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Coexistence of Algebraic Limit Cycles and Small Limit Cycles of Two Classes of Near-Hamiltonian Systems with a Nilpotent Singular Point

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Abstract: In this paper, two classes of near-Hamiltonian systems with a nilpotent center are considered: the coexistence of algebraic limit cycles and small limit cycles. For the first class of systems, there exist $2n+1$ limit cycles, which include an algebraic limit cycle and $2n$ small limit cycles. For the second class of systems, there exist $\frac{n^2+3n+2}{2}$ limit cycles, including an algebraic limit cycle and $\frac{n^2+3n}{2}$ small limit cycles.

Keywords: nilpotent center; bifurcation; melnikov function; limit cycles

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1. Introduction and Main Results

Hilbert's 16th problem has been a hot topic in the field of differential equations since it was proposed in 1900. Many methods have been proposed and developed to deal with this problem according to the kind of singular points. For elementary singularities, there are many methods for the bifurcation of the limit cycle around a singularity, such as the formal series method, successor function method, Melnikov function method, etc. There are also many methods for nilpotent singularities, such as the inverse integrating factor method, canonical method, Melnikov function method, etc. (for more details, see [1]). Regardless of the particular method, it will involve a large amount of computation, and even for some simple systems, the research difficulty is quite high. Therefore, many scholars pay more attention to systems with special structures.

For the maximal number of small-amplitude limit cycles bifurcating from either an elementary focus or the center of a planar differential system with degree n, when $n = 2$, there are three limit cycles in the small neighborhood of the origin [2]. When $n = 3$, there are 12 limit cycles in the small neighborhood of the elementary origin [3]. However, for nilpotent singularities, there are only nine limit cycles [4] for $n = 3$. For degenerate singularities, some new bifurcation behaviors can be generated. For nilpotent singular points, double bifurcation wasn studied in [5]. A new form of double bifurcation of nilpotent focus was subsequently discussed in [6]. Let $H(n)$ denote the maximal number of limit cycles that a planar differential system with degree n can have; the best results are $H(2) \geq 4$ and $H(3) \geq 13$ (see [7–11]). It is far from being solved for $n \geq 4$.

For cubic-order nilpotent singular points of planar dynamical systems, the central problem was solved by using the inverse integrating factor method in [12,13]. The existence of eight small-amplitude limit cycles bifurcating from a nilpotent critical point for a class of cubic systems was proved in [14]. The local behavior of an isolated nilpotent critical point for polynomial Hamiltonian systems was investigated in [15]. Furthermore, by using the Melnikov function method, some special systems were studied in [11,16]. For a double homoclinic loop passing through a nilpotent saddle, the number of limit cycles bifurcated in a neighborhood of the loop with seven different distributions was studied in [17].

Consider the following near-Hamiltonian system:

$$\begin{cases} \dot{x} = H_y + \varepsilon p(x, y, \delta), \\ \dot{y} = -H_x + \varepsilon q(x, y, \delta), \end{cases} \tag{1}$$

where $H(x, y)$, $p(x, y, \delta)$, and $q(x, y, \delta)$ are all polynomial functions in x and y; $|\varepsilon| > 0$ is a small parameter and vector parameter $\delta \in D \subset \mathbb{R}^m$ with D compact.

When $\varepsilon = 0$, system (1) becomes

$$\begin{cases} \dot{x} = H_y, \\ \dot{y} = -H_x, \end{cases} \tag{2}$$

which is a Hamiltonian system. Suppose that $H(x, y) = h$; $h \in J$, an open interval, defines a closed orbit L_h of (2). Then, the number of limit cycles for system (1) can be estimated by the number of zeros of the first-order Melnikov function of the form

$$M(h, \delta) = \oint_{L_h} q dx - p dy.$$

Thus, the first-order Melnikov function plays an important part in the study of the number of limit cycles (see, for instance, [1,18–22]).

