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On a Matrix Formulation of the Sequence of Bi-Periodic Fibonacci Numbers

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Abstract: In this study, we investigate some new properties of the sequence of bi-periodic Fibonacci numbers with arbitrary initial conditions, through an approach that combines the matrix aspect and the fundamental Fibonacci system. Indeed, by considering the properties of the eigenvalues of their related 2×2 matrix, we provide a new approach to studying the analytic representations of these numbers. Moreover, the similarity of the associated 2×2 matrix with a companion matrix, allows us to formulate the bi-periodic Fibonacci numbers in terms of a homogeneous linear recursive sequence of the Fibonacci type. Therefore, the combinatorial aspect and other analytic representations formulas of the Binet type for the bi-periodic Fibonacci numbers are achieved. The case of bi-periodic Lucas numbers is outlined, and special cases are exposed. Finally, some illustrative examples are given.

Keywords: bi-periodic Fibonacci numbers; matrix formulation; companion matrix; Fibonacci sequence; fundamental Fibonacci system; analytic formulas; combinatorial expressions

MSC: 11B39; 11B75; 11C20; 65Q10; 65Q30

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1. Introduction

The famous well-known sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by a linear recursive relation of order 2, $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Since its appearance in connection with the famous problem of the evolution of the population of rabbits in the work of Fibonacci, this sequence has been widely studied and has appeared as a powerful tool in various fields of mathematics and applied sciences. In his interesting manuscript [1,2], Koshy gave several properties and applications of the sequence of Fibonacci numbers and other known classical sequences of numbers, such as Lucas and Pell numbers. In addition, the sequence of Fibonacci numbers has been the subject of many generalizations, and it has been the source of several identities in additive number theory [3].

Recently, some generalizations of the sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ have been provided in the literature by considering a connection with a periodicity condition linked to a special parametrization (see, for example, [4–8]). Indeed, in [5], Edson and Yayenie introduced a new generalization for the sequence of Fibonacci numbers, labeled *bi-periodic Fibonacci sequence*, which is defined as follows. Let a and b be two non-zero real numbers, and consider the sequence denoted by $\{F_n^{(a,b)}\}_{n\geq 0}$ defined by

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & if \ \ n \ \text{is} \ \ \text{even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & if \ \ n \ \text{is} \ \ \text{odd} \end{cases}, \tag{1}$$

for $n \geq 2$, with initial conditions given by $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. We can easily observe that Expression (1) covers some known sequences of numbers. Indeed, for $a = b = 1$, we obtain the classical sequence of Fibonacci numbers, and for $a = b = 2$, Expression (1) is reduced to the sequence of Pell numbers. Moreover, if we consider $a = b = k$, for some positive integer k, we obtain the sequence of k-Fibonacci numbers. In addition, given that the Lucas numbers are defined by the same recurrence relation as the sequence of Fibonacci numbers, another important generalization of Lucas numbers is the sequence of bi-periodic Lucas numbers, introduced by Bilgici in [9]. The sequence of bi-periodic Lucas numbers, denoted by $\{l_n^{(a,b)}\}_{n\geq 0}$, is defined recursively by

$$l_n^{(a,b)} = \begin{cases} al_{n-1}^{(a,b)} + l_{n-2}^{(a,b)}, & if\ \ n\ \ is\ \ even \\ bl_{n-1}^{(a,b)} + l_{n-2}^{(a,b)}, & if\ \ n\ \ is\ \ odd \end{cases}, \tag{2}$$

for $n \geq 2$, with initial conditions $l_0^{(a,b)} = 2$, $l_1^{(a,b)} = a$. For $a = b = 1$, we obtain the classical sequence of Lucas numbers, and for $a = b = k$, with k being a positive integer, we obtain the sequence of k-Lucas numbers. Furthermore, using an analogous definition, another matrix method was considered in [6,8], where some properties of bi-periodic Fibonacci sequences were established. Motivated by the work of Horadam [10], the author of [7] obtained some basic properties of Horadam bi-periodic sequences that generalize the results known for Fibonacci and Lucas bi-periodic sequences. Currently, for the Fibonacci–Lucas bi-periodic sequences, it is worth noting that in the literature, the authors do not use Expressions (1)–(2), but they study these expressions using the parity function $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd, defined by $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ is the integer part of x (see [4–7]).

Our goal in this paper is to study the sequence of bi-periodic Fibonacci numbers, with arbitrary initial conditions $F_0^{(a,b)}$ and $F_1^{(a,b)}$. Our main tool is based on an equivalent matrix formulation of Expression (1). First, with the aid of eigenvalues of a specific 2×2 matrix, we manage to establish the analytic formula of the Binet type for the bi-periodic Fibonacci numbers $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$, with arbitrary initial conditions $F_0^{(a,b)}$ and $F_1^{(a,b)}$. Second, more properties of this sequence are provided using the Fibonacci fundamental system. More specifically, the similarity of 2×2 matrix defining the matrix formulation of the sequence of bi-periodic Fibonacci numbers with a companion 2×2 matrix allows us to formulate the numbers $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ in terms of a recursive sequence of the Fibonacci type of order two. Therefore, the numbers $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ are written in terms of the fundamental solution of the Fibonacci fundamental system, related to this sequence. Consequently, the explicit combinatorial formula for the sequence of bi-periodic Fibonacci numbers is provided. Moreover, when the roots of characteristic polynomial are simple, another analytical formula for the bi-periodic Fibonacci numbers is given. In addition, the case of bi-periodic Lucas numbers is outlined, and special cases are exposed. It should be noted that our approach and methods are different from those in the current literature on bi-periodic Fibonacci sequences.

This paper is organized as follows. In Section 2, we study the matrix formulation of the bi-periodic Fibonacci sequence (1), with arbitrary initial conditions $F_0^{(a,b)}$ and $F_1^{(a,b)}$. Section 3 is devoted to the analytic formula of the Binet type for the sequence of bi-periodic Fibonacci numbers. In addition, some special cases are studied. By considering the linear recursive sequence related to the companion 2×2 matrix, we provide in Section 4 the explicit formulas of bi-periodic Fibonacci numbers using three approaches: linear, analytic, and combinatorial. For illustrative purposes, special cases and significant examples are provided. Finally, some concluding remarks and perspectives are stated.

In this study, the two real numbers a and b are non-zero real numbers.

2. Matrix Formulation of the Sequence of Bi-Periodic Fibonacci Numbers and Their Analytic Formula

2.1. Matrix Formulation of the Bi-Periodic Fibonacci Sequence

Let a and b be two non-zero real numbers and consider the sequence of bi-periodic Fibonacci numbers defined by Expression (1). Let us consider the following matrices

$$Y_n^{(a,b)} = \begin{bmatrix} F_{2n}^{(a,b)} \\ F_{2n-1}^{(a,b)} \end{bmatrix}, Z_n^{(a,b)} = \begin{bmatrix} F_{2n+1}^{(a,b)} \\ F_{2n}^{(a,b)} \end{bmatrix}, A[a,1] = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A[b,1] = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix}.$$

We can observe that Equations (1) are equivalent to the following two simultaneous matrices equations $Y_n^{(a,b)} = A[a,1]Z_{n-1}^{(a,b)}$ and $Z_n^{(a,b)} = A[b,1]Y_n^{(a,b)}$, which implies that we have $Z_n^{(a,b)} = A[b,1]Y_n^{(a,b)} = A[b,1]A[a,1]Z_{n-1}^{(a,b)}$. Therefore, we derive that the bi-periodic Fibonacci numbers can be studied using the following matrix formulation:

$$Z_n^{(a,b)} = D[a,b]Z_{n-1}^{(a,b)}, \tag{3}$$

where $D[a,b] = A[b,1]A[a,1]$ is the matrix given by $D[a,b] = A[b,1]A[a,1] = \begin{bmatrix} ab+1 & b \\ a & 1 \end{bmatrix}$.

