(D^{\alpha_n}_{a_n,b_n} u(\cdot, ..., \cdot) \right) ... \right) \right) \right] (v_1, ..., v_n), \tag{5}$$

for any $(v_1, ..., v_n) \in \mathbb{N}_0^n$; note that the right-hand side of (5) is always well-defined. We call $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u$ the multidimensional generalized Hilfer $(\mathbf{a}, \mathbf{b}, \alpha)$-fractional derivative of the sequence $u(\cdot)$; the multidimensional Riemann–Liouville fractional difference operator $\mathbb{D}^{\alpha}_{R}$ and the multidimensional Caputo fractional difference operator $\mathbb{D}^{\alpha}_{C}$ are defined similarly.

We continue by providing certain illustrative examples.

Example 1.

(i) Suppose that $\emptyset \neq D \subseteq [0, +\infty)^n$ is a finite set, $c_\beta \in \mathbb{C}$ for all $\beta = (\beta_1, ..., \beta_n) \in D$ and

$$u(x_1, ..., x_n) := \sum_{\beta \in D} c_\beta g_{\beta_1}(x_1) \cdot ... \cdot g_{\beta_n}(x_n), \quad x_1 \geq 0, \ ..., \ x_n \geq 0.$$

Suppose further that $\alpha_j \geq 0$, $a_j(t) = g_{\gamma_j}(t)$ and $b_j(t) = g_{\delta_j}(t)$ for some non-negative numbers $\gamma_j \geq 0$ and $\delta_j \geq 0$ such that $\gamma_j + \beta_j \geq m_j$ $(1 \leq j \leq n)$. Set $f_j(t) := g_{\delta_j+\gamma_j+\beta_j-m_j}(t)$, $t > 0$, if $\gamma_j + \beta_j > m_j$ and $f_j(t) := 0$, $t \geq 0$, if $\gamma_j + \beta_j = m_j$. Then we have

$$\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u(x_1, ..., x_n) = \sum_{\beta \in D} c_\beta f_1(x_1) \cdot ... \cdot f_n(x_n), \quad x_1 \geq 0, \ ..., \ x_n \geq 0.$$

This formula enables one to clarify a great number of various partial fractional differential equations which do have the function $u(x_1, ..., x_n)$ as its solution; for example, we have

$$\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u(x_1, ..., x_n) = \left[\sum_{\beta \in D} c_\beta \frac{x_1^{\delta_1+\gamma_1-m_1}}{\Gamma(\delta_1+\beta_1+\gamma_1-m_1)} \cdot .... \cdot \frac{x_n^{\delta_n+\gamma_n-m_n}}{\Gamma(\delta_n+\beta_n+\gamma_n-m_n)} \right] \cdot u(x_1, ..., x_n),$$

for any $x_1 \geq 0, ..., x_n \geq 0$, provided that $\delta_j + \gamma_j > m_j$ for $1 \leq j \leq n$.

(ii) Suppose that $\emptyset \neq D \subseteq [0, +\infty)^n$ is a finite set, $c_\beta \in \mathbb{C}$ for all $\beta = (\beta_1, ..., \beta_n) \in D$ and

$$u(v_1, ..., v_n) := \sum_{\beta \in D} c_\beta k^{\beta_1}(v_1) \cdot ... \cdot k_{\beta_n}(v_n), \quad v_1 \in \mathbb{N}_0, \ ..., \ v_n \in \mathbb{N}_0.$$

Suppose further that $\alpha_j \geq 0$, $a_j(v) = k^{\gamma_j}(v)$ and $b_j(v) = k^{\delta_j}(v)$ for some non-negative numbers $\gamma_j \geq 0$ and $\delta_j \geq 0$ such that $\gamma_j + \beta_j \geq m_j$ $(1 \leq j \leq n)$. Set

$$f_j(v) := k^{\delta_j+\gamma_j+\beta_j-m_j}(v + m_j) - \sum_{l=v+1}^{v+m_j} k^{\gamma_j+\beta_j-m_j}(v + m_j - l) k^{\delta_j}(l), \quad v \in \mathbb{N}_0.$$

We know that (see [45] (Example 3)):

$$\Delta^{\alpha} k^{\beta}(\cdot) = k^{\beta-\alpha}(\cdot + \lceil \alpha \rceil), \quad \beta \geq \alpha > 0.$$

This simply implies

$$\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}}u(v_1,...,v_n)=\sum_{\beta\in D}c_{\beta}f_1(v_1)\cdot...\cdot f_n(v_n),\quad v_1\in\mathbb{N}_0,\ ...,\ v_n\in\mathbb{N}_0.$$

Remark 1.

(i) *Instead of the generalized Hilfer fractional derivatives and differences, we can consider here any other type of fractional derivatives of functions defined on the segment of the non-negative real axis ([46]). In such a way, we can extend the notion considered in this section and obtain much more general forms of the partial fractional derivatives.*

(ii) *It is well-known that the composition of the Riemann–Liouville (Caputo) fractional derivatives of orders $\alpha>0$ and $\beta>0$ is not the Riemann–Liouville (Caputo) fractional derivative of order $\alpha+\beta$; see [6] (Sections 2.3.5 and 2.3.6) for more details. We can further extend the notion of fractional derivative $\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}}u$ by replacing some terms $D^{\alpha_j}_{a_j,b_j}$ in its definition by the finite compositions $D^{\alpha_{j_1}}_{a_{j_1},b_{j_1}}D^{\alpha_{j_2}}_{a_{j_2},b_{j_2}}\cdot...\cdot D^{\alpha_{j_s}}_{a_{j_s},b_{j_s}}$ of terms with respect to the variable x_j $(1\leq j\leq n)$.*

Let us recall that Clairaut's theorem on equality of mixed partial derivatives states that if a function $u:\Omega\to\mathbb{R}$ defined on a non-empty set $\Omega\subseteq\mathbb{R}^n$ is given, as well as that $x\in\mathbb{R}^n$ is a point such that some neighborhood $O(x)$ of it belongs to Ω, and $u(\cdot,\cdot)$ has continuous second partial derivatives on $O(x)$, then we have

$$\frac{\partial^2u}{\partial x_i\partial x_j}(x)=\frac{\partial^2u}{\partial x_j\partial x_i}(x).$$

This equality cannot be so easily interpreted for the generalized Hilfer partial fractional derivatives, because the equality

$$\left[D^{\alpha_1}_{a_1,b_1}\left(D^{\alpha_2}_{a_2,b_2}u\right)\right](x_1,x_2)=\left[D^{\alpha_2}_{a_2,b_2}\left(D^{\alpha_1}_{a_1,b_1}u\right)\right](x_1,x_2),\tag{6}$$

is not true, in general (of course, it is true in the case that $b=d$, $a=c$ and $m_1=m_2$, at least almost everywhere). The Formula (6) does not hold even for the Riemann–Liouville fractional derivatives and the Caputo fractional derivatives, as the following simple counterexample shows.

Example 2. *Suppose that $0<\alpha_1<1$, $0<\alpha_2<1$ and $\alpha_1\neq\alpha_2$. Let us consider the Caputo approach, in which $a_1(t)=a_2(t)=\delta(t)$, $b_1(t)=g_{1-\alpha_1}(t)$, $b_2(t)=g_{1-\alpha_2}(t)$ and $m_1=m_2=1$. Then a simple computation shows that the equality (6) is equivalent with*

$$\begin{aligned}&\int_0^{x_1}g_{1-\alpha_2}(x_1-r)\frac{d}{dr}\int_0^{x_2}g_{1-\alpha_1}(x_2-l)\frac{\partial u}{\partial l}u(r,l)\,dl\,dr\\&=\int_0^{x_1}g_{1-\alpha_1}(x_1-r)\frac{d}{dr}\int_0^{x_2}g_{1-\alpha_2}(x_2-l)\frac{\partial u}{\partial l}u(r,l)\,dl\,dr.\end{aligned}\tag{7}$$

Take now $u(x_1,x_2):=x_1x_2$ for $x_1\geq0$ and $x_2\geq0$. Then (7) is equivalent with

$$g_{2-\alpha_2}(x_1)\cdot g_{2-\alpha_1}(x_2)=g_{2-\alpha_1}(x_1)\cdot g_{2-\alpha_2}(x_2),$$

which is wrong. In the discrete setting, we cannot expect the validity of nontrivial fractional analogues of Equation (2).

We continue with the observation that the formulae [1] (1.13, 1.21) can be straightforwardly extended to the multidimensional setting. For example, if $u \in L^1_{loc}([0,\infty)^n)$ and $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$, then we have

$$D_R^{\alpha_1} D_R^{\alpha_2} \cdot ... \cdot D_R^{\alpha_n} J_{t_n}^{\alpha_n} \cdot ... \cdot J_{t_2}^{\alpha_2} J_{t_1}^{\alpha_1} u = u$$

and

$$\mathbf{D}_C^{\alpha_1} \mathbf{D}_C^{\alpha_2} \cdot ... \cdot \mathbf{D}_C^{\alpha_n} J_{t_n}^{\alpha_n} \cdot ... \cdot J_{t_2}^{\alpha_2} J_{t_1}^{\alpha_1} u = u,$$

with the meaning clear. The situation is a little bit complicated if we consider the second formulae in Equations (1.13) and (1.21) from [1]; for example, in the two-dimensional setting, we have

$$\begin{aligned} &J_{t_2}^{\alpha_2} J_{t_1}^{\alpha_1} D_R^{\alpha_1} D_R^{\alpha_2} u(x_1,x_2) \\ &\quad = u(x_1,x_2) - \sum_{k=0}^{m_2-1} \frac{\partial^k}{\partial x_2^k} \left[J_{t_2}^{m_2-\alpha_2} *_0 u \right](x_1,0) \cdot g_{\alpha_2+k+1-m_2}(x_2) \\ &\quad - \sum_{k=0}^{m_1-1} \left\{ \int_0^{x_2} g_{\alpha_2}(x_2-s) \left[\frac{\partial^k}{\partial x_1^k} \left[J_{t_1}^{m_1-\alpha_1} *_0 D_R^{\alpha_2} u \right](x_1,x_2) \right]_{x_1=0, x_2=s} ds \right\} \cdot g_{\alpha_1+k+1-m_1}(x_1), \end{aligned} \tag{8}$$

for any $(x_1,x_2) \in [0,\infty)^2$, provided that $u \in L^1_{loc}([0,\infty)^2)$, $m_1 = \lceil \alpha_1 \rceil$, $m_2 = \lceil \alpha_2 \rceil$, for each $x_2 \geq 0$ the function $x_1 \mapsto D_R^{\alpha_2} u(x_1,x_2)$, $x_1 > 0$ is locally integrable and satisfies $J_{t_1}^{m_1-\alpha_1} *_0 D_R^{\alpha_2} u \in W^{m_1,1}_{loc}([0,\infty) : X)$, and for each $x_1 \geq 0$ we have $J_{t_2}^{m_2-\alpha_2} *_0 (\partial^{m_2-1}/\partial x_2^{m_2-1}) u \in W^{m_2,1}_{loc}([0,\infty) : X)$, as well as

