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# mKdV Equation on Time Scales: Darboux Transformation and *N*-Soliton Solutions

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**Abstract:** In this paper, the spectral problem of the mKdV equation satisfying the compatibility condition on time scales is directly constructed. By using the zero-curvature equation on time scales, the mKdV equation on time scales is obtained. When $x \in \mathbb{R}$ and $t \in \mathbb{R}$, the equation degenerates to the classical mKdV equation. Then, the single-soliton, two-soliton, and $N$-soliton solutions of the mKdV equation under the zero boundary condition on time scales are presented via employing the Darboux transformation (DT). Particularly, we obtain the corresponding single-soliton solutions expressed using the Cayley exponential function on four different time scales ($\mathbb{R}$, $\mathbb{Z}$, $q$-discrete, $\mathbb{C}$).

**Keywords:** modified Korteweg–de Vries equation; time scales; Cayley exponential function; Darboux transformation

**MSC:** 35C08; 26E70; 35R07

## 1. Introduction

The modified Korteweg–de Vries (mKdV) equation, namely

$$q_t + q_{xxx} + 6\gamma q^2 q_x = 0, \quad \gamma = \pm 1, \tag{1}$$

is an important integrable nonlinear soliton equation [1–4]. The $q$ is a real function dependent on the space variable $x$ and the time variable $t$. Based on the signs of the nonlinear coefficients $\gamma = 1$ and $\gamma = -1$, the mKdV equation can be divided into a focusing and defocusing mKdV equation, respectively. Additionally, by using supersymmetry and the Miura transformation, it is possible to relate the mKdV equation to other types of nonlinear equations, such as the KdV equation [5].

The mKdV equation is the third-order flow of the Ablowitz–Kaup–Newel–Segur (AKNS) system and has attracted much attention. For the focusing mKdV equation, Wadati derived its simple-pole, double-pole, and triple-pole solutions under zero boundary conditions [6,7]. Subsequently, its $N$-soliton and breather solutions were proposed by using the inverse scattering transform (IST) method [8]. And for the defocusing mKdV equation, Chen et al. developed the DT and obtained its dark $N$-soliton solution by employing the Binet–Cauchy formula [9]. Under the zero boundary conditions, Deift and Zhou utilized the powerful steepest descent method to present the long-time asymptotic behavior [10]. In [11], the Hamiltonian formalism for the defocusing mKdV equation with special non-zero boundary conditions $q^+ = q^-$ was provided. In 2017, three types of multi-soliton solutions for the coupled mKdV equation were presented by solving Riemann–Hilbert problems associated with reflectionless cases [12]. Additionally, data-driven solutions for the zero boundary of this equation were explored using the neural network algorithm. These solutions encompassed kink, complex, bright-bright, and interactions between solitons and kink types [13]. Reza Mokhtari proposed a numerical solution of the mKdV equation based on a radial basis function [14]. In 2023, the multi-pole solutions of the discrete mKdV

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equation were obtained by emerging the IST [15]. The aforementioned research on the mKdV equations belongs to only continuous and discrete systems. However, a single continuous or discrete model is challenging to accurately characterize mixed continuous and discrete systems. The equation constructed on time scales in this paper can be obtained simultaneously with both kinds of equations. Continuous mKdV equations can be obtained when the time scale is a set of real numbers, and semi-discrete mKdv equations can be obtained when the time scale is a set of integers. This naturally leads us to search for the form of the mKdV equation in the hybrid system.

In 1988, the time scale was proposed by Hilger, which is a special measure chain [16]. This concept enables the unification and generalization of continuous and discrete systems and enables the creation of more new physically meaningful hybrid dynamic models, such as inventory production models [17], predator–prey models [18], population reproduction models [19], and economic models [20]. The time scale has topological and order relations induced by $\mathbb{R}$ [21]. The time scale has also attracted a great deal of attention from experts and scholars in the field of mathematical physics. In 1996, Kaymakcalan et al. introduced the theory of measure chains in their study of dynamic measurement systems and developed the theory of Lyapunov stability [22]. In 2002, Ahlbrandt et al. introduced partial differential operators on the time scale, Euler–Lagrange equations, and the Picone constant for the double integral variational problem. They also presented the Sturm–Picone comparison theorem on time scales [23]. Then, Guseinov et al. investigated the relationship between the Riemann integral and the Lebesgue integral on time scales and established a criterion for Riemann integrability [24]. Liu et al. established a general theory for symmetry analysis of time scale dynamical systems with a first-order time $\Delta$-derivative and stability analysis of time scale dynamical systems with a first-order time $\nabla$ derivative [25,26]. Cieslinski developed and analyzed the integrable analogue of the Sine–Gordon equation on arbitrary time scales, which was achieved through the direct creation of Lax pairs that adhere to the compatibility condition [27]. This is different from the previous process of starting from a spectral problem and using the zero-curvature equation to construct the family of AKNS integrable equations.

