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# Research on New Interval-Valued Fractional Integrals with Exponential Kernel and Their Applications

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**Abstract:** This paper aims to introduce a new fractional extension of the interval Hermite–Hadamard ($\mathcal{HH}$), $\mathcal{HH}$–Fejér, and Pachpatte-type inequalities for left- and right-interval-valued harmonically convex mappings ($\mathcal{LR}IVH$ convex mappings) with an exponential function in the kernel. We use fractional operators to develop several generalizations, capturing unique outcomes that are currently under investigation, while also introducing a new operator. Generally, we propose two methods that, in conjunction with more generalized fractional integral operators with an exponential function in the kernel, can address certain novel generalizations of increasing mappings under the assumption of $\mathcal{LR}IV$ convexity, yielding some noteworthy results. The results produced by applying the suggested scheme show that the computational effects are extremely accurate, flexible, efficient, and simple to implement in order to explore the path of upcoming intricate waveform and circuit theory research.

**Keywords:** $\mathcal{LR}$-interval-valued harmonically convex; $\mathcal{LR}$-interval-valued convexity; generalized interval fractional integral with exponential kernel; $\mathcal{HH}$–Fejér; Pachpatte-type inequalities; physical systems

**MSC:** 26A33; 26A51; 26D10

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## 1. Introduction

The convexity of mapping stands as one of the most advantageous tools for addressing an array of issues in both pure and applied sciences. Recent research has significantly delved into exploring the attributes and inequalities associated with convexity across various directions; for further insights, refer to [1–3] and the cited references. $\mathcal{HH}$-inequality, extensively applied in numerous practical mathematical domains, notably in probability and optimization, ranks among the pivotal mathematical inequalities concerning convex mappings.

The inequality is written as if $Y : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is a convex mapping on $[\mathfrak{y}_1, \mathfrak{y}_2]$ and with $\mathfrak{y}_1 \leq \mathfrak{y}_2$ such that

$$Y\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} Y(\omega)d\omega \leq \frac{Y(\mathfrak{y}_1) + Y(\mathfrak{y}_2)}{2} \tag{1}$$

Numerous researchers have paid significant attention to the mean value of a continuous convex mapping $Y : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, which is endowed with error margins by this foundational inequality. Extensive investigations have explored $\mathcal{HH}$-type inequalities for various types of convex mappings. For instance, Kórus [4] delved into s-convex mappings, Abramovich and Persson [5] examined N-quasi-convex mappings, while Delavar and De

La Sen [6] investigated h-convex mappings, among others. Khan et al. [7], Marinescu and Monea [8], İşcan [9], Kadakal et al. [10], Kadakal and Bekar [11], and the relevant references therein offer valuable insights into the latest advancements in this dynamic field.

Ahmad et al. [12] introduced fractional integrals with exponential kernels as follows to establish a fractional version of the $\mathcal{HH}$-type inequality:

**Definition 1.** *Assume $Y$ is in $L_1([\mathfrak{y}_1, \mathfrak{y}_2], \mathbb{R})$. The fractional integrals $\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+}$ and $\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-}$ of order $\mathscr{a} > 0$ are expressed as follows:*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+} Y(\lambda) = \frac{1}{\mathscr{a}} \int_{\mathfrak{y}_1}^{\lambda} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\lambda-\tau)} Y(\tau) d\tau, \ \lambda > \mathfrak{y}_1 \tag{2}$$

*and*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-} Y(\lambda) = \frac{1}{\mathscr{a}} \int_{\lambda}^{\mathfrak{y}_2} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\tau-\lambda)} Y(\tau) \mathrm{d}\tau, \ \lambda < \mathfrak{y}_2 \tag{3}$$

Fractional calculus has become an essential tool in applied mathematics and sciences. This topic has attracted considerable interest from researchers. Consequently, many authors have explored various aspects, such as the expansions of trapezium inequalities for k-fractional integrals [13], the $\mathcal{HH}$–Fejér type inequality for Riemann–Liouville fractional integrals [14], and the $\mathcal{HH}$–Fejér type inequality for Katugampola fractional integrals [15]. This has fostered a productive interaction among different approaches to fractional calculus and important integral inequalities. For further significant conclusions about fractional integral operators, interested readers should consult [16,17] and their references.

One notable application of set-valued analysis is interval analysis, which plays a significant role in both applied and pure sciences. An interval analysis was initially used to calculate the error boundaries of numerical solutions for finite-state machines. Over the past 50 years, it has become essential in resolving interval uncertainty within computer and mathematical models. Specific applications include computer graphics [18], neural network output optimization [19], and automatic error analysis [20]. Furthermore, interval-valued mappings have been applied in optimization theory, as discussed in [21,22]. In interval fractional calculus, Zhou et al. [23] proposed the following version of interval-fractional integrals with exponential kernels:

**Definition 2.** *Let $Y : \mathcal{CO} \to \mathbb{R}_{\mathbb{I}}, (0 < \mathfrak{y}_1 < \mathfrak{y}_2)$ be an interval-valued mapping defined by $Y(\lambda) = [Y_*(\lambda), Y^*(\lambda)]$ such that $Y$ is an integrable over $[\mathfrak{y}_1, \mathfrak{y}_2]$. Then, the interval fractional integral operators $\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+}$ and $\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-}$ of order $\mathscr{a} > 0$ are stated as follows:*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+} Y(\lambda) = \frac{1}{\mathscr{a}} \int_{\mathfrak{y}_1}^{\lambda} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\lambda-\tau)} Y(\tau) d\tau, \ \lambda > \mathfrak{y}_1, \tag{4}$$

*and*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-} Y(\lambda) = \frac{1}{\mathscr{a}} \int_{\lambda}^{\mathfrak{y}_2} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\tau-\lambda)} Y(\tau) \mathrm{d}\tau, \ \lambda < \mathfrak{y}_2. \tag{5}$$

*Obviously, we have*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+} Y(\lambda) = \left[\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+} Y_*(\lambda), \mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_1+} Y^*(\lambda)\right], \tag{6}$$

*and*

$$\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-} Y(\lambda) = \left[\mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-} Y_*(\lambda), \mathcal{J}^{\mathscr{a}}_{\mathfrak{y}_2-} Y^*(\lambda)\right]. \tag{7}$$

Additionally, Zhou et al. [24] also found the new versions of $\mathcal{HH}$ inequalities over these generalized fractional integrals via exponential trigonometric convex mappings. Moreover, Khan et al. [25–27] also presented the fuzzy and interval versions of these frac-

tional integrals via an exponential in the kernel over fuzzy and $\mathcal{LR}$ exponential trigonometric convex mappings and their applications in the inequality theory.

In addition to other scientific disciplines, the electrical circuit theory utilizes Jensen harmonic convexity, which is particularly noteworthy. The total resistance of a group of parallel resistors is calculated by summing the reciprocals of each individual resistance value and then taking the reciprocal of that sum. For example, if the resistances of two parallel resistors are represented as $\Lambda_1$ and $\Lambda_2$, the total resistance is given by

$$Z = \frac{1}{\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2}} = \frac{\Lambda_1 \Lambda_2}{\Lambda_1 + \Lambda_2}, \tag{8}$$

This is half of the harmonic mean. Furthermore, the harmonic mean of the effective masses along the three crystallographic directions is the definition of a semiconductor's "conductivity effective mass." Moreover, undesirable higher frequencies may be overlaid on the fundamental waveform of harmonically convex mappings, leading to a deformed wave pattern [28]. See [29–37] and the references therein for applications.

Following the previously indicated pattern, we establish interval–$\mathcal{HH}$, $\mathcal{HH}$–Fejér, and Pachpatte-type integral inequalities for $\mathcal{LR}IVH$ convex mappings using the fractional integral operator for integrable mappings. Additionally, we discuss various generalizations using a more general fractional integral operator with an exponential kernel. Our findings are both more intriguing and practically valuable than the current results. Ultimately, there is perfect agreement regarding the performance and application of the more general operator, demonstrating the suggested approach's effectiveness in manifesting inequalities through $\mathcal{LR}IV$ convexity.