$$\begin{aligned} J_{t_2}^{\alpha_2} J_{t_1}^{\alpha_1} \mathbf{D}_C^{\alpha_1} \mathbf{D}_C^{\alpha_2} u(x_1,x_2) &= u(x_1,x_2) - \sum_{k=0}^{m_2-1} \left[\frac{\partial^k}{\partial x_2^k} u(x_1,0) \right] \cdot g_{k+1}(x_2) \\ &\quad - \sum_{k=0}^{m_1-1} \left[\int_0^{x_2} g_{\alpha_2}(x_2-s) \left[\frac{\partial^k}{\partial x_1^k} \mathbf{D}_C^{\alpha_2} u(x_1,x_2) \right]_{x_1=0, x_2=s} ds \right] \cdot g_{k+1}(x_1), \end{aligned} \tag{9}$$

for any $(x_1,x_2) \in [0,\infty)^2$, provided that $m_1 = \lceil \alpha_1 \rceil$, $m_2 = \lceil \alpha_2 \rceil$, $u \in L^1_{loc}([0,\infty)^2)$, for each $x_2 \geq 0$ the function $x_1 \mapsto f(x_1) := (\partial^{m_1-1}/\partial x_1^{m_1-1}) \mathbf{D}_C^{\alpha_2} u(x_1,x_2)$, $x_1 \geq 0$ is continuous, $g_{m_1-\alpha_1} *_0 f \in W^{m_1,1}_{loc}([0,\infty) : X)$, for each $x_1 \geq 0$ the function $x_2 \mapsto g(x_2) := (\partial^{m_2-1}/\partial x_2^{m_2-1}) u(x_1,x_2)$, $x_2 \geq 0$ is continuous and $g_{m_2-\alpha_2} *_0 g \in W^{m_2,1}_{loc}([0,\infty) : X)$. Here, $W^{m_1,1}_{loc}([0,\infty) : X)$ and $W^{m_2,1}_{loc}([0,\infty) : X)$ denote the usual Sobolev spaces; cf. [1] for the notation used.

3. Multidimensional Generalized Weyl Fractional Derivatives and Differences

In this section, which consists of two separate subsections, we investigate the multidimensional generalized Weyl fractional derivatives and differences.

3.1. Generalized Weyl Fractional Derivatives and Differences

If $u : \mathbb{R} \to X$ is a locally integrable function, $\alpha \geq 0$ and $m = \lceil \alpha \rceil$, then the Weyl fractional derivative $D_W^{\alpha} u$ of function $u(\cdot)$ of order α is well-defined if the mapping $x \mapsto \int_{-\infty}^{x} g_{m-\alpha}(x-s) u(s)\, ds$, $x \in \mathbb{R}$ is well-defined and m-times continuously differentiable, by

$$\left[D_W^{\alpha} u \right](x) := \frac{d^m}{dx^m} \int_{-\infty}^{x} g_{m-\alpha}(x-s) u(s)\, ds, \quad x \in \mathbb{R};$$

cf. [47] for more details. Now we would like to propose the following notion.

Definition 3. *Suppose that $a \in L^1_{loc}([0,\infty))$, $u : \mathbb{R} \to X$ is a locally integrable function, $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. The generalized Weyl fractional derivative $D^{\alpha,a}_W u$ of function $u(\cdot)$ is well-defined if the mapping $x \mapsto \int_{-\infty}^{x} a(x-s)u(s)\,ds$, $x \in \mathbb{R}$ is well-defined and m-times continuously differentiable, by*

$$\left[D^{\alpha,a}_W u\right](x) := \frac{d^m}{dx^m}\int_{-\infty}^{x} a(x-s)u(s)\,ds, \quad x \in \mathbb{R}.$$

We call the function $x \mapsto I_{W,a}(x) := \int_{-\infty}^{x} a(x-s)u(s)\,ds$, $x \in \mathbb{R}$, if it is well-defined, the generalized Weyl a-integral of function $u(\cdot)$. If $a(t) = g_\zeta(t)$ for some $\zeta \in (0,1)$, then the class of functions for which the above integral absolutely converges and behaves nicely was considered for the first time by M. J. Lighthill in [48], where it was called the class of "good functions". In the general case, we have

$$\int_{-\infty}^{x} a(x-s)u(s)\,ds = \int_{0}^{+\infty} a(s)u(x-s)\,ds, \quad x \in \mathbb{R}$$

and the dominated convergence theorem implies

$$\frac{d}{dx^n}\int_{-\infty}^{x} a(x-s)u(s)\,ds = \int_{0}^{+\infty} a(s)u^{(n)}(x-s)\,ds, \quad x \in \mathbb{R},\ n \in \mathbb{N},$$

provided that there exists $m \in \mathbb{N}$ such that $\int_0^{+\infty} |a(s)|(1+s)^{-m}\,ds < +\infty$, and the function $u(\cdot)$ and all its derivatives are differentiable almost everywhere and for each $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0$ there exists a finite real number $M_{n,\alpha} \geq 1$ such that $\|u^{(\alpha)}(x)\| \leq M_{n,\alpha}(1+|x|)^{-n}$, $x \in \mathbb{R}$; we call such functions "vector-valued good functions" and denote the corresponding class by $\mathbf{S}(X)$. If (G) holds, where

(G) There exists an integer $m \in \mathbb{N}$ such that $\int_0^{+\infty} |a(s)|(1+s)^{-m}\,ds < +\infty$ and $\int_0^{+\infty} |b(s)|(1+s)^{-m}\,ds < +\infty$,

then we can repeat verbatim the argumentation from [47] (Section 3, pp. 239–240) in order to see that the law of exponents for generalized Weyl integrals holds true:

$$I_{W,a}I_{w,b}u = I_{W,a*_0 b}u, \quad u \in \mathbf{S}(X); \tag{10}$$

Here, we we will only note that the Dirichlet integral formula given on [47] (p. 239, l.-7–l.-4) in our new framework takes the form

$$\int_t^w a(x-t)\left[\int_x^w b(s-x)f(s)\,ds\right]dx = \int_t^w \left(a *_0 b\right)(s-t)f(s)\,ds,$$

which follows from an elementary change of variables in the double integral. Furthermore, if (G) holds, then we can repeat verbatim the argumentation from [47] (Section 4, pp. 240–244) in order to see that the law of exponents for generalized Weyl derivatives holds true:

$$D^{\alpha,a}_W D^{\beta,b}_W u = D^{\lceil\alpha\rceil+\lceil\beta\rceil,a*_0 b}_W u, \quad u \in \mathbf{S}(X). \tag{11}$$

In connection with the above issue, we would like to note that the approach of R. Hilfer is insignificant for the definitions of Weyl fractional derivatives introduced above. Without going into full detail, we will only note here that the following formula holds true:

$$\int_{-\infty}^{x} b(x-s)\frac{d^m}{ds^m}\int_{-\infty}^{s} a(s-r)u(r)\,dr\,ds = \frac{d^m}{dx^m}\int_{-\infty}^{x}\left(a *_0 b\right)(x-s)u(s)\,ds, \ x \in \mathbb{R}, \tag{12}$$

provided that $u \in \mathbf{S}(X)$ and (G) holds; furthermore, the assumption $u \in \mathbf{S}(X)$ can be slightly relaxed and all abovementioned statements can be slightly generalized keeping in mind the concrete value of integer $m \in \mathbb{N}$ satisfying (G); details can be left to interested readers.

Suppose now that $a : \mathbb{N}_0 \to \mathbb{C}$ and $f : \mathbb{Z} \to X$ are given sequences. If the series $\sum_{s=0}^{+\infty} a(s)f(v-s)$ is absolutely convergent for all $v \in \mathbb{Z}$, then we define

$$\Big(\Delta_{W,a}f\Big)(v) := \sum_{s=-\infty}^{v} a(v-s)f(s) = \sum_{s=0}^{+\infty} a(s)f(v-s), \quad v \in \mathbb{Z}. \tag{13}$$

Assume that the sequence $\Delta_{W,a}f : \mathbb{Z} \to X$ is well-defined and $m \in \mathbb{N}$. Then we put

$$\Big(\Delta_{W,a,m}f\Big)(v) := \Big(\Delta^m \Delta_{W,a}f\Big)(v), \quad v \in \mathbb{Z}.$$

It is worth noting that if $m = \lceil \alpha \rceil$ and $a \equiv k^{m-\alpha}$ for some $\alpha > 0$, then the operator $\Delta_{a,m}$ reduces to the Weyl fractional derivative $D_W^{\alpha}f$ of sequence $f(\cdot)$ of order α; cf. [49] (Definition 2.3). Because of that, we will call the sequence $\Delta_{W,a,m}f$ the generalized Weyl (a,m)-fractional derivative of sequence $f(\cdot)$.

Concerning the discrete counterpart of Formula (12), let us first define ($0 \leq \beta \leq 1$; $b : \mathbb{N}_0 \to \mathbb{C}$)

$$\Delta_W^{\alpha,\beta}f := \Delta^{\beta(m-\alpha)}\Delta^m\Delta^{(1-\beta)(m-\alpha)}f \quad \text{and} \quad \Delta_{a,b}f := \Delta_a\Delta^m\Delta_b f.$$

Then, under certain logical assumptions, we have the following (the multidimensional analogues of these formulae can be also achieved):

$$\Delta_W^{\alpha,\beta}f = D_W^{\alpha}f \quad \text{and} \quad \Delta_{a,b}f = \Delta_{a*_0 b}f. \tag{14}$$

Both formulae can be proved in the same manner, with the help of the discrete Fubini theorem and the result established in [50] (Theorem 3.12(ii), (iii)). For the sake of brevity, we will prove here the first formula in (14) only, extending thus the result established in [49] (Remark 2.4):

$$\begin{aligned}
\Big[\Delta_W^{\alpha,\beta}f\Big](v) &= \sum_{s=-\infty}^{v} k^{\beta(m-\alpha)}(v-s)\Big[\Delta^m\Delta^{(1-\beta)(m-\alpha)}f\Big](s)\\
&= \sum_{s=-\infty}^{v} k^{\beta(m-\alpha)}(v-s)\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\Big[\Delta^{(1-\beta)(m-\alpha)}f\Big](s+i)\\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{s=-\infty}^{v+i} k^{\beta(m-\alpha)}(v+i-s)\Big[\Delta^{(1-\beta)(m-\alpha)}f\Big](s)\\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{s=-\infty}^{v+i} k^{\beta(m-\alpha)}(v+i-s)\sum_{l=-\infty}^{s} k^{(1-\beta)(m-\alpha)}(s-l)f(l)\\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{s=-\infty}^{v+i} k^{m-\alpha}(v+i-s)f(s) = \Big[D_W^{\alpha}f\Big](v), \quad v \in \mathbb{Z}.
\end{aligned}$$

