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Semi-Classical Limit and Quantum Corrections in Non-Coincidence Power-Law $f(Q)$-Cosmology

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Abstract: Within the framework of symmetric teleparallel $f(Q)$-gravity, using a connection defined in the non-coincidence gauge, we derive the Wheeler–DeWitt equation of quantum cosmology. The gravitational field equation in $f(Q)$-gravity permits a minisuperspace description, rendering the Wheeler–DeWitt equation a single inhomogeneous partial differential equation. We use the power-law $f(Q) = f_0 Q^{\mu}$ model, and with the application of linear quantum observables, we calculate the wave function of the universe. Finally, we investigate the effects of the quantum correction terms in the semi-classical limit.

Keywords: symmetric teleparallel *f(Q)*-gravity; nonmetricity gravity; Wheeler-DeWitt equation; exact solutions

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1. Introduction

Symmetric teleparallel general relativity [1–6] (STEGR) is a gravitational theory that is equivalent to general relativity (GR). In STEGR, the geometry of physical space is described by a metric tensor as in GR, but the auto-parallels are defined by a symmetric and flat connection that inherits the symmetries of the metric tensor. This leads to different auto-parallels than those of GR, which are constructed by the Levi–Civita connection.

Assume that $\tilde{\Gamma}^{\kappa}_{\mu\nu}$ is a general connection. This can be decomposed [7] as follows: $\tilde{\Gamma}^{\kappa}_{\mu\nu} = \left\{ {\kappa \atop \mu\ \nu} \right\} + 2\Gamma^{\kappa}_{[\mu\nu]} + \Delta^{\kappa}_{\mu\nu}$, where $\left\{ {\kappa \atop \mu\ \nu} \right\}$ is the Levi–Civita connection, $\Gamma^{\kappa}_{[\mu\nu]}$ is the torsion tensor, and $\Delta^{\kappa}_{\mu\nu}$ denotes the symmetric and flat nonmetricity part [8]. In STEGR, the connection is flat, so it leads to a zero-valued curvature tensor $R^{\kappa}{}_{\lambda\mu\nu} = 0$ and it is symmetric, i.e., $\mathrm{T}^{\kappa}_{\mu\nu} = 0$. Thus, only the nonmetricity tensor survives, which leads to the nonmetricity scalar Q. The latter scalar substitutes the Ricci scalar of the Einstein–Hilbert Action integral, leading to STEGR. As the Ricci scalar R and the nonmetricity scalar Q differ by a boundary term [1], it follows that STEGR is equivalent to GR.

Nowadays, GR is challenged by the analysis of recent cosmological data [9–14]. This has led the scientific community to introduce alternative and modified theories of gravity. Within the symmetry and teleparallel theory, the simplest modification is the $f(Q)$-gravity [15,16], where the Lagrangian function for the gravitational action integral is a nonlinear function of the nonmetricity scalar Q. In the linear limit of the function, f, the STEGR theory, with or without the cosmological constant term, is recovered. $f(Q)$-gravity is the analog in the STEGR framework of other modified f-theories defined by the Levi–Civita connection or in teleparallelism; see Refs. [17–24] and references therein.

The dynamical degrees of freedom introduced by the nonlinear $f(Q)$ can be attributed to scalar fields [25]. In the scalar field description, the $f(Q)$-theory is equivalent to a specific

case of the scalar nonmetricity theory, where the scalar field is non-minimally coupled to gravity. Therefore, $f(Q)$-gravity has properties similar to the Machian theory, even if the theory is not purely Machian. For more details, refer to [26,27].

The $f(Q)$-theory suffers from two major problems in the cosmological perturbations of a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background geometry. In particular, it suffers from strong coupling and the appearance of ghosts [28,29]. Despite the fact that the $f(Q)$-theory fails to explain the global evolution of the universe, it provides a unique mechanism leading to the construction of different toy models with interesting applications in gravitational physics. See, for instance, [30–41] and references therein. For other modifications of the STEGR theory, we refer the reader to [42–48]. Moreover, the $f(Q)$-theory is the simplest mathematical theory that can be applied to understand the effects of the selection of the connection in modified symmetric teleparallel theories.

This investigation deals with quantum cosmology within the framework of the $f(Q)$-theory and the effects of the quantum correction in the semiclassical limit [49]. Specifically, for FLRW geometry, we assume a connection defined in the non-coincidence gauge where the field equations admit a minisuperspace description. For this specific connection, the field equations are derived from a point-like Lagrangian with three dependent variables: the scale factor of the FLRW geometry and two scalar fields, which attribute the degrees of freedom provided by the definition of the connection and the nonlinear function $f(Q)$. For this two-scalar field cosmological model, we write the Wheeler–DeWitt (WDW) equation of quantum cosmology [50,51]. We employ the theory of symmetries of differential equations [52] to define quantum observables. These are applied to define similarity transformations and derive the wave function of the universe.

In the hydrodynamic description of quantum cosmology [53,54], we make use of the quantum observables to write down conservation laws for the classical field equations and to calculate the classical solutions, because for the power-law $f(Q)$-model, the classical field equations form an superintegrable Hamiltonian system. The effects of the quantum correction terms related to the nonzero Ricci scalar term of the minisuperspace [50,51] and of the Bohmian potential term [55–59] are discussed. It follows that the quantum corrections do not affect the general evolution and dynamics of the classical gravitational model. The structure of the paper is as follows:

In Section 2, we briefly discuss STEGR and its modification, $f(Q)$-gravity. This gravitational theory has a scalar field description; the theory is equivalent to a specific case of the scalar-nonmetricity gravitational model. Furthermore, the effects of conformal transformations in the theory are investigated, where we can understand the degrees of freedom introduced by the theory.

The case of an isotropic and homogeneous universe in the framework of $f(Q)$-gravity is presented in Section 3. We review previous results on the different families of connections in the FLRW background and explore how the selection of the connection affects the gravitational model and the evolution of physical space. Moreover, by applying scalar fields, we introduce the minisuperspace description for the field equations.

For the connection where the Lagrangian of the field equations is point-like—describing the motion of classical particles in curved space under a conservative force term—in Section 4, we derive the WDW equation, which is a single inhomogeneous partial differential equation. For the power-law $f(Q)$ model in Section 5, we present the quantum observables for the WDW equation. The latter is used to construct similarity transformations aimed at reducing the WDW equation into an ordinary differential equation, providing a closed-form expression for the wave function of the universe. In Section 6, we employ the Madelung representation of quantum mechanics to derive the classical and semiclassical limits where quantum correction terms are introduced in the gravitational field equations. For the two models, we explicitly solve the reduced gravitational field equations using the Hamilton–Jacobi theory. It follows that the quantum correction terms do not affect the main dynamics of the classical solution. The two asymptotic limits of the classical cosmological solution describe the self-similar solution related to the power-law model,

as well as a cosmological solution with two perfect fluids that do not interact and have constant equations of state parameters. Finally, in Section 7, we draw our conclusions.

2. Symmetric Teleparallel $f(Q)$-Gravity

In this section, we discuss the main properties and definitions of STEGR and introduce the gravitational model under our consideration, which is that of symmetric teleparallel $f(Q)$-gravity.

2.1. STEGR

In the STEGR framework, the physical space is described by a four-dimensional metric tensor $g_{\mu\nu}$, and the connection $\Gamma^{\kappa}_{\mu\nu}$, which is symmetric, i.e., $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$, and flat. Hence, there exists a point transformation $x^{\mu} \to x^{\mu'}$, such that all the components of the connection in the new coordinate system are zero, i.e., $\Gamma^{\kappa'}_{\mu'\nu'} = 0$ [8].