## 2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_I$ contain all bounded and closed intervals within $\mathbb{R}$. $\Pi \in \mathbb{R}_I$ should be defined as follows:

$$\Pi = [\Pi_*, \Pi^*] = \{y \in \mathbb{R} | \Pi_* \leq y \leq \Pi^*\}, (\Pi_*, \Pi^* \in \mathbb{R}). \tag{9}$$

It is argued that $\Pi$ is degenerate if $\Pi_* = \Pi^*$. The interval $[\Pi_*, \Pi^*]$ is referred to as non-negative if $\Pi_* \geq 0$, $\mathbb{R}_I^+$ represents the set of all non-negative intervals and is defined as $\mathbb{R}_I^+ = \{[\Pi_*, \Pi^*] : [\Pi_*, \Pi^*] \in \mathbb{R}_I \text{ and } \Pi_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$, $\Pi, \Theta \in \mathbb{R}_I$ be defined with $\Theta = [\Theta_*, \Theta^*]$ and $\Pi = [\Pi_*, \Pi^*]$, so we may define the interval arithmetic as follows:

- Scaler multiplication:

$$\varrho.\Pi = \begin{cases} [\varrho\Pi_*, \varrho\Pi^*] & \text{if } \varrho > 0, \\ \{0\} & \text{if } \varrho = 0, \\ [\varrho\Pi^*, \varrho\Pi_*] & \text{if } \varrho < 0. \end{cases} \tag{10}$$

- Minkowski difference

$$[\Theta_*, \Theta^*] - [\Pi_*, \Pi^*] = [\Theta_* - \Pi_*, \Theta^* - \Pi^*], \tag{11}$$

- Addition:

$$[\Theta_*, \Theta^*] + [\Pi_*, \Pi^*] = [\Theta_* + \Pi_*, \Theta^* + \Pi^*], \tag{12}$$

- Multiplication:

$$[\Theta_*, \Theta^*] \times [\Pi_*, \Pi^*] = [min\{\Theta_*\Pi_*, \Theta^*\Pi_*, \Theta_*\Pi^*, \Theta^*\Pi^*\}, max\{\Theta_*\Pi_*, \Theta^*\Pi_*, \Theta_*\Pi^*, \Theta^*\Pi^*\}]. \tag{13}$$

The inclusion " $\supseteq$ " means that
$\Pi \supseteq \Theta$ if and only if $\left[\Pi_*, \Pi^*\right] \supseteq \left[\Theta_*, \Theta^*\right]$, and if and only if

$$\Pi_* \leq \Theta_*,\ \Theta^* \leq \Pi^*. \tag{14}$$

**Remark 1.** ([32]) *(i) The relation " $\leq_p$" is defined on $\mathbb{R}_I$ by*
*$[\Theta_*, \Theta^*] \leq_p [\Pi_*, \Pi^*]$ if and only if $\Theta_* \leq \Pi_*$, $\Theta^* \leq \Pi^*$,*
*for all $[\Theta_*, \Theta^*]$, $[\Pi_*, \Pi^*] \in \mathbb{R}_I$, and it is a pseudo-order relation. The relation $[\Theta_*, \Theta^*] \leq_p [\Pi_*, \Pi^*]$ coincident to $[\Theta_*, \Theta^*] \leq [\Pi_*, \Pi^*]$ on $\mathbb{R}_I$ when it is " $\leq_p$".*
*(ii) It can be easily seen that " $\leq_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " "left and right" (or "$\mathcal{LR}$" order, in short).*

**Definition 3.** *According to [32], the mapping $Y : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}_I^+$ is referred to be $\mathcal{LR}IV$ convex mapping on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if*

$$Y(g\mathfrak{n} + (1-g)\mathfrak{t}) \leq_p gY(\mathfrak{n}) + (1-g)Y(\mathfrak{t}), \tag{15}$$

*for all $\mathfrak{n}, \mathfrak{t} \in [\mathfrak{y}_1, \mathfrak{y}_2]$, and $g \in [0, 1]$. If the inequality (15) is reversed, then $Y$ is referred to be $\mathcal{LR}IV$ concave mapping on $[\mathfrak{y}_1, \mathfrak{y}_2]$.*

**Definition 4.** *According to [31], the mapping $Y : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}_I^+$ is referred to be $\mathcal{LR}$-interval-valued harmonically convex mapping ($\mathcal{LR}IVH$ convex mapping) on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if*

$$Y\left(\frac{\mathfrak{n}\mathfrak{t}}{g\mathfrak{t} + (1-g)\mathfrak{y}_1}\right) \leq_p gY(\mathfrak{n}) + (1-g)Y(\mathfrak{t}), \tag{16}$$

*for all $\mathfrak{n}, \mathfrak{t} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ and $g \in [0, 1]$. If the inequality (16) is reversed, then $Y$ is referred to be $\mathcal{LR}IVH$ concave mapping on $[\mathfrak{y}_1, \mathfrak{y}_2]$.*

## 3. Generalized Interval Fractional Integrals and Related Hermite–Hadamard Inequalities

Exponential kernels are a feature of new operators for interval-valued fractional integrals. The performance and application of the more general operator are highlighted by the way the suggested method precisely matches inequalities pertaining to $\mathcal{LR}IVH$ convexity and $\mathcal{LR}IV$ convexity. We now present a generalized form of interval fractional integrals related to fractional integrals with an exponential kernel.

We introduced the more general concept of the fractional integral operator with an exponential in the kernel as follows:

**Definition 5.** *Let $Y : \mathcal{CO} \to \mathbb{R}_{\mathbb{I}}, (0 < \mathfrak{y}_1 < \mathfrak{y}_2)$ be an interval-valued mapping defined by $Y(\lambda) = [Y_*(\lambda), Y^*(\lambda)]$ such that $Y$ is a non-negative and integrable, and let be a differentiable and strictly non-decreasing on $(\mathfrak{y}_1, \mathfrak{y}_2)$. Then, the interval fractional integral operators $\mathcal{J}_{\mathfrak{y}_1}^{\alpha,\mathfrak{h}}$ and $\mathcal{J}_{\mathfrak{y}_2}^{\alpha,\mathfrak{h}}$ of order $\alpha > 0$ are stated as*

$$\mathcal{J}_{\mathfrak{y}_1^+}^{\alpha,\mathfrak{h}} Y(\lambda) = \frac{1}{\alpha}\int_{\mathfrak{y}_1}^{\lambda} e^{-\frac{1-\alpha}{\alpha}((\lambda)-\psi(\tau))'}(\tau)Y(\tau)d\tau,\ \lambda > \mathfrak{y}_1, \tag{17}$$

*and*

$$\mathcal{J}_{\mathfrak{y}_2^-}^{\alpha,\mathfrak{h}} Y(\lambda) = \frac{1}{\alpha}\int_{\lambda}^{\mathfrak{y}_2} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{h}(\tau)-\mathfrak{h}(\lambda))}\mathfrak{h}'(\tau)Y(\tau)\mathrm{d}\tau, \lambda < \mathfrak{y}_2. \tag{18}$$

*Obviously, we have*

$$\mathcal{J}_{\mathfrak{y}_1^+}^{\alpha,\mathfrak{h}} Y(\lambda) = \left[\mathcal{J}_{\mathfrak{y}_1^+}^{\alpha,\mathfrak{h}} Y_*(\lambda), \mathcal{J}_{\mathfrak{y}_1^+}^{\alpha,\mathfrak{h}} Y^*(\lambda)\right], \tag{19}$$

*and*

$$\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{\mathfrak{y}_2^-}\Upsilon(\lambda) = \left[\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{\mathfrak{y}_2^-}\Upsilon_*(\lambda), \mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{\mathfrak{y}_2^-}\Upsilon^*(\lambda)\right]. \tag{20}$$

Next, we give the following description of the one-sided definition of a more extended fractional integral operator with an exponential kernel.

**Definition 6.** *Let $\Upsilon : \mathcal{CO} \to \mathbb{R}_{\mathbb{I}}, (0 < \mathfrak{y}_1 < \mathfrak{y}_2)$ be a mapping such that $\Upsilon$ is a non-negative and integrable, and let be a differentiable and strictly non-decreasing on $(\mathfrak{y}_1, \mathfrak{y}_2)$. Then, the one-sided fractional integral operator $\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{0^+,\lambda}$ is stated as*

$$\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{0^+,\lambda}\Upsilon(\lambda) = \frac{1}{\mathfrak{a}}\int_0^{\lambda} e^{-\frac{1-\mathfrak{a}}{\mathfrak{a}}((\lambda)-\psi(\tau))'(\tau)}\Upsilon(\tau)d\tau,\ \lambda > \tau. \tag{21}$$

*Obviously, we have*

$$\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{0^+,\lambda}\Upsilon(\lambda) = \left[\mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{0^+,\lambda}\Upsilon_*(\lambda), \mathcal{J}^{\mathfrak{a},\mathfrak{h}}_{0^+,\lambda}\Upsilon^*(\lambda)\right]. \tag{22}$$

*Throughout the upcoming results, we set $\varphi = \frac{1-\mathfrak{a}}{\mathfrak{a}}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2\mathfrak{y}_1}\right)$.*

For the applications of newly defined integrals, we now provide the following derivation of the $\mathcal{HH}$ inequality for $\mathcal{LR}IVH$ convex mappings in the frame of a new fractional integral operator.