The matrix $D[a,b]$ will play a central role in this study. Indeed, several important results of the sequence of bi-periodic Fibonacci numbers will be provided, with the aid of some properties of the matrix $D[a,b]$. Equation (3) implies that we have

$$Z_n^{(a,b)} = D[a,b]^n Z_0^{(a,b)}, \text{ where } Z_0^{(a,b)} = \begin{bmatrix} F_1^{(a,b)} \\ F_0^{(a,b)} \end{bmatrix}. \tag{4}$$

Formula (4) permits us to see that the computation of the entries of the powers $D[a,b]^n$ is required for studying the sequence of bi-periodic Fibonacci numbers.

2.2. The Analytic Binet Formula of Bi-Periodic Fibonacci Numbers via the Canonical Jordan Form

A direct computation shows that the characteristic polynomial of the matrix $D[a,b]$ is

$$P(z) = \det(D[a,b] - zI_{2\times 2}) = z^2 - (ab+2)z + 1.$$

In this subsection, we present the analytic formula of the Binet type for the bi-periodic Fibonacci numbers by considering two cases: the case when $P(z)$ owns two distinct simple roots, and the case when it owns only a unique root with multiplicity two. First, for $a^2b^2 + 4ab \neq 0$, the eigenvalues of the matrix $D[a,b]$ are given by $\lambda_1 = \dfrac{ab+2-\sqrt{a^2b^2+4ab}}{2}$ and $\lambda_2 = \dfrac{ab+2+\sqrt{a^2b^2+4ab}}{2}$. Therefore, a long straightforward computation allows us to obtain the diagonal form of this matrix as follows:

$$D[a,b] = \begin{bmatrix} ab+1 & b \\ a & 1 \end{bmatrix} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}, \tag{5}$$

where

$$P = \begin{bmatrix} \frac{ab-\sqrt{ab(ab+4)}}{2a} & \frac{ab+\sqrt{ab(ab+4)}}{2a} \\ 1 & 1 \end{bmatrix}, \; P^{-1} = \begin{bmatrix} -\frac{a}{\sqrt{ab(ab+4)}} & \frac{ab+\sqrt{ab(ab+4)}}{2\sqrt{ab(ab+4)}} \\ \frac{a}{\sqrt{ab(ab+4)}} & \frac{-ab+\sqrt{ab(ab+4)}}{2\sqrt{ab(ab+4)}} \end{bmatrix}. \tag{6}$$

The diagonal form (5) and (6) of the matrix $D[a,b]$ makes it possible to calculate explicitly the entries of the powers $D[a,b]^n$ in terms of the two distinct eigenvalues λ_1 and λ_2. Indeed, for every $n \geq 0$, we have $D[a,b]^n = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}$. Hence, with the

aid of Equations (4), we can obtain $Z_n^{(a,b)}$ in terms of the two distinct eigenvalues λ_1 and λ_2. That is, for every $n \geq 0$, we have $Z_n^{(a,b)} = P\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}P^{-1}Z_0^{(a,b)}$, where $Z_0^{(a,b)}$ is as in (4). Therefore, the preceding formula permits us to compute the analytic formula of the Binet type for the bi-periodic Fibonacci numbers $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$. Indeed, a direct computation implies that $P^{-1}Z_0^{(a,b)} = \begin{bmatrix}-\beta & \gamma_1\\ \beta & \gamma_2\end{bmatrix}\begin{bmatrix}F_1^{(a,b)}\\ F_0^{(a,b)}\end{bmatrix} = \begin{bmatrix}-\beta F_1^{(a,b)} + \gamma_1 F_0^{(a,b)}\\ \beta F_1^{(a,b)} + \gamma_2 F_0^{(a,b)}\end{bmatrix}$. For the reason of simplicity, we set

$$\begin{cases}\alpha_1 = \frac{ab-\sqrt{ab(ab+4)}}{2a} \text{ and } \alpha_2 = \frac{ab+\sqrt{ab(ab+4)}}{2a};\\ \beta = \frac{a}{\sqrt{ab(ab+4)}},\ \gamma_1 = \frac{ab+\sqrt{ab(ab+4)}}{2\sqrt{ab(ab+4)}} \text{ and } \gamma_2 = \frac{-ab+\sqrt{ab(ab+4)}}{2\sqrt{ab(ab+4)}}.\end{cases} \tag{7}$$

Therefore, we obtain $\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}P^{-1}Z_0^{(a,b)} = \begin{bmatrix}(-\beta F_1^{(a,b)} + \gamma_1 F_0^{(a,b)})\lambda_1^n\\ (\beta F_1^{(a,b)} + \gamma_2 F_0^{(a,b)})\lambda_2^n\end{bmatrix}$. Hence, we obtain $Z_n^{(a,b)} = \begin{bmatrix}\alpha_1 & \alpha_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}(-\beta F_1^{(a,b)} + \gamma_1 F_0^{(a,b)})\lambda_1^n\\ (\beta F_1^{(a,b)} + \gamma_2 F_0^{(a,b)})\lambda_2^n\end{bmatrix}$. Finally, a straightforward computation allows us to obtain the following result.

Theorem 1. *Let a and b be two non-zero real numbers, and consider the sequence of the bi-periodic Fibonacci $\{F_n^{(a,b)}\}_{n\geq 0}$ numbers defined by (1). Suppose that the eigenvalues of the matrix $D[a,b]$ are simple, namely, $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2} \neq \lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$. Then, the analytic formula of the Binet type of bi-periodic Fibonacci numbers is given by*

$$\begin{cases}F_{2n+1}^{(a,b)} = (\alpha_1\gamma_1\lambda_1^n + \alpha_2\gamma_2\lambda_2^n)F_0^{(a,b)} - \beta(\alpha_1\lambda_1^n - \alpha_2\lambda_2^n)F_1^{(a,b)}\\ F_{2n}^{(a,b)} = (\gamma_1\lambda_1^n + \gamma_2\lambda_2^n)F_0^{(a,b)} - \beta(\lambda_1^n - \lambda_2^n)F_1^{(a,b)}\end{cases}, \tag{8}$$

where the scalars α_1, α_2, β, γ_1, γ_2 are given as in (7), and $F_0^{(a,b)}$ and $F_1^{(a,b)}$ are the arbitrary initial conditions.

Expression (8) shows that the analytic formula of the Binet type for the bi-periodic Fibonacci numbers $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ is given in terms of arbitrary initial conditions $F_0^{(a,b)}$ and $F_1^{(a,b)}$. Generally, in the current literature, the usual initial conditions are $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Therefore, the result of Theorem 1, namely, (8), can be applied to the sequence of bi-periodic Fibonacci numbers, with the usual initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. That is, a direct computation allows us to obtain the following corollary.

Corollary 1. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers, with the initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Then, the analytic formula of the Binet type of $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ is*

$$\begin{cases}F_{2n+1}^{(a,b)} = -\frac{a}{\sqrt{ab(ab+4)}}\left(\frac{ab-\sqrt{ab(ab+4)}}{2a}\lambda_1^n - \frac{ab+\sqrt{ab(ab+4)}}{2a}\lambda_2^n\right)\\ F_{2n}^{(a,b)} = -\frac{a}{\sqrt{ab(ab+4)}}(\lambda_1^n - \lambda_2^n)\end{cases}, \tag{9}$$

where $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$ and $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$.