Because of these two fundamental properties for the connection $\Gamma^{\kappa}_{\mu\nu}$, the curvature tensor is defined as follows:

$$R^{\kappa}{}_{\lambda\mu\nu}(\Gamma) = \frac{\partial \Gamma^{\kappa}{}_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\kappa}{}_{\lambda\mu}}{\partial x^{\nu}} + \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\kappa}{}_{\mu\sigma}, \tag{1}$$

and the torsion tensor reads as follows:

$$\mathrm{T}^{\kappa}_{\mu\nu}(\Gamma) = \frac{1}{2}\Big(\Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\nu\mu}\Big), \tag{2}$$

are always zero.

The fundamental scalar of STEGR is the nonmetricity scalar [1], as follows:

$$Q = Q_{\kappa\mu\nu}P^{\kappa\mu\nu}, \tag{3}$$

where $Q_{\kappa\mu\nu}$ is the nonmetricity scalar $Q_{\kappa\mu\nu} = \nabla_{\kappa} g_{\mu\nu}$, as follows:

$$Q_{\kappa\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} - \Gamma^{\sigma}{}_{\kappa\mu}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\kappa\nu}g_{\mu\sigma}, \tag{4}$$

and $P^{\kappa}_{\mu\nu}$ is defined as follows:

$$P^{\kappa}_{\mu\nu} = \frac{1}{4}Q^{\kappa}_{\mu\nu} + \frac{1}{2}Q_{(\mu\ \ \nu)}^{\ \ \kappa} + \frac{1}{4}\big(Q^{\kappa} - \tilde{Q}^{\kappa}\big)g_{\mu\nu} - \frac{1}{4}\delta^{\kappa}_{(\mu}Q_{\nu)} \tag{5}$$

where the vector fields Q_{μ} and $\tilde{Q}_{\mu}$ are given by the following expressions:

$$Q_{\mu} = Q_{\mu\nu}{}^{\nu}\ ,\ \tilde{Q}_{\mu} = Q^{\nu}{}_{\mu\nu}. \tag{6}$$

The gravitational action integral in STEGR is defined [1] as follows:

$$S = \int d^4x\sqrt{-g}Q. \tag{7}$$

The nonmetricity scalar Q for the connection $\Gamma^{\kappa}_{\mu\nu}$ and the Ricci scalar $\hat{R}\big(\hat{\Gamma}\big)$ for the Levi–Civita connection $\hat{\Gamma}^{\kappa}_{\mu\nu}$ of the metric tensor are as follows:

$$\hat{\Gamma}^{\kappa}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\big(g_{\mu\kappa,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\big) \tag{8}$$

and are related to $Q = \hat{R} + B$, where B is a boundary term [6], as follows:

$$B = -\frac{1}{2}\mathring{\nabla}_{\lambda}P^{\lambda}, \tag{9}$$

and $P^{\lambda} = P^{\lambda}_{\mu\nu}\bar{g}^{\mu\nu}$. Consequently, the variation of the action integral (7) leads to the field equations of Einstein's GR, which follow from the Einstein–Hilbert Action. Finally, operator $\mathring{\nabla}_{\lambda}$ in (9) denotes the covariant derivative with respect to the Levi–Civita connection $\hat{\Gamma}^{\kappa}_{\mu\nu}$ (8).

2.2. $f(Q)$-Gravity

The simplest modification of the STEGR action integral (7) is the introduction of the cosmological constant. Nevertheless, inspired by other GR modifications, the introduction of nonlinear terms in the nonmetricity scalar Q in (7) leads to the introduction of dynamical degrees of freedom, which can influence the dynamics and describe the cosmic evolution.

In this general concept, we assume the gravitational action integral to be [15,16], as follows:

$$S_{f(Q)} = \int d^4x\sqrt{-g}f(Q), \tag{10}$$

where $f(Q)$ is a smooth differentiable function, and when $f''(Q) = 0$, the limit of STEGR with or without the cosmological constant term is recovered. With a prime $'$, we note the total derivative with respect to the scalar Q, i.e., $f'(Q) = \frac{df(Q)}{dQ}$ and $f''(Q) = \frac{d^2f(Q)}{dQ^2}$.

The modified gravitational field equations related to the action integral (10) follow from the variation with respect to the metric tensor $g_{\mu\nu}$.

The gravitational field equations are as follows [15,16] :

$$\frac{2}{\sqrt{-g}}\nabla_{\lambda}\left(\sqrt{-g}f'(Q)P^{\lambda}_{\mu\nu}\right) - \frac{1}{2}f(Q)g_{\mu\nu} + f'(Q)\left(P_{\mu\rho\sigma}Q_{\nu}^{\rho\sigma} - 2Q_{\rho\sigma\mu}P^{\rho\sigma}{}_{\nu}\right) = 0. \tag{11}$$

equivalently, we have the following:

$$f'(Q)G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(f(Q) - f'(Q)Q\right) + 2f''(Q)P^{\lambda}{}_{\mu\nu}\nabla_{\lambda}Q = 0, \tag{12}$$

where $G_{\mu\nu}$ is the equivalent of the Einstein tensor in STEGR, as follows:

$$G_{\mu\nu} = \left(P_{\mu\rho\sigma}Q_{\nu}^{\rho\sigma} - 2Q_{\rho\sigma\mu}P^{\rho\sigma}{}_{\nu}\right) + \frac{2}{\sqrt{-g}}\nabla_{\lambda}\left(\sqrt{-g}P^{\lambda}_{\mu\nu}\right). \tag{13}$$

When $Q = Q_0$, expression (12) reads as follows:

$$G_{\mu\nu} + \Lambda_{eff}g_{\mu\nu} = 0, \tag{14}$$

where Λ_{eff} is an effective cosmological constant term defined as $\Lambda_{eff} = \frac{1}{2}\frac{f(Q_0) - f'(Q_0)Q_0}{f'(Q_0)}$.

Therefore, for the case where $Q = Q_0$ and $f(Q_0) - f'(Q_0)Q_0 = 0$, the vacuum solutions of STEGR are recovered. On the other hand, for the case where $Q = Q_0$ and $f(Q_0) - f'(Q_0)Q_0 \neq 0$, the effects of a nonzero cosmological constant are present in STEGR.

Furthermore, varying the action integral (10) with respect to the symmetric and flat connection $\Gamma^{\kappa}_{\mu\nu}$ yields the following equation of motion:

$$\nabla_{\mu}\nabla_{\nu}\left(\sqrt{-g}f'(Q)P^{\mu\nu}{}_{\sigma}\right) = 0. \tag{15}$$

The connection is characterized as being defined in the "coincidence gauge" when Equation (15) is identically satisfied. On the other hand, we refer to the connection as being defined in the "non-coincidence gauge" [15,16].

The transformation rule for the metric tensor follows that of a tensor field. However, the connection is not a tensor and, thus, has a different transformation rule. Specifically, the connection is coordinate dependent, although the tensors defined by the connection, such as the curvature tensor, the torsion tensor, and the nonmetricity tensor, are coordinate-independent.

When we consider a specific line element for the metric tensor, this corresponds to the definition of a proper coordinate system. Thus, the components of the connection may not be identically zero in these coordinates, and Equation (15) is not trivially satisfied. This means that dynamical degrees of freedom are introduced due to the connection defined in the non-coincidence gauge. These dynamical degrees of freedom associated with the non-coincidence gauge are of geometric origin. As we shall see in the following section, they can be attributed to scalar fields.