3.2. Continuation: Multidimensional Generalized Weyl Fractional Calculus

Suppose now that $a_j \in L^1_{loc}([0,\infty))$ for all $j \in \mathbb{N}_n$, $u : \mathbb{R}^n \to X$ is a locally integrable function and $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$. Define $\alpha := (\alpha_1, ..., \alpha_n)$ and

$$\mathbb{D}_W^{\alpha,\mathbf{a}}u(x_1,...,x_n) := \Big[D_W^{\alpha_1,a_1}\Big(D_W^{\alpha_2,a_2}\Big(...\Big(D_W^{\alpha_n,a_n}u(\cdot,...,\cdot)\Big)...\Big)\Big)\Big](x_1,...,x_n), \tag{15}$$

for a.e. $(x_1,...,x_n) \in \mathbb{R}^n$, provided that the right-hand side of (15) is well-defined. Here, we assume that the variables $x_1,\ x_2, ..., x_{n-1}$ are fixed in the computation of the term $D_W^{\alpha_n,a_n}u(x_1,...,x_n)$, ..., as well as that the variables $x_2,\ x_3, ..., x_n$ are fixed in the computation of the final term on the right-hand side of (15). We call $\mathbb{D}_W^{\alpha,\mathbf{a}}u$ the multidimensional generalized Weyl $(\alpha,\mathbf{a})$-fractional derivative of the function $u(\cdot)$. If $a_j \equiv g_{m_j-\alpha_j}$, where

$m_j = \lceil \alpha_j \rceil$ for all $j \in \mathbb{N}_n$, then we call $\mathbb{D}^{\alpha}_{W} u := \mathbb{D}^{\alpha,\mathbf{a}}_{W} u$ the multidimensional generalized Weyl α-fractional derivative of function $u(\cdot)$; cf. also [7] (p. 343) for the scalar-valued version of this notion. We call the function

$$\mathbf{x} \mapsto I_{W,\mathbf{a}}(\mathbf{x}) := \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdot ... \cdot \int_{-\infty}^{x_n} a_1(x_1 - s_1) a_2(x_2 - s_2) \cdot ... \cdot a_n(x_n - s_n) \times u(s_1, s_2, ..., s_n)\, ds_1\, ds_2 \cdot ... \cdot ds_n,\ \mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,$$

if it is well-defined, the generalized Weyl **a**-integral of function $u(\cdot)$.

Suppose now that $u : \mathbb{Z}^n \to X$, $a_j : \mathbb{N}_0 \to \mathbb{C}$ are given sequences and $m_j \in \mathbb{N}$ are given integers ($1 \leq j \leq n$). Then we introduce the following multidimensional fractional difference operator

$$\mathbb{D}_{W,\mathbf{a},\mathbf{m}} u(v_1, ..., v_n) := \left[\Delta_{W,a_1,m_1} \left(\Delta_{W,a_2,m_2} \left(... \left(\Delta_{W,a_n,m_n} u(\cdot, ..., \cdot) \right) ... \right) \right) \right] (v_1, ..., v_n), \tag{16}$$

for any $(v_1, ..., v_n) \in \mathbb{Z}^n$, provided that the right-hand side of (16) is well-defined. We call $\mathbb{D}_{W,\mathbf{a},\mathbf{m}} u$ the generalized multidimensional Weyl $(\mathbf{a}, \mathbf{m})$-fractional derivative of $u(\cdot)$. If $m_j = \lceil \alpha_j \rceil$ and $a_j \equiv k^{m_j - \alpha_j}$ for $1 \leq j \leq n$, then we call $\mathbb{D}_{W,\mathbf{a},\mathbf{m}} u$ the generalized multidimensional Weyl α-fractional derivative of $u(\cdot)$, where $\alpha = (\alpha_1, ..., \alpha_n)$.

Remark 2. *It is clear that in place of the generalized Weyl fractional derivatives and differences, we can consider here any other type of fractional derivatives of functions defined on the whole real axis (see, e.g., [7] (Chapter 5) and [46,51]).*

The formulae [47] ((7.4), (7.6), (7.10), (7.12), (7.13)) can be simply formulated in the multidimensional setting. For example, we have

$$D^{\alpha_1}_{W} D^{\alpha_2}_{W} ... D^{\alpha_n}_{W} e^{a_1 x_1 + a_2 x_2 + ... + a_n x_n} = a_1^{\alpha_1} a_2^{\alpha_2} \cdot ... \cdot a_n^{\alpha_n} e^{a_1 x_1 + a_2 x_2 + ... + a_n x_n}, \tag{17}$$

provided that $a_j > 0$ and $\alpha_j > 0$ for $1 \leq j \leq n$, with the meaning clear.

If all partial derivatives of a function $u : \mathbb{R}^n \to X$ are continuous almost everywhere and for each $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ there exists a finite real number $M_{m,\alpha} \geq 1$ such that $\|u^{(\alpha)}(x)\| \leq M_{m,\alpha}(1 + |x|)^{-n}$, $x \in \mathbb{R}^n$, then we say that $u(\cdot)$ is a vector-valued good function of several variables; the corresponding class of vector-valued good functions will be denoted by $\mathbf{S}_n(X)$ henceforth. If $u \in \mathbf{S}_n(X)$, then the function $I_{W,\mathbf{a}}(\cdot)$ is infinitely differentiable and for each $\alpha \in \mathbb{N}_0^n$ and $\mathbf{x} \in \mathbb{R}^n$ we have

$$I^{(\alpha)}_{W,\mathbf{a}}(\mathbf{x}) = \int_{[0,+\infty)^n} a_1(s_1) a_2(s_2) \cdot ... \cdot a_n(s_n) \frac{\partial^{\alpha} u}{\partial x_1^{\alpha_1} \cdot ... \cdot \partial x_n^{\alpha_n}} (\mathbf{x} - \mathbf{s})\, d\mathbf{s}.$$

Furthermore, if the following condition holds:

(G1) There exists an integer $m \in \mathbb{N}$ such that $\int_0^{+\infty} |a_j(s)|(1+s)^{-m}\, ds < +\infty$ and $\int_0^{+\infty} |b_j(s)|(1+s)^{-m}\, ds < +\infty$ for all $j \in \mathbb{N}_n$,

then we can apply the Fubini theorem and (10) in order to see that the law of exponents for generalized multidimensional Weyl integrals holds true:

$$I_{W,\mathbf{a}} I_{w,\mathbf{b}} u = I_{W,\mathbf{a} *_0 \mathbf{b}} u, \quad u \in \mathbf{S}_n(X), \tag{18}$$

where $\mathbf{a} *_0 \mathbf{b} := (a_1 *_0 b_1, ..., a_n *_0 b_n)$. If (G1) is valid, then the following multidimensional analogue of (11) holds:

$$\mathbb{D}^{\alpha,\mathbf{a}}_{W} \mathbb{D}^{\beta,\mathbf{b}}_{W} u = \mathbb{D}^{\lceil \alpha \rceil + \lceil \beta \rceil, \mathbf{a} *_0 \mathbf{b}}_{W} u, \quad u \in \mathbf{S}_n(X), \tag{19}$$

where $\lceil \alpha \rceil + \lceil \beta \rceil := (\lceil \alpha_1 \rceil + \lceil \beta_1 \rceil, ..., \lceil \alpha_n \rceil + \lceil \beta_n \rceil)$; in particular, we can clarify Clairaut's theorem on equality of mixed partial Weyl fractional derivatives of type (6).

The generation of C-regularized solution operator families in $L^p(\mathbb{R}^n)$ by the Weyl fractional differential operators of the form

$$A = \sum_{\alpha \in D} c_\alpha \mathbb{D}_W^\alpha u,$$

where D is a non-empty subset of $\mathbb{N}_0^n$ and $c_\alpha \in \mathbb{C}$ for all $\alpha \in D$, is a rather nontrivial problem. We will consider this issue elsewhere.

4. Multidimensional Fractional Calculus on Some Special Regions of $\mathbb{R}^n$

Keeping in mind the notion introduced in the previous two sections, we have an open door to consider the partial fractional derivatives of functions defined on the subsets $I \subseteq \mathbb{R}^n$ which have the form $I = I_1 \times I_2 \times ... \times I_n$, where $I_j = [0, T_j)$, $I_j = [0, T_j]$, $I_j = [0, +\infty)$ or $I_j = \mathbb{R}$ for $1 \leq j \leq n$; for example, in the two-dimensional setting, we can consider functions defined on the half-space $I = [0, \infty) \times \mathbb{R}$ or the closed rectangle $[0, T] \times \mathbb{R}$, where $T > 0$.

Suppose that $f : I \to X$ and I has the above form. Suppose, further, that $\alpha_j \geq 0$ for all $j \in \mathbb{N}_n$ and $\alpha = (\alpha_1, ..., \alpha_n)$. We define

$$\mathbb{D}^\alpha u(x_1, ..., x_n) := \left[D^{\alpha_1} \left(D^{\alpha_2} \left(... \left(D^{\alpha_n} u(\cdot, ..., \cdot) \right) ... \right) \right) \right] (x_1, ..., x_n), \tag{20}$$

for a.e. $(x_1, ..., x_n) \in I$, provided that the right-hand side of (20) is well-defined, where $D^{\alpha_j} = D_{a_j, b_j}^{\alpha_j}$ for some $a_j \in L^1_{loc}(I_j)$ or $a_j(t) = \delta(t)$, and $b_j \in L^1_{loc}(I_j)$ or $b_j(t) = \delta(t)$, provided that $I_j = [0, T_j)$, $I_j = [0, T_j]$ or $I_j = [0, +\infty)$, and $D^{\alpha_j} = D_W^{\alpha, a_j}$ with some $a_j \in L^1_{loc}([0, \infty))$, if $I_j = \mathbb{R}$. We will not consider here the partial fractional derivatives of functions defined on some other regions of $\mathbb{R}^n$; for example, it could be interesting to consider the partial fractional derivatives of functions defined on convex polyhedrals in $\mathbb{R}^n$.

In the discrete setting, we will only consider the sets $I \subseteq \mathbb{Z}^n$ which have the form $I = I_1 \times I_2 \times ... \times I_n$, where $I_j = \mathbb{N}_0$ or $I_j = \mathbb{Z}$ for $1 \leq j \leq n$. If I has such a form and $u : I \to X$, then we define the partial fractional derivative $\mathbb{D}^\alpha u(v_1, ..., v_n)$ similar to the continuous setting; for example, in the two-dimensional setting, we can consider sequences defined on the set $I = \mathbb{N}_0 \times \mathbb{Z}$ or $I = \mathbb{Z} \times \mathbb{N}_0$.

We continue by providing the following illustrative example.

Example 3. *Suppose that $n \geq 2$, $\emptyset \neq D \subseteq [0, +\infty)^n$ is a finite set, $c_\beta \in \mathbb{C}$ for all $\beta = (\beta_1, ..., \beta_n) \in D$, $\beta_n > 0$ and*

$$u(x_1, ..., x_n) := \sum_{\beta \in D} c_\beta g_{\beta_1}(x_1) \cdot ... \cdot g_{\beta_{n-1}}(x_{n-1}) e^{\beta_n x_n}, \ x_1 \geq 0, \ ..., \ x_{n-1} \geq 0, \ x_n \in \mathbb{R}.$$

Suppose further that $\alpha_j \geq 0$, $a_j(t) = g_{\gamma_j}(t)$ and $b_j(t) = g_{\delta_j}(t)$ for some non-negative numbers $\gamma_j \geq 0$ and $\delta_j \geq 0$ such that $\gamma_j + \beta_j \geq m_j$ $(1 \leq j \leq n-1)$. Let $D^{\alpha_j} u = D_{a_j, b_j}^{\alpha_j} u$ for $1 \leq j \leq n-1$, and let $D^{\alpha_n} u = D_W^{\alpha_n}$. If we define the functions $f_j(\cdot)$, for $1 \leq j \leq n-1$, as in Example 1(i), then we have

$$\mathbb{D}^\alpha u(x_1, ..., x_n) = \sum_{\beta \in D} c_\beta \beta_n^{\alpha_n} f_1(x_1) \cdot ... \cdot f_{n-1}(x_{n-1}) e^{\beta_n x_n},$$

for any $x_1 \geq 0, \ ..., \ x_{n-1} \geq 0$ and $x_n \in \mathbb{R}$; cf. also (17).