**Theorem 1.** *For $\mathfrak{a} > 0$, there is a non-negative mapping $\Upsilon : \mathcal{CO} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}_{\mathbb{I}}$ with $\mathfrak{y}_2 > \mathfrak{y}_1$ and $\Upsilon \in L_1([\mathfrak{y}_1, \mathfrak{y}_2], \mathbb{R}_{\mathbb{I}})$. If $\Upsilon$ is a $\mathcal{LR}IVH$ convex mapping on $\mathcal{CO}$, then*

$$\Upsilon\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) \leq_p \frac{1-\mathfrak{a}}{2(1-e^{-\varphi})}\left[\mathcal{J}^{\mathfrak{a}}_{\frac{1}{\mathfrak{y}_1}^-}(\Upsilon\circ Q)\left(\frac{1}{\mathfrak{y}_2}\right) + \mathcal{J}^{\mathfrak{a}}_{\frac{1}{\mathfrak{y}_2}^+}(\Upsilon\circ Q)\left(\frac{1}{\mathfrak{y}_1}\right)\right] \leq_p \frac{\Upsilon(\mathfrak{y}_1)+\Upsilon(\mathfrak{y}_2)}{2}, \tag{23}$$

*where $Q(\lambda) = \frac{1}{\lambda}, \lambda \in \left[\frac{1}{\mathfrak{y}_2}, \frac{1}{\mathfrak{y}_1}\right]$.*

**Proof.** By utilizing the $\mathcal{LR}IVH$ convexity of $\Upsilon$ on $I$, we obtain the following for every $\mathfrak{K}_1, \mathfrak{K}_2 \in I$ with $g = \frac{1}{2}$:

$$\begin{aligned} \Upsilon_*\left(\tfrac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) &\leq \tfrac{\Upsilon_*(\mathfrak{K}_1)+\Upsilon_*(\mathfrak{K}_2)}{2} \\ \Upsilon^*\left(\tfrac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) &\leq \tfrac{\Upsilon^*(\mathfrak{K}_1)+\Upsilon^*(\mathfrak{K}_2)}{2}. \end{aligned} \tag{24}$$

By choosing $\mathfrak{K}_1 = \frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}, \mathfrak{K}_2 = \frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}$, and (24) takes the following form:

$$\begin{aligned} 2\Upsilon_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) &\leq \Upsilon_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right) \\ 2\Upsilon^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) &\leq \Upsilon^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right). \end{aligned} \tag{25}$$

By multiplying both sides by $e^{-\varphi g}$, and then integrating them with respect to $g$ from 0 to 1, we obtain

$$\begin{aligned}
\int_0^1 e^{-\varphi g} Y_*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathrm{d}g &\leq \left[\int_0^1 e^{-\varphi g} Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g\right.\\
&\quad \left.+\int_0^1 e^{-\varphi g} Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g\right]\\
\frac{2(1-e^{-\varphi})}{\varphi} Y_*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) &\leq \left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}} e^{\left(-\frac{\mathfrak{y}_1\mathfrak{y}_2(1-\mathscr{a})}{\mathscr{a}(\mathfrak{y}_2-\mathfrak{y}_1)}\left(\lambda-\frac{1}{\mathfrak{y}_2}\right)\right)} Y_*\left(\frac{1}{\lambda}\right)\mathrm{d}\lambda\right.\\
&\quad \left.+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}} e^{\left(-\frac{\mathfrak{y}_1\mathfrak{y}_2(1-\mathscr{a})}{\mathscr{a}(\mathfrak{y}_2-\mathfrak{y}_1)}\left(\frac{1}{\mathfrak{y}_1}-\lambda\right)\right)} Y_*\left(\frac{1}{\lambda}\right)\mathrm{d}\lambda\right]\\
&= \mathscr{a}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{y}_1}-}(Y_*\circ Q)\left(\frac{1}{\mathfrak{y}_2}\right)\right.\\
&\quad \left.+\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{y}_2}+}(Y_*\circ Q)\left(\frac{1}{\mathfrak{y}_1}\right)\right].
\end{aligned} \tag{26}$$

Similarly, for $Y^*$, we have

$$\begin{aligned}
\frac{2(1-e^{-\varphi})}{\varphi} Y^*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) &\leq \left[\int_0^1 e^{-\varphi g} Y^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g\right.\\
&\quad \left.+\int_0^1 e^{-\varphi g} Y^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g\right]\\
&\leq \left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}} e^{\left(-\frac{\mathfrak{y}_1\mathfrak{y}_2(1-\mathscr{a})}{\mathscr{a}(\mathfrak{y}_2-\mathfrak{y}_1)}\left(\lambda-\frac{1}{\mathfrak{y}_2}\right)\right)} Y^*\left(\frac{1}{\lambda}\right)\mathrm{d}\lambda\right.\\
&\quad \left.+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}} e^{\left(-\frac{\mathfrak{y}_1\mathfrak{y}_2(1-\mathscr{a})}{\mathscr{a}(\mathfrak{y}_2-\mathfrak{y}_1)}\left(\frac{1}{\mathfrak{y}_1}-\lambda\right)\right)} Y^*\left(\frac{1}{\lambda}\right)\mathrm{d}\lambda\right]\\
&= \mathscr{a}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{y}_1}-}(Y^*\circ Q)\left(\frac{1}{\mathfrak{y}_2}\right)\right.\\
&\quad \left.+\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{y}_2}+}(Y^*\circ Q)\left(\frac{1}{\mathfrak{y}_1}\right)\right].
\end{aligned} \tag{27}$$

From (26) and (27), we obtain the first inequality.

First, we observe that if $Y$ is a $\mathcal{LR}IVH$ convex mapping, then for $g\in[0,1]$, it provides the second inequality in (23).

$$Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\leq gY_*(\mathfrak{y}_1)+(1-g)Y_*(\mathfrak{y}_2)$$

and

$$Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\leq gY_*(\mathfrak{y}_2)+(1-g)Y_*(\mathfrak{y}_1),$$

By adding the above inequalities, we have

$$\begin{aligned} Y_*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)+Y_*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right) &\le Y_*(\mathfrak{Y}_1)+Y_*(\mathfrak{Y}_2)\\ Y^*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)+Y^*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right) &\le Y^*(\mathfrak{Y}_1)+Y^*(\mathfrak{Y}_2). \end{aligned} \tag{28}$$

After that, one obtains the following by multiplying both sides of (28) by $e^{-\varphi g}$ and integrating the inequality from 0 to 1:

$$\int_0^1 e^{-\varphi g}Y_*\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g+\int_0^1 e^{-\varphi g}Y_*\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right)\mathrm{d}g\le[Y_*(\mathfrak{Y}_1)+Y_*(\mathfrak{Y}_2)]\int_0^1 e^{-\varphi g}\mathrm{d}g$$

$$\int_0^1 e^{-\varphi g}Y^*\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g+\int_0^1 e^{-\varphi g}Y^*\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right)\mathrm{d}g\le[Y^*(\mathfrak{Y}_1)+Y^*(\mathfrak{Y}_2)]\int_0^1 e^{-\varphi g}\mathrm{d}g.$$

As a result, we have

$$\mathscr{a}\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\right)\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_1}^-}(Y_*\circ Q)\left(\frac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_2}^+}(Y_*\circ Q)\left(\frac{1}{\mathfrak{Y}_1}\right)\right]\le\frac{1-e^{-\varphi}}{\varphi}[Y_*(\mathfrak{Y}_1)+Y_*(\mathfrak{Y}_2)]$$

$$\mathscr{a}\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\right)\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_1}^-}(Y^*\circ Q)\left(\frac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_2}^+}(Y^*\circ Q)\left(\frac{1}{\mathfrak{Y}_1}\right)\right]\le\frac{1-e^{-\varphi}}{\varphi}[Y^*(\mathfrak{Y}_1)+Y^*(\mathfrak{Y}_2)].$$

Which implies that

$$\begin{aligned}\mathscr{a}\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\right)\Bigg(\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_1}^-}(Y_*\circ Q)\left(\frac{1}{\mathfrak{Y}_2}\right),\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_1}^-}(Y^*\circ Q)\left(\frac{1}{\mathfrak{Y}_2}\right)\right]+\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_2}^+}(Y_*\circ Q)\left(\frac{1}{\mathfrak{Y}_1}\right),\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_2}^+}(Y^*\circ Q)\left(\frac{1}{\mathfrak{Y}_1}\right)\right]\Bigg)\\ \le_p\frac{1-e^{-\varphi}}{\varphi}([Y_*(\mathfrak{Y}_1),Y^*(\mathfrak{Y}_1)]+[Y_*(\mathfrak{Y}_2),Y^*(\mathfrak{Y}_2)]),\end{aligned}$$

Hence,

$$\mathscr{a}\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\right)\left[\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_1}^-}(Y\circ Q)\left(\frac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{\frac{1}{\mathfrak{Y}_2}^+}(Y\circ Q)\left(\frac{1}{\mathfrak{Y}_1}\right)\right]\le_p\frac{1-e^{-\varphi}}{\varphi}[Y(\mathfrak{Y}_1)+Y(\mathfrak{Y}_2)]$$

The proof is concluded. □

**Remark 2.** *As suggested by Iscan in [34], in the limiting situation,*

$$\lim_{\mathscr{a}\to 1}\frac{1-\mathscr{a}}{2\left(1-e^{-\frac{1-\mathscr{a}}{\mathscr{a}}\left(\frac{\mathfrak{Y}_2-\mathfrak{Y}_1}{\mathfrak{Y}_1\mathfrak{Y}_2}\right)}\right)}=\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{2(\mathfrak{Y}_2-\mathfrak{Y}_1)}. \tag{29}$$

*and* $Y_*(\curlywedge)=Y^*(\curlywedge)$.