2.3. Equivalency with Scalar Field Theory

We introduce the Lagrange multiplier λ_m, such that the gravitational action integral (10) reads as follows:

$$S_{f(Q)} = \int d^4x\sqrt{-g}\big(f(Q) + \lambda_m(Q - \hat{Q})\big), \tag{16}$$

where $\hat{Q} = \hat{Q}(x^\kappa)$ and is the functional expression for the nonmetricity scalar Q.

Variation with respect to the scalar Q leads to the equation of motion $\frac{\delta S_{f(Q)}}{\delta Q} = 0$, that is, $\lambda_m = -f'(Q)$.

By replacing expression (16), we have the following:

$$S_{f(Q)} = \int d^4x\sqrt{-g}\big(f'(Q)\hat{Q} + \big(f(Q) - Qf'(Q)\big)\big). \tag{17}$$

We introduce the scalar field $\phi = f'(Q)$, and the potential function $V(\phi) = (Qf'(Q) - f(Q))$, and the latter action integral is expressed in the following simpler form:

$$S_{f(Q)} = \int d^4x\sqrt{-g}(\phi\hat{Q} - V(\phi)). \tag{18}$$

This is analogous to the O'Hanlon gravity [60] in the framework of the symmetric teleparallel theory. Hence, we can say that nonmetricity $f(Q)$-gravity has properties similar to those of a Machian theory [61], although the theory is not purely Machian [26,27].

The action integral (18) is a particular case of the more general theory with a scalar field minimally coupled to gravity—that is, of the scalar-nonmetricity theory—defined as follows:

$$S_{ST\phi} = \int d^4x\sqrt{-g}\left(\frac{F(\phi)}{2}Q - \frac{\omega(\phi)}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right), \tag{19}$$

where the gravitational field equations are as follows:

$$F(\phi)G_{\mu\nu} + 2F_{,\phi}\phi_{,\lambda}P^{\lambda}{}_{\mu\nu} + g_{\mu\nu}V(\phi) + \frac{\omega(\phi)}{2}\left(g_{\mu\nu}g^{\lambda\kappa}\phi_{,\lambda}\phi_{,\kappa} - \phi_{,\mu}\phi_{,\nu}\right) = 0. \tag{20}$$

The equation of motion for the scalar field reads as follows:

$$\frac{\omega(\phi)}{\sqrt{-g}}g^{\mu\nu}\partial_\mu\big(\sqrt{-g}\partial_\nu\phi\big) + \frac{\omega_{,\phi}}{2}g^{\lambda\kappa}\phi_{,\lambda}\phi_{,\kappa} + \frac{1}{2}F_{,\phi}Q - V_{,\phi} = 0, \tag{21}$$

while the equation of motion for the connection is as follows:

$$\nabla_\mu\nabla_\nu\left(\sqrt{-g}F(\phi)P^{\mu\nu}{}_{\sigma}\right) = 0. \tag{22}$$

For a linear function, $F(\phi) = 2\phi$, and $\omega(\phi) = 0$, the $f(Q)$-gravity is recovered, and the field Equation (12) is as follows:

$$\phi G_{\mu\nu} + 2\phi_{,\lambda}P^{\lambda}{}_{\mu\nu} + g_{\mu\nu}V(\phi) = 0. \tag{23}$$

Moreover, Equation (21) is simultaneously satisfied by the definition of the scalar field potential $V(\phi)$, while the equation of motion for the connection is simplified to the following:

$$\nabla_\mu \nabla_\nu \left(\sqrt{-g} \phi P^{\mu\nu}{}_\sigma \right) = 0. \tag{24}$$

2.4. Conformal Transformation

In the previous lines, we learned that $f(Q)$ is a partially Machian theory, where, in the scalar field description, the theory is defined in the so-called Jordan frame. In the following lines, we discuss the effects of conformal transformations in $f(Q)$-gravity and introduce the equivalent theory in the Einstein frame.

Let the two conformally related four-dimensional metrics be $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, defined as follows:

$$\bar{g}_{\mu\nu} = e^{2\Omega(x^\kappa)} g_{\mu\nu} \ , \ \bar{g}^{\mu\nu} = e^{-2\Omega(x^\kappa)} g^{\mu\nu}, \tag{25}$$

where $\Omega(x^\kappa)$ is a smooth differentiable function. $\Omega(x^\kappa)$ is known as the conformal factor and defines the transformations.

The geometric quantities, which define $f(Q)$-gravity for the conformally related metrics are related as follows [27]:

$$\bar{Q}_{\lambda\mu\nu} = e^{2\Omega} Q_{\lambda\mu\nu} + 2\Omega_{,\lambda} \bar{g}_{\mu\nu}. \tag{26}$$

and we have the following:

$$\bar{P}^\lambda = \bar{P}^\lambda_{\mu\nu} \bar{g}^{\mu\nu} = e^{-2\Omega} P^\lambda + 3\Omega'^\lambda. \tag{27}$$

Therefore, the nonmetricity scalars Q and $\bar{Q}$ are as follows:

$$\bar{Q} = \bar{Q}_{\lambda\mu\nu} \bar{P}^{\lambda\mu\nu} = e^{-2\Omega} Q + \left(2\Omega_{,\lambda} P^\lambda + 6\Omega_\lambda \Omega'^\lambda \right). \tag{28}$$

Assume now the action integral (18) for the $f(\bar{Q})$-theory in the framework of the space with metric $\bar{g}_{\mu\nu}$. Then, the equivalent theory for the metric tensor $g_{\mu\nu}$ is given by the following action integral [62]:

$$\bar{S}_{f(\bar{Q})} = \int d^n x \sqrt{-g} \left(Q - \frac{1}{2} B \ln \phi + \frac{3}{2\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{V(\varphi)}{\phi^2} \right), \tag{29}$$

where we assume that $\Omega = -\frac{1}{2} \ln \phi$, and scalar B is the boundary term that relates the Ricci scalar $\hat{R}(\hat{\Gamma})$ for the Levi–Civita connection to the nonmetricity scalar, given by expression (9). Indeed, the boundary term is defined as [62] $B = -\frac{1}{2} \mathring{\nabla}_\lambda P$, equivalently, [62]

$$B = \left(\tilde{Q}^\kappa - Q^\kappa \right). \tag{30}$$

We observe that the boundary term B plays an important role in the conformal equivalent description of the theory, and it is another scalar field. The definition of the boundary term is directly related to the definition of the connection Γ and the equation of motion (15).

At this point, it is important to mention that the conformal equivalency of the gravitational theory in the two frames relates only to the trajectory solutions for the field equations. In general, it is not an equivalence of physical properties. Nevertheless, for the scalar-nonmetricity theory, it has been found that conformal transformations preserve the main eras of the cosmological history and evolution.

3. Isotropic and Homogeneous Cosmology

In cosmological scales, physical space is assumed to be homogeneous and isotropic, as described by the FLRW line element. The latter (in spherical coordinates) is expressed as follows:

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{31}$$

where k is the spatial curvature of three-dimensional space; $N(t)$ is the lapse function, $a(t)$ is the scale factor, and $H = \frac{1}{N}\frac{\dot{a}}{a}$, $\dot{a} = \frac{da}{dt}$, is the Hubble function. We assume the comoving observer $u^\mu = \frac{1}{N}\delta^\mu_t$; thus, the Hubble function is related to the expansion rate $\theta = u^\mu_{;\mu}$ by the expression $H = \frac{1}{3}\theta$.