Similarly, using the result of Theorem 1, we show that the analytic formula for the bi-periodic Lucas numbers (2) is formulated as follows.

Corollary 2. *Consider* $\{l_n^{(a,b)}\}_{n\geq 0}$ *the sequence of bi-periodic Lucas numbers with initial conditions* $l_0^{(a,b)} = 2$ *and* $l_1^{(a,b)} = a$*. Then, the analytic formula of the Binet type of* $l_{2n}^{(a,b)}$ *and* $l_{2n+1}^{(a,b)}$ *is given by*

$$\begin{cases} l_{2n+1}^{(a,b)} = 2(\alpha_1\gamma_1\lambda_1^n + \alpha_2\gamma_2\lambda_2^n) - a\beta(\alpha_1\lambda_1^n - \alpha_2\lambda_2^n) \\ l_{2n}^{(a,b)} = 2(\gamma_1\lambda_1^n + \gamma_2\lambda_2^n) - a\beta(\lambda_1^n - \lambda_2^n) \end{cases}, \tag{10}$$

where $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$ *and* $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$.

We illustrate the result of Corollary 1 by the following generic numerical example.

Example 1. *Let* $\{F_n^{(1,2)}\}_{n\geq 0}$ *be the sequence of bi-periodic Fibonacci numbers defined by*

$$F_n^{(1,2)} = \begin{cases} F_{n-1}^{(1,2)} + F_{n-2}^{(1,2)}, & if\ \ n\ \ is\ \ even \\ 2F_{n-1}^{(1,2)} + F_{n-2}^{(1,2)}, & if\ \ n\ \ is\ \ odd \end{cases}, \tag{11}$$

for $n \geq 2$*, with initial conditions given by* $F_0^{(1,2)} = 0$ *and* $F_1^{(1,2)} = 1$*. The first values of* $\{F_n^{(1,2)}\}_{n\geq 0}$ *are given in Table 1 below. Equations (11) are equivalent to the following matrix equation* $Z_n^{(1,2)} = D[1,2]Z_{n-1}^{(1,2)}$*, where* $Z_n^{(1,2)} = \begin{bmatrix} F_{2n+1}^{(1,2)} \\ F_{2n}^{(1,2)} \end{bmatrix}$ *and* $D[1,2] = A[2,1]A[1,1] = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$*. The characteristic polynomial of the matrix* $D[1,2]$ *is* $P(z) = \det(D[1,2] - z\mathbf{I}_{2\times 2}) = z^2 - 4z + 1$*. Thus, the eigenvalues of the matrix* $D[1,2]$ *are* $\lambda_1 = 2 - \sqrt{3}$ *and* $\lambda_2 = 2 + \sqrt{3}$*. By direct application of the preceding process, we derive* $Z_n^{(1,2)} = P\begin{bmatrix} (2-\sqrt{3})^n & 0 \\ 0 & (2+\sqrt{3})^n \end{bmatrix} P^{-1}Z_0^{(1,2)}$*, where* $P = \begin{bmatrix} (1-\sqrt{3}) & (1+\sqrt{3}) \\ 1 & 1 \end{bmatrix}$*,* $P^{-1} = \begin{bmatrix} -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{-3+\sqrt{3}}{6} \end{bmatrix}$ *and* $Z_0^{(1,2)} = [F_1^{(1,2)}, F_0^{(1,2)}]^t = [1,0]^t$*. Therefore, a direct computation allows us to obtain that the analytic formula of the Binet type of the sequence of bi-periodic Fibonacci numbers (11) is given as follows*

$$\begin{cases} F_{2n+1}^{(a,b)} = -\frac{\sqrt{3}}{6}((1-\sqrt{3})(2-\sqrt{3})^n - (1+\sqrt{3})(2+\sqrt{3})^n) \\ F_{2n}^{(a,b)} = -\frac{\sqrt{3}}{6}((2-\sqrt{3})^n - (2+\sqrt{3})^n) \end{cases}.$$

Table 1. Some values for $F_n^{(1,2)}$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$F_n^{(1,2)}$	0	1	1	3	4	11	15	41	56	153	209	571
$F_{2n}^{(1,2)}$	0	1	4	15	56	209	780	2911	10,864	40,545	151,316	564,719
$F_{2n+1}^{(1,2)}$	1	3	11	41	153	571	2131	7953	29,681	110,771	413,403	1,542,841

For $a^2b^2 + 4ab = 0$, the matrix $D[a,b]$ owns only one eigenvalue given by $\lambda = \frac{ab+2}{2}$. A direct computation allows us to obtain the matrix canonical Jordan form of the matrix $D[a,b]$, given by $D[a,b] = \begin{bmatrix} ab+1 & b \\ a & 1 \end{bmatrix} = S\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}S^{-1}$, where

$$S = \begin{bmatrix} b & 0 \\ -\frac{ab}{2} & 1 \end{bmatrix} \text{ and } S^{-1} = \begin{bmatrix} \frac{1}{b} & 0 \\ \frac{a}{2} & 1 \end{bmatrix}. \tag{12}$$

Then, the entries of the powers $D[a,b]^n$ can be calculated explicitly in terms of the eigenvalue λ, and the analytic formula of the Binet type for the bi-periodic Fibonacci

numbers $F_n^{(a,b)}$ is derived. That is, for every $n \geq 0$, we have $D[a,b]^n = S\begin{bmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{bmatrix}S^{-1}$. Therefore, using the matrix Equation (4), the vector $Z_n^{(a,b)}$ is stated as follows:

$$Z_n^{(a,b)} = S\begin{bmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{bmatrix}S^{-1}Z_0^{(a,b)}, \tag{13}$$

for every $n \geq 0$, where $\lambda = \frac{ab+2}{2}$, S, S^{-1} are as in (12) and $Z_0^{(a,b)} = [F_1^{(a,b)}, F_0^{(a,b)}]^t$. As a result, the matrix Equation (13) permits us to obtain the analytic formula of the Binet type for the bi-periodic Fibonacci numbers. Indeed, for every $n \geq 1$, we have

$$\begin{aligned}\begin{bmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{bmatrix}S^{-1}Z_0^{(a,b)} &= \begin{bmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{bmatrix}\begin{bmatrix}\frac{1}{b} & 0\\ -\frac{a}{2} & 1\end{bmatrix}\begin{bmatrix}F_1^{(a,b)}\\ F_0^{(a,b)}\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{b}F_1^{(a,b)}\lambda^n + n(\frac{a}{2}F_1^{(a,b)} + F_0^{(a,b)})\lambda^{n-1}\\ (\frac{a}{2}F_1^{(a,b)} + F_0^{(a,b)})\lambda^n\end{bmatrix}.\end{aligned}$$

Finally, a direct calculation gives

$$Z_n^{(a,b)} = S\begin{bmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{bmatrix}S^{-1}Z_0^{(a,b)} = \begin{bmatrix}nb\lambda^{n-1}F_0^{(a,b)} + (\lambda^n + \frac{ab}{2}n\lambda^{n-1})F_1^{(a,b)}\\ (\lambda^n - \frac{ab}{2}n\lambda^{n-1})F_0^{(a,b)} - \frac{a^2b}{4}n\lambda^{n-1}F_1^{(a,b)}\end{bmatrix}.$$

for every $n \geq 1$. In summary, this former expression provides the following result.