We now go over the Pachpatte-type inequalities proved in Theorems 2 and 3 for $\mathcal{LR}IVH$ convex mappings:

**Theorem 2.** *For* $\mathscr{a}>0$, *there are two* $\mathcal{LR}IVH$ *convex mappings,* $Y,\mathcal{P}:\mathfrak{CO}\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}_{\mathbb{I}}$ *such that* $Y,\mathcal{P}\in L_1([\mathfrak{Y}_1,\mathfrak{Y}_2],\mathbb{R}_{\mathbb{I}})$ *with* $\mathfrak{Y}_2>\mathfrak{Y}_1$ *and* $\mathfrak{Y}_1,\mathfrak{Y}_2\in\mathfrak{CO}$, *then*

$$\begin{aligned}\tfrac{\mathscr{a}\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\quad&\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{Y}_1})^-}Y\left(\frac{1}{\mathfrak{Y}_2}\right)\mathcal{P}\left(\frac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{Y}_2})^+}Y\left(\frac{1}{\mathfrak{Y}_1}\right)\mathcal{P}\left(\frac{1}{\mathfrak{Y}_1}\right)\right]\\ &\le_p Y_1(\mathfrak{Y}_1,\mathfrak{Y}_2)\tfrac{\varphi^2-2\varphi+4-e^{-\varphi}(\varphi^2+2\varphi+4)}{2\varphi^3}.\\ &+Y_2(\mathfrak{Y}_1,\mathfrak{Y}_2)\tfrac{\varphi-2+e^{-\varphi}(\varphi+2)}{\varphi^3},\end{aligned} \tag{30}$$

*where*

$$\Upsilon_1(\mathfrak{y}_1,\mathfrak{y}_2)=[Y(\mathfrak{y}_1)\mathcal{P}(\mathfrak{y}_1)+Y(\mathfrak{y}_2)\mathcal{P}(\mathfrak{y}_2)] \tag{31}$$

*and*

$$\Upsilon_2(\mathfrak{y}_1,\mathfrak{y}_2)=[Y(\mathfrak{y}_1)\mathcal{P}(\mathfrak{y}_2)+Y(\mathfrak{y}_2)\mathcal{P}(\mathfrak{y}_1)]. \tag{32}$$

**Proof.** Since $Y$ and $\mathcal{P}$ are $\mathcal{LR}IVH$ convex mappings on $\mathfrak{CO}$, then for $g\in[0,1]$,

$$\begin{aligned}
&Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\\
&\quad\le g^2Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1)+(1-g)^2Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2)\\
&\qquad+g(1-g)[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1)]\\
&Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\\
&\quad\le g^2Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_1)+(1-g)^2Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_2)\\
&\qquad+g(1-g)[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_2)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_1)].
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\\
&\quad\le g^2Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2)+(1-g)^2Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1)\\
&\qquad+g(1-g)[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1)]\\
&Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\\
&\quad\le g^2Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_2)+(1-g)^2Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_1)\\
&\qquad+g(1-g)[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_2)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_1)].
\end{aligned} \tag{34}$$

Adding (34) and (35), we obtain

$$\begin{aligned}
&Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\\
&\quad\le\left(2g^2-2g+1\right)[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2)]\\
&\qquad+2g(1-g)[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1)]\\
&Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\\
&\quad\le\left(2g^2-2g+1\right)[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_1)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_2)]\\
&\qquad+2g(1-g)[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_2)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_1)].
\end{aligned} \tag{35}$$

After that, one obtains the following by multiplying both sides of (35) by $e^{-\varphi g}$ and integrating the inequality with respect to $g$ from 0 to 1:

$$\begin{aligned}
&\int_0^1 e^{-\varphi g}Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g\\
&\qquad+\int_0^1 e^{-\varphi g}Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g\\
&\qquad\le[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2)]\int_0^1 e^{-\varphi g}\left(2g^2-2g+1\right)\mathrm{d}g\\
&\qquad+2[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2)+Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1)]\int_0^1 e^{-\varphi g}\left(2g^2-2g+1\right)\times g(1-g)\mathrm{d}g\\
&\int_0^1 e^{-\varphi g}Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g\\
&\qquad+\int_0^1 e^{-\varphi g}Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g\\
&\qquad\le[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_1)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_2)]\int_0^1 e^{-\varphi g}\left(2g^2-2g+1\right)\mathrm{d}g\\
&\qquad+2[Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_2)+Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_1)]\int_0^1 e^{-\varphi g}\left(2g^2-2g+1\right)\times g(1-g)\mathrm{d}g.
\end{aligned}$$

Consequently, we obtain

$$\frac{a\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\left[\mathcal{J}^a_{(\frac{1}{\mathfrak{y}_1})^-}Y_*\left(\frac{1}{\mathfrak{y}_2}\right)\mathcal{P}_*\left(\frac{1}{\mathfrak{y}_2}\right),\mathcal{J}^a_{(\frac{1}{\mathfrak{y}_1})^-}Y^*\left(\frac{1}{\mathfrak{y}_2}\right)\mathcal{P}^*\left(\frac{1}{\mathfrak{y}_2}\right)\right]+\left[\mathcal{J}^a_{(\frac{1}{\mathfrak{y}_2})^+}Y_*\left(\frac{1}{\mathfrak{y}_1}\right)\mathcal{P}_*\left(\frac{1}{\mathfrak{y}_1}\right),\mathcal{J}^a_{(\frac{1}{\mathfrak{y}_2})^+}Y^*\left(\frac{1}{\mathfrak{y}_1}\right)\mathcal{P}^*\left(\frac{1}{\mathfrak{y}_1}\right)\right]\right)$$

$$\begin{aligned}
&\leq_p \left([Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1), Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_1)] + [Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2), Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_2)]\right)\frac{(1-e^{-\varphi})(\varphi^2+2\varphi+4)}{2\varphi^3}\\
&+\left([Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2), Y^*(\mathfrak{y}_1)\mathcal{P}^*(\mathfrak{y}_2)] + [Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1), Y^*(\mathfrak{y}_2)\mathcal{P}^*(\mathfrak{y}_1)]\right)\frac{\varphi-2+e^{-\varphi}(\varphi+2)}{\varphi^3}\\
&= [\Upsilon_{*1}(\mathfrak{y}_1,\mathfrak{y}_2), \Upsilon_1^*(\mathfrak{y}_1,\mathfrak{y}_2)]\frac{\varphi^2-2\varphi+4-e^{-\varphi}(\varphi^2+2\varphi+4)}{2\varphi^3} + [\Upsilon_{*2}(\mathfrak{y}_1,\mathfrak{y}_2), \Upsilon_2^*(\mathfrak{y}_1,\mathfrak{y}_2)]\frac{\varphi-2+e^{-\varphi}(\varphi+2)}{\varphi^3}.
\end{aligned}$$

This concludes the proof of (30). □

**Theorem 3.** *For $\mathscr{a} > 0$, there are two $\mathcal{LR}IVH$ convex mappings $Y, \mathcal{P} : \mathfrak{CO} \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}_{\mathbb{I}}$, such that $Y, \mathcal{P} \in L_1([\mathfrak{y}_1, \mathfrak{y}_2], \mathbb{R}_{\mathbb{I}})$ with $\mathfrak{y}_2 > \mathfrak{y}_1$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathfrak{CO}$, then*

$$\begin{aligned}
2Y\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) \leq_p \tfrac{1-\mathscr{a}}{2(1-e^{-\varphi})}&\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}Y\left(\tfrac{1}{\mathfrak{y}_2}\right)\mathcal{P}\left(\tfrac{1}{\mathfrak{y}_2}\right) + \mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}Y\left(\tfrac{1}{\mathfrak{y}_1}\right)\mathcal{P}\left(\tfrac{1}{\mathfrak{y}_1}\right)\right]\\
&+\tfrac{\varphi^2-2\varphi+4-e^{-\varphi}(\varphi^2+2\varphi+4)}{2\varphi^2(1-e^{-\varphi})}Y_2(\mathfrak{y}_1,\mathfrak{y}_2) + \tfrac{\varphi-2+(\varphi+2)e^{-\varphi}}{\varphi^2(1-e^{-\varphi})}Y_1(\mathfrak{y}_1,\mathfrak{y}_2),
\end{aligned} \tag{36}$$

*where*

$$\mathfrak{y}_1(\mathfrak{y}_1,\mathfrak{y}_2) = [Y(\mathfrak{y}_1)\mathcal{P}(\mathfrak{y}_1) + Y(\mathfrak{y}_2)\mathcal{P}(\mathfrak{y}_2)], \tag{37}$$

*and*

$$\Upsilon_2(\mathfrak{y}_1,\mathfrak{y}_2) = [Y(\mathfrak{y}_1)\mathcal{P}(\mathfrak{y}_2) + Y(\mathfrak{y}_2)\mathcal{P}(\mathfrak{y}_1)]. \tag{38}$$

**Proof.** By utilizing the $\mathcal{LR}IVH$ convexity of the mappings $Y$ and $\mathcal{P}$ on $I$, we obtain the following for all $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathfrak{CO}$:

$$\begin{aligned}
Y_*\left(\tfrac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) &\leq \left(\tfrac{Y_*(\mathfrak{K}_1)+Y_*(\mathfrak{K}_2)}{2}\right)\\
Y^*\left(\tfrac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) &\leq \left(\tfrac{Y^*(\mathfrak{K}_1)+Y^*(\mathfrak{K}_2)}{2}\right),
\end{aligned} \tag{39}$$

and

$$\begin{pmatrix}
\mathcal{P}_*\left(\frac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) \leq \left(\frac{\mathcal{P}_*(\mathfrak{K}_1)+\mathcal{P}_*(\mathfrak{K}_2)}{2}\right)\\
\mathcal{P}^*\left(\frac{2\mathfrak{K}_1\mathfrak{K}_2}{\mathfrak{K}_1+\mathfrak{K}_2}\right) \leq \left(\frac{\mathcal{P}^*(\mathfrak{K}_1)+\mathcal{P}^*(\mathfrak{K}_2)}{2}\right).
\end{pmatrix} \tag{40}$$