3.1. Symmetries

In the coordinate system (t, r, θ, φ), the six isometries of the line element (31) are as follows:

$$K^1 = \sin\varphi\partial_\theta + \frac{\cos\varphi}{\tan\theta}\partial_\varphi, \quad K^2 = -\cos\varphi\partial_\theta + \frac{\sin\varphi}{\tan\theta}\partial_\varphi, \quad K^3 = \partial_\varphi,$$

$$K^4 = \sqrt{1-kr^2}\left(\sin\theta\cos\varphi\partial_r + \frac{1}{r}\left(\cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\right)\partial_\varphi\right),$$

$$K^5 = \sqrt{1-kr^2}\left(\sin\theta\sin\varphi\partial_r + \frac{1}{r}\left(\cos\theta\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\right)\partial_\varphi\right),$$

$$K^6 = \sqrt{1-kr^2}\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\varphi\right).$$

For $k = 0$, the latter symmetry vectors form the $E^3 \otimes SO(2)$ Lie algebra, otherwise, for $k \neq 0$, the six symmetry vectors are the elements of the $SO(4)$ Lie algebra.

The requirement for the connection to inherit the symmetry of the background geometry involves the following set of constraints:

$$\mathcal{L}_{K^I}\Gamma^\kappa_{\mu\nu} = 0, \tag{32}$$

where $\mathcal{L}_{K^I}$ is the Lie derivative with respect the vector field K^I and $I = 1, 2, 3, 4, 5, 6$.

The definition of the Lie derivative depends on the transformation rule. Because the connection has a different transformation rule from that of tensor fields, the Lie derivative differs accordingly. Specifically, in terms of coordinates, the Lie derivative of the connection reads as follows:

$$L_{K^I}\Gamma^\kappa_{\mu\nu} = K^{I\ \kappa}_{,\ \mu\nu} + \Gamma^\kappa_{\mu\nu,r}K^{I\ r} - K^{I\ \kappa}_{,\ r}\Gamma^r_{\mu\nu} + K^{I\ s}_{,\ \mu}\Gamma^\kappa_{s\nu} + K^{I\ s}_{,\ \nu}\Gamma^\kappa_{\mu s}. \tag{33}$$

Because the connection is symmetric, expression (33) is simplified as follows:

$$L_{K^I}\Gamma^\kappa_{\mu\nu} = \nabla_\nu\nabla_\mu\left(K^{I\ \kappa}\right) - R^\kappa_{\mu\nu\lambda}K^{I\ \lambda}. \tag{34}$$

However, in symmetric teleparallel theory, the connection is flat; that is,

$$R^\kappa_{\mu\nu\lambda}(\Gamma) = 0. \tag{35}$$

Consequently, the requirement for the connection to inherit the symmetries of the background space is equivalent to the following set of differential equations:

$$\nabla_\nu\nabla_\mu\left(K^{I\ \kappa}\right) = 0. \tag{36}$$

3.2. Symmetric and Flat Connection

For the FLRW line element (31) and the requirements (35), (36), it follows that there are different families of symmetric connections $\Gamma^{\kappa}_{\mu\nu}$. One family of connections is defined for $k \neq 0$ and three families of connections are defined for $k = 0$ [63,64].

The common nonzero components for the four families of connections are as follows:

$$\Gamma^{r}{}_{tr} = \Gamma^{r}{}_{rt} = \Gamma^{\theta}{}_{t\theta} = \Gamma^{\theta}{}_{\theta t} = \Gamma^{\varphi}{}_{t\varphi} = \Gamma^{\varphi}{}_{\varphi t} = -\frac{k}{\gamma(t)},$$
$$\Gamma^{r}{}_{rr} = \frac{kr}{1-kr^2}\,,\ \Gamma^{r}_{\theta\theta} = -r\left(1-kr^2\right)\,,\ \Gamma^{r}{}_{\varphi\varphi} = -r\sin^2\theta\left(1-\kappa r^2\right),$$
$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \Gamma^{\varphi}{}_{r\varphi} = \Gamma^{\varphi}{}_{\varphi r} = \frac{1}{r}\,,\ \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta\,,\ \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta.$$

For $k \neq 0$, the additional nonzero components for the family Γ^{k} are as follows:

$$\Gamma^{t}{}_{tt} = -\frac{k+\dot{\gamma}(t)}{\gamma(t)}, \quad \Gamma^{t}{}_{rr} = \frac{\gamma(t)}{1-kr^2} \quad \Gamma^{t}_{\theta\theta} = \gamma(t)r^2, \quad \Gamma^{t}{}_{\varphi\varphi} = \gamma(t)r^2\sin^2(\theta),$$

by comparing the latter connections with the notation presented in [63], it follows that $\gamma(t) = C_2(t)$.

Moreover, for $k = 0$, connection Γ^{A} has the additional nonzero components

$$\Gamma^{t}{}_{tt} = \gamma(t),$$

where Γ^{A} is connection $\Gamma^{(III)}_{Q}$ in [63] and $\gamma(t) = C_1(t)$.

Connection Γ^{B} is defined for $k = 0$, with the nonzero components

$$\Gamma^{t}{}_{tt} = \frac{\dot{\gamma}(t)}{\gamma(t)} + \gamma(t), \quad \Gamma^{r}{}_{tr} = \Gamma^{r}{}_{rt} = \Gamma^{\theta}{}_{t\theta} = \Gamma^{\theta}{}_{\theta t} = \Gamma^{\varphi}{}_{t\varphi} = \Gamma^{\varphi}{}_{\varphi t} = \gamma(t),$$

and it is connection $\Gamma^{(I)}_{Q}$ of [63] with $\gamma(t) = C_3(t)$

Finally, for $k = 0$, family Γ^{C} has nonzero components

$$\Gamma^{t}{}_{tt} = -\frac{\dot{\gamma}(t)}{\gamma(t)}, \quad \Gamma^{t}{}_{rr} = \gamma(t), \quad \Gamma^{t}{}_{\theta\theta} = \gamma(t)r^2, \quad \Gamma^{t}{}_{\varphi\varphi} = \gamma(t)r^2\sin^2\theta,$$

which is compared with connection $\Gamma^{(II)}_{Q}$ of [63] and $\gamma(t) = C_2(t)$.

The connection Γ^{A} is the unique connection defined in the coincidence gauge, and the function $\gamma(t)$ plays no role in the gravitational field equations. Nevertheless, the remaining three families of connections, Γ^{k}, Γ^{B}, and Γ^{C}, are defined in the non-coincidence gauge. In the framework of $f(Q)$-gravity, the function $\gamma(t)$ is constrained by the equation of motion (15).

3.3. Field Equations in $f(Q)$-Gravity

In the following lines, we present the field equations for $f(Q)$-gravity in an FLRW background geometry. The selection of the connection is crucial in the theory; thus, for each family of connections, there is a different set of results of field equations.