Theorem 2. *Let a and b be two non-zero real numbers, and consider $\{F_n^{(a,b)}\}_{n\geq 0}$ the bi-periodic Fibonacci sequence (1). Suppose that the characteristic polynomial of the matrix $D[a,b]$ owns double roots, namely, $\lambda = \lambda_1 = \lambda_2 = \frac{ab+2}{2}$. Then, the analytic formula of the Binet type for bi-periodic Fibonacci numbers is given by*

$$\begin{cases}F_{2n+1}^{(a,b)} = nb\lambda^{n-1}F_0^{(a,b)} + (\lambda^n + \frac{ab}{2}n\lambda^{n-1})F_1^{(a,b)}\\ F_{2n}^{(a,b)} = (\lambda^n - \frac{ab}{2}n\lambda^{n-1})F_0^{(a,b)} - \frac{a^2b}{4}n\lambda^{n-1}F_1^{(a,b)}\end{cases}.$$

By applying the result of Theorem 2 to the bi-periodic Fibonacci numbers with the usual initial conditions of the literature, $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$, and also to the bi-periodic Lucas numbers with the usual initial conditions $l_0^{(a,b)} = 2$ and $l_1^{(a,b)} = a$, we come to have the following corollary.

Corollary 3. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers (1) with initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Then, the analytic formula of the Binet type for $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ is*

$$\begin{cases}F_{2n+1}^{(a,b)} = \left(\frac{ab+2}{2}\right)^n + \frac{ab}{2}n\left(\frac{ab+2}{2}\right)^{n-1}\\ F_{2n}^{(a,b)} = -\frac{a^2b}{4}n\left(\frac{ab+2}{2}\right)^{n-1}\end{cases}.$$

Regarding the sequence $\{l_n^{(a,b)}\}_{n\geq 0}$ of the bi-periodic Lucas numbers (2), where the initial conditions are $l_0^{(a,b)} = 2$ and $l_1^{(a,b)} = a$, then the analytic formula of the Binet type for $l_{2n}^{(a,b)}$ and $l_{2n+1}^{(a,b)}$ is given by

$$\begin{cases}l_{2n+1}^{(a,b)} = 2nb\left(\frac{ab+2}{2}\right)^{n-1} + a(\left(\frac{ab+2}{2}\right)^n + \frac{ab}{2}n\left(\frac{ab+2}{2}\right)^{n-1})\\ l_{2n}^{(a,b)} = 2(\left(\frac{ab+2}{2}\right)^n - \frac{ab}{2}n\left(\frac{ab+2}{2}\right)^{n-1}) - \frac{a^3b}{4}n\left(\frac{ab+2}{2}\right)^{n-1}\end{cases}.$$

The next numerical example illustrates the result of Corollary 3.

Example 2. *Consider the sequence of bi-periodic Fibonacci numbers* $\{F_n^{(1,-4)}\}_{n\geq 0}$ *given by*

$$F_n^{(1,-4)} = \begin{cases} F_{n-1}^{(1,-4)} + F_{n-2}^{(1,-4)}, & if\ \ n\ \ is\ \ even \\ -4F_{n-1}^{(1,-4)} + F_{n-2}^{(1,-4)}, & if\ \ n\ \ is\ \ odd \end{cases}, \tag{14}$$

for $n \geq 2$, *with initial conditions* $F_0^{(1,-4)} = 0$ *and* $F_1^{(1,-4)} = 1$. *The first values of* $\{F_n^{(1,-4)}\}_{n\geq 0}$ *are given in Table 2 below. Equations* (14) *are equivalent to the following matrix equation* $Z_n^{(1,-4)} = D[1,-4]Z_{n-1}^{(1,-4)}$, *where* $Z_n^{(1,-4)} = \begin{bmatrix} F_{2n+1}^{(1,-4)} \\ F_{2n}^{(1,-4)} \end{bmatrix}$ *and* $D[1,-4] = A[-4,1]A[1,1] = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$. *Its characteristic polynomial is* $P(z) = z^2 + 2z + 1$. *Thus, the eigenvalue of the matrix* $D[1,-4]$ *is* $\lambda = -1$. *Hence, Expression* (13) *shows that* $Z_n^{(1,-4)} = S\begin{bmatrix} (-1)^n & n(-1)^{n-1} \\ 0 & (-1)^n \end{bmatrix} S^{-1} Z_0^{(1,-4)}$, *where* $S = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} -\frac{1}{4} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$ *and* $Z_0^{(1,2)} = [F_1^{(1,-4)}, F_0^{(1,-4)}]^t$. *Therefore, a straightforward computation implies that the analytic Binet expression of the bi-periodic Fibonacci sequence* (14) *is given as follows:*

$$\begin{cases} F_{2n+1}^{(1,-4)} = (-1)^n - 2(-1)^{n-1}n \\ F_{2n}^{(1,-4)} = (-1)^{n+1}n \end{cases}.$$

Table 2. Some values for $F_n^{(1,-4)}$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$F_n^{(1,-4)}$	0	1	1	−3	−2	5	3	−7	−4	9	5	−11
$F_{2n}^{(1,-4)}$	0	1	−2	3	−4	5	−6	7	−8	9	−10	11
$F_{2n+1}^{(1,-4)}$	1	−3	5	−7	9	−11	13	−15	17	−19	21	−23

2.3. Another Compact Analytic Formula of the Binet Type for the Special Initial Conditions

For the special initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$, the sequence of the bi-periodic Fibonacci numbers was described using a parity function $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ (see [9]). That is, for $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$, it was established in [Theorem 2, [5]] that the analytic formula of the Binet type for the bi-periodic Fibonacci numbers is $F_n^{(a,b)} = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\theta_2^n - \theta_1^n}{\theta_2 - \theta_1}$, where $\theta_1 = \frac{ab - \sqrt{a^2b^2+4ab}}{2}$ and $\theta_2 = \frac{ab + \sqrt{a^2b^2+4ab}}{2}$ are the roots of the characteristic equation $x^2 - abx - ab = 0$. Similarly, following the same method as in [5], it was established in [Theorem 2, [9]] that the analytic formula of the Binet type for the bi-periodic Lucas numbers with initial conditions $l_0^{(a,b)} = 2$ and $l_1^{(a,b)} = a$, is given by $l_n^{(a,b)} = \left(\frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right)(\theta_1^n + \theta_2^n)$.

The parity function $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer smaller than $\frac{n}{2}$, can be also considered for expressing the analytic formula of the Binet type for the bi-periodic Fibonacci numbers (9) and the bi-periodic Lucas numbers (10). That is, Expression (9) of the bi-periodic Fibonacci numbers can be formulated with the aid of the parity function as follows:

$$F_n^{(a,b)} = -\beta\alpha_1^{\xi(n)}\lambda_1^{\lfloor \frac{n}{2} \rfloor} + \beta\alpha_2^{\xi(n)}\lambda_2^{\lfloor \frac{n}{2} \rfloor},$$

where the scalars α_1, α_2 and β are as in (7) and $F_0^{(a,b)}, F_1^{(a,b)}$ are the arbitrary initial conditions. Since $\lambda_1 = \theta_1 + 1$ and $\lambda_2 = \theta_2 + 1$, the former formula can be written under the form

$$F_n^{(a,b)} = -\beta\alpha_1^{\xi(n)}(\theta_1 + 1)^{\lfloor \frac{n}{2} \rfloor} + \beta\alpha_2^{\xi(n)}(\theta_2 + 1)^{\lfloor \frac{n}{2} \rfloor},$$

where $\theta_1 = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ and $\theta_2 = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ are the roots of the equation $x^2 - abx - ab = 0$.