Substituting $\mathfrak{K}_1 = \frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}$ and $\mathfrak{K}_2 = \frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}$, we have

$$\begin{aligned}
&4Y_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\\
&\leq \left[Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right) + Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\right.\\
&\left.+Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right) + Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\right]\\
&= \left[Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right) + Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\right]\\
&+\left(g^2+(1-g)^2\right)[Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_1) + Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_2)]\\
&+2g(1-g)[Y_*(\mathfrak{y}_1)\mathcal{P}_*(\mathfrak{y}_1) + Y_*(\mathfrak{y}_2)\mathcal{P}_*(\mathfrak{y}_2)]\\
&= \left[Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right) + Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\right]\\
&+\left(2g^2-2g+1\right)\Upsilon_{*2}(\mathfrak{y}_1,\mathfrak{y}_2) + 2g(1-g)\Upsilon_{*1}(\mathfrak{y}_1,\mathfrak{y}_2),
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&4Y^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{P}^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\\
&\leq \left[Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right) + Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\right]\\
&+\left(2g^2-2g+1\right)\Upsilon_2^*(\mathfrak{y}_1,\mathfrak{y}_2) + 2g(1-g)\Upsilon_1^*(\mathfrak{y}_1,\mathfrak{y}_2)
\end{aligned} \tag{42}$$

After that, one obtains the following by multiplying both sides of (35) by $e^{-\varphi g}$ and integrating the inequality with respect to $g$ from 0 to 1:

$$\begin{aligned}
&\left(\frac{4\left(1-e^{-\varphi}\right)}{\varphi}\right)\left[Y_*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{P}_*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right),\ Y^*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{P}^*\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\right]\\
\leq&\left[\int_0^1 e^{-\varphi g}Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g,\int_0^1 e^{-\varphi g}Y^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathcal{P}^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g\right]\\
&+\left[\int_0^1 e^{-\varphi g}Y_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}_*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g,\int_0^1 e^{-\varphi g}Y^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathcal{P}^*\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g\right]\\
=&\left[\mathrm{Y}_{*2}(\mathfrak{y}_1,\mathfrak{y}_2),\mathrm{Y}_2^*(\mathfrak{y}_1,\mathfrak{y}_2)\right]\frac{\varphi^2-2\varphi+4-e^{-\varphi}\left(\varphi^2+2\varphi+4\right)}{2\varphi^3}+\left[\mathrm{Y}_{*1}(\mathfrak{y}_1,\mathfrak{y}_2),\mathrm{Y}_1^*(\mathfrak{y}_1,\mathfrak{y}_2)\right]\frac{\varphi-2+e^{-\varphi}(\varphi+2)}{\varphi^3}.
\end{aligned}$$

following appropriate reorganizations, we obtain the intended inequality (36). □

We need the following Lemma 1 in order to produce our next important conclusion, which will let us prove the $\mathcal{HH}$–Fejér-type inequality.

**Lemma 1.** *For $\mathscr{a}>0$, there is a mapping $Q:I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}_{\mathbb{I}}$ integrable and harmonically symmetric with respect to $\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}$, then*

$$\begin{aligned}
\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)&=\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right)\\
&=\frac{1}{2}\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)+\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})}-(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right)\right].
\end{aligned}\tag{43}$$

*where $\mathscr{a}>0$ and $\mathcal{D}(\Lambda)=\frac{1}{\Lambda},\Lambda\in\left[\frac{1}{\mathfrak{y}_2},\frac{1}{\mathfrak{y}_1}\right]$.*

**Proof.** Using the given assumption, we can substitute $g=\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\Lambda$ in the integral below and compute the result.

$$\begin{aligned}
\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)&=\frac{1}{\mathscr{a}}\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}e^{-\varphi(\frac{1}{\mathfrak{y}_1}-g)}Q\left(\frac{1}{g}\right)dg\\
&=-\frac{1}{\mathscr{a}}\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}e^{-\varphi(g-\frac{1}{\mathfrak{y}_2})}Q\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\Lambda}\right)\mathrm{d}\\
&=\frac{1}{\mathscr{a}}\int_{\frac{1}{\mathfrak{y}_1}}^{\frac{1}{\mathfrak{y}_2}}e^{-\varphi(g-\frac{1}{\mathfrak{y}_2})}Q\left(\frac{1}{\Lambda}\right)\mathrm{d}\Lambda\\
&=\frac{1}{\mathscr{a}}\int_{\frac{1}{\mathfrak{y}_1}}^{\frac{1}{\mathfrak{y}_2}}e^{-\varphi(g-\frac{1}{\mathfrak{y}_2})}Q\left(\frac{1}{\Lambda}\right)\mathrm{d}\Lambda\\
&=\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right),
\end{aligned}$$

the desired outcome. □

**Theorem 4.** *For $\mathscr{a}>0$, there is a $\mathcal{LR}IVH$ convex mapping $Y:\mathfrak{CO}\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}_{\mathbb{I}}$, such that $Y\in L_1([\mathfrak{y}_1,\mathfrak{y}_2],\mathbb{R}_{\mathbb{I}})$ with $\mathfrak{y}_2>\mathfrak{y}_1$ and $\mathfrak{y}_1,\mathfrak{y}_2\in\mathfrak{CO}$. Also, if there is a non-negative that is integrable and harmonically symmetric with respect to $\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}$, then*

$$\begin{aligned}
&Y\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(\mathcal{Q}\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(\mathcal{Q}\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)\right]\\
\leq_p&\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(YQ\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(YQ\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)\right]\\
\leq_p&\left[Y(\mathfrak{y}_1)+Y(\mathfrak{y}_2)\right]\left[\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathscr{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Q\circ\mathcal{D})\left(\frac{1}{\mathfrak{y}_1}\right)\right].
\end{aligned}\tag{44}$$

*where $\mathcal{D}(\Lambda)=\frac{1}{\Lambda},\Lambda\in\left[\frac{1}{\mathfrak{y}_2},\frac{1}{\mathfrak{y}_1}\right]$.*

**Proof.** For all $g \in [0,1]$, we obtain inequality (24) because $Y$ is an $\mathcal{LR}IVH$ convex mapping on $\mathfrak{CO}$. After multiplying both sides of (25) by $e^{-\varphi g}Q\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)$, we integrate the inequality from 0 to 1 with respect to $g$.

$$\begin{aligned} &2Y_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g \\ \le{}& \int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g \\ &+\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)Y_*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d} \end{aligned} \tag{45}$$

and

$$\begin{aligned} &2Y^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\int_0^1 e^{-\varphi g}\mathcal{D}\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g \\ \le{}& \int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)\mathrm{d}g \\ &+\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}\right)Y^*\left(\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{g\mathfrak{y}_1+(1-g)\mathfrak{y}_2}\right)\mathrm{d}g. \end{aligned} \tag{46}$$

By setting $\lambda = \frac{g\mathfrak{y}_2+(1-g)\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}$ and utilizing $Q$, which is harmonically symmetric, we obtain

$$\begin{aligned} &\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}Y_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\lambda}\right)\mathrm{d}\lambda \\ &\le \tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\lambda}\right)Y_*\left(\tfrac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\lambda}\right)\mathrm{d}\lambda\right. \\ &\left.+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\lambda}\right)Y_*\left(\tfrac{1}{\lambda}\right)\mathrm{d}\lambda\right], \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}Y^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\lambda}\right)\mathrm{d}\lambda \\ &\le \tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\lambda}\right)Y^*\left(\tfrac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_2}-\lambda}\right)\mathrm{d}\lambda\right. \\ &\left.+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}} e^{\left(\frac{-\varphi\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}(\lambda-\frac{1}{\mathfrak{y}_2})\right)}Q\left(\tfrac{1}{\lambda}\right)Y^*\left(\tfrac{1}{\lambda}\right)\mathrm{d}\lambda\right]. \end{aligned} \tag{48}$$

It follows that

$$\tfrac{\mathfrak{y}_1\mathfrak{y}_2\mathfrak{a}}{\mathfrak{y}_2-\mathfrak{y}_1}Y_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right) \le \tfrac{\mathfrak{y}_1\mathfrak{y}_2\mathfrak{a}}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Y_*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Y_*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_1}\right)\right], \tag{49}$$

and

$$\tfrac{\mathfrak{y}_1\mathfrak{y}_2\mathfrak{a}}{\mathfrak{y}_2-\mathfrak{y}_1}Y^*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right) \le \tfrac{\mathfrak{y}_1\mathfrak{y}_2\mathfrak{a}}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Y^*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Y^*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_1}\right)\right]. \tag{50}$$