For the nonzero spatial curvature and connection Γ^{k}, we calculate the nonmetricity scalar as follows:

$$Q\left(\Gamma^{k}\right) = -\frac{6\dot{a}^2}{N^2a^2} + \frac{3\gamma}{a^2}\left(\frac{\dot{a}}{a} + \frac{\dot{N}}{N}\right) + \frac{3\dot{\gamma}}{a^2} + k\left(\frac{6}{a^2} + \frac{3}{\gamma N^2}\left(\frac{\dot{N}}{N} + \frac{\dot{\gamma}}{\gamma} - \frac{3\dot{a}}{a}\right)\right). \tag{37}$$

The gravitational field equations in $f(Q)$-gravity are as follows:

$$0 = 3f'(Q)H^2 + \frac{1}{2}\big(f(Q) - Qf'(Q)\big) - \frac{3\gamma\dot{Q}f''(Q)}{2a^2} + 3k\left(\frac{f'(Q)}{a^2} - \frac{\dot{Q}f''(Q)}{2\gamma N^2}\right), \tag{38}$$

$$\begin{aligned} 0 = &-\frac{2}{N}\big(f'(QH)\big)^{\cdot} - 3H^2 f'(Q) - \frac{1}{2}\big(f(Q) - Qf'(Q)\big) \\ &+ \frac{\gamma\dot{Q}f''(Q)}{2a^2} - k\left(\frac{f'(Q)}{a^2} + \frac{3\dot{Q}f''(Q)}{2\gamma N^2}\right), \end{aligned} \tag{39}$$

and the equation of motion for the connection is as follows:

$$0 = \big(f''\dot{Q}\big)^{\cdot}\left(1 + k\frac{a^2}{N^2\gamma^2}\right) + f''\dot{Q}\left(\left(1 + 3k\frac{a^2}{N^2\gamma^2}\right)NH + \left(1 - k\frac{a^2}{N^2\gamma^2}\right)\frac{\dot{N}}{N} + 2\frac{\dot{\gamma}}{\gamma}\right). \tag{40}$$

In the previous field equations, if we set $k = 0$, we recover the gravitational field equations for the connection Γ^C.

Furthermore, for the Γ^A connection defined in the coincidence gauge, the nonmetricity scalar is as follows:

$$Q\Big(\Gamma^A\Big) = -6H^2, \tag{41}$$

and the gravitational field equations are as follows:

$$0 = 3H^2 f'(Q) + \frac{1}{2}\big(f(Q) - Qf'(Q)\big), \tag{42a}$$

$$0 = -\frac{2}{N}\big(f'(Q)H\big)^{\cdot} - 3H^2 f'(Q) - \frac{1}{2}\big(f(Q) - Qf'(Q)\big). \tag{42b}$$

Finally, for the family of connection Γ^B, the nonmetricity scalar reads as follows:

$$Q = -6H^2 + \frac{3\gamma}{N}\left(3H - \frac{\dot{N}}{N^2}\right) + \frac{3\dot{\gamma}}{N^2}, \tag{43}$$

while the gravitational field equations are as follows:

$$0 = 3H^2 f'(Q) + \frac{1}{2}\big(f(Q) - Qf'(Q)\big) + \frac{3\gamma\dot{Q}f''(Q)}{2N^2} \tag{44a}$$

$$0 = -\frac{2}{N}\big(f'(Q)H\big)^{\cdot} - 3H^2 f'(Q) - \frac{1}{2}\big(f(Q) - Qf'(Q)\big) + \frac{3\gamma\dot{Q}f''(Q)}{2N^2}. \tag{44b}$$

Because Γ^B is defined in the non-coincidence gauge, the equation of motion for the connection is as follows:

$$0 = \big(f''\dot{Q}\big)^{\cdot} + N\dot{Q}\left(3H - \frac{\dot{N}}{N^2}\right)f''(Q). \tag{45}$$

3.4. Minisuperspace Description

A novel property of these cosmological models is that they admit a "minisuperspace" description. In particular, for each connection, there exists a Lagrangian function whose variation leads to the corresponding field equations.

For the Γ^k connection (and Γ^C for $k = 0$) the corresponding field equations follow from the variation of the (non-canonical) Lagrangian function:

$$L\Big(\Gamma^k\Big) = -\frac{3}{N}a\phi\dot{a}^2 + 3kNa\phi - \frac{N}{2}V(\phi) + \frac{3k}{2N}a^3\dot{\Psi}\dot{\phi} - \frac{3}{2}\frac{aN}{\dot{\Psi}}\dot{\phi}, \tag{46}$$

where $\phi = f'(Q)$, $V(\phi) = (Qf'(Q) - f(Q))$ and $\dot{\Psi} = \frac{1}{\gamma}$. We observe that the latter Lagrangian has kinetic components in the denominator.

Similarly, for the Γ^A connection, the Lagrangian function is as follows:

$$L\left(\Gamma^A\right) = -\frac{6}{N}a\phi\dot{a}^2 - Na^3V(\phi). \tag{47}$$

Finally, for the Γ^B connection, the corresponding Lagrangian of the field equation is as follows:

$$L\left(\Gamma^B\right) = -\frac{3}{N}\phi a\dot{a}^2 - \frac{3}{2N}a^3\dot{\phi}\dot{\psi} - \frac{N}{2}a^3V(\phi), \tag{48}$$

where the scalar field ψ is now related to the connection as $\dot{\psi} = \gamma$. The Lagrangian function (48) is in the form of a point-like dynamical system. In particular, it describes a constraint dynamical system where the scale factor and two scalar fields play the roles of particles with interaction $U_{eff} = a^3V(\phi)$.

The lapse function in the Lagrangians is necessary to reconstruct the constraint equation in each case.

Between the above Lagrangian functions, only that that correspond to connections Γ^A and Γ^B describe dynamical system where expressed in the form $L = \frac{1}{N}K_E - NU_{eff}$, where K is the kinetic energy, $K_E = \frac{1}{2}\mathcal{G}_{AB}\dot{q}^A\dot{q}^B$ and U_{eff} is the effective potential. Thus, for these two dynamical systems, we can write the Hamiltonian function in the following form:

$$H \equiv N\left(K_E + U_{eff}\right) = 0. \tag{49}$$

We make use of this property in order to continue with the quantization of the field equations. The quantification of the field equations, corresponding to the Γ^A connection, has been studied in detail. Because of the nature of the Lagrangian, Dirac's method for the quantization of dynamical constraint systems was applied in [65]. Due to the existence of second-class constraints, the quantization process was different from the usual WDW formalism, where only first-class constraints existed. See the more recent study [66].

The field equations for the Γ_B connection described by the point-like Lagrangian function (48) possess only first-class constraints, which means that the quantization process is similar to that of the WDW formalism and different from that of the corresponding system for the connection Γ_A.

4. The Wheeler–DeWitt Equation

In GR, the Wheeler–DeWitt equation follows from the quantization of the constraint equation in the ADM formalism (for further discussion, we refer the reader to [50,51,67]). The WDW equation is a hyperbolic functional differential equation on superspace and represents a family of differential equations at different points. However, when there exists a minisuperspace description, the WDW reduces to a single differential equation.

In the minisuperspace description, the Hamiltonian constraint is expressed as follows:

$$H = N\left[\frac{1}{2}\mathcal{G}^{AB}P_AP_B + \mathcal{U}(\mathbf{q})\right] = N\mathcal{H} \equiv 0, \tag{50}$$

where $\mathcal{G}^{AB}$ is the minisuperspace.

The classical field equations are invariant under conformal transformations. This property should also hold for the WDW equation of quantum cosmology. The quantization $P_A = i\hbar\partial_A$ leads to a differential equation of the Klein–Gordon type, which is generally not conformally invariant. Therefore, the Klein–Gordon equation is replaced by the Yamabe equation to ensure conformal invariance. This is achieved by modifying the potential with the term $\frac{n-2}{8(n-1)}\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar of the minisuperspace $\mathcal{G}^{AB}$ with dimension n.