Suppose that system (2) has a nilpotent singular point at the origin; that is to say, the function H satisfies $H_x(0, 0) = H_y(0, 0) = 0$ and

$$\frac{\partial(H_y, -H_x)}{\partial(x, y)}(0, 0) \neq 0,$$
$$\det \frac{\partial(H_y, -H_x)}{\partial(x, y)}(0, 0) = 0.$$

Further, without loss of generality, we can suppose that

$$H_{yy}(0, 0) = 1, \quad H_{xx}(0, 0) = H_{xy}(0, 0) = 0.$$

It is found that the function H at the origin can be expanded as

$$H(x, y) = \frac{1}{2}y^2 + \sum_{i+j\geq 3} h_{ij} x^i y^j. \tag{3}$$

The implicit function theorem demonstrates the existence of a unique C^∞ function $\varphi(x) = \sum_{j\geq 2} e_j x^j$, which satisfies $H_y(x, \varphi(x)) = 0$ for $|x|$ small. Consequently, $H(x, \varphi(x))$ assumes the form

$$H(x, \varphi(x)) = \sum_{j\geq 3} h_j x^j. \tag{4}$$

Let $k \geq 3$ be an integer such that

$$h_k \neq 0, \quad h_j = 0 \text{ for } j < k. \tag{5}$$

Under the conditions (3)–(5), from [15], we know that the origin is (i) a cusp if k is odd; (ii) a saddle if k is even with $h_k < 0$; and (iii) a center if k is even with $h_k > 0$.

Let (5) hold with $h_k > 0$ and $k = 2m$. In this case, the origin is a nilpotent center. There have been many studies on limit cycle bifurcations near a nilpotent center. For example, ref. [23] studied the bifurcation of limit cycles for a quartic near-Hamiltonian system by perturbing a nilpotent center and obtained six limit cycles. Ref. [24] provided an algorithm for calculating the first coefficients of the expansion of the first-order Melnikov function near a nilpotent center. Ref. [25] researched the Hopf bifurcation of an analytic Liénard

system whose unperturbed system has a nilpotent center. The authors [26] studied the limit cycle bifurcations in planar cubic near-Hamiltonian systems with a nilpotent center and proved that there exist at least nine limit cycles near the nilpotent center. Ref. [27] gave a general form of expansion of the first-order Melnikov function near a nilpotent center.

However, there were toofew results to consider the bifurcation of limit cycles from systems with arbitrary degrees because the computation of the Melnikov function becomes more difficult with the increase in degree n. There are few results concerning the coexistence of algebraic limit cycles and the small limit cycles of near-Hamiltonian systems with a nilpotent singular point. In this paper, we mainly study the limit cycle bifurcations of two classes of near-Hamiltonian systems with a nilpotent center. In these two classes of systems, we prove the coexistence of small limit cycles and algebraic limit cycles. To be more specific, we first consider the following near-Hamiltonian system:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = 2x^3 + \varepsilon y(1 - x^4 - y^2)\left(b_{0,0} + \sum\limits_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{6}$$

where $0 < |\varepsilon| \ll 1$ and $b_{2i,2j}$, $i, j = 0, 1, 2, \cdots, n$ are bounded. With the time transformation $t \to -t$, system (6) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x^3 + \varepsilon y(x^4 + y^2 - 1)\left(b_{0,0} + \sum\limits_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{7}$$

where $0 < |\varepsilon| \ll 1$.

For $\varepsilon = 0$, system (7) has a nilpotent center at the origin, and it is Hamiltonian with associated Hamiltonian function $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^4$. In this case, (7) has a family of periodic orbits of the form

$$L_h : \ \frac{1}{2}y^2 + \frac{1}{2}x^4 = h, \quad h \in (0, +\infty)$$

surrounding the nilpotent center, where as h tends towards 0, the limit of L_h is the origin.

Using the first order Melnikov function method, we obtain our main results, which are stated below.

Theorem 1. *For system (7), there exist coefficients $b_{2i,2j}$, $i, j = 0, 1, 2, \cdots, n$ such that it has precisely $2n$ limit cycles near the origin for $0 < |\varepsilon| \ll 1$.*

Clearly, we can derive the following result.