There are many classical methods for solving integrable nonlinear equations, such as the DT [28,29], the IST [30,31], commutation methods [32], and the Wronskian technique [33–35]. For high-dimensional or complex nonlinear equations, it is often difficult to find exact analytical solutions, and numerical solutions are often used to approximate these equations [36–38]. Among others, the DT is instrumental in studying nonlinear equations with Lax pairs on time scales. In 2016, Hovhannisyan et al. generalized the DT method on time scales and derived a multi-soliton solution for the KdV equation on the time scale [39]. Dong et al. employed the gauge transformation to derive the DT of Gerdjikov–Ivanov equation and coupled Cubic-Quintic NLS equation on time scales [40,41]. Previous research has offered a valuable approach for investigating the mKdV equation on time scales. Thus, we obtain the mKdV equation on time scales by directly constructing its spectral problem and solving it by employing the DT.

The structure of this paper is as follows: the spectral problem for the mKdV equation on time scales is constructed directly in Section 2. From this spectral problem, the mKdV equation is obtained combined with the zero-curvature equation on time scales by using the time scale differentiation and rules for operating with $\nabla$ derivatives. Then, by choosing different time scales, the mKdV equation on the time scale will degenerate into different forms. Therefore, when $x \in \mathbb{R}$, $t \in \mathbb{R}$, the equation will degenerate to the classical mKdV equation. In Section 3, based on the Darboux theory, the DT of the time scale mKdV equation is constructed, and the zero-seed solution is chosen to obtain the single-soliton, two-soliton, and $N$-soliton solutions on time scales. Particularly, the single-soliton solution is derived for four different time scales ($\mathbb{R}$, $\mathbb{Z}$, $q$-discrete, and $\mathbb{C}$) using the Cayley exponential function.

## 2. The mKdV Equation on Time Scales

In this section, we investigate the construction of the mKdV equation on the time scale using the time scale theory and verify its correctness.

Firstly, we introduce several concepts related to the time scale [20]. Let $\mathbb{T}$ (time scale) and $\mathbb{X}$ (space scale) be an arbitrary nonempty closed subset of real numbers $\mathbb{R}$.

**Definition 1.** *Backward jump operators are given for* $(t, x) \in \mathbb{T} \times \mathbb{X}$ *[42]*

$$\rho_1 : \mathbb{T} \to \mathbb{T},\ \rho_2 : \mathbb{X} \to \mathbb{X},$$

$$\rho_1(t) = \sup\{m \in \mathbb{T} : m < t\},\ \rho_2(x) = \sup\{n \in \mathbb{X} : n < x\}.$$

*(i) If* $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$, *then*

$$\rho_1(t) = t,\ \rho_2(x) = x.$$

*(ii) If* $\mathbb{T} \times \mathbb{X} = h\mathbb{Z} \times h\mathbb{Z}$ *(h is a constant), then*

$$\rho_1(t) = \sup\{t - h, t - 2h, \cdots\} = t - h,$$

$$\rho_2(x) = \sup\{x - h, x - 2h, \cdots\} = x - h.$$

**Definition 2.** *The* $\nabla$ *derivatives with t and x variables are defined as*

$$\nabla_t G(t, x) = \lim_{q_1 \to \mu(t)} \frac{G(t, x) - G^{\rho_1}(t, x)}{q_1},$$

$$\nabla_x G(t, x) = \lim_{q_2 \to \nu(x)} \frac{G(t, x) - G^{\rho_2}(t, x)}{q_2},$$

*where the graininess functions* $\nu, \mu$ *are given by*

$$\nu(x) = x - \rho_2(x), \mu(t) = t - \rho_1(t)\ .$$

*In addition,*

$$G^{\rho_1}(t, x) := G(\rho_1(t), x) = G(t, x) - \mu(t)\nabla_t G(t, x),$$
$$G^{\rho_2}(t, x) := G(t, \rho_2(s)) = G(t, x) - \nu(x)\nabla_x G(t, x).$$

**Definition 3.** *The Cayley exponential function on time scales is defined as*

$$E_\beta(t, t_0) := \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}(\beta(s))\Delta s\right),\ E_\beta(t) := E_\beta(t, 0),$$

*where* $\beta = \beta(t)$ *is a rd-continuous regressive function; thus,*

$$\xi_l(z) := \frac{1}{l} \log \frac{1 + \frac{1}{2}zl}{1 - \frac{1}{2}zl},\ l > 0,\ \xi_0(z) := z.$$