The WDW equation reads as follows:

$$\mathcal{H}\Psi = \left[\hbar^2\left(\frac{1}{2}\Delta - \frac{n-2}{8(n-1)}\mathcal{R}\right) - \mathcal{U}(\mathbf{q})\right]\Phi(\mathbf{q}) = 0, \tag{51}$$

where $\Delta = \frac{1}{\sqrt{-\mathcal{G}}}\partial_A\left(\sqrt{-\mathcal{G}}\mathcal{G}^{AB}\partial_B\right)$ is the Laplace operator with respect to the minisuperspace $\mathcal{G}^{AB}$.

As we shall see in the following, a nonzero Ricci scalar $\mathcal{R}$ introduces the Hamiltonian function of the semiclassical system—a potential term that corresponds to quantum corrections.

4.1. Connection Γ^B

From Lagrangian function (48), we calculate the three-dimensional minisuperspace as follows:

$$\mathcal{G}_{AB} = \begin{pmatrix} 6\phi a & 0 & 0 \\ 0 & 0 & \frac{3}{2}a^3 \\ 0 & \frac{3}{2}a^3 & 0 \end{pmatrix}, \ \mathcal{G}^{AB} = \begin{pmatrix} \frac{1}{6\phi a} & 0 & 0 \\ 0 & 0 & \frac{2}{3a^3} \\ 0 & \frac{2}{3a^3} & 0 \end{pmatrix}. \tag{52}$$

Thus, the corresponding Hamiltonian function is expressed as follows:

$$\mathcal{H} = \frac{1}{12\phi a}P_a^2 + \frac{2}{3a^3}P_\phi P_\psi - a^3 V(\phi), \tag{53}$$

with

$$P_a = 6\phi a\dot{a}\ , \ P_\phi = \frac{3}{2}a^3\dot{\psi} \text{ and } P_\psi = \frac{3}{2}a^3\dot{\phi}. \tag{54}$$

Moreover, the Ricci scalar for the minisuperspace is derived as follows:

$$\mathcal{R} = -\frac{3}{4a^3\phi}, \tag{55}$$

Therefore, the WDW Equation (51) for the Hamiltonian function (53), $\mathcal{H}\Phi = 0$, is expressed as follows:

$$\frac{1}{6a\phi}\Phi_{,aa} + \frac{4}{3a^3}\Phi_{,\phi\psi} + \frac{5}{12a^2\phi}\Phi_{,a} + \frac{1}{3\phi a^3}\Phi_{,\psi} + \left(\frac{3}{32a^3\phi} - \frac{2}{\hbar^2}a^3V(\phi)\right)\Phi = 0. \tag{56}$$

4.2. Power-Law Theory

We assume that the function $f(Q)$ is a power-law, i.e., $f(Q) = f_0Q^\mu$. The power-law function provides a cosmological history that depends on the selection of the connection [68]. However, for all connections, the power-law $f(Q)$ function is associated with the existence of self-similar cosmological solutions [69]. For the cosmological model defined by the connection Γ^B, the self-similar scaling solution is always unstable, while the unique attractor describes the accelerated de Sitter universe. This physical property remains the same in the presence of matter [70].

In the scalar field description of the $f(Q)$ theory, the power-law model $f(Q) = f_0Q^\mu$ leads to the power-law potential function $V(\phi) = V_0\phi^\kappa$, where constants V_0 and κ are related to f_0, μ as follows:

$$V_0 = (\mu - 1)f_0^{-\frac{1}{\mu-1}}\mu^{-\frac{\mu}{\mu-1}}\ , \ \kappa = \frac{\mu}{\mu - 1}\ . \tag{57}$$

Recall that for values of μ close to 1, the gravitational model describes small deviations from general relativity. We refer the reader to the interesting discussion on the power-law $f(R)$ theory presented in [71].

Therefore, the WDW Equation (56) for the power-law model is simplified as follows:

$$\frac{1}{6a\phi}\Phi_{,aa} + \frac{4}{3a^3}\Phi_{,\phi\psi} + \frac{5}{12a^2\phi}\Phi_{,a} + \frac{1}{3a^3\phi}\Phi_{,\psi} + \left(\frac{3}{32a^3\phi} - \frac{V_0}{\hbar^2}a^3\phi^\kappa\right)\Phi = 0. \tag{58}$$

The solution of the WDW Equation (58) provides the wave function $\Phi(a, \phi, \psi)$ for this specific cosmological model.

The WDW equation is a linear inhomogeneous partial differential equation. In the following, we employ the method of similarity transformations, also known as Lie symmetry analysis, to determine expressions for the wave function Φ, satisfying the partial differential Equation (58).

5. Similarity Transformations

Lie symmetry analysis is a robust method for the study of nonlinear differential equations. It provides a systematic way to investigate the algebraic properties and compute solutions for nonlinear and inhomogeneous differential equations [72,73]. The main characteristic of Lie symmetry analysis is that it allows for the construction of one-parameter point transformations, which leave the given differential equation invariant. From these point transformations, it is possible to derive similarity transformations used to facilitate solutions. This procedure is known as the reduction process, and in the case of partial differential equations, the existence of a Lie symmetry is equivalent to the existence of a quantum operator.

For a review of the wide applications of symmetry analysis in classical gravitational physics and cosmology, we refer the reader to [52]. In the framework of the WDW equation, the mathematical approach for constructing quantum operators from Lie symmetry analysis is analytically described in [74]. This method has been widely used for constructing the wave function of the universe in a broad range of gravitational models [75–82].

The application of the Lie symmetry analysis for the WDW Equation (58), i.e., the Yamabe equation, leads to the following quantum operators, as follows:

$$\Xi_1 : \left(\partial_\psi - \frac{\beta_1}{\hbar} i\right)\Phi = 0, \tag{59}$$

$$\Xi_2 : \left(-\frac{(\kappa+1)}{6} a\partial_a + \phi\partial_\phi - \frac{\beta_2}{\hbar} i\right)\Phi = 0, \tag{60}$$

$$\Xi_3 : \left(\frac{(\kappa+1)}{6}\ln\phi a\partial_a - \ln\phi\ \phi\partial_\phi + \left(\psi - \frac{2(\kappa+1)}{3}\ln a\right)\partial_\psi - \left(\frac{(K-1)}{8}\ln\phi + \frac{\beta_3}{\hbar} i\right)\right)\Phi = 0. \tag{61}$$

With the use of the latter operators, we can construct the following solution for the wave function of the universe, as follows:

$$\Phi(a,\phi,\psi) = a^{-\frac{3}{4}+\frac{2}{3}\frac{i}{\hbar}\beta_1(\kappa+1)}\phi^{-\frac{\kappa+1}{8}+\frac{i}{\hbar}\lambda}e^{-i\hbar\beta_1\psi}\left(\Phi_0^1 J_{\bar\lambda}\left(\frac{2}{3\hbar} i\sqrt{3V_0}a^3\phi^{\frac{\kappa+1}{2}}\right) + \Phi_0^2 Y_{\bar\lambda}\left(\frac{2}{3\hbar} i\sqrt{3V_0}a^3\phi^{\frac{\kappa+1}{2}}\right)\right), \tag{62}$$

where $J_{\bar\lambda}$, $Y_{\bar\lambda}$ are the Bessel functions of the first and second kinds, respectively, and constants λ, $\bar\lambda$ are defined as follows:

$$\lambda = \frac{1}{9}(\beta_1 + 9\beta_2 + \beta_1\kappa(2+\kappa)), \tag{63}$$

$$\bar\lambda = \frac{i}{9}\sqrt{\frac{\beta_1}{\hbar}\left(4\frac{\beta_1}{\hbar}\kappa(\kappa+2) + 9i(\kappa-1) - 72\frac{\beta_2}{\hbar}\right)}. \tag{64}$$

Wave functions of the form in expression (62) have been explicitly derived before in the literature [81,82]. Thus, the same analysis for the construction of a Hilbert space can be applied. We conclude the discussion here and shift our focus to the classical limit.