Theorem 2. $x^4 + y^2 = 1$ *is an algebraic limit cycle of (6).*

Next, we consider system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = (n+1)x^{2n+1} + \varepsilon y(1 - x^{2n+2} - y^2)\left(b_{0,0} + \sum\limits_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{8}$$

where $0 < |\varepsilon| \ll 1$ and $b_{2i,2j}$, $i, j = 0, 1, 2, \cdots, n$ are bounded. With the time transformation $t \to -t$, system (8) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(n+1)x^{2n+1} + \varepsilon y(x^{2n+2} + y^2 - 1)\left(b_{0,0} + \sum\limits_{i+j=1}^{n} b_{2i,2j}x^{2i}y^{2j}\right), \end{cases} \tag{9}$$

where $0 < |\varepsilon| \ll 1$.

For $\varepsilon = 0$, system (9) has a nilpotent center at the origin, and it is Hamiltonian with associated Hamiltonian function $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^{2n+2}$. In this case, (9) has a family of periodic orbits of the form

$$L_h : \frac{1}{2}y^2 + \frac{1}{2}x^{2n+2} = h, \quad h \in (0, +\infty).$$

Surrounding the nilpotent center, where h tends towards 0, the limit of L_h is the origin.

Using the first-order Melnikov function method, we have the following main results.

Theorem 3. *For system (9), there exist coefficients $b_{2i,2j}$, $i, j = 0, 1, 2, \cdots, n$ such that it has precisely $\frac{n^2+3n}{2}$ limit cycles near the origin for $0 < |\varepsilon| \ll 1$.*

Obviously, we can obtain the following result.

Theorem 4. *$x^{2n+2} + y^2 = 1$ is an algebraic limit cycle of (8).*

The rest of this paper is organized as follows. In Section 2, we provide an expansion of the first-order Melnikov function of (7) and prove Theorems 1 and 2. In Section 3, an expansion of the first-order Melnikov function of (9) is presented, and Theorems 3 and 4 are proved. In Section 4, we provide two examples as applications of our main results.

2. Proof of Theorems 1 and 2

In this section, we will prove Theorems 1 and 2. Before we do that, we first establish a preliminary.

From [27], we have the following lemma.

Lemma 1. *Let (5) hold with $h_k > 0$ and $k = 2m$. Then, there exists a C^∞ function $N(v, \delta)$ such that*

$$M(h, \delta) = h^{\frac{1+m}{2m}} N(h^{\frac{1}{m}}, \delta).$$

Moreover, if (1) is analytic, then so is $N(v, \delta)$. Thus, if

$$N(v, \delta) = \sum_{l \geq 0} b_l(\delta) v^l$$

for $|v|$ small, then

$$M(h, \delta) = h^{\frac{1+m}{2m}} \sum_{l \geq 0} b_l(\delta) h^{\frac{l}{m}}. \tag{10}$$

for $0 < h \ll 1$.

Thus, one can use the coefficients $b_l(\delta)$ in (10) to study the number of limit cycles of (1) near the origin. From [27], we have

Lemma 2. *Under the condition of Lemma 1, if there exist $l \geq 1$ and $\delta_0 \in \mathbb{R}^l$ such that*

$$b_j(\delta_0) = 0, \ \ j = 0, \cdots, l-1, \ \ b_l(\delta_0) \neq 0,$$
$$\det \frac{\partial(b_0, \cdots, b_{l-1})}{\partial(\delta_1, \cdots, \delta_l)}(\delta_0) \neq 0,$$

then, for some (ε, δ) near $(0, \delta_0)$, (1) has precisely l limit cycles near the origin.

For the expansion of the first-order Melnikov function $M(h)$ of (7), we have the following result.

Lemma 3. *For $0 < h \ll 1$,*

$$M(h) = (2h-1)(2h)^{\frac{3}{4}} \sum_{r=0}^{2n} b_r (2h)^{\frac{r}{2}}, \tag{11}$$

where

$$b_r = \begin{cases} \left(\dfrac{\Gamma(\frac{1}{4} + \frac{r}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4} + \frac{r}{2})} \right) b_{2r,0}, & r = 0, 1, \\ \displaystyle\sum_{j=1}^{[\frac{r}{2}]} \left(\frac{\Gamma(\frac{r}{2} - j + \frac{1}{4})\Gamma(j + \frac{3}{2})}{\Gamma(\frac{r}{2} + \frac{7}{4})} \right) b_{2r-4j,2j} + \left(\frac{\Gamma(\frac{r}{2} + \frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{r}{2} + \frac{7}{4})} \right) b_{2r,0}, & r = 2, \cdots, n, \\ \displaystyle\sum_{j=r-n}^{[\frac{r}{2}]} \left(\frac{\Gamma(\frac{r}{2} - j + \frac{1}{4})\Gamma(j + \frac{3}{2})}{\Gamma(\frac{r}{2} + \frac{7}{4})} \right) b_{2r-4j,2j}, & r = n+1, \cdots, 2n, \end{cases} \tag{12}$$

with Γ being defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{13}$$

for $Re(z) > 0$.