Similarly, the analytic formula of the Binet type for bi-periodic Lucas numbers (10) can be rewritten in terms of the parity function and using the fact that $\lambda_1 = \alpha + 1$ and $\lambda_2 = \beta + 1$. More precisely, we have

$$l_n^{(a,b)} = 2(\alpha_1^{\xi(n)}\gamma_1(\theta_2+1)^{\lfloor \frac{n}{2} \rfloor} + \alpha_2^{\xi(n)}\gamma_2(\theta_1+1)^{\lfloor \frac{n}{2} \rfloor}) - a\beta(\alpha_1^{\xi(n)}(\theta_2+1)^{\lfloor \frac{n}{2} \rfloor} - \alpha_2^{\xi(n)}(\theta_1+1)^{\lfloor \frac{n}{2} \rfloor}),$$

where $\theta_1 = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$, $\theta_2 = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ are the roots of the characteristic equation $x^2 - abx - ab = 0$, the scalars $\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2$ are given as in (7), and $l_0^{(a,b)} = 2$, $l_1^{(a,b)} = a$.

Observe that, in the previous discussion, the results of the literature consider the analytic formula of the Binet type for the bi-periodic Fibonacci numbers and the bi-periodic Lucas numbers only when $a^2b^2 + 4ab \neq 0$ and with specific initial conditions. However, we show that Theorem 2 and Corollary 3 extend the result of the literature to the case where $a^2b^2 + 4ab = 0$ and for arbitrary initial conditions.

In the next section, we study the linear and combinatorial formulas for the bi-periodic Fibonacci numbers. More precisely, through the similarity of the matrix $D[a,b]$ matrix with a companion matrix, we establish the linear form, as well as the combinatorial form of the bi-periodic Fibonacci numbers. In addition, another process can help us to furnish the analytic formula of the Binet type for the bi-periodic Fibonacci numbers.

3. Linear and Combinatorial Expressions of the Bi-Periodic Fibonacci Numbers and the Fibonacci Fundamental System

3.1. Linear Expression of the Bi-Periodic Fibonacci Numbers

In general, matrices having a companion form play a central role in the study of sequences defined by a linear recurrence relation and vice versa. The matrix $D[a,b]$ is also similar to a matrix having a companion form. For every two non-zero real numbers a and b, a straightforward computation allows us to have

$$D[a,b] = \begin{bmatrix} ab+1 & b \\ a & 1 \end{bmatrix} = QA[(ab+2), -1]Q^{-1}, \tag{15}$$

where

$$A[-(ab+2), -1] = \begin{bmatrix} ab+2 & -1 \\ 1 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & b \\ 1 & -(ab+1) \end{bmatrix}, \; Q^{-1} = \begin{bmatrix} \frac{(ab+1)}{b} & 1 \\ \frac{1}{b} & 0 \end{bmatrix}. \tag{16}$$

The companion similarities (15) and (16) of the matrix $D[a,b]$ make it possible to calculate explicitly the entries of the powers $D[a,b]^n$, in terms of the related fundamental Fibonacci system of order 2×2. Indeed, let $\{v_n\}_{n\geq 0}$ be the sequence defined by the recursive relation of order two:

$$v_{n+1} = (ab+2)v_n - v_{n-1}, \text{ for } n \geq 1, \tag{17}$$

with the initial condition $v_0 = \alpha_0$, $v_1 = \alpha_1$. Let $\{v_n^{(0)}\}_{n\geq 0}$, $\{v_n^{(1)}\}_{n\geq 0}$ be the two sequences defined by

$$\begin{cases} v_{n+1}^{(s)} = (ab+2)v_n^{(s)} - v_{n-1}^{(s)}, \text{ for } n \geq 1 \\ v_n^{(s)} = \delta_{s,n} \text{ for } n = 0, \; 1 \end{cases}, \tag{18}$$

where $\delta_{s,n}$ means the Kronecker symbol. The set $\mathcal{V} = \left\{ \{v_n^{(0)}\}_{n\geq 0}; \{v_n^{(1)}\}_{n\geq 0} \right\}$ is called the *fundamental Fibonacci sequence*. The sequence $\{v_n^{(0)}\}_{n\geq 0}$ is described in terms of the sequence $\{v_n^{(1)}\}_{n\geq 0}$, as follows.

Lemma 1. *Let* $\mathcal{V} = \left\{\{v_n^{(0)}\}_{n\geq 0}; \{v_n^{(1)}\}_{n\geq 0}\right\}$ *be the Fibonacci fundamental system of the sequence* (17). *Then, for* $n \geq 1$, *we have*

$$v_n^{(0)} = -v_{n-1}^{(1)}. \tag{19}$$

Proof. Let $\{w_n\}_{n\geq 0}$ be the sequence defined by $w_n = -v_{n-1}^{(1)}$. For $n = 1, 2$ we have $w_1 = v_0^{(1)} = 0$, $w_2 = -v_1^{(1)} = -1$ and $v_2^{(1)} = -1$. Hence, we obtain $w_1 = v_0^{(1)} = 0$ and $w_2 = -v_1^{(1)} = v_2^{(1)} = -1$. Moreover, $w_{n+1} = v_n^{(1)} = (ab+2)v_{n-1}^{(1)} - v_{n-2}^{(1)} = (ab+2)w_n - w_{n-1}$. In other terms, the sequence $\{w_n\}_{n\geq 1}$ satisfies the recurrence relation (17). Therefore, we obtain $w_n = v_n^{(0)} = -v_{n-1}^{(1)}$, for $n \geq 1$. □

It is well known that the matrix formulation of the sequence (17) is given in terms of the companion matrix $A[(ab+2), -1] = \begin{bmatrix} ab+2 & -1 \\ 1 & 0 \end{bmatrix}$ under the form

$$\mathbb{V}_{n+1} = A[(ab+2), -1]\mathbb{V}_n, \text{ for } n \geq 1,$$

where $\mathbb{V}_n = \begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix}$, which implies that $\mathbb{V}_{n+1} = A[(ab+2), -1]^n\mathbb{V}_1$, for $n \geq 1$. On the other side, it was established in [11], (see also [12]) that the powers $A[(ab+2), -1]^n$, can be expressed in terms of the fundamental system $\mathcal{V} = \{\{v_n^{(0)}\}_{n\geq 0}; \{v_n^{(1)}\}_{n\geq 0}\}$. More precisely, for every $n \geq 0$, we have $A[(ab+2), -1]^n = \begin{bmatrix} v_{n+1}^{(1)} & v_{n+1}^{(0)} \\ v_n^{(1)} & v_n^{(0)} \end{bmatrix}$. Therefore, the n-th power of the matrix $D[a, b]$ is $D[a, b]^n = (-1)^n Q \begin{bmatrix} v_{n+1}^{(1)} & v_{n+1}^{(0)} \\ v_n^{(1)} & v_n^{(0)} \end{bmatrix} Q^{-1}$, where Q, Q^{-1} are as in (16). Using Equation (19), we derive that $D[a, b]^n$ can be furnished only in terms of the sequence $\{v_n^{(1)}\}_{n\geq 0}$ under the form