Lemma 1 is applied to the left side of (49), giving us

$$\begin{aligned} &\tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}Y_*\left(\tfrac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)\left[\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_1}\right)\right] \\ &\le \tfrac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_1})^-}(Y_*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_2}\right)+\mathcal{J}^{\mathfrak{a}}_{(\frac{1}{\mathfrak{y}_2})^+}(Y_*Q\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{y}_1}\right)\right] \end{aligned} \tag{51}$$

and

$$\begin{aligned}&\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}Y^*\left(\tfrac{2\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_1+\mathfrak{Y}_2}\right)\left[\mathcal{J}^{\alpha}_{(\frac{1}{\mathfrak{Y}_1})^-}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\alpha}_{(\frac{1}{\mathfrak{Y}_2})^+}(\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{Y}_1}\right)\right]\\&\leq\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{\mathfrak{Y}_2-\mathfrak{Y}_1}\left[\mathcal{J}^{\alpha}_{(\frac{1}{\mathfrak{Y}_1})^-}(Y^*\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{Y}_2}\right)+\mathcal{J}^{\alpha}_{(\frac{1}{\mathfrak{Y}_2})^+}(Y^*\mathcal{Q}\circ\mathcal{D})\left(\tfrac{1}{\mathfrak{Y}_1}\right)\right].\end{aligned}\tag{52}$$

By multiplying both sides of (25) by $e^{-\varphi g}Q\left(\frac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)$ and integrating the inequality with respect to $g$ from 0 to 1, we can prove the second inequality in (44).

$$\begin{aligned}&\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y_*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g\\&+\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y_*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right)\mathrm{d}g\\&\leq[Y_*(\mathfrak{Y}_1)+Y_*(\mathfrak{Y}_2)]\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g,\end{aligned}\tag{53}$$

and

$$\begin{aligned}&\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y^*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g\\&+\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y^*\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right)\mathrm{d}g\\&\leq[Y^*(\mathfrak{Y}_1)+Y^*(\mathfrak{Y}_2)]\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g.\end{aligned}\tag{54}$$

From (53) and (54), we have

$$\begin{aligned}&\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g+\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)Y\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_1+(1-g)\mathfrak{Y}_2}\right)\mathrm{d}g\\&\leq_p[Y(\mathfrak{Y}_1)+Y(\mathfrak{Y}_2)]\int_0^1 e^{-\varphi g}Q\left(\tfrac{\mathfrak{Y}_1\mathfrak{Y}_2}{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}\right)\mathrm{d}g.\end{aligned}$$

Again, by setting $\lambda=\frac{g\mathfrak{Y}_2+(1-g)\mathfrak{Y}_1}{\mathfrak{Y}_1\mathfrak{Y}_2}$ and after simple calculations, the second inequality is concluded in (44). □

**Remark 3.** *From Theorems 3 and 4, if $Y_*(\lambda)=Y^*(\lambda)$, then*

*(1) Theorem 4 in [35] is obtained if one chooses $\alpha\to 1$.*
*(2) Theorem 2 in [35] can be obtained if one accepts $Q(\lambda)=1$ and $\alpha\to 1$.*

## 4. New Extensions for Convex Mappings via Interval Fractional Integral with an Exponential in the Kernel

In this article, we will assume that $\mathfrak{h}(g)$ is a monotone, non-decreasing, non-negative mapping defined on $[0,\infty)$ such that $\mathfrak{h}(0)=0$, and $\mathfrak{h}'(g)$ is continuous on $[0,\infty)$.

**Theorem 5.** *For $\alpha>0$, let $Y$ and $\mathcal{D}$ be two non-negative mappings with $Y\leq\mathcal{D}$ defined on $[0,\infty)$. Additionally, if an $\mathcal{LRIV}$ convex mapping $\mathfrak{S}$ with $\mathfrak{S}(0)=0$ and a non-decreasing mapping $Y$ and a decreasing mapping $\frac{Y}{\mathcal{D}}$ defined on $[0,\infty)$, then the following inequality is satisfied by the fractional integral operator stated in (21) such that*

$$\frac{\mathcal{J}^{\alpha,\mathfrak{h}}_{0^+,\lambda}[Y(\lambda)]}{\mathcal{J}^{\alpha,\mathfrak{h}}_{0^+,\lambda}[\mathcal{D}(\lambda)]}\geq_p\frac{\mathcal{J}^{\alpha,\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))]}{\mathcal{J}^{\alpha,\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(\mathcal{D}(\lambda))]}.\tag{55}$$

**Proof.** The mapping $\frac{\mathfrak{S}(\lambda)}{\lambda}$ is non-decreasing according to the hypothesis that is provided. $Y$ and the mapping $\frac{\mathfrak{S}(\lambda)}{\lambda}$ are both non-decreasing. There is an obvious decrease in the mapping $\frac{Y(\lambda)}{\mathcal{D}(\lambda)}$. Thus, we obtain the following for any $g, \omega \in [0, \infty)$:

$$\begin{array}{c} \left(\frac{\mathfrak{S}_*(Y(g))}{Y(g)} - \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\right)\left(\frac{Y(\omega)}{\mathcal{D}(\omega)} - \frac{Y(g)}{\mathcal{D}(g)}\right) \geq 0 \\ \left(\frac{\mathfrak{S}^*(Y(g))}{Y(g)} - \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\right)\left(\frac{Y(\omega)}{\mathcal{D}(\omega)} - \frac{Y(g)}{\mathcal{D}(g)}\right) \geq 0. \end{array} \tag{56}$$

It follows that

$$\begin{array}{c} \frac{\mathfrak{S}_*(Y(g))}{Y(g)}\frac{Y(\omega)}{\mathcal{D}(\omega)} + \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\frac{Y(g)}{\mathcal{D}(g)} - \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\frac{Y(\omega)}{\mathcal{D}(\omega)} - \frac{\mathfrak{S}_*(Y(g))}{Y(g)}\frac{Y(g)}{\mathcal{D}(g)} \geq 0 \\ \frac{\mathfrak{S}^*(Y(g))}{Y(g)}\frac{Y(\omega)}{\mathcal{D}(\omega)} + \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\frac{Y(g)}{\mathcal{D}(g)} - \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\frac{Y(\omega)}{\mathcal{D}(\omega)} - \frac{\mathfrak{S}^*(Y(g))}{Y(g)}\frac{Y(g)}{\mathcal{D}(g)} \geq 0. \end{array} \tag{57}$$

When we multiply (57) by $\mathcal{D}(g)\mathcal{D}(\omega)$, we obtain

$$\begin{array}{c} \frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g) + \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega) - \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(g) - \frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(g)\mathcal{D}(\omega) \geq 0, \\ \frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g) + \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega) - \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(g) - \frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(g)\mathcal{D}(\omega) \geq 0. \end{array} \tag{58}$$

After integrating the inequality from 0 to $\lambda$ and multiplying (58) by $\frac{1}{a}e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)$, which is non-negative because $g \in (0, \lambda), \lambda > 0$, we obtain

$$\begin{array}{l} \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)\mathrm{d}g + \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega)\mathrm{d}g \\ -\frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(g)\mathrm{d}g - \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(g)\mathcal{D}(\omega)\mathrm{d}g \geq 0 \\ \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)\mathrm{d}g + \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega)\mathrm{d}g \\ -\frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(g)\mathrm{d}g - \frac{1}{a}\int_0^\lambda e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(g)\mathcal{D}(\omega)\mathrm{d}g \geq 0. \end{array}$$

This follows that

$$\begin{array}{l} Y(\omega)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \left(\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda)) \\ \quad -\left(\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(\omega)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)) - \mathcal{D}(\omega)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}Y(\lambda)\right) \geq 0, \\ Y(\omega)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \left(\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda)) \\ \quad -\left(\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(\omega)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)) - \mathcal{D}(\omega)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}Y(\lambda)\right) \geq 0. \end{array} \tag{59}$$

Once more, we derive the following by multiplying (59) by $\frac{1}{a}e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(\omega))}\mathfrak{h}'(\omega)$, which is non-negative because $\omega \in (0, \lambda), \omega > 0$, and we integrate the resulting identity from 0 to $\lambda$:

$$\begin{array}{l} \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda) \\ \quad \geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}_*(Y(\lambda))) + \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}_*(Y(\lambda)))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)), \\ Y\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right)\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda) \\ \quad \geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}^*(Y(\lambda))) + \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}^*(Y(\lambda)))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)). \end{array} \tag{60}$$

It follows that

$$\begin{array}{c} \frac{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))}{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))} \leq \frac{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}_*(Y(\lambda)))}{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))}, \\ \frac{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))}{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))} \leq \frac{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}^*(Y(\lambda)))}{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))}. \end{array} \tag{61}$$

Now, since $Y \leq \mathcal{D}$ is defined on $[0,\infty)$ and $\frac{\mathfrak{S}(\lambda)}{\lambda}$ is a non-decreasing mapping, for $g \in [0,\lambda)$, we have

$$\begin{aligned} \frac{\mathfrak{S}_*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}, \\ \frac{\mathfrak{S}^*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}. \end{aligned} \tag{62}$$

When we multiply (62) by $\frac{1}{a}e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\mathcal{D}(g)$, we obtain a non-negative result because $g \in (0,\lambda), \lambda > 0$. We then integrate the inequality from 0 to $\lambda$.