Proof. It is easy to see that the first-order Melnikov function $M(h)$ of (7) can be written as

$$M(h) \ = \ \oint_{L_h} y(x^4 + y^2 - 1) \left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j} \right) dx.$$

Due to symmetry and $y^2 = 2h - x^4$ along the curve L_h, we obtain

$$\begin{aligned} M(h) &= 4 \int_0^{(2h)^{\frac{1}{4}}} (2h - x^4)^{\frac{1}{2}} (2h-1) \left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} (2h - x^4)^j \right) dx \\ &= 4(2h-1)(2h)^{\frac{1}{2}} \int_0^{(2h)^{\frac{1}{4}}} \left(1 - \frac{x^4}{2h}\right)^{\frac{1}{2}} \left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} (2h)^j \left(1 - \frac{x^4}{2h}\right)^j \right) dx. \end{aligned}$$

Let $x = (2hv)^{\frac{1}{4}}$. Then, by direct calculation, we have

$$\begin{aligned} M(h) &= (2h-1)(2h)^{\frac{3}{4}} \int_0^1 v^{-\frac{3}{4}} (1-v)^{\frac{1}{2}} \left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} (2h)^{\frac{i}{2}+j} v^{\frac{i}{2}} (1-v)^j \right) dv \\ &= (2h-1)(2h)^{\frac{3}{4}} \left(\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})} b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} (2h)^{\frac{i}{2}+j} \frac{\Gamma(\frac{i}{2} + \frac{1}{4})\Gamma(j + \frac{3}{2})}{\Gamma(\frac{i}{2} + j + \frac{7}{4})} \right). \end{aligned}$$

It yields (11) and (12). This finishes the proof. □

Proof of Theorem 1. For $j = 0, 1, \cdots, \left[\frac{n}{2}\right]$, $i = 0, 1, \cdots, n-2j$, and for $j = 1, 2, \cdots, n-1$, $i = n-2j+1, n-2j+2, \cdots, 2n-2j$, we let $b_{2i,2j} = 0$, and $b_{0,2n} \neq 0$. Then, we have $b_0 = b_1 = b_2 = \cdots = b_{2n-1} = 0$, $b_{2n} \neq 0$.

Note that

$$\begin{aligned}
&\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2},\\
&\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4},\\
&\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{7}{4})\Gamma(\frac{9}{4})} = \frac{64}{15},\\
&\frac{\Gamma(\frac{1}{4}+n)\Gamma(\frac{3}{4}+n)}{\Gamma(\frac{7}{4}+n)\Gamma(\frac{9}{4}+n)} = \frac{64}{(4n+1)(4n+3)(4n+5)},\\
&\frac{\Gamma(-\frac{3}{4}+\frac{n}{2})\Gamma(-\frac{3}{4}+\frac{n+1}{2})}{\Gamma(\frac{7}{4}+\frac{n}{2})\Gamma(\frac{7}{4}+\frac{n+1}{2})} = \frac{1024}{(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}
\end{aligned}$$

for any positive integer n. Then, from (12), one can obtain the following:

$$\begin{aligned}
&\frac{\partial(b_0, b_1, b_2, \cdots, b_{2n-1})}{\partial(b_{0,0}, b_{2,0}, b_{4,0}, \cdots, b_{2n,0}, b_{2n-2,2}, b_{2n,2}, \cdots, b_{2,2n-2})}\\
= &\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{4})}\frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{4})}\cdots\frac{\Gamma(\frac{n}{2}+\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+\frac{7}{4})}\frac{\Gamma(\frac{n+1}{2}-\frac{3}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{n+1}{2}+\frac{7}{4})}\\
&\frac{\Gamma(\frac{n+2}{2}-\frac{3}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{n+2}{2}+\frac{7}{4})}\frac{\Gamma(\frac{n+3}{2}-\frac{3}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{n+3}{2}+\frac{7}{4})}\cdots\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{3}{4})}{\Gamma(n+\frac{5}{4})} \neq 0.
\end{aligned}$$