$$D[a, b]^n = (-1)^n Q \begin{bmatrix} v_{n+1}^{(1)} & -v_n^{(1)} \\ v_n^{(1)} & -v_{n-1}^{(1)} \end{bmatrix} Q^{-1}, \tag{20}$$

for every $n \geq 0$, where Q, Q^{-1} are as in (16). The sequence $\{v_n^{(1)}\}_{n\geq 0}$ is called the *fundamental sequence*. Combining the matrix Expressions (4) and (20), we show that the vector $Z_n^{(a,b)}$ can be formulated in terms of the fundamental sequence $\{v_n^{(1)}\}_{n\geq 0}$ as follows:

$$Z_n^{(a,b)} = (-1)^n Q \begin{bmatrix} v_{n+1}^{(1)} & -v_n^{(1)} \\ v_n^{(1)} & -v_{n-1}^{(1)} \end{bmatrix} Q^{-1} Z_0^{(a,b)}, \tag{21}$$

for every $n \geq 0$, where Q, Q^{-1} are as in (16), and $Z_0^{(a,b)}$ is the vector column given by (4). Expression (21) permits us to write the bi-periodic Fibonacci numbers $F_{2n+1}^{(a,b)}$, $F_{2n}^{(a,b)}$ in terms of the fundamental sequence $\{v_n^{(1)}\}_{n\geq 0}$. More precisely, we have $Q^{-1}Z_0^{(a,b)} = \begin{bmatrix} \frac{(ab+1)}{b} & 1 \\ \frac{1}{b} & 0 \end{bmatrix} \begin{bmatrix} F_1^{(a,b)} \\ F_0^{(a,b)} \end{bmatrix} = \begin{bmatrix} \frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)} \\ \frac{F_1^{(a,b)}}{b} \end{bmatrix}$, which implies that we have

$$\begin{bmatrix} v_{n+1}^{(1)} & -v_n^{(1)} \\ v_n^{(1)} & -v_{n-1}^{(1)} \end{bmatrix} Q^{-1} Z_0^{(a,b)} = \begin{bmatrix} (\frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)})v_{n+1}^{(1)} - \frac{F_1^{(a,b)}v_n^{(1)}}{b} \\ (\frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)})v_n^{(1)} - \frac{F_1^{(a,b)}v_{n-1}^{(1)}}{b} \end{bmatrix},$$

for every $n \geq 1$. Therefore, we obtain

$$Z_n^{(a,b)} = \begin{bmatrix} F_{2n+1}^{(a,b)} \\ F_{2n}^{(a,b)} = \end{bmatrix} = (-1)^n \begin{bmatrix} ab+1 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} (\frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)})v_{n+1}^{(1)} - \frac{F_1^{(a,b)}v_n^{(1)}}{b} \\ (\frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)})v_n^{(1)} - \frac{F_1^{(a,b)}v_{n-1}^{(1)}}{b} \end{bmatrix}.$$

Finally, we derive

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = (ab+1)\Delta(a,b)v_{n+1}^{(1)} + (F_0^{(a,b)})v_n^{(1)} - \Omega(a,b)v_{n-1}^{(1)} \\ (-1)^n F_{2n}^{(a,b)} = a\Delta(a,b)v_{n+1}^{(1)} - a\Omega(a,b)v_n^{(1)} \end{cases},$$

where $\Omega(a,b) = \frac{F_1^{(a,b)}}{b}$ and $\Delta(a,b) = \frac{(ab+1)F_1^{(a,b)}}{b} + F_0^{(a,b)}$. Thus, another computational process allows us to obtain the formula of the bi-periodic Fibonacci numbers F_{2n+1} and F_{2n} under the form

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \left[(ab+1)v_{n+1}^{(1)} + v_n^{(1)}\right]F_0^{(a,b)} + \left[\frac{(ab+1)v_{n+1}^{(1)}}{b} - \frac{v_{n-1}^{(1)}}{b}\right]F_1^{(a,b)} \\ (-1)^n F_{2n}^{(a,b)} = \left[av_{n+1}^{(1)}\right]F_0^{(a,b)} + \left[\frac{a(ab+1)}{b}v_{n+1}^{(1)} - \frac{av_n^{(1)}}{b}\right]F_1^{(a,b)} \end{cases}.$$

In summary, we have the following result.

Theorem 3. *Let a and b be two non-zero real numbers, and consider $\{F_n^{(a,b)}\}_{n\geq 0}$, the bi-periodic Fibonacci sequence (1). Let $\{v_n^{(1)}\}_{n\geq 0}$ be the fundamental sequence defined by (18). Then, the linear formula of the bi-periodic Fibonacci numbers is given by*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \left[\varphi_1 v_{n+1}^{(1)} + v_n^{(1)}\right]F_0^{(a,b)} + \frac{1}{b}\left[\varphi_1 v_n^{(1)} - v_{n-1}^{(1)}\right]F_1^{(a,b)} \\ (-1)^n F_{2n}^{(a,b)} = av_{n+1}^{(1)}F_0^{(a,b)} + \frac{1}{b}\left[a\varphi_1 v_{n+1}^{(1)} - av_n^{(1)}\right]F_1^{(a,b)} \end{cases}, \tag{22}$$

where $\varphi_1 = ab+1$ and $F_0^{(a,b)}$, $F_1^{(a,b)}$ are the arbitrary initial conditions.

For the usual initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$, we have the following corollary.

Corollary 4. *Consider the sequence of bi-periodic Fibonacci numbers defined by Expression (1), with initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Let $\{v_n^{(1)}\}_{n\geq 0}$ be the fundamental sequence defined by (18). Then, the linear formulas of F_{2n} and F_{2n+1} are given by*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \frac{1}{b}\left[(ab+1)v_n^{(1)} - v_{n-1}^{(1)}\right] \\ (-1)^n F_{2n}^{(a,b)} = \frac{1}{b}\left[a(ab+1)v_{n+1}^{(1)} - av_n^{(1)}\right] \end{cases}.$$

Similarly, for the bi-periodic Lucas numbers, we arrive at the result.

Corollary 5. *Consider the sequence of bi-periodic Lucas numbers defined by Expression (2), with initial conditions $l_0^{(a,b)} = 2$ and $l_1^{(a,b)} = a$. Let $\{v_n^{(1)}\}_{n\geq 0}$ be the fundamental sequence defined by (18). Then, the linear formulas $l_n^{(a,b)}$ and $l_{2n+1}^{(a,b)} = a$ are given by*

$$\begin{cases} (-1)^n l_{2n+1}^{(a,b)} = 2\left[(ab+1)v_{n+1}^{(1)} + v_n^{(1)}\right] + \frac{a}{b}\left[(ab+1)v_n^{(1)} - v_{n-1}^{(1)}\right] \\ (-1)^n l_{2n}^{(a,b)} = 2av_{n+1}^{(1)} + \frac{a}{b}\left[a(ab+1)v_{n+1}^{(1)} - av_n^{(1)}\right] \end{cases}.$$

We illustrate Corollary 4 by the following numerical example.