As in Example 1(i), we can construct a great number of various partial fractional differential equations having the function $u(x_1, ..., x_n)$ as their solution; for example, we have

$$\mathbb{D}^{\alpha}_{\mathbf{a},\mathbf{b}} u(x_1, ..., x_n) = \left[\sum_{\beta \in D} c_\beta \beta_n^{\alpha_n} \frac{x_1^{\delta_1+\gamma_1-m_1}}{\Gamma(\delta_1+\beta_1+\gamma_1-m_1)} \cdot .... \cdot \frac{x_{n-1}^{\delta_{n-1}+\gamma_{n-1}-m_{n-1}}}{\Gamma(\delta_{n-1}+\beta_{n-1}+\gamma_{n-1}-m_{n-1})} \right] \cdot u(x_1, ..., x_n),$$

for any $x_1 \geq 0, \ ..., \ x_{n-1} \geq 0$ and $x_n \in \mathbb{R}$, provided that $\delta_j + \gamma_j > m_j$ for $1 \leq j \leq n-1$.

5. Multidimensional Vector-Valued Laplace Transform

The multidimensional vector-valued Laplace transform has not attracted as much attention of the authors to date. Suppose that $f : [0, +\infty)^n \to X$ is a locally integrable function. Then the multidimensional vector-valued Laplace transform of $f(\cdot)$, denoted by $F(\cdot) = \tilde{f} = \mathcal{L}f$, is defined through

$$F(\lambda_1, ..., \lambda_n) := \int_0^{+\infty} ... \int_0^{+\infty} e^{-\lambda_1 t_1 - ... - \lambda_n t_n} f(t_1, ..., t_n) \, dt_1 ... dt_n, \tag{21}$$

if it is well-defined. We say that $f(\cdot)$ is Laplace transformable if and only if there exist real constants $\omega_1 \in \mathbb{R}, \ ..., \ \omega_n \in \mathbb{R}$ such that $F(\lambda_1, ..., \lambda_n)$ is well-defined for $\Re\lambda_1 > \omega_1, \ ..., \ \Re\lambda_n > \omega_n$. This is always the case if there exist finite real constants $M \geq 1$ and $\omega_1 \in \mathbb{R}, \ ..., \ \omega_n \in \mathbb{R}$ such that $\|f(t_1, ..., t_n)\| \leq M \exp(\omega_1 t_1 + ... + \omega_n t_n)$ for a.e. $t_1 \geq 0, ..., t_n \geq 0$, when we say that $f(\cdot)$ is exponentially bounded; then $F(\lambda_1, ..., \lambda_n)$ is well-defined for $\Re\lambda_1 > \omega_1, \ ..., \ \Re\lambda_n > \omega_n$ and $F(\cdot)$ is analytic in this region of $\mathbb{C}^n$ (see L. Hörmander [52] for the basic introduction to the theory of analytic functions of several complex variables). The uniqueness theorem for Laplace transform holds in the multidimensional framework.

The numerical inversion of a multidimensional vector-valued Laplace transform has been considered in many research articles to date (these papers can be easily located online and we will not quote them here). On the other hand, it seems that the complex inversion theorem for the multidimensional Laplace transform in both the scalar-valued setting and the vector-valued setting has not been properly formulated by now. Concerning this issue, we will state and prove the following extension of [53] (Theorem 2.5.1):

Theorem 1. *Suppose that $M > 0, \omega_1 \geq 0, \ ..., \ \omega_n \geq 0, \epsilon_1 > 0, \ ..., \ \epsilon_n > 0$ and $F : \{\lambda \in \mathbb{C} : \Re\lambda > \omega_1\} \times ... \times \{\lambda \in \mathbb{C} : \Re\lambda > \omega_n\} \to X$ is an analytic function such that*

$$\left\| F(\lambda_1, ..., \lambda_n) \right\| \leq M|\lambda_1|^{-1-\epsilon_1} \cdot ... \cdot |\lambda_n|^{-1-\epsilon_n}, \quad \Re\lambda_j > \omega_j \ \ (1 \leq j \leq n). \tag{22}$$

Then there exist a real number $M_1 > 0$ and a continuous function $f : [0, +\infty)^n \to X$ such that

$$\left\| f(t_1, ..., t_n) \right\| \leq M_1 [t_1^{\epsilon_1} e^{\omega_1 t_1} \cdot ... \cdot t_n^{\epsilon_n} e^{\omega_n t_n}] \text{ for all } t_1 \geq 0, \ ..., \ t_n \geq 0 \tag{23}$$

and $F(\lambda_1, ..., \lambda_n) = (\mathcal{L}f)(\lambda_1, ..., \lambda_n)$ for $\Re\lambda_j > \omega_j$ $(1 \leq j \leq n)$.

Proof. We present the main details of the proof only. Let $a_j > \omega_j$ be pairwisely distinct numbers $(1 \leq j \leq n)$, and let

$$f(t_1, ..., t_n) := \frac{1}{(2\pi i)^n} \int_{a_1 - i\infty}^{a_1 + i\infty} ... \int_{a_n - i\infty}^{a_n + i\infty} e^{\lambda_1 t_1 + ... + \lambda_n t_n} F(\lambda_1, ..., \lambda_n) \, d\lambda_1 ... d\lambda_n, \tag{24}$$

for any $t_1 \geq 0, \ ..., \ t_n \geq 0$; it can be easily shown that the integral appearing in (24) is absolutely convergent so that $f(\cdot)$ is well-defined. The dominated convergence theorem implies that $f(\cdot)$ is continuous; moreover, we can use the Fubini theorem, the growth rate

of $F(\cdot)$ and the computation carried out in the proof of the last mentioned theorem in order to see that there exists a constant $M_1 > 0$, independent of $a_1, ..., a_n$, such that

$$\left\| f(t_1, ..., t_n) \right\| \leq M_1 [t_1^{\epsilon_1} e^{a_1 t_1} \cdot ... \cdot t_n^{\epsilon_n} e^{a_n t_n}] \text{ for all } t_1 \geq 0, \ ..., \ t_n \geq 0.$$

On the other hand, an elementary contour argument shows that the definition of function $f(\cdot)$ does not depend on the choice of numbers $a_1 > \omega_1, ..., a_n > \omega_n$. In actual fact, we can fix the numbers $a_1 > \omega_1, ..., a_{n-1} > \omega_{n-1}$ and prove first that the definition of function $f(\cdot)$ does not depend on the choice of number $a_n > \omega_n$; after that, we can repeat this procedure $(n-1)$ times. Using this fact and letting $a_j \to \omega_j +$ for $1 \leq j \leq n$, we obtain (23). It remains to be proved that $F(\lambda_1, ..., \lambda_n) = (\mathcal{L}f)(\lambda_1, ..., \lambda_n)$ for $\Re\lambda_j > \omega_j$ $(1 \leq j \leq n)$. Let the numbers $\lambda_1, ..., \lambda_n$ enjoy the above properties and let $\omega_j < a_j < \Re\lambda_j$ for $1 \leq j \leq n$. Then the Fubini theorem and an elementary argumentation shows that

$$(\mathcal{L}f)(\lambda_1, ..., \lambda_n) = \frac{1}{(2\pi i)^n} \int_{a_1 - i\infty}^{a_1 + i\infty} ... \int_{a_n - i\infty}^{a_n + i\infty} \frac{F(z_1, ..., z_n)}{(\lambda_1 - z_1) \cdot ... \cdot (\lambda_n - z_n)} \, dz_1 ... dz_n.$$

Using the residue theorem and deforming the line $[a_n - i\infty, a_n + i\infty]$ into the union of the segment $[a_n - iR, a_n + iR]$ and the semi-circle $a_n + \{Re^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\}$, we obtain

$$\begin{aligned}(\mathcal{L}f)(\lambda_1, ..., \lambda_n) &= \frac{1}{(2\pi i)^{n-1}} \\ &\times \int_{a_1 - i\infty}^{a_1 + i\infty} ... \int_{a_{n-1} - i\infty}^{a_{n-1} + i\infty} \frac{F(z_1, ..., z_{n-1}, \lambda_n)}{(\lambda_1 - z_1) \cdot ... \cdot (\lambda_{n-1} - z_{n-1})} \, dz_1 ... dz_{n-1}.\end{aligned}$$

Repeating this argument, we simply obtain the required equality. □

6. Some Classes of Fractional Partial Differential-Difference Inclusions

In this section, we investigate some classes of the fractional partial differential-difference inclusions. We will divide the material of this section into three separate subsections.

6.1. Fractional Partial Differential Inclusions with Riemann–Liouville and Caputo Derivatives

Suppose that $\alpha_1 \in [0, 2)$, $\alpha_2 \in [0, 2)$, $m_1 = \lceil \alpha_1 \rceil$, $m_2 = \lceil \alpha_2 \rceil$ and $\mathcal{A}$ is a closed MLO in X (the precise assumptions about $\mathcal{A}$ will be clarified a little bit later). In this subsection, we will provide certain results about the well-posedness of the following abstract two-dimensional Cauchy inclusions:

$$D_R^{\alpha_1} D_R^{\alpha_2} u(x_1, x_2) \in \mathcal{A}u(x_1, x_2) + f(x_1, x_2), \quad x_1 \geq 0, \ x_2 \geq 0, \tag{25}$$

subjected to the initial conditions of the form

$$\frac{\partial^k}{\partial x_2^k} \left[J_{t_2}^{m_2 - \alpha_2} *_0 u \right] (x_1, 0) = f_k(x_1), \quad 0 \leq k \leq m_2 - 1; \tag{26}$$

$$\int_0^{x_2} g_{\alpha_2}(x_2 - s) \left[\frac{\partial^k}{\partial x_1^k} \left[J_{t_1}^{m_1 - \alpha_1} *_0 D_R^{\alpha_2} u \right] (x_1, x_2) \right]_{x_1 = 0, x_2 = s} ds = h_k(x_2), \tag{27}$$

for $0 \leq k \leq m_1 - 1$, and

$$\mathbf{D}_C^{\alpha_1} \mathbf{D}_C^{\alpha_2} u(x_1, x_2) \in \mathcal{A}u(x_1, x_2) + f(x_1, x_2), \quad x_1 \geq 0, \ x_2 \geq 0, \tag{28}$$

subjected to the initial conditions of the form

$$\frac{\partial^k}{\partial x_2^k} u(x_1, 0) = f_k(x_1), \quad 0 \le k \le m_2 - 1; \tag{29}$$

$$\int_0^{x_2} g_{\alpha_2}(x_2 - s) \left[\frac{\partial^k}{\partial x_1^k} \mathbf{D}_C^{\alpha_2} u(x_1, x_2) \right]_{x_1=0, x_2=s} ds = h_k(x_2), \quad 0 \le k \le m_1 - 1. \tag{30}$$

Our basic assumption will be that $f(\cdot, \cdot)$ is a Laplace transformable function.