Recall that for large values of the argument of the Bessel functions, the asymptotic solution is as follows:

$$\Phi(a,\phi,\psi) \simeq a^{-\frac{3}{2}+\frac{2}{3}\frac{i}{\hbar}\beta_1(\kappa+1)}\phi^{-\frac{\kappa+1}{4}+\frac{i}{\hbar}\lambda}e^{-\frac{i}{\hbar}\beta_1\psi}\left(\bar\Phi_0^1\cos(K) + \Phi_0^2\sin(K)\right), \tag{65}$$

with $K = \frac{2}{3\hbar} i\sqrt{3V_0} a^3 \phi^{\frac{\kappa+1}{2}} - \frac{\bar{\lambda}\pi}{2} - \frac{\pi}{4}$; while in the limit $a^3\phi^{\frac{\kappa+1}{2}} \to 0$, the asymptotic behavior of the wave function reads as follows:

$$\Phi(a,\phi,\psi) \simeq a^{-\frac{3}{4}+\frac{2}{3}i\hbar\beta_1(\kappa+1)+3\bar{\lambda}} \phi^{-\frac{\kappa+1}{8}+\frac{i}{\hbar}\lambda-\frac{\kappa+1}{2}\bar{\lambda}} e^{-i\frac{\beta_1}{\hbar}\psi}. \tag{66}$$

In the following lines, we focus on the analysis of the semiclassical limit of quantum cosmology.

6. Semiclassical Limit

In the Madelung representation [53] of quantum mechanics; that is, in the hydrodynamic approach, we write the wave function for $\Phi(q) = \Omega(q)e^{\frac{i}{\hbar}S(q)}$, where Ω is the amplitude of the wave function. By replacing the WDW Equation (51) and separating the real and imaginary parts, we have the following:

$$\frac{1}{2}\mathcal{G}^{AB}\left(\frac{\partial S}{\partial q^A}\right)\left(\frac{\partial S}{\partial q^B}\right) + \mathcal{U}(\mathbf{q}) + \hbar^2\left(\frac{n-2}{8(n-1)}\mathcal{R} - \frac{1}{2\Omega}\Delta(\Omega)\right) = 0. \tag{67}$$

In the limit where $\hbar^2 \to 0$; that is, in the WKB approximation, the Hamilton–Jacobi equation for the classical gravitational field equations is recovered.

The new term $V_Q = -\frac{1}{2\Omega}\Delta(\Omega)$ is known as the quantum potential in the de Broglie–Bohm representation of quantum mechanics [55,56] and it depends only on the amplitude of the wave function. Furthermore, from the above expression, it is clear how the curvature of the minisuperspace contributes to the semiclassical limit.

The observables, Q_1, Q_2, and Q_3, lead to the conservation laws for the classical system, as follows:

$$I_1 = P_\psi, \tag{68}$$

$$I_2 = -\frac{(\kappa+1)}{6} a P_a + \phi P_\phi, \tag{69}$$

$$I_3 = \ln\phi\left(\frac{(\kappa+1)}{6} a P_a - \phi P_\phi\right) + \left(\psi - \frac{2(\kappa+1)}{3}\ln a\right) P_\psi - \frac{(K-1)}{8}\ln\phi. \tag{70}$$

where $P_a = \frac{\partial S}{\partial a}$, $P_\phi = \frac{\partial S}{\partial \phi}$ and $P_\psi = \frac{\partial S}{\partial \psi}$.

6.1. Classical Solution

The Hamilton–Jacobi equation for the classical system reads as follows:

$$\frac{1}{6a\phi}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{4}{3a^3}\left(\frac{\partial S}{\partial \phi}\right)\left(\frac{\partial S}{\partial \psi}\right) + 2V_0 a^3 \phi^\kappa = 0.$$

By using the conservation laws, we determine the closed-form solution for the action $S(a,\phi,\psi)$, as follows:

$$\begin{aligned} S(a,\phi,\psi) = {} & \frac{1}{9}\int \frac{9I_2 + \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6 \phi^{\kappa+1}\right)} - (\kappa+1)^2 I_1}{\phi} d\phi \\ & + \frac{2}{3(\kappa+1)}\int \frac{\sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6 \phi^{\kappa+1}\right)} - (\kappa+1)^2 I_1}{a} da \\ & + 9U_0(\kappa+1)\int\int \frac{\phi^\kappa}{\sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6 \phi^{\kappa+1}\right)}} d\phi\, da + I_1\psi. \end{aligned} \tag{71}$$

Thus,

$$P_a = -\frac{2\left(I_1(\kappa+1)^2 - \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6\phi^{\kappa+1}\right)}\right)}{3a(\kappa+1)}, \tag{72}$$

$$P_\phi = -\frac{I_1(\kappa+1)^2 - 9I_2 - \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6\phi^{\kappa+1}\right)}}{9\phi}, \tag{73}$$

$$P_\psi = I_1, \tag{74}$$

By replacing the momentum terms from (54), we end with a system of three nonlinear first-order differential equations.

In the limit where $a^6\phi^{\kappa+1} \to 0$, the field equations are described by the following system (we have assumed the lapse function to be a constant), as follows:

$$\phi a^2 \dot{a} \simeq -\frac{A_1(I_1, I_2, \kappa)}{6},$$
$$\phi a^3 \dot{\psi} \simeq -\frac{2A_2(I_1, I_2, \kappa)}{27},$$
$$a^3 \dot{\phi} \simeq \frac{2}{3} I_1.$$

where

$$A_1(I_1, I_2, \kappa) = \frac{2}{3(\kappa+1)}\left(I_1(\kappa+1)^2 - \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right)\right)}\right),$$
$$A_2(I_1, I_2, \kappa) = I_1(\kappa+1)^2 - 9I_2 - \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right)\right)}.$$

We write the following equivalent system:

$$\int \frac{1}{\phi} d\phi \simeq 4\frac{I_1}{A_1}\int \frac{1}{a} da, \quad \int d\psi \simeq \frac{4A_2}{9A_1}\int \frac{1}{a} da, \tag{75}$$

where the analytic solution in terms of the scale factor reads as follows:

$$\phi(a) \simeq a^{\frac{4I_1}{A_1}}, \ \psi(a) \simeq \frac{4A_2}{9A_1}\ln a. \tag{76}$$

On the other hand, when the term $a^6\phi^{\kappa+1}$ dominates, we end with the following system:

$$\dot{a} \simeq -\frac{\sqrt{-\frac{3}{2}V_0}}{6(\kappa+1)} a\phi^{\frac{\kappa-1}{2}}, \tag{77}$$

$$\dot{\psi} \simeq -\frac{\sqrt{-2V_0}}{\sqrt{27}}\phi^{\frac{\kappa-1}{2}}, \tag{78}$$

$$\dot{\phi} \simeq \frac{2}{3} I_1 a^{-3}. \tag{79}$$

The solution is real only when $V_0 < 0$. Thus, in this case, in a similar way as before, we end with the reduced system, as follows:

$$\int \phi^{\frac{\kappa-1}{2}} d\phi \simeq -\frac{4\sqrt{2}(\kappa+1)}{\sqrt{3|V_0|}} I_1 \int a^{-4} da\,, \tag{80}$$

$$\psi(a) \simeq \int \frac{4(\kappa+1)}{3}\phi^{\frac{\kappa-1}{2}}\frac{1}{a} da\,. \tag{81}$$

which means that the asymptotic solution is as follows:

$$\phi(a) \simeq \left(-\frac{4\sqrt{2}(\kappa+1)I_1}{\sqrt{3|V_0|}}\right)^{\frac{2}{\kappa+1}} a^{-\frac{6}{\kappa+1}}\ ,\ \psi(a) \simeq a^{3\frac{1-\kappa}{1+\kappa}}. \tag{82}$$

Recall that this solution is valid when $a^3\phi^{\frac{\kappa+1}{2}} \to 0$, i.e., $\left(-\frac{4\sqrt{2}(\kappa+1)I_1}{\sqrt{3|V_0|}}\right)^{\frac{2}{\kappa+1}} \to 0$.