$$\begin{aligned} &\tfrac{1}{a}\textstyle\int_0^{\lambda} e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}\mathcal{D}(g)\mathrm{d}g \\ &\quad\leq \tfrac{1}{a}\textstyle\int_0^{\lambda} e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)\mathrm{d}g, \\ &\tfrac{1}{a}\textstyle\int_0^{\lambda} e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}\mathcal{D}(g)\mathrm{d}g \\ &\quad\leq \tfrac{1}{a}\textstyle\int_0^{\lambda} e^{-\frac{1-a}{a}(\mathfrak{h}(\lambda)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)\mathrm{d}g. \end{aligned}$$

Using (21), it is evident that

$$\begin{aligned} \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) &\leq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}_*(\mathcal{D}(\lambda))), \\ \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) &\leq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}^*(\mathcal{D}(\lambda))). \end{aligned} \tag{63}$$

Thus, we have the necessary outcome from (61) and (63). □

**Theorem 6.** *For $a > 0$, let $Y$ and $\mathcal{D}$ be two non-negative mappings with $Y \leq \mathcal{D}$ defined on $[0,\infty)$. Additionally, if an $\mathcal{LR}IV$ convex mapping $\mathfrak{S}$ with $\mathfrak{S}(0) = 0$ and a non-decreasing mapping $Y$ and a decreasing mapping $\frac{Y}{\mathcal{D}}$ is defined on $[0,\infty)$, then the following inequality is satisfied by the fractional integral operator stated in (21) such that*

$$\frac{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}[Y(\lambda)]\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(\mathcal{D}(\lambda))] + \mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[Y(\lambda)]\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(\mathcal{D}(\lambda))]}{\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}[\mathcal{D}(\lambda)]\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))] + \mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathcal{D}(\lambda)]\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))]} \geq_p 1. \tag{64}$$

**Proof.** The mapping $\frac{\mathfrak{S}(\lambda)}{\lambda}$ is non-decreasing, just as $Y$ is. For every $g, \omega \in [0,\lambda)$, it is evident that the mapping $\frac{Y(\lambda)}{\mathcal{D}(\lambda)}$ is decreasing. $\frac{1}{\mathscr{b}}e^{-\frac{1-\mathscr{b}}{\mathscr{b}}(\mathfrak{h}(\lambda)-\mathfrak{h}(\omega))}\mathfrak{h}'(\omega)$ is multiplied, which is non-negative since $\omega \in (0,\lambda), \omega > 0$, and the resulting identity is integrated from 0 to $\lambda$.

$$\begin{aligned} &\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) \\ &\quad\geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}Y(\lambda)\right)\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)), \\ &\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) + \mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)\right) \\ &\quad\geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}Y(\lambda)\right)\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda)). \end{aligned} \tag{65}$$

Now, since $Y \leq \mathcal{D}$ is defined on $[0,\infty)$ and $\frac{\mathfrak{S}_*(\lambda)}{\lambda}$ is a non-decreasing mapping, for $g, \omega \in [0,\lambda)$, we have

$$\begin{aligned} \frac{\mathfrak{S}_*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}, \\ \frac{\mathfrak{S}^*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}. \end{aligned} \tag{66}$$

The mapping $\frac{1}{\mathscr{a}}e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\mathfrak{h}(\curlywedge)-\psi(g))}\mathfrak{h}'(g)\mathcal{D}(g)$ is the product of (66), which is non-negative since $g \in (0,\curlywedge), \curlywedge > 0$, and integrating the inequality from 0 to $\curlywedge$ produces

$$\begin{array}{l} \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\mathfrak{h}(\curlywedge)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}\mathcal{D}(g)\mathrm{d}g \\ \quad \leq \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\mathfrak{h}(\curlywedge)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)\mathrm{d}g, \\ \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\mathfrak{h}(\curlywedge)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}\mathcal{D}(g)\mathrm{d}g \\ \quad \leq \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{1-\mathscr{a}}{\mathscr{a}}(\mathfrak{h}(\curlywedge)-\psi(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)\mathrm{d}g. \end{array} \tag{67}$$

Using (21), it is evident that

$$\begin{array}{l} \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}\left(\frac{\mathfrak{S}_*(Y(\curlywedge))}{Y(\curlywedge)}\mathcal{D}(\curlywedge)\right) \leq \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}(\mathfrak{S}_*(\mathcal{D}(\curlywedge))), \\ \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}\left(\frac{\mathfrak{S}^*(Y(\curlywedge))}{Y(\curlywedge)}\mathcal{D}(\curlywedge)\right) \leq \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}(\mathfrak{S}^*(\mathcal{D}(\curlywedge))). \end{array} \tag{68}$$

Hence, we obtain the necessary outcomes from (67) and (68). □

**Remark 4.** *Theorem 6 becomes Theorem 5 if one assumes that* $\mathscr{b} = \mathscr{a}$.

**Theorem 7.** *For* $\mathscr{a} > 0$, *let* $Y$, $\mathcal{D}$, *and* $Q$ *be two non-negative mappings with* $Y \leq \mathcal{D}$ *defined on* $[0,\infty)$. *Additionally, if an* $\mathcal{LR}IV$ *convex mapping* $\mathfrak{S}$ *with* $\mathfrak{S}(0) = 0$ *and two non-decreasing mappings* $Y$ *and* $Q$ *and a decreasing mapping* $\frac{Y}{\mathcal{D}}$ *are defined on* $[0,\infty)$, *then the following inequality is satisfied by the fractional integral operator stated in (21) such that*

$$\frac{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}[Y(\curlywedge)]}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}[\mathcal{D}(\curlywedge)]} \geq_p \frac{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}[\mathfrak{S}(Y(\curlywedge))Q(\curlywedge)]}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}[\mathfrak{S}(\mathcal{D}(\curlywedge))Q(\curlywedge)]}. \tag{69}$$

**Proof.** Since $Y \leq \mathcal{D}$ is defined on $[0,\infty)$ and $\frac{\mathfrak{S}(\curlywedge)}{\curlywedge}$ is non-decreasing, for $g, \omega \in [0,\curlywedge)$, we have

$$\begin{array}{l} \frac{\mathfrak{S}_*(Y(g))}{Y(g)} \leq \frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}, \\ \frac{\mathfrak{S}^*(Y(g))}{Y(g)} \leq \frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}. \end{array} \tag{70}$$

This is non-negative since $g \in (0,\curlywedge), \curlywedge > 0$ and the inequality's integration from 0 to $\curlywedge$ produces

$$\begin{array}{l} \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\curlywedge)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}\mathcal{D}(g)Q(g)\mathrm{d}g \\ \quad \leq \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\curlywedge)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)Q(g)\mathrm{d}g, \\ \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\curlywedge)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}\mathcal{D}(g)Q(g)\mathrm{d}g \\ \quad \leq \frac{1}{\mathscr{a}}\int_0^{\curlywedge} e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\curlywedge)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}\mathcal{D}(g)Q(g)\mathrm{d}g. \end{array} \tag{71}$$

Using (21), it is evident that

$$\begin{array}{l} \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}\left(\frac{\mathfrak{S}_*(Y(\curlywedge))}{Y(\curlywedge)}\mathcal{D}(\curlywedge)Q(\curlywedge)\right) \leq \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}(\mathfrak{S}_*(\mathcal{D}(\curlywedge))Q(\curlywedge)), \\ \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}\left(\frac{\mathfrak{S}^*(Y(\curlywedge))}{Y(\curlywedge)}\mathcal{D}(\curlywedge)Q(\curlywedge)\right) \leq \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\curlywedge}(\mathfrak{S}^*(\mathcal{D}(\curlywedge))Q(\curlywedge)). \end{array} \tag{72}$$

Also, since the mapping $\mathfrak{S}$ is $\mathcal{LR}IVH$ convex mapping and $\mathfrak{S}(0) = 0$, $\frac{\mathfrak{S}(g)}{g}$ is non-decreasing. Since $Y$ is non-decreasing, so is $\frac{\mathfrak{S}(Y(g))}{Y(g)}$. Clearly, the mapping $\frac{Y(g)}{\mathcal{D}(g)}$ is decreasing for all $g, \omega \in [0,\curlywedge), \curlywedge > 0$. Thus,

$$\begin{array}{l} \left(\frac{\mathfrak{S}_*(Y(g))}{Y(g)}Q(g) - \frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Q(\omega)\right)(Y(\omega)\mathcal{D}(g) - Y(g)\mathcal{D}(\omega)) \geq 0, \\ \left(\frac{\mathfrak{S}^*(Y(g))}{Y(g)}Q(g) - \frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Q(\omega)\right)(Y(\omega)\mathcal{D}(g) - Y(g)\mathcal{D}(\omega)) \geq 0. \end{array} \tag{73}$$