Let us consider first the problem (28) equipped with the initial conditions (29)–(30). Assuming that $f \in L^1_{loc}([0,\infty)^2 : X)$, all conditions for applying the Formula (9) are satisfied and using the fact that for every locally integrable function $u \in L^1_{loc}([0,\infty)^2 : X)$, the assumption

$$J_{t_2}^{\alpha_2} J_{t_1}^{\alpha_1} u(x_1, x_2) = 0, \quad x_1 \ge 0, \ x_2 \ge 0$$

implies $u \equiv 0$, we obtain that the problem [(28)–(30)] is equivalent with

$$\begin{aligned} u(x_1, x_2) - \sum_{k=0}^{m_2-1} g_{k+1}(x_2) \cdot f_k(x_1) - \sum_{k=0}^{m_1-1} g_{k+1}(x_1) \cdot h_k(x_2) \\ \in \mathcal{A} \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) u(s, r) \, ds \, dr \\ + \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) f(s, r) \, ds \, dr, \quad x_1 \ge 0, \ x_2 \ge 0, \end{aligned} \tag{31}$$

since $\mathcal{A}$ is closed. Similarly, if $f \in L^1_{loc}([0,\infty)^2 : X)$ and all conditions for applying the Formula (8) are satisfied, the problem [(25)–(27)] is equivalent with

$$\begin{aligned} u(x_1, x_2) - \sum_{k=0}^{m_2-1} g_{\alpha_2+k+1-m_2}(x_2) \cdot f_k(x_1) - \sum_{k=0}^{m_1-1} g_{\alpha_1+k+1-m_1}(x_1) \cdot h_k(x_2) \\ \in \mathcal{A} \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) u(s, r) \, ds \, dr \\ + \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) f(s, r) \, ds \, dr, \quad x_1 \ge 0, \ x_2 \ge 0. \end{aligned} \tag{32}$$

We will use the following notion (cf. also [5] (Definition 3.1.1(i))).

Definition 4. *It is said that a locally integrable function $u : [0,\infty)^2 \to X$ is*

(i) A solution of [(28)–(30)] if and only if

$$\int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) u(s, r) \, ds \, dr \in D(\mathcal{A})$$

and (32) holds for a.e. $x_1 \ge 0$ and $x_2 \ge 0$.

(ii) A strong solution of [(28)–(30)] if and only if there exists a locally integrable function $u_{\mathcal{A},\alpha_1,\alpha_2} : [0,\infty)^2 \to X$ such that

$$\begin{aligned} \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) u_{\mathcal{A},\alpha_1,\alpha_2}(s, r) \, ds \, dr \\ \in \mathcal{A} \int_0^{x_2} g_{\alpha_2}(x_2 - r) \int_0^{x_1} g_{\alpha_1}(x_1 - s) u(s, r) \, ds \, dr \text{ for a.e. } x_1 \ge 0 \text{ and } x_2 \ge 0, \end{aligned}$$

and

$$\begin{aligned}
&u(x_1,x_2)-\sum_{k=0}^{m_2-1} g_{k+1}(x_2)\cdot f_k(x_1)-\sum_{k=0}^{m_1-1} g_{k+1}(x_1)\cdot h_k(x_2)\\
&=\int_0^{x_2} g_{\alpha_2}(x_2-r)\int_0^{x_1} g_{\alpha_1}(x_1-s)u_{\mathcal{A},\alpha_1,\alpha_2}(s,r)\,ds\,dr\\
&+\int_0^{x_2} g_{\alpha_2}(x_2-r)\int_0^{x_1} g_{\alpha_1}(x_1-s)f(s,r)\,ds\,dr \text{ for a.e. } x_1\geq 0 \text{ and } x_2\geq 0.
\end{aligned}$$

We similarly define the notion of a (strong) solution of problem [(25)–(27)].

It is clear that any strong solution of [(28)–(30)] ([(25)–(27)]) is likewise a solution of the same problem and that the converse statement is not true, in general.

Let us now take a closer look at the abstract Cauchy inclusions (31) and (32). Applying the two-dimensional Laplace transform and the Fubini theorem, we obtain that the problem (31) is equivalent with

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}u(x_1,x_2)\,dx_1\,dx_2-\sum_{k=0}^{m_2-1}\lambda^{-1-k}\int_0^{+\infty} e^{-zx_1}f_k(x_1)\,dx_1\\
&\quad-\sum_{k=0}^{m_1-1} z^{-1-k}\int_0^{+\infty} e^{-\lambda x_2}h_k(x_2)\,dx_2\\
&\quad\in\mathcal{A}\left[z^{-\alpha_1}\lambda^{-\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}u(x_1,x_2)\,dx_1\,dx_2\right]\\
&\quad+z^{-\alpha_1}\lambda^{-\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}f(x_1,x_2)\,dx_1\,dx_2,
\end{aligned}\tag{33}$$

for all $z\in\mathbb{C}$ with $\Re z>\omega_1$ for some $\omega_1>0$ and $\lambda\in\mathbb{C}$ with $\Re\lambda>\omega_2$ for some $\omega_2>0$, under certain logical assumptions, as well as that the problem (32) is equivalent with

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}u(x_1,x_2)\,dx_1\,dx_2-\sum_{k=0}^{m_2-1}\lambda^{m_2-1-k-\alpha_2}\int_0^{+\infty} e^{-zx_1}f_k(x_1)\,dx_1\\
&\quad-\sum_{k=0}^{m_1-1} z^{m_1-1-k-\alpha_1}\int_0^{+\infty} e^{-\lambda x_2}h_k(x_2)\,dx_2\\
&\quad\in\mathcal{A}\left[z^{-\alpha_1}\lambda^{-\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}u(x_1,x_2)\,dx_1\,dx_2\right]\\
&\quad+z^{-\alpha_1}\lambda^{-\alpha_2}\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}f(x_1,x_2)\,dx_1\,dx_2,
\end{aligned}\tag{34}$$

for all $z\in\mathbb{C}$ with $\Re z>\omega_1$ for some $\omega_1>0$ and $\lambda\in\mathbb{C}$ with $\Re\lambda>\omega_2$ for some $\omega_2>0$, under certain logical assumptions. After setting

$$\widetilde{u}(z,\lambda):=\int_0^{+\infty}\int_0^{+\infty} e^{-zx_1-\lambda x_2}u(x_1,x_2)\,dx_1\,dx_2,$$

we obtain that the problem (33) is equivalent with

$$\begin{aligned}
&\left(z^{\alpha_1}\lambda^{\alpha_2}-\mathcal{A}\right)\widetilde{u}(z,\lambda)\ni\sum_{k=0}^{m_2-1} z^{\alpha_1}\lambda^{\alpha_2-1-k}\int_0^{+\infty} e^{-zx_1}f_k(x_1)\,dx_1\\
&\quad-\sum_{k=0}^{m_1-1} z^{\alpha_1-1-k}\lambda^{\alpha_2}\int_0^{+\infty} e^{-\lambda x_2}h_k(x_2)\,dx_2+\widetilde{f}(z,\lambda),
\end{aligned}\tag{35}$$

for all $z \in \mathbb{C}$ with $\Re z > \omega_1$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_2$, while the problem (34) is equivalent with

$$\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)\widetilde{u}(z,\lambda) \ni \sum_{k=0}^{m_2-1} z^{\alpha_1}\lambda^{m_2-1-k}\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1 - \sum_{k=0}^{m_1-1} z^{m_1-1-k}\lambda^{\alpha_2}\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2 + \widetilde{f}(z,\lambda), \tag{36}$$

for all $z \in \mathbb{C}$ with $\Re z > \omega_1$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_2$. In the case that there exists an injective operator $C \in L(X)$ which commutes with $\mathcal{A}$ and condition (C1) clarified below holds, then the inclusion (35), resp., (36), is equivalent with:

$$\widetilde{u}(z,\lambda) = \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C \sum_{k=0}^{m_2-1} z^{\alpha_1}\lambda^{\alpha_2-1-k}\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1 - \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C \sum_{k=0}^{m_1-1} z^{\alpha_1-1-k}\lambda^{\alpha_2}\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2 + \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\widetilde{f}(z,\lambda), \tag{37}$$

for all $z \in \mathbb{C}$ with $\Re z > \omega_1$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_2$, resp.,

$$\widetilde{u}(z,\lambda) = \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C \sum_{k=0}^{m_2-1} z^{\alpha_1}\lambda^{m_2-1-k}\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1 - \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C \sum_{k=0}^{m_1-1} z^{m_1-1-k}\lambda^{\alpha_2}\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2 + \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\widetilde{f}(z,\lambda), \tag{38}$$

for all $z \in \mathbb{C}$ with $\Re z > \omega_1$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_2$.

Now we will formalize all this and state the following result by assuming some special conditions on the multivalued linear operator $\mathcal{A}$.

Theorem 2. *Suppose that $C \in L(X)$ is injective and commutes with $\mathcal{A}$, $f(\cdot;\cdot)$ is Laplace transformable and the following condition holds:*

(C1) There exist real numbers $\omega_1 > 0$ and $\omega_2 > 0$ such that $z^{\alpha_1}\lambda^{\alpha_2} \in \rho_C(\mathcal{A})$ for all $z \in \mathbb{C}$ with $\Re z > \omega_1$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_2$.

Denote by D_1 the set of all indexes $k \in \mathbb{N}_{m_2-1}^0$ such that $f_k(\cdot)$ is not identically equal to the zero function and by D_2 the set of all indexes $k \in \mathbb{N}_{m_1-1}^0$ such that $h_k(\cdot)$ is not identically equal to the zero function. If the following conditions hold:

(i) For every $k \in D_1$, there exists a Laplace transformable function $u_k^1(\cdot;\cdot)$ such that

$$\widetilde{u_k^1}(z,\lambda) = z^{\alpha_1}\lambda^{\alpha_2-1-k}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1,$$

resp.

$$\widetilde{u_k^1}(z,\lambda) = z^{\alpha_1}\lambda^{m_2-1-k}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1,$$

for $\Re z > \omega_1$ and $\Re\lambda > \omega_2$.

(ii) For every $k \in D_2$, there exists a Laplace transformable function $u_k^2(\cdot;\cdot)$ such that

$$\widetilde{u_k^2}(z,\lambda) = z^{\alpha_1-1-k}\lambda^{\alpha_2}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2,$$

resp.

$$\widetilde{u_k^2}(z,\lambda) = z^{m_1-1-k}\lambda^{\alpha_2}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2,$$

for $\Re z > \omega_1$ and $\Re\lambda > \omega_2$.