*(i) The Cayley exponential function satisfies*

$$E_\alpha^\Delta = \frac{1}{2}\alpha\left(E_\alpha + E_\alpha^{\rho_1}\right),\ E_\alpha(0) = 1,\ E_\alpha^{\rho_1} = \frac{1 + \frac{\alpha\mu}{2}}{1 - \frac{\alpha\mu}{2}} E_\alpha.$$

*(ii) When* $\mathbb{T} = \mathbb{R}$ *and* $\mathbb{T} = h\mathbb{Z}$, *the Cayley exponential function is given as follows:*

$$E_\beta(t) = e^{\alpha t} \text{ and } E_\beta(t) = \left(\frac{1 + \frac{1}{2}\beta h}{1 - \frac{1}{2}\beta h}\right)^{\frac{t}{h}},$$

*respectively.*

**Definition 4.** *The product $gh$ is $\nabla$ differentiable at $x$, $t$; thus, the product rules are given by*

$$\nabla_x(gh) = (\nabla_x g)h + g^{\rho_2}(\nabla_x h),$$
$$\nabla_t(gh) = (\nabla_t g)h + g^{\rho_1}(\nabla_t h).$$

Then, we reconsider the spectral problem of the focusing mKdV Equation (1) ($\gamma = 1$):

$$\begin{cases} \varphi_x = \tilde{M}\varphi, \\ \varphi_t = \tilde{N}\varphi, \end{cases} \tag{2}$$

where

$$\tilde{M} = \begin{pmatrix} \lambda & q \\ -q & \lambda \end{pmatrix},$$
$$\tilde{N} = \begin{pmatrix} -4\lambda^3 - 2\lambda q^2 & -4\lambda^2 q - 2\lambda q_x - 2q^3 - q_{xx} \\ 4\lambda^2 q - 2\lambda q_x + 2q^3 + q_{xx} & 4\lambda^3 + 2\lambda q^2 \end{pmatrix},$$

$\varphi(x,t)$ is a wave function, and $q(x,t)$ is a potential function, where $\lambda$ is a spectral parameter.

In the following, based on the Lax pairs (2) of the classical mKdV equation and the time scale theory, the spectral problem of the mKdV equation on time scales is presented as follows:

$$\begin{cases} \nabla_x \varphi = M\varphi, \\ \nabla_t \varphi = N\varphi, \end{cases} \tag{3}$$

where

$$M = \begin{pmatrix} -i\lambda & \frac{1}{2}q + \frac{1}{2}q^{\rho_2} \\ \frac{1}{2}r + \frac{1}{2}r^{\rho_2} & i\lambda \end{pmatrix} = \lambda e_3 + \frac{1}{2}(Q + Q^{\rho_2}),$$
$$N = \begin{pmatrix} -4i\lambda^3 - 2i\lambda q r^{\rho_1} & 2\lambda^2(q + q^{\rho_1}) + 2i\lambda\nabla_x q + 2qq^{\rho_1}r - \nabla_{xx}q \\ 2\lambda^2(r + r^{\rho_1}) - 2i\lambda\nabla_x r + 2qr^{\rho_1}r - \nabla_{xx}r & 4i\lambda^3 + 2i\lambda q^{\rho_1}r \end{pmatrix},$$
$$= 4\lambda^3 e_3 + 2\lambda^2(Q + Q^{\rho_1}) + 2\lambda(QQ^{\rho_1} + \nabla_x Q)e_3 - \nabla_{xx}Q + 2Q^2Q^{\rho_1}$$
$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad e_3 = -i\sigma_3.$$

Firstly, taking $A = Q + Q^{\rho_2}$, $B = Q + Q^{\rho_1}$, $C = QQ^{\rho_1}$, Equation (3) is reduced to

$$\begin{aligned} M &= \lambda e_3 + \frac{1}{2}A, \\ N &= 4\lambda^3 e_3 + 2\lambda^2 B + 2\lambda(C + \nabla_x Q)e_3 - \nabla_{xx}Q + 2QC. \end{aligned} \tag{4}$$

Based on the $\nabla$-derivative algorithm, we obtain the following zero-curvature equation on time scales via $\nabla_{xt}\varphi = \nabla_{tx}\varphi$.