Thus, according to Lemma 2, the conclusion of Theorem 1 holds. This ends the proof. □

Proof of Theorem 2. Let $f_1 = x^4 + y^2 - 1$; a direct computation shows that

$$\frac{df_1}{dt} = 2\varepsilon y^2(x^4 + y^2 - 1)(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j}x^{2i}y^{2j}),$$

which demonstrates that $x^4 + y^2 - 1 = 0$ is an invariant algebraic curve of system (6). Moreover, when $h = \frac{1}{2}$ in the Melnikov function $M(h)$ of (7), it means that $x^4 + y^2 = 1$ is a limit cycle of system (6). So, $x^4 + y^2 = 1$ is an algebraic limit cycle of (6). □

3. Proof of Theorems 3 and 4

In this section, we will provide a proof of Theorems 3 and 4. To achieve this, we first demonstrate an expansion of the first-order Melnikov function of (9) near the origin as follows.

Lemma 4. *For* $0 < h \ll 1$,

$$M(h) = \frac{2}{n+1}(2h-1)(2h)^{\frac{1}{2n+2}+\frac{1}{2}}\sum_{r=0}^{n(n+1)} b_r(2h)^{\frac{r}{n+1}}, \tag{14}$$

where

$$b_r = \begin{cases} 0, & r = j(n+1) - i,\ j = 2, 3, \cdots, n,\ i = 1, 2, \cdots, j-1,\\ b_{2(r-(n+1)\left[\frac{r}{n+1}\right]),2\left[\frac{r}{n+1}\right]}B\left(\frac{r}{n+1} - \left[\frac{r}{n+1}\right] + \frac{1}{2n+2}, \left[\frac{r}{n+1}\right] + \frac{3}{2}\right), & \\ & r = i + j(n+1),\ j = 0, 1, \cdots, n,\ i = 0, 1, \cdots, n-j, \end{cases} \tag{15}$$

with $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, *and* Γ *being defined by (13).*

Proof. It is obvious that the first-order Melnikov function $M(h)$ of (9) can be written as

$$M(h) = \oint_{L_h} y(x^{2n+2}+y^2-1)\left(b_{0,0}+\sum_{i+j=1}^{n} b_{2i,2j}x^{2i}y^{2j}\right)dx.$$

Due to symmetry and $y^2 = 2h - x^{2n+2}$ along the curve L_h, we obtain

$$\begin{aligned} M(h) &= 4\int_0^{(2h)^{\frac{1}{2n+2}}} (2h-x^{2n+2})^{\frac{1}{2}}(2h-1)\left(b_{0,0}+\sum_{i+j=1}^{n} b_{2i,2j}x^{2i}(2h-x^{2n+2})^j\right)dx \\ &= \int_0^{(2h)^{\frac{1}{2n+2}}} \left(1-\frac{x^{2n+2}}{2h}\right)^{\frac{1}{2}}\left(b_{0,0}+\sum_{i+j=1}^{n} b_{2i,2j}x^{2i}(2h)^j\left(1-\frac{x^{2n+2}}{2h}\right)^j\right)dx \\ &\times\ 4(2h-1)(2h)^{\frac{1}{2}}. \end{aligned}$$

Let $x = (2hv)^{\frac{1}{2n+2}}$. Then, by direct calculation, we have

$$\begin{aligned} M(h) &= \int_0^1 v^{\frac{1}{2n+2}-1}(1-v)^{\frac{1}{2}}\left(b_{0,0}+\sum_{i+j=1}^{n} b_{2i,2j}(2h)^{\frac{i}{n+1}+j}v^{\frac{i}{n+1}}(1-v)^j\right)dv \\ &\times\ \frac{2}{n+1}(2h-1)(2h)^{\frac{1}{2n+2}+\frac{1}{2}} \\ &= \frac{2}{n+1}(2h-1)(2h)^{\frac{1}{2n+2}+\frac{1}{2}}\left(B(\frac{1}{2n+2},\frac{3}{2})b_{0,0}\right. \\ &\left.+\sum_{i+j=1}^{n} b_{2i,2j}(2h)^{\frac{i}{n+1}+j}B(\frac{i}{n+1}+\frac{1}{2n+2},j+\frac{3}{2})\right). \end{aligned}$$

Then, we obtain (14) and (15). This finishes the proof. □

Next, we leverage Lemmas 2 and 4 to elucidate Theorem 3.