Example 3. *Let us consider the bi-periodic Fibonacci sequence* $\{F_n^{(2,3)}\}_{n\geq 0}$ *defined by*

$$F_n^{(2,3)} = \begin{cases} 2F_{n-1}^{(2,3)} + F_{n-2}^{(2,3)}, & if \ n \ is \ even \\ 3F_{n-1}^{(2,3)} + F_{n-2}^{(2,3)}, & if \ n \ is \ odd \end{cases}, \tag{23}$$

for $n \geq 2$, *with initial conditions* $F_0^{(2,3)} = 0$ *and* $F_1^{(2,3)} = 1$. *A direct computation implies that* $D[2,3] = \begin{bmatrix} 0 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{7}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}$. *Thus, the related fundamental Fibonacci system* $\mathcal{V} = \left\{ \{v_n^{(0)}\}_{n\geq 0}; \{v_n^{(1)}\}_{n\geq 0} \right\}$ *is as follows:*

$$\begin{cases} v_{n+1}^{(s)} = 8v_n^{(s)} - v_{n-1}^{(s)}, \text{ for } n \geq 1, \\ v_n^{(s)} = \delta_{s,n} \text{ for } n = 0, \ 1, \end{cases}, \tag{24}$$

Thus, by a direct application of Expression (21), *we show that, for every* $n \geq 1$, $Z_n^{(a,b)} = (-1)^n \begin{bmatrix} 0 & 3 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} v_{n+1}^{(1)} & -v_n^{(1)} \\ v_n^{(1)} & -v_{n-1}^{(1)} \end{bmatrix} \begin{bmatrix} \frac{7}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *holds. Finally, a straightforward computation implies that the linear expression of the sequence of bi-periodic Fibonacci numbers* (23), *is given by*

$$\begin{cases} F_{2n+1}^{(1,-4)} = 7v_n^{(1)} - v_{n-1}^{(1)} \\ F_{2n}^{(1,-4)} = \frac{7v_{n+1}^{(1)} - 50v_n^{(1)} + 7v_{n-1}^{(1)}}{3} \end{cases}.$$

The combinatorial and analytical Binet formulas of the fundamental sequence $\{v_n^{(1)}\}_{n\geq 0}$ combined with Expression (22) will permit us to elaborate the combinatorial and analytical Binet formulas of the bi-periodic Fibonacci numbers, which are given in the next two subsections. Moreover, the assertions of Corollaries 4 and 5 allow us to obtain the combinatorial and analytical Binet formulas for the bi-periodic Fibonacci and Lucas numbers with the usual initial conditions.

3.2. Combinatorial Expression for the Bi-Periodic Fibonacci Numbers

Let $\{v_n^{(1)}\}_{n\geq 0}$ be the fundamental sequence and $\{w_n\}_{n\geq 0}$ be the sequence defined by

$$\begin{cases} w_n = \rho(n+1,2) = \sum_{k_0+2k_1=n-1} \frac{(k_0+k_1)!}{k_0!k_1!} a_0^{k_0} a_1^{k_1}, \\ w_0 = 0, w_1 = 1 \end{cases},$$

where $a_0 = -(ab+2)$ and $a_1 = 1$. It was established in [13] that the sequence $\{w_n\}_{n\geq 0}$ satisfies the linear recursive relation $w_{n+1} = (ab+2)w_n - w_{n-1}$. In addition, the two sequences $\{v_n^{(1)}\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ own the same initial conditions $v_0^{(1)} = w_0 = 0$ and $v_1^{(1)} = w_1 = 1$. Therefore, we obtain

$$v_n^{(1)} = w_n = \rho(n+1,2) = \sum_{k_0+2k_1=n-1} \frac{(k_0+k_1)!}{k_0!k_1!} a_0^{k_0} a_1^{k_1}, \text{ for } n \geq 1, \tag{25}$$

where $a_0 = -(ab+2a)$ and $a_1 = 1$. Combining Equation (22) and Equation (25), the combinatorial formula of the bi-periodic Fibonacci numbers is formulated in the following theorem.

Theorem 4. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of the bi-periodic Fibonacci numbers defined by (1). Then, the combinatorial formulas of $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ are given by*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = [\varphi_1\rho(n+2,2)+\rho(n+1,2)]F_0^{(a,b)} + \frac{1}{b}[\varphi_1\rho(n+1,2)-\rho(n,2)]F_1^{(a,b)} \\ (-1)^n F_{2n}^{(a,b)} = a\rho(n+2,2)F_0^{(a,b)} + \frac{1}{b}[a\varphi_1\rho(n+2,2)-a\rho(n+1,2)]F_1^{(a,b)} \end{cases}, \tag{26}$$

with $\varphi_1 = ab+1$, and $\rho(n+1,2) = \sum\limits_{k_0+2k_1=n-1}(-1)^{k_0}\dfrac{(k_0+k_1)!}{k_0!k_1!}(ab+2a)^{k_0}$, for $n\geq 1$.

For the usual initial conditions $F_0^{(a,b)}=0$ and $F_1^{(a,b)}=1$ Theorem 4 allows us to obtain the following corollary.

Corollary 6. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers defined by (1), with initial conditions $F_0^{(a,b)}=0$ and $F_1^{(a,b)}=1$. Then, the combinatorial formulas of $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ are given under the form*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \frac{1}{b}[\varphi_1\rho(n+1,2)-\rho(n,2)] \\ (-1)^n F_{2n}^{(a,b)} = \frac{1}{b}[a\varphi_1\rho(n+2,2)-a\rho(n+1,2)] \end{cases},$$

with $\varphi_1 = ab+1$ and $\rho(n+1,2) = \sum\limits_{k_0+2k_1=n-1}(-1)^{k_0}\dfrac{(k_0+k_1)!}{k_0!k_1!}(ab+2a)^{k_0}$, for $n\geq 1$.

For the bi-periodic Lucas numbers, Theorem 4 permits us to formulate the following corollary.

Corollary 7. *Let $\{l_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Lucas numbers defined by (2), with initial conditions $l_0^{(a,b)}=2$ and $l_1^{(a,b)}=a$. Then, the combinatorial formulas of $l_{2n}^{(a,b)}$ and $l_{2n+1}^{(a,b)}$ are given as follows:*

$$\begin{cases} (-1)^n l_{2n+1}^{(a,b)} = 2[\varphi_1\rho(n+2,2)+\rho(n+1,2)] + \frac{a}{b}[\varphi_1\rho(n+1,2)-\rho(n,2)] \\ (-1)^n l_{2n}^{(a,b)} = 2a\rho(n+2,2) + \frac{a}{b}[a\varphi_1\rho(n+2,2)-a\rho(n+1,2)]F_1^{(a,b)} \end{cases},$$

with $\varphi_1 = ab+1$ and $\rho(n+1,2) = \sum\limits_{k_0+2k_1=n-1}(-1)^{k_0}\dfrac{(k_0+k_1)!}{k_0!k_1!}(ab+2a)^{k_0}$, for $n\geq 1$.

3.3. Another Approach for the Analytical Formula for the Bi-Periodic Fibonacci Numbers

When the roots $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$, $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$ of the characteristic polynomial $P(z) = \det(D[a,b]-zI_{2\times 2}) = z^2-(ab+2)z+1$ are simple, Equations (22) and (26) allow us to obtain another analytic formula of the Binet type for the bi-periodic Fibonacci numbers. In order to succeed in such an analytic Binet formula, we need to recall the following general result established in [12].