$$\nabla_t M - \nabla_x N + M^{\rho_1}N - N^{\rho_2}M = 0. \tag{5}$$

Then, Equation (4) is substituted into Equation (5) to compare the coefficients of the powers of $\lambda$:

$$\begin{aligned}
&\lambda^0 : \frac{1}{2}\nabla_t A + \nabla_{xxx} Q - 2(\nabla_x Q)C - 2Q^{\rho_2}\nabla_x C - \frac{1}{2}A^{\rho_1}\nabla_{xx} Q \\
&+ A^{\rho_1} QC + \frac{1}{2}(\nabla_{xx} Q^{\rho_2})A - (QC)^{\rho_2} A = 0, \\
&\lambda : -2(\nabla_x C)e_3 - 3(\nabla_{xx} Q)e_3 + 2QCe_3 + (\nabla_{xx} Q^{\rho_2})e_3 \\
&- 2(QC)^{\rho_2} e_3 + A^{\rho_1}(C + \nabla_x Q)e_3 - (C^{\rho_2} + \nabla_x Q^{\rho_2})Ae_3 = 0, \\
&\lambda^2 : -2\nabla_x B + 2(C + \nabla_x Q) + A^{\rho_1} B - 2(C^{\rho_2} + \nabla_x Q^{\rho_2}) - B^{\rho_2} A = 0, \\
&\lambda^3 : 2Be_3 + 2A^{\rho_1} e_3 - 2B^{\rho_2} e_3 - 2Ae_3 = 0 \\
&\lambda^4 : 4 - 4 = 0.
\end{aligned} \tag{6}$$

By assuming $\mu(t)$, $\nu(x)$ are nonzero and by expanding the first equation of Equation (6), the coupled mKdV equation is given on the time scale, as follows:

$$\begin{cases} \nabla_t(q + q^{\rho_2}) - 4(q^{\rho_1} r \nabla_x q + q^{\rho_1} q^{\rho_2} \nabla_x r + (qr)^{\rho_2} \nabla_x q^{\rho_1}) + 2\nabla_{xxx} q = 0, \\ \nabla_t(r + r^{\rho_2}) - 4(qr^{\rho_1} \nabla_x r + r^{\rho_1} r^{\rho_2} \nabla_x q + (qr)^{\rho_2} \nabla_x r^{\rho_1}) + 2\nabla_{xxx} r = 0. \end{cases} \tag{7}$$

When $r = -q$, we obtain the following mKdV equation on the time scale:

$$\nabla_t(q + q^{\rho_2}) - 4(qq^{\rho_1} - q^{\rho_1} q^{\rho_2})\nabla_x q - 4q^{2\rho_2}\nabla_x q^{\rho_1} + 2\nabla_{xx} q = 0. \tag{8}$$

Next, we present the mKdV equation on two special time scales.

**Case1: The mKdV equation on the discrete time scale.**

Taking the semi-discrete time scale $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z}$, we have

$$\nu(x) = 1, \quad \mu(t) = 0$$

and

$$G^{\rho_1}(x,t) = G(x,t), \quad G^{\rho_2}(x,t) = G(x,t) - (1 - E)G(x,t),$$

where $E$ is a shift operator. Then, Equation (7) is reduced to the coupled discrete mKdV equation, as follows:

$$\begin{cases} (1 + E)q_t - 4[(1 + E)(qr)(1 - E)q + qEq(1 - E)r] + 2(1 - E)^3 q = 0, \\ (1 + E)r_t - 4[(1 + E)(qr)(1 - E)r + rEr(1 - E)q] + 2(1 - E)^3 r = 0. \end{cases} \tag{9}$$

In addition, when $r = -q$, the mKdV equation on the discrete time scale is obtained, as follows:

$$(1 + E)q_t + 4\Big[(1 + E)q^2(1 - E)q + qEq(1 - E)q\Big] + 2(1 - E)^3 q = 0. \tag{10}$$

**Case2: The mKdV equation on the continuous time scale.**

When we take the continuous time scale, we have

$$\nu(x) = \mu(t) = 0.$$

Then, Equation (7) is simplified to the following coupled mKdV equation:

$$\begin{cases} q_t - 4qrq_x - 2q^2 r_x + q_{xxx} = 0, \\ r_t - 4qrr_x - 2q_x r^2 - r_{xxx} = 0. \end{cases} \tag{11}$$

Specially, the classical mKdV equation is derived when $r = -q$.

$$q_t + q_{xxx} + 6q^2 q_x = 0.$$

## 3. The Darboux Transformation and $N$-Soliton Solution on Time Scales

In this section, based on the spectral problem (3) in Section 2, the single-soliton, two-soliton, and $N$-soliton solutions of the coupled mKdV equation are presented using the DT on the time scale. Furthermore, the specific forms of the solutions in different time scale cases are then discussed.