Proof of Theorem 3. For $j = 0,1,\cdots,n-1$, $i = 0,1,\cdots,n-j$, we define $b_{2i,2j} = 0$, and $b_{0,2n} \neq 0$. Consequently, we observe that $b_0 = b_1 = b_2 = \cdots = b_{n^2} = 0$, $b_{n(n+1)} \neq 0$. Following a similar proof strategy as that of Theorem 1, it is readily apparent from (15) that

$$\begin{aligned} &\frac{\partial(b_0,b_1,\cdots,b_n,b_{n+1},\cdots,b_{2n},b_{2n+2},\cdots,b_{3n},b_{3n+3},\cdots,b_{n^2})}{\partial(b_{0,0},b_{2,0},\cdots,b_{2n,0},b_{0,2},\cdots,b_{2(n-1),2},b_{04},\cdots,b_{2(n-2),4},b_{0,6},\cdots,b_{2,2(n-1)})} \\ =\ & B(\frac{1}{2n+2},\frac{3}{2})B(\frac{3}{2n+2},\frac{3}{2})\cdots B(\frac{2n+1}{2n+2},\frac{3}{2})B(\frac{1}{2n+2},\frac{5}{2})\cdots \\ & B(\frac{2n-1}{2n+2},\frac{5}{2})B(\frac{1}{2n+2},\frac{7}{2})\cdots B(\frac{2n-3}{2n+2},\frac{7}{2})B(\frac{1}{2n+2},\frac{9}{2})\cdots \\ & B(\frac{3}{2n+2},n+\frac{1}{2}) \\ \neq\ & 0. \end{aligned}$$

Thus, according to Lemma 2, the conclusion of Theorem 1 holds. Then, we complete the proof. □

Proof of Theorem 4. Let $f_2 = x^{2n+2}+y^2-1$; a direct computation shows that

$$\frac{df_2}{dt} = 2\varepsilon y^2(x^{2n+2}+y^2-1)(b_{0,0}+\sum_{i+j=1}^{n} b_{2i,2j}x^{2i}y^{2j}),$$

which reveals that $x^{2n+2}+y^2-1=0$ is an invariant algebraic curve of system (8). When $h=\frac{1}{2}$ in the Melnikov function $M(h)$ of (9), this means that $x^{2n+2}+y^2=1$ is a limit cycle of system (8). Thus, $x^{2n+2}+y^2=1$ is an algebraic limit cycle of (8). □

4. Examples

In this section, we present two examples as applications of our main results.

Example 1. *In system (7), we take $n=3$ to obtain*

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-2x^3+\varepsilon y(x^4+y^2-1)\left(b_{0,0}+\sum\limits_{i+j=1}^{3} b_{2i,2j}x^{2i}y^{2j}\right), \end{cases} \tag{16}$$

where $0<|\varepsilon|\ll 1$ and $b_{2i,2j}$, $i,j=0,1,2,3$ are bounded.

From Lemma 3, we can obtain

$$M(h)=(2h-1)(2h)^{\frac{3}{4}}\sum_{r=0}^{6} b_r(2h)^{\frac{r}{2}}$$

for $0<h\ll 1$, where

$$\begin{aligned}
b_0 &= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}b_{0,0},\\
b_1 &= \frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{4})}b_{2,0},\\
b_2 &= \frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{4})}b_{4,0}+\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{4})}b_{0,2},\\
b_3 &= \frac{\Gamma(\frac{7}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{13}{4})}b_{6,0}+\frac{\Gamma(\frac{3}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{13}{4})}b_{2,2},\\
b_4 &= \frac{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{15}{4})}b_{4,2}+\frac{\Gamma(\frac{1}{4})\Gamma(\frac{7}{2})}{\Gamma(\frac{15}{4})}b_{0,4},\\
b_5 &= \frac{\Gamma(\frac{3}{4})\Gamma(\frac{7}{2})}{\Gamma(\frac{17}{4})}b_{2,4},\\
b_6 &= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{9}{2})}{\Gamma(\frac{19}{4})}b_{0,6}.
\end{aligned}$$