(iii) There exists a Laplace transformable function $u_k^2(\cdot;\cdot)$ *such that*

$$\widetilde{u_f}(z,\lambda) = \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\widetilde{f}(z,\lambda),$$

for $\Re z > \omega_1$ *and* $\Re\lambda > \omega_2$.

Then there exists a unique solution of problem $u(x_1,x_2)$ *of* [(28)–(30)], *resp.*, [(25)–(27)], *which is given by*

$$u(x_1,x_2) = \sum_{k\in D_1} u_k^1(x_1,x_2) + \sum_{k\in D_2} u_k^2(x_1,x_2) + u_f(x_1,x_2) \text{ for a.e. } x_1 \geq 0,\ x_2 \geq 0. \tag{39}$$

Furthermore, suppose that (i)–(iii) *and the following conditions hold:*

(is) For every $k \in D_1$, *there exists a Laplace transformable function* $u_k^1(\cdot;\cdot)$ *such that*

$$\widetilde{u_k^1}(z,\lambda) = z^{2\alpha_1}\lambda^{2\alpha_2-1-k}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1 - z^{\alpha_1}\lambda^{\alpha_2-1-k}\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1,$$

resp.

$$\widetilde{u_k^1}(z,\lambda) = z^{2\alpha_1}\lambda^{\alpha_2+m_2-1-k}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1 - z^{\alpha_1}\lambda^{m_2-1-k}\int_0^{+\infty} e^{-zx_1} f_k(x_1)\,dx_1,$$

for $\Re z > \omega_1$ *and* $\Re\lambda > \omega_2$.

(iis) For every $k \in D_2$, *there exists a Laplace transformable function* $u_k^2(\cdot;\cdot)$ *such that*

$$\widetilde{u_k^2}(z,\lambda) = z^{2\alpha_1-1-k}\lambda^{2\alpha_2}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2 - z^{\alpha_1-1-k}\lambda^{\alpha_2} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2,$$

resp.

$$\widetilde{u_k^2}(z,\lambda) = z^{\alpha_1+m_1-1-k}\lambda^{2\alpha_2}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2 - z^{m_1-1-k}\lambda^{\alpha_2} C\int_0^{+\infty} e^{-\lambda x_2} h_k(x_2)\,dx_2,$$

for $\Re z > \omega_1$ *and* $\Re\lambda > \omega_2$.

(iiis) There exists a Laplace transformable function $u_k^2(\cdot;\cdot)$ *such that*

$$\widetilde{u_f}(z,\lambda) = z^{\alpha_1}\lambda^{\alpha_2}\left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A}\right)^{-1} C\widetilde{f}(z,\lambda) - C\widetilde{f}(z,\lambda),$$

for $\Re z > \omega_1$ *and* $\Re\lambda > \omega_2$.

Then there exists a unique solution of problem $u(x_1,x_2)$ *of* [(28)–(30)], *resp.*, [(25)–(27)], *which is given by*

$$u(x_1,x_2) = \sum_{k\in D_1} u_k^1(x_1,x_2) + \sum_{k\in D_2} u_k^2(x_1,x_2) + u_f(x_1,x_2) \text{ for a.e. } x_1 \geq 0,\ x_2 \geq 0. \tag{40}$$

Then the function $u(x_1,x_2)$, *given by* (40), *is a strong solution of problem* $u(x_1,x_2)$ *of* [(28)–(30)], *resp.*, [(25)–(27)].

Proof. Since we assume the conditions (i)–(iii), we simply infer that the function $u(x_1, x_2)$, given by (40), satisfies (37), resp., (38). Arguing reversely, we obtain that (35), resp., (36), holds true. Applying the inverse double Laplace transform, we obtain that (33), resp., (34), holds true, which simply completes the proof of the first part of theorem. The second part of theorem follows similarly since, in this case, there exists a locally integrable function $u_{\mathcal{A},\alpha_1,\alpha_2}(\cdot;\cdot)$ such that $u_{\mathcal{A},\alpha_1,\alpha_2}(\cdot;\cdot) \in \mathcal{A}u(\cdot;\cdot)$ a.e. on $[0,+\infty)^2$, which can be proved by performing the double Laplace transform and (is)–(iiis); see also [5] (Theorem 1.2.4(i)). □

The subsequent result follows immediately from Theorems 1 and 2 (we can similarly clarify the corresponding conditions ensuring the existence of a unique strong solution of problems under our consideration; we use the symbol $\widetilde{\cdot}$ to denote both the one-dimensional and the two-dimensional Laplace transform here, which will not cause any confusion).

Theorem 3. *Suppose that $f(\cdot;\cdot)$ is Laplace transformable and the following condition holds: (C1s)*(C1) *holds and there exist real numbers $M > 0$ and $\beta \in (0,1]$ such that*

$$\left\| \left(z^{\alpha_1}\lambda^{\alpha_2} - \mathcal{A} \right)^{-1} C \right\| \leq \frac{M}{(1+|z|^{\alpha_1}|\lambda|^{\alpha_2})^{\beta}}, \quad \Re z > \omega_1,\ \Re\lambda > \omega_2. \tag{41}$$

Suppose, further, that the following conditions hold:

(i) For every $k \in D_1$, there exist real numbers $M_{k,1} > 0$, $\epsilon_{1,1}^k > 0$ and $\epsilon_{1,2}^k > 0$ such that

$$\left\| |z|^{\alpha_1}|\lambda|^{\alpha_2-1-k} \frac{\|\widetilde{f}_k(z)\|}{(1+|z|^{\alpha_1}|\lambda|^{\alpha_2})^{\beta}} \right\| \leq M_{k,1}|z|^{-1-\epsilon_{1,1}^k}|\lambda|^{-1-\epsilon_{1,2}^k}, \quad \Re z > \omega_1,\ \Re\lambda > \omega_2.$$

(ii) For every $k \in D_2$, there exist real numbers $M_{k,2} > 0$, $\epsilon_{2,1}^k > 0$ and $\epsilon_{2,2}^k > 0$ such that

$$\left\| |z|^{\alpha_1-1-k}|\lambda|^{\alpha_2} \frac{\|\widetilde{h}_k(\lambda)\|}{(1+|z|^{\alpha_1}|\lambda|^{\alpha_2})^{\beta}} \right\| \leq M_{k,1}|z|^{-1-\epsilon_{2,1}^k}|\lambda|^{-1-\epsilon_{2,2}^k}, \quad \Re z > \omega_1,\ \Re\lambda > \omega_2.$$

(iii) There exist real numbers $M' > 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\left\| \frac{\|\widetilde{f}(z,\lambda)\|}{(1+|z|^{\alpha_1}|\lambda|^{\alpha_2})^{\beta}} \right\| \leq M'|z|^{-1-\epsilon_1}|\lambda|^{-1-\epsilon_2}, \quad \Re z > \omega_1,\ \Re\lambda > \omega_2.$$

Then there exists a unique continuous solution $u(x_1, x_2)$ of problem [(28)–(30)], *resp.,* [(25)–(27)], *and we have*

$$\begin{aligned} \left\| u(x_1,x_2) \right\| \leq M'' \Bigg[& \sum_{k\in D_1} x_1^{\epsilon_{1,1}} x_2^{\epsilon_{1,2}} e^{\omega_1 x_1+\omega_2 x_2} \\ & + \sum_{k\in D_2} x_1^{\epsilon_{2,1}} x_2^{\epsilon_{2,2}} e^{\omega_1 x_1+\omega_2 x_2} + x_1^{\epsilon_1} x_2^{\epsilon_2} e^{\omega_1 x_1+\omega_2 x_2} \Bigg], \quad x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

If $0 \notin D_1 \cup D_2$, then the requirements of Theorem 3 are satisfied in many important real situations, even for the degenerate Poisson heat operator $\Delta \cdot m(x)^{-1}$; cf. [5] and references cited therein for further information in this direction.

Remark 3. *Suppose that $\alpha_1 + \alpha_2 < 2$. Then it is clear that the estimate* (41) *holds if $\Sigma_{(\alpha_1+\alpha_2)\pi/2} \subseteq \rho_C(\mathcal{A})$ and there exists $\beta \in (0,1]$ such that*

$$\left\| (\lambda - \mathcal{A})^{-1} C \right\| \leq \frac{M}{(1+|\lambda|)^{\beta}}, \quad \lambda \in \Sigma_{(\alpha_1+\alpha_2)\pi/2}.$$

Unfortunately, we cannot prove that (41) *holds if there exists a positive real number* $a > 0$ *such that* $a + \Sigma_{(\alpha_1+\alpha_2)\pi/2} \subseteq \rho_C(\mathcal{A})$ *and*

$$\left\| (\lambda - \mathcal{A})^{-1} C \right\| \leq \frac{M}{(1+|\lambda|)^{\beta}}, \quad \lambda \in a + \Sigma_{(\alpha_1+\alpha_2)\pi/2}.$$

The main problem lies in the fact that for every real number $\omega_1 > 0$, *we have*

$$\lim_{x \to \pm\infty} dist\Big(\{ re^{i} \alpha_1 : r \geq 0 \}, \left(\omega_1 + ix \right)^{\alpha_1} \Big) = 0.$$

Remark 4. *Suppose that* $\alpha_1 + \alpha_2 \geq 2$. *Then we can apply Theorem 3, with* $C \neq I$, *to a class of two-dimensional partial fractional differential equations involving the single-valued linear operators* $\mathcal{A} = A$ *whose C-resolvent is bounded by* $(1 + |\cdot|)^{-1}$ *on the set of form* $\mathbb{C} \setminus K$, *where K is compact; see [5] for the corresponding examples. In particular, if* $\alpha_1 = \alpha_2 = 1$, *then we can analyze the well-posedness of the problem*

$$\frac{\partial^2}{\partial x_1 \partial x_2} u(x_1, x_2) = Au(x_1, x_2) + f(x_1, x_2), \quad x_1 \geq 0, \ x_2 \geq 0,$$

subjected to the initial conditions $u(x_1, 0) = f_0(x_1)$, $x_1 \geq 0$ *and* $u(0, x_2) = u(0,0) + h_0(x_2)$, $x_2 \geq 0$.

Using the multidimensional generalizations of the Formulae (8) and (9), we can similarly analyze the well-posedness of the abstract fractional Cauchy inclusions

$$D_R^{\alpha_1} D_R^{\alpha_2} \cdot ... \cdot D_R^{\alpha_n} u(\mathbf{x}) \in \mathcal{A}u(\mathbf{x}) + f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, ..., x_n) \in [0, +\infty)^n$$

and

$$\mathbf{D}_C^{\alpha_1} \mathbf{D}_C^{\alpha_2} \cdot ... \cdot \mathbf{D}_C^{\alpha_n} u(\mathbf{x}) \in \mathcal{A}u(\mathbf{x}) + f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, ..., x_n) \in [0, +\infty)^n,$$

subjected to certain initial conditions (for the scalar-valued case, see also [54] (Section 3)). Details can be left to interested readers.