### *3.1. The Darboux Transformation on Time Scales*

First, we introduce a gauge transformation $T[1]$, such that

$$\varphi[1] = T[1]\varphi, \tag{12}$$

where

$$T[1] = \lambda T_1 + T_0, \quad T_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad T_1 = \begin{pmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{pmatrix}, \tag{13}$$

and $\varphi[1]$ is a fundamental solution of the spectral problem (3) of the mKdV equation on time scales. Thus, taking $r = -q$, we have

$$\begin{cases} \nabla_x \varphi[1] = M[1]\varphi[1], \\ \nabla_t \varphi[1] = N[1]\varphi[1], \end{cases} \tag{14}$$

where

$$\begin{aligned} &M[1] = \lambda e_3 + \frac{1}{2}A[1], \\ &N[1] = 4\lambda^3 e_3 + 2\lambda^2 B[1] + 2\lambda(C[1] + \nabla_x Q[1])e_3 - \nabla_{xx} Q[1] + 2Q[1]C[1], \\ &C[1] = Q[1]Q[1]^{\rho_1}, B[1] = Q[1] + Q[1]^{\rho_1}, \\ &A[1] = Q[1] + Q[1]^{\rho_2}, Q[1] = \begin{pmatrix} 0 & q[1] \\ -q[1] & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

where $A[1], B[1], C[1], Q[1]$ are the matrices after a Darboux transformation.

Furthermore, we substitute Equation (12) into Equation (14); thus, the following can be obtained:

$$\begin{aligned} \nabla_x \varphi[1] &= M[1]T[1]\varphi, \\ \nabla_t \varphi[1] &= N[1]T[1]\varphi, \end{aligned} \tag{16}$$

Then, according to Definition 4, the derivatives of Equation (12) with respect to x and t are obtained, respectively, as follows:

$$\begin{aligned} \nabla_x \varphi[1] &= \nabla_x T[1]\varphi + T[1]^{\rho_2} \nabla_x \varphi, \\ \nabla_t \varphi[1] &= \nabla_t T[1]\varphi + T[1]^{\rho_1} \nabla_t \varphi, \end{aligned} \tag{17}$$

Then, we substitute $\nabla_x \varphi = M\varphi, \nabla_t \varphi = N\varphi$ into the above formula to get

$$\begin{aligned} \nabla_x \varphi[1] &= \nabla_x T[1]\varphi + T[1]^{\rho_2} M\varphi, \\ \nabla_t \varphi[1] &= \nabla_t T[1]\varphi + T[1]^{\rho_1} N\varphi, \end{aligned} \tag{18}$$

Thus, two constraint relations for $T[1]$ are yielded by Equation (16) into Equation (18):

$$M[1]T[1] = \nabla_x T[1] + T[1]^{\rho_2} M, \tag{19}$$

$$N[1]T[1] = \nabla_t T[1] + T[1]^{\rho_1} N. \tag{20}$$

Consequently, we substitute Equation (13) into Equations (19) and (20).

$$\begin{aligned}
&i\lambda a - i\lambda^2 a_{11} + \frac{1}{2}q[1]c + \frac{1}{2}q[1]c_{21} = -i\lambda a^{\rho_2} - i\lambda^2 a_{11}^{\rho_2} - \frac{1}{2}qb^{\rho_2} + \frac{1}{2}q\lambda b_{21}^{\rho_2},\\
&i\lambda b - i\lambda^2 b_{11} + \frac{1}{2}q[1]d + \frac{1}{2}q[1]d_{22} = i\lambda b^{\rho_2} + i\lambda^2 b_{12}^{\rho_2} + \frac{1}{2}qa^{\rho_2} + \frac{1}{2}q\lambda a_{21}^{\rho_2},\\
&i\lambda c + i\lambda^2 c_{21} - \frac{1}{2}q[1]a - \frac{1}{2}\lambda q[1]a_{21} = -i\lambda c^{\rho_2} - i\lambda^2 c_{21}^{\rho_2} - \frac{1}{2}qd^{\rho_2} - \frac{1}{2}q\lambda d_{22}^{\rho_2},\\
&i\lambda d + i\lambda^2 d_{22} - \frac{1}{2}q[1]b - \frac{1}{2}\lambda b_{12}q[1] = \frac{1}{2}qc^{\rho_2} + \frac{1}{2}q\lambda c_{21}^{\rho_2} + i\lambda d^{\rho_2} + i\lambda^2 d_{22}^{\rho_2},
\end{aligned} \tag{21}$$

Subsequently, the coefficients of the powers of $\lambda$ are compared:

$$\begin{aligned}
\lambda^0 : q[1]c &= -qb^{\rho_2},\\
q[1]d &= qa^{\rho_2},\\
q[1]a &= qd^{\rho_2},\\
q[1]b &= -qc^{\rho_2}.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\lambda : -ia + \frac{1}{2}q[1]c_{21} &= -ia^{\rho_2} - \frac{1}{2}qb_{12}^{\rho_2},\\
-ib + \frac{1}{2}q[1]d_{22} &= ib^{\rho_2} + \frac{1}{2}qa_{11}^{\rho_2},\\
ic - \frac{1}{2}q[1]a_{11} &= -ic^{\rho_2} - \frac{1}{2}qd_{22}^{\rho_2},\\
id - \frac{1}{2}q[1]b_{21} &= id^{\rho_2} + \frac{1}{2}qc_{21}^{\rho_2}.
\end{aligned} \tag{23}$$

$$\begin{aligned}
\lambda^2 : a_{11} = a_{11}^{\rho_2}, \quad -b_{12} &= b_{12}^{\rho_2},\\
-c_{21} = c_{21}^{\rho_2}, \quad d_{22} &= d_{22}^{\rho_2},
\end{aligned} \tag{24}$$