Take $b_{0,0}=b_{2,0}=b_{4,0}=b_{6,0}=b_{0,2}=b_{2,2}=b_{4,2}=b_{0,4}=b_{2,4}=0$ and $b_{0,6}\neq 0$. Thus, we have $b_0=b_1=b_2=b_3=b_4=b_5=0$, $b_6=\frac{\Gamma(\frac{1}{4})\Gamma(\frac{9}{2})}{\Gamma(\frac{19}{4})}b_{0,6}\neq 0$, and

$$\begin{aligned}
&\det\frac{\partial(b_0,b_1,b_2,b_3,b_4,b_5)}{\partial(b_{0,0},b_{2,0},b_{4,0},b_{6,0},b_{4,2},b_{2,4})}\\
&=\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{4})}\frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{4})}\frac{\Gamma(\frac{7}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{13}{4})}\frac{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})}{\Gamma(\frac{15}{4})}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{7}{2})}{\Gamma(\frac{17}{4})}\\
&=\frac{32768}{14189175}\pi^3.
\end{aligned}$$

It follows from Lemma 2 that (16) has precisely six limit cycles near the origin for $0<|\varepsilon|\ll 1$. It is easy to check that $x^4+y^2=1$ is an algebraic limit cycle of (16). Then, we obtain the conclusions of Theorems 1 and 2.

Example 2. *In system (9), we take $n = 2$ to obtain*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -3x^5 + \varepsilon y(x^6 + y^2 - 1)\left(b_{0,0} + \sum\limits_{i+j=1}^{2} b_{2i,2j}x^{2i}y^{2j}\right), \end{cases} \tag{17}$$

where $0 < |\varepsilon| \ll 1$ and $b_{2i,2j}$, $i, j = 0, 1, 2$ are bounded.

By Lemma 4, we have

$$M(h) = \frac{2}{3}(2h-1)(2h)^{\frac{2}{3}} \sum_{r=0}^{6} b_r (2h)^{\frac{r}{3}}$$

for $0 < h \ll 1$, where

$$\begin{aligned} b_0 &= B(\tfrac{1}{6}, \tfrac{3}{2})b_{0,0}, \quad b_1 = B(\frac{1}{2}, \frac{3}{2})b_{2,0}, \\ b_2 &= B(\tfrac{5}{6}, \tfrac{3}{2})b_{4,0}, \quad b_3 = B(\frac{1}{6}, \frac{5}{2})b_{0,2}, \\ b_4 &= B(\tfrac{1}{2}, \tfrac{5}{2})b_{2,2}, \quad b_5 = 0, \\ b_6 &= B(\tfrac{1}{6}, \tfrac{7}{2})b_{0,4}. \end{aligned}$$

Take $b_{0,0} = b_{2,0} = b_{4,0} = b_{0,2} = b_{2,2} = 0$, $b_{0,4} \neq 0$. Then we have $b_0 = b_1 = b_2 = b_3 = b_4 = 0$, $b_6 \neq 0$, and

$$\begin{aligned} &\det \frac{\partial(b_0, b_1, b_2, b_3, b_4)}{\partial(b_{0,0}, b_{2,0}, b_{4,0}, b_{0,2}, b_{2,2})} \\ &= \frac{\Gamma(\frac{1}{6})\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{3})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} \frac{\Gamma(\frac{5}{6})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{3})} \frac{\Gamma(\frac{1}{6})\Gamma(\frac{5}{2})}{\Gamma(\frac{8}{3})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)} \\ &= \frac{2187\sqrt{3}\pi^{\frac{9}{2}}}{10240\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}. \end{aligned}$$

It follows from Lemma 2 that (17) has precisely five limit cycles near the origin for $0 < |\varepsilon| \ll 1$. It is easy to check that $x^6 + y^2 = 1$ is an algebraic limit cycle of (17). Then, we obtain the conclusions of Theorems 3 and 4.

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