Physical Properties of the Asymptotic Solutions

In the following lines, we investigate the physical properties of the asymptotic solutions derived before.

From the field equations of connection Γ^B, we have the following:

$$H^2 = \frac{1}{6\phi}(V(\phi) - 3\dot{\phi}\dot{\psi}) \tag{83}$$

Hence, for the asymptotic solution at the limit $a^6\phi^{\kappa+1} \to 0$, we derive the following:

$$H^2 \simeq H_0^1(I_1, I_2, \kappa)a^{-6-8\frac{I_1}{I_2}} + H_0^1(I_1, I_2, \kappa, V_0)a^{4\frac{I_1}{I_2}(1+\kappa)-8\frac{I_1}{I_2}}. \tag{84}$$

This corresponds to a cosmological solution where the effective fluid in the STEGR framework corresponds to two perfect fluids with constant equation parameters.

On the other hand, in the limit where the term $a^6\phi^{\kappa+1}$ dominates the asymptotic behavior for the Hubble function, we have the following:

$$H^2 \simeq \bar{H}_0^1(I_1, I_2, \kappa, V_0)a^{-6+\frac{12}{1+\kappa}}. \tag{85}$$

The latter describes the perfect fluid solution with the constant equation of state parameter, which leads to a self-similar spacetime [69]. The equation of state parameter is defined as $w_{eff} = 3 - \frac{8}{1+\kappa}$, from which it follows that acceleration occurs when $-1 < \kappa < \frac{5}{3}$.

6.2. Quantum Potential

We determine the solution of the field equations in the semiclassical regime where the quantum effects take place and affect the cosmological dynamics.

From the wave function (62), we calculate the quantum potential, as follows:

$$V_Q(a, \phi, \psi) = \frac{V_Q^0(I_1, I_2, \kappa)}{\phi a^3}, \tag{86}$$

where $V_Q^0(I_1, I_2, \kappa)$ is a constant related to the parameters I_1 , I_2 and κ.

Consequently, the Hamilton–Jacobi Equation (67) is as follows:

$$\frac{1}{6a\phi}\left(\frac{\partial\hat{S}}{\partial a}\right)^2 + \frac{4}{3a^3}\left(\frac{\partial\hat{S}}{\partial\phi}\right)\left(\frac{\partial\hat{S}}{\partial\psi}\right) + 2V_0a^3\phi^\kappa + \frac{2\bar{V}_Q^0(I_1, I_2, \kappa)}{a^3\phi}\hbar^2 = 0.$$

The conservation laws presented earlier also apply to the Hamiltonian system with quantum corrections.

Thus, the action reads as follows :

$$\hat{S}(a,\phi,\psi) = \frac{1}{9}\int \frac{9I_2 + \sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6\phi^{\kappa+1} - 27\bar{V}_Q^0\hbar^2\right)} - (\kappa+1)^2 I_1}{\phi} d\phi$$
$$+ \frac{2}{3(\kappa+1)}\int \frac{\sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6\phi^{\kappa+1} - 27\bar{V}_Q^0\hbar^2\right)} - (\kappa+1)^2 I_1}{a} da$$
$$+ 9U_0(\kappa+1)\int\int \frac{\phi^\kappa}{\sqrt{(\kappa+1)^2\left(I_1\left((\kappa+1)^2 I_1 - 18I_2\right) - 27V_0 a^6\phi^{\kappa+1} - 27\bar{V}_Q^0\hbar^2\right)}} d\phi\, da + I_1\psi. \quad (87)$$

We observe that the latter action is of the same function form as (71), resulting in the same gravitational field equations as before, with a rescale on the constants I_1, I_2, and κ.

Consequently, the behavior of the semiclassical solution is similar to that derived before for the classical solution without the quantum correction term.

7. Conclusions

In this study, we reviewed the basic mathematical properties of nonmetricity $f(Q)$-gravity. This gravitational model represents the simplest generalization of STEGR by introducing nonlinear terms of the nonmetricity scalar Q into the gravitational action integral. $f(Q)$-gravity is a particular case of the more general nonmetricity-scalar theory, where a scalar field is non-minimally coupled to gravity. $f(Q)$-gravity exhibits Machian properties, even though it is not purely Machian. Due to this characteristic, we discussed the effects of conformal transformations on the gravitational action integral and introduced the analogy of the Jordan and Einstein frames in nonmetricity theory.

Within the cosmological framework of FLRW geometry, the theory provides four different sets of gravitational field equations. This arises because the connection in STEGR theory is not uniquely defined. The choice of connection leads to the introduction of geometrodynamical degrees of freedom in the field equations, affecting the dynamics of the cosmological parameters. For the coordinate system where the line element for FLRW is expressed in the usual form of (31), there is a family of connections defined in the so-called coincidence gauge. In this case, the connection depends on a gauge function, which does not affect the cosmological dynamics. However, for the remaining three families of connections, the geometrodynamical degrees of freedom of the resulting gravitational equations can be attributed to two scalar fields. These two scalar fields have a geometric origin related to the dynamical degrees of freedom introduced by the nonlinear function f, and the degrees of freedom introduced by the connection.

For the connection where the two scalar fields define a canonical kinetic term, such that the gravitational field equations admit a minisuperspace description, we employed the quantization process to derive the WDW equation of quantum cosmology. We considered power-law $f(Q) = f_0 Q^\mu$ gravity, thus quantum observables were calculated using the theory of Lie symmetries. In this consideration, we focused on a multi-scalar field cosmological model, regardless of whether the origin of the scalar field was geometric.

The quantum operators constructed by the Lie symmetry analysis were used to reduce the WDW equation and write a closed-form solution for the wave function. We focused on the effects of quantum correction terms in the semiclassical limit. Therefore, we calculated the action by solving the Hamilton–Jacobi equation for the classical system, both with and without quantum correction terms. From the derivation of the action, we were able to write the equivalent reduced classical system.

We found that the quantum corrections did not affect the general evolution of the classical solution. However, the quantum potential can rescale the integration constants of the classical solutions. This is an important observation because it states that the initial value problem can be overcome by using the quantum corrections in the semi-classical limit.

Although $f(Q)$-cosmology is challenged by strong coupling and the presence of ghosts, this geometric model provides a mathematical framework for a better understanding of the effects of connection choice in gravitational physics. The nonmetricity $f(Q)$-theory distinguishes the connection from the metric, in contrast to GR, leading to new physics. For example, there exists the gravitational model of dipole cosmology in a Kantowski–Sachs background [83]. In this model, the tilted parameter interacts with the dynamical degrees of the connection.

In future work, we plan to extend this analysis to study quantum corrections in the semiclassical limit in the context of the nonmetricity-scalar theory and other extensions or modifications of STEGR.

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