Then, we get the following relations:

$$\begin{aligned}
&a = -d,\\
&b = c,\\
&a_{11} = d_{22} = 1,\\
&b_{12} = c_{21} = 0,\\
&q[1] = q + 2ib.
\end{aligned}$$

Thus, the gauge transform $T[1]$ is reduced to

$$T[1] = \lambda I + T_0,$$

$$T_0 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

In order to find the relationship between the eigenvectors and the gauge transform, we need to further determine the specific form of $T_0$. Assuming that

$$T_0 = H_1 \Lambda_1 H_1^{-1},$$

where

$$\begin{aligned} H_1 &= \begin{pmatrix} \phi(\lambda_1,x,t) & -\psi(\lambda_1,x,t) \\ \psi(\lambda_1,x,t) & \phi(\lambda_1,x,t) \end{pmatrix}, \\ \Lambda_1 &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \\ H_1^{-1} &= \frac{1}{\Delta_1}\begin{pmatrix} \phi(\lambda_1,x,t) & \psi(\lambda_1,x,t) \\ -\psi(\lambda_1,x,t) & \phi(\lambda_1,x,t) \end{pmatrix}, \\ \Delta_1 &= \phi^2+\psi^2, \end{aligned} \tag{25}$$

$\begin{pmatrix} \phi & \psi \end{pmatrix}^T, \begin{pmatrix} -\psi & \phi \end{pmatrix}^T$ are the eigenvectors of the spectral problem (3) when the eigenvalues $\lambda = \lambda_1$ and $\lambda = -\lambda_1$, respectively; moreover, $\Delta_1 := \det H_1 \neq 0$, the Lax pairs about $H_1$, and the specific expression for $T_0$ are obtained, as follows:

$$\begin{aligned} &\begin{cases} \nabla_x H_1 = e_3 H_1 \Lambda + \frac{1}{2} A H_1, \\ \nabla_t H_1 = 4e_3 H_1 \Lambda^3 + 2BH_1\Lambda^2 + 2(C+\nabla_x Q)e_3 H_1 \Lambda - (\nabla_{xx} Q)H_1 + 2QCH_1, \end{cases} \\ &T_0 = H_1 \Lambda_1 H_1^{-1} = \frac{1}{\Delta_1}\begin{pmatrix} \lambda_1(\phi^2-\psi^2) & 2\lambda_1\phi\psi \\ 2\lambda_1\phi\psi & \lambda_1(\psi^2-\phi^2) \end{pmatrix}. \end{aligned} \tag{26}$$

Combined with Equation (26), we can verify the invariance of the gauge transform to satisfy both constraint Equations (19) and (20); thus, we obtain the single-soliton solution of the mKdV equation and the one-fold DT on time scales, as follows:

$$\begin{aligned} T[1] &= \lambda I + T_0, \\ q[1] &= q + 4i\frac{\lambda_1\phi\psi}{\Delta_1}. \end{aligned} \tag{27}$$

where $T[1]$ is called the Darboux matrix.

Similarly, we perform two iterative computations to construct the two-fold DT on time scales. Let

$$\varphi[2] = T[2]\varphi[1] = T[2]T[1]\varphi,$$

$\varphi[2]$ satisfies the Lax pairs of the same form as the spectral problem (3).

$$\begin{cases} \nabla_x \varphi[2] = M[2]\varphi[2], \\ \nabla_t \varphi[2] = N[2]\varphi[2], \end{cases} \tag{28}$$

where

$$\begin{aligned} &M[2] = \lambda e_3 + \frac{1}{2}A[2], \\ &N[2] = 4\lambda^3 e_3 + 2\lambda^2 B[2] + 2\lambda(C[2]+\nabla_x Q[2])e_3 - \nabla_{xx}Q[2] + 2Q[2]C[2], \\ &C = Q[2]Q[2]^{\rho_1}, \\ &B[2] = Q[2] + Q[2]^{\rho_1}, \\ &A[2] = Q[2] + Q[2]^{\rho_2}, \\ &Q[2] = \begin{pmatrix} 0 & q[2] \\ -q[2] & 0 \end{pmatrix}. \end{aligned}$$