6.2. The Abstract Multiterm Fractional Partial Differential Equations with Riemann–Liouville and Caputo Derivatives

In this subsection, we investigate the following operator extensions of the partial fractional differential Equation (1):

$$\sum_{k=1}^{n} A_k D_R^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n) = f(x_1, ..., x_n), \quad x_1 \geq 0, \ ..., \ x_n \geq 0, \tag{42}$$

subjected to the initial conditions

$$\left[\frac{\partial^j}{\partial x_k^j} J_{t_k}^{m_k - \alpha_k} u(x_1, ..., x_n) \right]_{x_k = 0} = f_{k,j}(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n), \tag{43}$$

for $1 \leq k \leq n$, $0 \leq j \leq m_k - 1$, and

$$\sum_{k=1}^{n} A_k D_C^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n) = f(x_1, ..., x_n), \quad x_1 \geq 0, \ ..., \ x_n \geq 0, \tag{44}$$

subjected to the initial conditions

$$\left[\frac{\partial^j}{\partial x_k^j}u(x_1,...,x_n)\right]_{x_k=0} = f_{k,j}(x_1,...,x_{k-1},x_{k+1},...,x_n), \tag{45}$$

for $1 \leq k \leq n$, $0 \leq j \leq m_k - 1$, where A_k is a closed linear operator and $\alpha_k \geq 0$ for $1 \leq k \leq n$. In order to do that, we essentially apply the multidimensional vector-valued Laplace transform.

We will use the following notion.

Definition 5.

(i) By a mild LT-solution $u(x_1,...,x_n)$ of [(42) and (43)], resp. [(44) and (45)], we mean any Laplace transformable function $u(x_1,...,x_n)$ such that the terms $D_R^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$, resp. $D_C^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$, are well-defined and Laplace transformable for $1 \leq k \leq n$ as well as that the terms $(\partial^j/\partial x_k^j)J_{t_k}^{m_k-\alpha_k}u(x_1,...,x_n)$, resp. $(\partial^j/\partial x_k^j)u(x_1,...,x_n)$, are well-defined and continuous with respect to the variable x_j for $1 \leq k \leq n$, $0 \leq j \leq m_k - 1$,

$$\sum_{k=1}^{n} A_k\left(\mathcal{L}D_R^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)\right)(\lambda_1,...,\lambda_n) = \tilde{f}(\lambda_1,...,\lambda_n), \tag{46}$$

for $\Re\lambda_j > \omega_j$ $(1 \leq j \leq n)$ and some non-negative real numbers $\omega_1 \geq 0$, ..., $\omega_n \geq 0$, resp. (46) holds with the term $D_R^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$ replaced with the term $D_C^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$ therein, and (43), resp. (45), holds.

(ii) By a strong LT-solution $u(x_1,...,x_n)$ of [(42) and (43)], resp. [(44) and (45)], we mean any mild LT-solution $u(x_1,...,x_n)$ of this problem which additionally satisfies that the terms $A_kD_R^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$, resp. $A_kD_C^{(0,...,\alpha_k,...,0)}u(x_1,...,x_k,...,x_n)$, are well-defined and Laplace transformable for $1 \leq k \leq n$.

The uniqueness theorem for Laplace transform and the closedness of operators A_k for $1 \leq k \leq n$ show that any strong LT-solution of [(42) and (43)], resp. [(44) and (45)], satisfies that (42), resp. (44), holds for a.e. $x_1 \geq 0$, ..., $x_n \geq 0$.

Our main result concerning the well-posedness of Equations (42)–(45) reads as follows

Theorem 4. *Suppose that $C \in L(X)$ is injective, A_k is a closed linear operator commuting with C and $\alpha_k \geq 0$ for $1 \leq k \leq n$. Suppose, further, that there exist non-negative real numbers $\omega_1 \geq 0$, ..., $\omega_n \geq 0$ such that the operator $\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k$ is injective and $(\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k)^{-1}C \in L(X)$ for $\Re\lambda_1 > \omega_1$, ..., $\Re\lambda_n > \omega_n$. Let the following conditions also hold:*

(i) There exists a locally integrable, exponentially bounded function $h(x_1,...,x_n)$ for $x_1 \geq 0,...,$ $x_n \geq 0$ satisfying that $D_R^{(0,...,\alpha_k,...,0)}h(x_1,...,x_k,...,x_n)$, resp. $D_C^{(0,...,\alpha_k,...,0)}h(x_1,...,x_k,...,x_n)$, is well-defined, locally integrable and exponentially bounded $(1 \leq k \leq n)$, the terms $(\partial^j/\partial x_k^j)J_{t_k}^{m_k-\alpha_k}h(x_1,...,x_n)$, resp. $(\partial^j/\partial x_k^j)h(x_1,...,x_n)$, are well-defined and continuous with respect to the variable x_j for $1 \leq k \leq n$, $0 \leq j \leq m_k - 1$, and

$$\widetilde{h}(\lambda_1,...,\lambda_n) = \left(\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k\right)^{-1}C\widetilde{f_0}(\lambda_1,...,\lambda_n), \quad \Re\lambda_1 > \omega_1, \ ..., \ \Re\lambda_n > \omega_n, \tag{47}$$

where $f = Cf_0$.

(ii) If $1 \leq k \leq n$ and $0 \leq j \leq m_k - 1$, then there exists a locally integrable, exponentially bounded function $h_{k,j}(x_1,...,x_{k-1},x_{k+1},...,x_n)$ for $x_1 \geq 0,...,$ $x_{k-1} \geq 0$, $x_{k+1} \geq 0,...,x_n \geq 0$ satisfying that the terms $D_R^{(0,...,\alpha_v,...,0)}h_{k,j}(x_1,...,x_{k-1},x_{k+1},...,x_n)$, resp. $D_C^{(0,...,\alpha_v,...,0)}h_{k,j}$ $(x_1,...,x_{k-1},x_{k+1}...,x_n)$, are well-defined, locally integrable and exponentially bounded for

$1 \leq v \leq n$, *the terms* $(\partial^j/\partial x_v^j) J_{t_k}^{m_k-\alpha_k} h_{k,j}(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$, *resp.* $(\partial^j/\partial x_v^j) h_{k,j}$ $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$ *are well-defined and continuous with respect to the variable* x_v *for* $1 \leq v \leq n$, *and*

$$\widetilde{h_{k,j}}(\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n) = \left(\sum_{k=1}^{n} \lambda_k^{\alpha_k} A_k \right)^{-1} C A_k \widetilde{f_{k,j,0}}(\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n), \tag{48}$$

provided that $\Re\lambda_1 > \omega_1, ..., \Re\lambda_{k-1} > \omega_{k-1}, \Re\lambda_{k+1} > \omega_{k+1}, ..., \Re\lambda_n > \omega_n$, *where* $f_{k,j} = C f_{k,j,0}$.

Then there exists a unique mild LT*-solution* $u(x_1, ..., x_n)$ *of* [(42) and (43)], *resp.* [(44) and (45)], *and we have*

$$u(x_1, ..., x_n) = \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} h_{k,j}(x_1, ..., x_n) + h(x_1, ..., x_n), \quad x_1 \geq 0, \ ..., \ x_n \geq 0. \tag{49}$$

Furthermore, if the following conditions hold:

(is) If $1 \leq v \leq n$, *then the terms* $A_v h(x_1, ..., x_v, ..., x_n)$ *and* $D_R^{(0,...,\alpha_v,...,0)} A_v h(x_1, ..., x_v, ..., x_n)$, *resp.*$D_C^{(0,...,\alpha_v,...,0)} A_v h(x_1, ..., x_v, ..., x_n)$, *are well-defined, locally integrable and exponentially bounded;*

(iis) If $1 \leq v \leq n$, $1 \leq k \leq n$ *and* $0 \leq j \leq m_k - 1$, *then the terms* $A_v h_{k,j}(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$ *and* $D_R^{(0,...,\alpha_v,...,0)} A_v h_{k,j}(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)$, *resp.* $D_C^{(0,...,\alpha_v,...,0)} A_v h_{k,j}(x_1, ..., x_{k-1}, x_{k+1}..., x_n)$, *are well-defined, locally integrable and exponentially bounded,*

then the function $u(x_1, ..., x_n)$, *given by* (49), *is a strong* LT*-solution of* [(42) and (43)], *resp.* [(44) and (45)].

Proof. Let $u(x_1, ..., x_n)$ be given by (49), and let $\Re\lambda_1 > \omega_1, \ ..., \Re\lambda_n > \omega_n$. Our assumptions imply that the term $D_R^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n)$, resp. $D_C^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n)$, is well-defined as well as that we have the following (see also Equations (1.22)–(1.23) [1] and Equation (16) [55]):

$$\begin{aligned} D_R^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n) &= \lambda_k^{\alpha_k} \tilde{u}(\lambda_1, ..., \lambda_n) \\ &- \sum_{j=0}^{m_k-1} \left[\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n} f_{k,j} \right] (\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n) \lambda_k^{m_k-1-j}, \end{aligned} \tag{50}$$

resp.

$$\begin{aligned} D_C^{(0,...,\alpha_k,...,0)} u(x_1, ..., x_k, ..., x_n) &= \lambda_k^{\alpha_k} \tilde{u}(\lambda_1, ..., \lambda_n) \\ &- \sum_{j=0}^{m_k-1} \left[\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n} f_{k,j} \right] (\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n) \lambda_k^{\alpha_k-1-j}, \end{aligned} \tag{51}$$

where $\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n}$ denotes the multidimensional Laplace transform with respect to the variables $t_1, ..., t_{k-1}, t_{k+1}, ..., t_n$. Furthermore, our assumptions simply imply that

$$\begin{aligned} \tilde{u}(\lambda_1, ..., \lambda_n) = \sum_{k=1}^{n} \left(\sum_{k=1}^{n} \lambda_k^{\alpha_k} A_k \right)^{-1} C A_k \sum_{j=0}^{m_k-1} \left[\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n} f_{k,j,0} \right] \\ (\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n) + \left(\sum_{k=1}^{n} \lambda_k^{\alpha_k} A_k \right)^{-1} C \widetilde{f_0}(\lambda_1, ..., \lambda_n). \end{aligned}$$

This simply implies

$$\left[\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k\right]\tilde{u}(\lambda_1,...,\lambda_n)-\sum_{k=1}^{n}A_k \times\sum_{j=0}^{m_k-1}\left[\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n}f_{k,j}\right](\lambda_1,...,\lambda_{k-1},\lambda_{k+1},...,\lambda_n)\lambda_k^{m_k-1-j}=\tilde{f}(\lambda_1,...,\lambda_n),$$

resp.

$$\left[\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k\right]\tilde{u}(\lambda_1,...,\lambda_n)-\sum_{k=1}^{n}A_k \times\sum_{j=0}^{m_k-1}\left[\mathcal{L}_{t_1,...,t_{k-1},t_{k+1},...,t_n}f_{k,j}\right](\lambda_1,...,\lambda_{k-1},\lambda_{k+1},...,\lambda_n)\lambda_k^{\alpha_k-1-j}=\tilde{f}(\lambda_1,...,\lambda_n).$$

Keeping in mind Equations (50) and (51), it readily follows that Equation (46) and its analogue with Caputo fractional derivatives hold good. Therefore, the function $u(x_1,...,x_n)$ is a mild LT-solution of problem [(42) and (43)], resp. [(44) and (45)]. The uniqueness of mild LT-solutions of this problem follows from a simple argumentation involving the injectiveness of the operator $\sum_{k=1}^{n}\lambda_k^{\alpha_k}A_k$ for $\Re\lambda_1>\omega_1$, ..., $\Re\lambda_n>\omega_n$ and the uniqueness theorem for the Laplace transform. Finally, if the conditions (is) and (iis) hold, then we can simply prove that the function $A_vD_R^{(0,...,\alpha_v,...,0)}u(x_1,...x_n)$ is Laplace transformable and

$$\mathcal{L}\left[A_vD_R^{(0,...,\alpha_v,...,0)}u(x_1,...x_n)\right]=A_v\left[\mathcal{L}u(x_1,...x_n)\right],$$

which simply completes the proof. □

Keeping in mind Theorem 1, we can apply Theorem 4 in many concrete situations, even if $\alpha_k>2$ for some indexes $k\in\mathbb{N}_n$; cf. [5,55] for more details. Let us finally observe that we can similarly analyze some generalizations of the problems [(42)–(45)] with various types of generalized Laplace fractional derivatives, especially with the generalized Hilfer (a,b,α)-fractional derivatives [43].