It follows that

$$\begin{array}{l}\frac{\mathfrak{S}_*(Y(g))Q(g)}{Y(g)}Y(\omega)\mathcal{D}(g)-\frac{\mathfrak{S}_*(Y(\omega))Q(\omega)}{Y(\omega)}Y(g)\mathcal{D}(\omega)\\ -\frac{\mathfrak{S}_*(Y(\omega))Q(\omega)}{Y(\omega)}Y(\omega)\mathcal{D}(g)-\frac{\mathfrak{S}_*(Y(g))Q(g)}{Y(g)}Y(\omega)\mathcal{D}(g)\geq 0,\\ \frac{\mathfrak{S}^*(Y(g))Q(g)}{Y(g)}Y(\omega)\mathcal{D}(g)-\frac{\mathfrak{S}^*(Y(\omega))Q(\omega)}{Y(\omega)}Y(g)\mathcal{D}(\omega)\\ -\frac{\mathfrak{S}^*(Y(\omega))Q(\omega)}{Y(\omega)}Y(\omega)\mathcal{D}(g)-\frac{\mathfrak{S}^*(Y(g))Q(g)}{Y(g)}Y(\omega)\mathcal{D}(g)\geq 0.\end{array} \tag{74}$$

We multiply (74) by $\frac{1}{\mathscr{a}}e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)$, which is non-negative since $g\in(0,\lambda), \lambda>0$. By integrating the final identity between 0 to $\lambda$, we obtain

$$\begin{array}{l}\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)Q(g)\mathrm{d}g\\ +\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega)Q(\omega)\mathrm{d}g\\ -\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(\omega)Q(g)\mathrm{d}g\\ -\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}_*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)Q(\omega)\mathrm{d}g\geq 0,\\ \frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)Q(g)\mathrm{d}g\\ +\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(g)\mathcal{D}(\omega)Q(\omega)\mathrm{d}g\\ -\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}Y(\omega)\mathcal{D}(\omega)Q(g)\mathrm{d}g\\ -\frac{1}{\mathscr{a}}\int_0^\lambda e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(g))}\mathfrak{h}'(g)\frac{\mathfrak{S}^*(Y(g))}{Y(g)}Y(\omega)\mathcal{D}(g)Q(\omega)\mathrm{d}g\geq 0.\end{array}$$

It follows that

$$\begin{array}{l}Y(\omega)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\left(\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)Q(\omega)\right)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\\ -\left(\frac{\mathfrak{S}_*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)Q(\omega)\right)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))-\mathcal{D}(\omega)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)\geq 0,\\ Y(\omega)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\left(\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)Q(\omega)\right)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\\ -\left(\frac{\mathfrak{S}^*(Y(\omega))}{Y(\omega)}\mathcal{D}(\omega)Q(\omega)\right)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))-\mathcal{D}(\omega)\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)\geq 0.\end{array} \tag{75}$$

Again, we multiply (75) by $\frac{1}{\mathscr{a}}e^{-\frac{\mathscr{a}-1}{a}(\mathfrak{h}(\lambda)-\mathfrak{h}(\omega))}\mathfrak{h}'(\omega)$, which is non-negative since $g\in(0,\lambda)$, $\lambda>0$. By integrating the final identity between 0 to $g$, we obtain

$$\begin{array}{l}\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)\\ \geq\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right),\\ \mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)\\ \geq\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)+\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(Y(\lambda))\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)\end{array} \tag{76}$$

It follows that

$$\begin{array}{l}\frac{\mathcal{J}^{\mathscr{a},\lambda}_{0^+}(Y(\lambda))}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))}\geq\frac{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}_*(Y(\lambda)))Q(\lambda)}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}_*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)},\\ \frac{\mathcal{J}^{\mathscr{a},\lambda}_{0^+}(Y(\lambda))}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathcal{D}(\lambda))}\geq\frac{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}(\mathfrak{S}^*(Y(\lambda)))Q(\lambda)}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}\left(\frac{\mathfrak{S}^*(Y(\lambda))}{Y(\lambda)}\mathcal{D}(\lambda)Q(\lambda)\right)}.\end{array} \tag{77}$$

From (76) and (77), we thus obtain the necessary outcome. □

**Theorem 8.** *For $\mathscr{a}>0$, let $Y$, $\mathcal{D}$, and $Q$ be two non-negative mappings with $Y\leq\mathcal{D}$ defined on $[0,\infty)$. Additionally, if an $\mathcal{LRIV}$ convex mapping $\mathfrak{S}$ with $\mathfrak{S}(0)=0$ and two non-decreasing mappings $Y$ and $Q$ and a decreasing mapping $\frac{Y}{\mathcal{D}}$ are defined on $[0,\infty)$, then the following inequality is satisfied by the fractional integral operator stated in (21) such that*

$$\frac{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}[Y(\lambda)]\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))Q(\lambda)]+\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[Y(\lambda)]\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))Q(\lambda)]}{\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}[\mathcal{D}(\lambda)]\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))Q(\lambda)]+\mathcal{J}^{\mathscr{b},\mathfrak{h}}_{0^+,\lambda}[\mathcal{D}(\lambda)]\mathcal{J}^{\mathscr{a},\mathfrak{h}}_{0^+,\lambda}[\mathfrak{S}(Y(\lambda))Q(\lambda)]}\geq_p 1. \tag{78}$$

**Proof.** The inequality from 0 to $\nwarrow$ can be integrated to obtain the following result by multiplying both sides of (75) by $\frac{1}{\ell}e^{-\frac{\ell-1}{\ell}(\mathfrak{h}(\nwarrow)-\mathfrak{h}(\omega))}\mathfrak{h}'(\omega)$, which is non-negative because $\omega \in (0,\nwarrow), \omega > 0$.

$$\begin{aligned}
&\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(Y(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}_*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right)+\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(Y(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}_*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right)\\
&\geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathcal{D}(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}_*(Y(\nwarrow))Q(\nwarrow))+\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}_*(Y(\nwarrow))Q(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathcal{D}(\nwarrow)),\\
&\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(Y(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}^*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right)+\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(Y(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}^*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right)\\
&\geq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathcal{D}(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}^*(Y(\nwarrow))Q(\nwarrow))+\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}^*(Y(\nwarrow))Q(\nwarrow))\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathcal{D}(\nwarrow))
\end{aligned} \tag{79}$$

Given that $Y \leq \mathcal{D}$ on $[0,\infty)$ and $\frac{\mathfrak{S}(\nwarrow)}{\nwarrow}$ is non-decreasing, we obtain the following for $g, \omega \in [0,\nwarrow)$:

$$\begin{aligned}
\frac{\mathfrak{S}_*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}_*(\mathcal{D}(g))}{\mathcal{D}(g)},\\
\frac{\mathfrak{S}^*(Y(g))}{Y(g)} &\leq \frac{\mathfrak{S}^*(\mathcal{D}(g))}{\mathcal{D}(g)}.
\end{aligned} \tag{80}$$

Once we multiply both sides of (80) by $\frac{1}{a}e^{-\frac{a-1}{a}}(\mathfrak{h}(\nwarrow)-\mathfrak{h}(g))\mathfrak{h}'(g)\mathcal{D}(g)Q(g), g \in (0,\nwarrow)$, $g > 0$ and integrate the resulting identity from 0 to $g$, we have

$$\begin{aligned}
\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}_*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right) &\leq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}_*(\mathcal{D}(\nwarrow))Q(\nwarrow)),\\
\mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}^*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right) &\leq \mathcal{J}^{a,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}^*(\mathcal{D}(\nwarrow))Q(\nwarrow)).
\end{aligned} \tag{81}$$

In the same way, if we multiply both sides of (80) by $\frac{1}{\ell}e^{-\frac{\ell-1}{\ell}(\mathfrak{h}(\nwarrow)-\mathfrak{h}(g))}\mathfrak{h}'(g)\mathcal{D}(g)Q(g)$, $g \in (0,\nwarrow), g > 0$ and integrate the inequality from 0 to $\nwarrow$, we obtain

$$\begin{aligned}
\mathcal{J}^{\ell,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}_*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right) &\leq \mathcal{J}^{\ell,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}_*(\mathcal{D}(\nwarrow))Q(\nwarrow)),\\
\mathcal{J}^{\ell,\mathfrak{h}}_{0^+,\nwarrow}\left(\frac{\mathfrak{S}^*(Y(\nwarrow))}{Y(\nwarrow)}\mathcal{D}(\nwarrow)Q(\nwarrow)\right) &\leq \mathcal{J}^{\ell,\mathfrak{h}}_{0^+,\nwarrow}(\mathfrak{S}^*(\mathcal{D}(\nwarrow))Q(\nwarrow)).
\end{aligned} \tag{82}$$

Therefore, we obtain the desired result. □

**Remark 5.** *If we set $\ell$ equal to $a$, then Theorem 8 will transform into Theorem 7.*

## 5. Conclusions

The interval fractional integral operators with an exponential kernel have been effectively used in this work to derive interval $\mathcal{HH}$, $\mathcal{HH}$–Fejér, and Pachpatte-type integral inequalities involving the interval fractional integral operator. These inequalities are primarily derived from mappings exhibiting $\mathcal{LR}IVH$ convexity. Several generalizations of the $\mathcal{LR}IV$ convexity theory can be achieved through the key process of adapting the extended form with an exponential in the kernel to the more interval-general fractional integral operator. This study highlights the efficiency of this approach. We validated the findings by presenting two distinct schemes and demonstrating that the results of the proposed method align closely with those of the interval–Riemann–Liouville fractional integral operator. The results clearly indicate that both strategies presented are reliable and effective for addressing various nonlinear problems in science and engineering. We conclude that the findings in this work contribute broadly to the understanding of complex waveforms and circuit theory. Further research is needed to confirm this potential connection.

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