Similar steps are taken as in the first iteration, so

$$
\begin{aligned}
H_2 &= \begin{pmatrix} \phi[1](\lambda_2,x,t) & -\psi[1](\lambda_2,x,t) \\ \psi[1](\lambda_2,x,t) & \phi[1](\lambda_2,x,t) \end{pmatrix}, \\
H_2^{-1} &= \frac{1}{\Delta_2}\begin{pmatrix} \phi[1](\lambda_2,x,t) & \psi[1](\lambda_2,x,t) \\ -\psi[1](\lambda_2,x,t) & \phi[1](\lambda_2,x,t) \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \\
\Delta_2 &= \phi[1]^2 + \psi[1]^2,
\end{aligned}
$$

where $\begin{pmatrix} -\psi[1] & \phi[1] \end{pmatrix}^T$, $\begin{pmatrix} \phi[1] & \psi[1] \end{pmatrix}^T$ are the eigenvectors of the spectral problem (14) when the eigenvalues $\lambda = \lambda_2$ and $\lambda = -\lambda_2$, respectively. Thus, we get

$$
T_0[1] = H_2\Lambda_2 H_2^{-1} = \frac{1}{\Delta_2}\begin{pmatrix} \lambda_2(\phi[1]^2 - \psi[1]^2) & 2\lambda_2\phi[1]\psi[1] \\ 2\lambda_2\phi[1]\psi[1] & \lambda_2(\psi[1]^2 - \phi[1]^2) \end{pmatrix},
$$

Moreover, the two-soliton solution of the mKdV equation is obtained on time scales, as follows:

$$
\begin{aligned}
T[2] &= \lambda I + T_0[1], \\
q[2] &= q[1] + 4i\frac{\lambda_2\phi[1]\psi[1]}{\Delta_2}.
\end{aligned} \tag{29}
$$

Based on the above iterative process, we can derive the $N$-fold DT and then the $N$-soliton solution for the mKdV equation on time scales, as follows:

$$
\begin{aligned}
T[N] &= \lambda I + T_0[N-1], \\
q[N] &= q[1] + 4i\sum_{j=2}^{N}\frac{\lambda_j\phi[j-1]\psi[j-1]}{\Delta_j}.
\end{aligned}
$$

where

$$
\begin{aligned}
T_0[N-1] &= H_N\Lambda_N {H_N}^{-1} \\
&= \frac{1}{\Delta_N}\begin{pmatrix} \lambda_N(\phi[N-1]^2 - \psi[N-11]^2) & 2\lambda_N\phi[N-1]\psi[N-1] \\ 2\lambda_N\phi[N-1]\psi[N-1] & \lambda_N(\psi[N-1]^2 - \phi[N-1]^2) \end{pmatrix}.
\end{aligned}
$$

### *3.2. Soliton Solutions of the mKdV Equation on Time Scales*

The explicit form of the single-soliton solution for the coupled mKdV equation on time scales is given using the DT (27) in this section. The soliton solution of the mKdV equation provides a fundamental understanding of soliton dynamics and nonlinear wave interactions, which could inspire researchers with new ways to manage solitons in nonlinear fibers. However, in practical applications, soliton transmission in nonlinear fiber is usually studied and solved with the nonlinear Schrodinger equation. The application of mKdV equations is more common in the study of theoretical physics and mathematical models than directly in the design of optical fiber communication systems. They have potential applications in areas such as fluid dynamics and plasma physics.

Firstly, taking the seed solution "$q(x,t) = 0$", the explicit form of the eigenfunctions for the spectral problem (3) with eigenvalue $\lambda = \lambda_1$ is

$$
\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} E_{-i\lambda_1}(x,x_0)E_{-4i\lambda_1^3}(t,t_0) \\ E_{i\lambda_1}(x,x_0)E_{4i\lambda_1^3}(t,t_0) \end{pmatrix}, \tag{30}
$$

where $E_\alpha(x, x_0)$, $E_\beta(t, t_0)$ are the Cayley exponential functions. Taking the purely imaginary eigenvalue $\lambda_1 = i\eta(\eta \in \mathbb{R})$, Equation (30) can be reduced to

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} E_\eta(x, x_0)E_{-4\eta^3}(t, t_0) \\ E_{-\eta}(x, x_0)E_{4\eta^3}(t, t_0) \end{pmatrix}. \tag{31}$$

Substituting eigenfunctions (31) into Equation (27), the single-soliton solution of the mKdV equation is derived on time scales, as follows:

$$q[1]_{\mathbb{T}\times\mathbb{X}} = -\frac{4\eta}{M + M^{-1}}, \tag{32}$$

where $M = E_{2\eta}(x, x_0)E_{-8\eta^3}(t, t_0)$.