Lemma 2. *(Rachidi et al.) Suppose that the roots $\lambda_1,\ldots,\lambda_r$ of characteristic polynomial $P(z) = z^r - a_1z^{r-1} - \ldots - a_{r-2}z - a_r$ ($a_{r-1}\neq 0$) are simple, namely, $\lambda_i\neq\lambda_j$ for $i\neq j$. Then, we have*

$$\rho(n,r) = \sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i-\lambda_k)}, \text{ for every } n\geq r, \tag{27}$$

where $P'(z) = \frac{dP}{dz}(z)$ and $\rho(n,r) = \sum\limits_{k_1+2k_2+\cdots+rk_r=n-r+1}\frac{(k_1+\cdots+k_r)!}{k_1!k_2!\ldots k_r!}a_1^{k_1}a_2^{k_2}\ldots a_r^{k_r}$, with $\rho(r,r)=1$ and $\rho(n,r)=0$ for $n\leq r-1$, where k_i is a natural number for all $0\leq i\leq r-1$.

Formula (27) shows that the analytic formula of the Binet type of the fundamental sequence $\{v_n^{(1)}\}_{n\geq 0}$ is given by

$$v_n^{(1)} = \rho(n+1,r) = \sum_{i=1}^{2} \frac{\lambda_i^n}{P'(\lambda_i)} = \frac{\lambda_1^n}{\lambda_1-\lambda_2} + \frac{\lambda_2^n}{\lambda_2-\lambda_1} = \frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}, \tag{28}$$

for all $n \geq 0$. Therefore, combining Expression (28) with Formulas (22) and (26), we obtain another Binet-type analytical formula for the bi-periodic Fibonacci numbers.

Theorem 5. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers (1). Then, the Binet-type analytical formulas for $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ are given by*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \left[\frac{\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})+(\lambda_1^n-\lambda_2^n)}{\lambda_1-\lambda_2}\right] F_0^{(a,b)} + \frac{1}{b}\left[\frac{\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1^{n-1}-\lambda_2^{n-1})}{\lambda_1-\lambda_2}\right] F_1^{(a,b)} \\ (-1)^n F_{2n}^{(a,b)} = a\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2} F_0^{(a,b)} + \frac{1}{b}\left[\frac{a\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-a(\lambda_1^n-\lambda_2^n)}{\lambda_1-\lambda_2}\right] F_1^{(a,b)} \end{cases},$$

with $\varphi_1 = ab+1$ and $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$, $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$ are the roots of the polynomial $P(z) = \det(D[a,b] - zI_{2\times 2}) = z^2 - (ab+2)z + 1$.

When the usual initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$ are considered, Theorem 5 allows us to arrive at the following corollary.

Corollary 8. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers, with initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. Then, the Binet-type analytical formulas for $F_{2n}^{(a,b)}$ and $F_{2n+1}^{(a,b)}$ are*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = \frac{1}{b}\left[\frac{\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1^{n-1}-\lambda_2^{n-1})}{\lambda_1-\lambda_2}\right] \\ (-1)^n F_{2n}^{(a,b)} = \frac{1}{b}\left[\frac{a\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-a(\lambda_1^n-\lambda_2^n)}{\lambda_1-\lambda_2}\right] \end{cases},$$

where $\varphi_1 = ab+1$ and $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$, $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$ are the roots of the polynomial $P(z) = \det(D[a,b] - zI_{2\times 2}) = z^2 - (ab+2)z + 1$.

For the bi-periodic Lucas numbers, Theorem 5 permits us to formulate the following corollary.

Corollary 9. *Let $\{F_n^{(a,b)}\}_{n\geq 0}$ be the sequence of bi-periodic Lucas numbers, with initial conditions $l_0^{(a,b)} = 2$ and $l_1^{(a,b)} = a$. Then, the Binet-type analytical formulas for $l_{2n}^{(a,b)}$ and $l_{2n+1}^{(a,b)}$ are*

$$\begin{cases} (-1)^n F_{2n+1}^{(a,b)} = 2\left[\frac{\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})+(\lambda_1^n-\lambda_2^n)}{\lambda_1-\lambda_2}\right] + \frac{a}{b}\left[\frac{\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1^{n-1}-\lambda_2^{n-1})}{\lambda_1-\lambda_2}\right] \\ (-1)^n F_{2n}^{(a,b)} = 2a\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2} + \frac{a}{b}\left[\frac{a\varphi_1(\lambda_1^{n+1}-\lambda_2^{n+1})-a(\lambda_1^n-\lambda_2^n)}{\lambda_1-\lambda_2}\right] \end{cases},$$

where $\varphi_1 = ab+1$ and $\lambda_1 = \frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$, $\lambda_2 = \frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$ are the roots of the polynomial $P(z) = \det(D[a,b] - zI_{2\times 2}) = z^2 - (ab+2)z + 1$.

The next numerical example illustrates Corollary 8.

Example 4. *Consider the numerical case studied in Example 1 namely, the bi-periodic Fibonacci numbers* $\{F_n^{(1,2)}\}_{n\geq 0}$*, with initial conditions* $F_0^{(1,2)} = 0$ *and* $F_1^{(1,2)} = 1$*. The related fundamental Fibonacci system* $\mathcal{V} = \left\{\{v_n^{(0)}\}_{n\geq 0}; \{v_n^{(1)}\}_{n\geq 0}\right\}$ *is*

$$\begin{cases} v_{n+1}^{(s)} = -4v_n^{(s)} + v_{n-1}^{(s)}, \text{ for } n \geq 1 \\ v_n^{(s)} = \delta_{s,n} \text{ for } n = 0,\ 1 \end{cases}.$$

Therefore, the combinatorial formula of the general term $v_n^{(1)}$ *is given by* $v_n^{(1)} = \rho(n+1,2) = \sum_{k_0+2k_1=n-1} \frac{(k_0+k_1)!}{k_0!k_1!}(-4)^{k_0}$*, for every* $n \geq 1$*. The characteristic polynomial* $P(z) = z^2 + 4z + 1$ *owns two simple roots* $\lambda_1 = -2+\sqrt{3}$ *and* $\lambda_2 = -2-\sqrt{3}$*. Then, the analytic Binet formula for the bi-periodic Fibonacci numbers* $\{F_n^{(1,2)}\}_{n\geq 0}$ *is as follows*

$$\begin{cases} (-1)^n F_{2n+1}^{(1,2)} = \frac{1}{2}\left[\frac{3(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1^{n-1}-\lambda_2^{n-1})}{\lambda_1-\lambda_2}\right] \\ (-1)^n F_{2n}^{(1,2)} = \frac{1}{2}\left[\frac{3(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1^{n}-\lambda_2^{n})}{\lambda_1-\lambda_2}\right] \end{cases}.$$

4. Concluding Remarks and Perspectives

In this study, some new properties of the sequence of the bi-periodic Fibonacci numbers, with arbitrary initial conditions, were established, through their matrix formulation. Indeed, results concerning the combinatorial formulas and analytic representations of this sequence of numbers were furnished using an approach combining the matrix theory and the Fibonacci fundamental system. In addition, our results were applied for the usual initial conditions, providing new results for the sequence of bi-periodic Fibonacci numbers and the sequence of bi-periodic Lucas numbers. More precisely, it was shown that our theorems are general, and we applied their results to the bi-periodic Fibonacci–Lucas sequences from the literature, where the usual initial conditions are $F_0^{(1,2)} = 0$ and $F_1^{(1,2)} = 1$. Furthermore, some illustrative numerical examples were provided, where we showed the efficiency of our results, which work for arbitrary initial conditions.

To the best of our knowledge, it seems to us that our approach and results, as well as our matrix method, are not common in the literature under this form.

This article is innovative in elaborating a matrix approach for the sequences of bi-periodic Fibonacci numbers. Finally, the matrix approach presented here can be extended to other sequences of bi-periodic numbers of type (1).

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