6.3. Fractional Partial Difference Equations with Generalized Weyl Derivatives

In our recent research article [56], we investigated various classes of the abstract non-scalar Volterra difference equations of several variables. In order to do that, we introduced and analyzed the notion of a discrete $(k,C,B,(A_i)_{1\leq i\leq n},(\mathrm{v}_i)_{1\leq i\leq n})$-existence family (cf. [56] (Definition 2.1)); the generation of discrete $(k,C,B,(A_i)_{1\leq i\leq n},(\mathrm{v}_i)_{1\leq i\leq n})$-existence families was analyzed in [56] (Theorem 2.1) under certain very mild assumptions.

In [56], (Theorem 2.2(i)), we proved the following result:

Lemma 1. *Suppose that* $\mathrm{v}_1\in\mathbb{N}_0^n$, ..., $\mathrm{v}_m\in\mathbb{N}_0^n$, $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n}\subseteq L(X)$ *is a discrete* $(k,C,B,(A_i)_{1\leq i\leq m},(\mathrm{v}_i)_{1\leq i\leq m})$*-existence family,* $\sum_{\mathrm{v}\in\mathbb{N}_0^n}\|S(\mathrm{v})\|<+\infty$ *and the following holds:*

(a) $f:\mathbb{Z}^n\to X$ *is a bounded sequence,* $k\in l^1(\mathbb{N}_0^n:\mathbb{C})$ *and* $\sum_{\mathrm{v}\in\mathbb{N}_0^n}|a_i(\mathrm{v})|<+\infty$ *for* $1\leq i\leq m$*, or*

(b) $f\in l^1(\mathbb{Z}^n:X)$, $k:\mathbb{N}_0^n\to\mathbb{C}$ *is a bounded sequence and* $a_i:\mathbb{Z}^n\to\mathbb{C}$ *is a bounded sequence for* $1\leq i\leq m$.

Define

$$u(\mathrm{v}):=\sum_{l\in\mathbb{Z}^n;l\leq\mathrm{v}}S(\mathrm{v}-l)f(l),\quad \mathrm{v}\in\mathbb{Z}^n \tag{52}$$

and

$$g(\mathrm{v}) := A_1\left(\sum_{l\leq \mathrm{v}+\mathrm{v}_1} - \sum_{l\leq \mathrm{v}}\right)(a_1 *_0 S)(\mathrm{v}+\mathrm{v}_1-l)f(l)+...$$
$$+ A_m\left(\sum_{l\leq \mathrm{v}+\mathrm{v}_m} - \sum_{l\leq \mathrm{v}}\right)(a_m *_0 S)(\mathrm{v}+\mathrm{v}_m-l)f(l), \quad v\in\mathbb{Z}^n. \tag{53}$$

Then $u(\cdot)$ *is bounded if* (a) *holds,* $u\in l^1(\mathbb{Z}^n:X)$ *if* (b) *holds, and we have*

$$Bu(\mathrm{v}) = A_1 \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_1} a_1(\mathrm{v}+\mathrm{v}_1-l)u(l)+...$$
$$+ A_m \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_m} a_1(\mathrm{v}+\mathrm{v}_m-l)u(l)+g(\mathrm{v}),\ v\in\mathbb{Z}^n.$$

For some concrete applications of Lemma 1 to the fractional partial difference equations with generalized Weyl derivatives, we will particularly consider the situation in which the sequences $a_i(\cdot)$ have the following form:

$$a_i(v_1,...,v_n) = a_1^i(v_1)\cdot...\cdot a_n^i(v_n), \quad (v_1,...,v_n)\in\mathbb{N}_0^n \ \ (1\leq i\leq m). \tag{54}$$

Suppose now that $\mathrm{v}_1\in\mathbb{N}_0^n$, ..., $\mathrm{v}_m\in\mathbb{N}_0^n$, $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n}\subseteq L(X)$ is a discrete $(k,C,B,(A_i)_{1\leq i\leq m},(\mathrm{v}_i)_{1\leq i\leq m})$-existence family, $\sum_{\mathrm{v}\in\mathbb{N}_0^n}\|S(\mathrm{v})\|<+\infty$, (54) and the following conditions hold:

(a1) $f:\mathbb{Z}^n\to X$ is a bounded sequence, $k\in l^1(\mathbb{N}_0^n:\mathbb{C})$ and $\sum_{v=0}^{+\infty}|a_j^i(v)|<+\infty$ for $1\leq j\leq n$ and $1\leq i\leq m$, or

(b1) $f\in l^1(\mathbb{Z}^n:X)$, $k:\mathbb{N}_0^n\to\mathbb{C}$ is a bounded sequence and $a_j^i:\mathbb{Z}\to\mathbb{C}$ is a bounded sequence for $1\leq j\leq n$ and $1\leq i\leq m$.

Let $\mathbf{m}=(m_1,...,m_n)\in\mathbb{N}^n$ be fixed, and let the sequences $u(\cdot)$ and $g(\cdot)$ be defined by (52) and (53), respectively. Then $u(\cdot)$ is bounded if (a1) holds, $u\in l^1(\mathbb{Z}^n:X)$ if (b1) holds, and a simple computation shows that we have

$$\left(\Delta^{m_1+...+m_n}_{v_1^{m_1}\cdot...\cdot v_n^{m_n}}Bu\right)(\mathrm{v})$$
$$= A_1\left(\Delta_{W,a_1,\mathbf{m}}u\right)(\mathrm{v}+\mathrm{v}_i)+...+A_m\left(\Delta_{W,a_m,\mathbf{m}}u\right)(\mathrm{v}+\mathrm{v}_m), \quad \mathrm{v}\in\mathbb{Z}^n. \tag{55}$$

Further on, if $\alpha_1=(\alpha_1^1,...,\alpha_n^1)\in[0,+\infty)^n$ ..., $\alpha_m=(\alpha_1^m,...,\alpha_n^m)\in[0,+\infty)^n$, $m_j^i=\lceil\alpha_j^i\rceil$ for $1\leq j\leq n$ and $1\leq i\leq m$, $a_j^i(v_j)=k^{m_j^i-\alpha_j^i}(v_j)$ for $1\leq j\leq n$, $1\leq i\leq m$, and

$$m_j=m_j^1=...=m_j^n, \quad 1\leq j\leq n,$$

then we have

$$\left(\Delta^{m_1+...+m_n}_{v_1^{m_1}\cdot...\cdot v_n^{m_n}}Bu\right)(\mathrm{v}) = A_1\left(\Delta_W^{\alpha_1}u\right)(\mathrm{v}+\mathrm{v}_i)+...+A_m\left(\Delta_W^{\alpha_m}u\right)(\mathrm{v}+\mathrm{v}_m),\ \mathrm{v}\in\mathbb{Z}^n. \tag{56}$$

We can also analyze some other relatives of (55) and (56) as well as the existence and uniqueness of almost periodic-type solutions to (55) and (56); cf. [56] for more details.

7. Conclusions

In this paper, we introduced and analyzed several new types of partial fractional derivatives in the continuous setting and the discrete setting. We investigated the well-posedness of some classes of the abstract fractional differential equations and the abstract

fractional difference equations depending on several variables, providing also many illustrative examples and useful remarks. We also provided some new applications of the multidimensional vector-valued Laplace transform.

We can also consider several new types of partial fractional derivatives using the multidimensional convolution products

$$\big(\mathbf{a} *_0 \mathbf{b}\big)(\mathbf{x}) := \int_0^{x_1} \cdot ... \cdot \int_0^{x_n} \mathbf{a}\big(x_1 - s_1, ..., x_n - s_n\big)\mathbf{b}\big(s_1, ..., s_n\big)\, ds_1 ... ds_n,$$

for $\mathbf{x} = \big(x_1, ..., x_n\big) \in [0, +\infty)^n$, where $\mathbf{a},\ \mathbf{b} \in L^1_{loc}([0, +\infty)^n)$, and

$$\big(\mathbf{a} \circ \mathbf{b}\big)(\mathbf{x}) := \int_{-\infty}^{x_1} \cdot ... \cdot \int_{-\infty}^{x_n} \mathbf{a}\big(x_1 - s_1, ..., x_n - s_n\big)\mathbf{b}\big(s_1, ..., s_n\big)\, ds_1 ... ds_n, \tag{57}$$

for $\mathbf{x} = \big(x_1, ..., x_n\big) \in \mathbb{R}^n$, where $\mathbf{a} \in L^1_{loc}([0, +\infty)^n)$ and $\mathbf{b} \in L^1_{loc}(\mathbb{R}^n)$. It is clear that Equation (57) presents an extension of the generalized Weyl $\mathbf{a}$-integral; if $\mathbf{a} \in L^1_{loc}([0, +\infty)^n)$, $\mathbf{u} \in L^1_{loc}(\mathbb{R}^n)$, $\alpha_j \geq 0$ for $1 \leq j \leq n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, then we also define

$$\mathbb{D}^{\alpha, \mathbf{a}, 1}_W \mathbf{u} := \frac{\partial^m}{\partial x_1^{m_1} \cdot ... \cdot \partial x_n^{m_n}} \big(\mathbf{a} \circ \mathbf{u}\big)(\mathbf{x}), \quad \mathbf{x} = \big(x_1, ..., x_n\big) \in \mathbb{R}^n,$$

where $m_j = \lceil \alpha_j \rceil$ for $1 \leq j \leq n$ and $m = m_1 + ... + m_n$. It is worth noting that the Formulae (18) and (19) continue to hold in this framework.

In the discrete framework, several new types of fractional partial difference operators can be introduced and analyzed using the multidimensional convolution products $*_0$, $\circ$ and the sequences $a : \mathbb{N}_0^n \to \mathbb{C}$ which do not have the form (54). We will consider such operators elsewhere.

Let us finally note that the multidimensional fractional calculus is still a very unexplored field of mathematics. It is our strong belief that the partial fractional differential-difference equations will receive the considerable attention of authors in the near future. Without any doubt, this will reinforce the significance of our research and greatly enhance the impact of this paper.

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