In the following section, several different time scales of single-soliton solutions are discussed.

**Case1: Single-soliton solution of the mKdV equation on the discrete time scale.**

Taking the discrete time scale $\mathbb{T} \times \mathbb{X} = h\mathbb{Z} \times h\mathbb{Z}$, we have

$$\mu(t) = h, \quad \nu(x) = h,$$

and

$$E_\alpha(x) = \left(\frac{1 + \frac{1}{2}\alpha h}{1 - \frac{1}{2}\alpha h}\right)^{\frac{x}{h}},$$

where $h$ is a constant. Therefore, the single-soliton solution on the discrete time scale is given by

$$q[1]_{h\mathbb{Z}\times h\mathbb{Z}} = -\frac{4\eta}{M_1 + M_1^{-1}}, \tag{33}$$

where $M_1 = \left(\frac{1+\eta h}{1-\eta h}\right)^{\frac{x}{h}}\left(\frac{1-4\eta^3 h}{1+4\eta^3 h}\right)^{\frac{t}{h}}$.

As can be seen from Figures 1–3, with the increase in $\eta$, the soliton's amplitude becomes larger, the soliton becomes sharper, and it propagates faster.

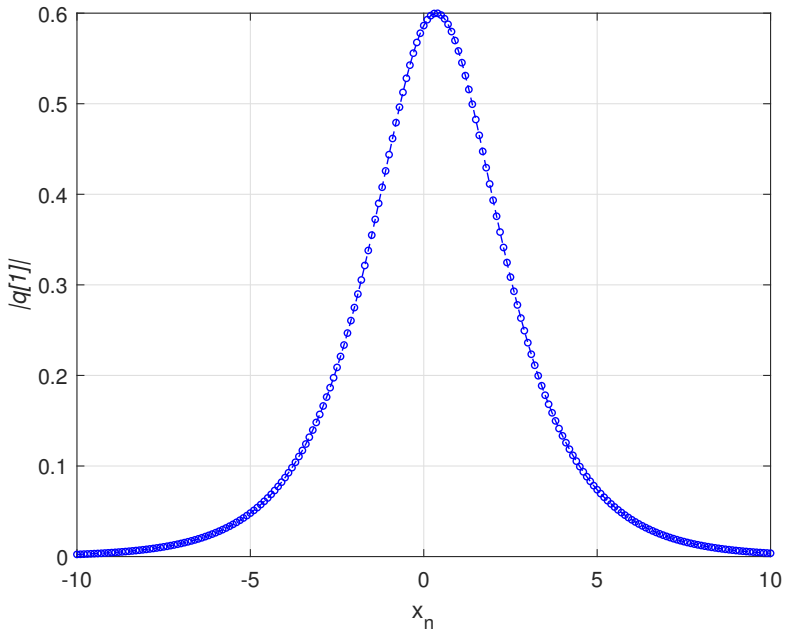



**Figure 1.** Single-soliton solution of (33) with $\eta = 0.3, h = 0.1$.

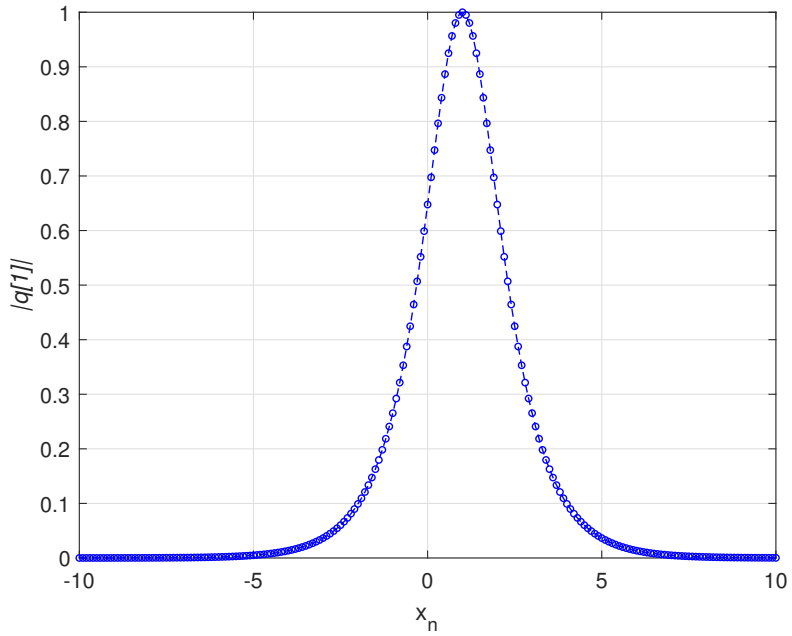



**Figure 2.** Single-soliton solution of (33) with $\eta = 0.5, h = 0.1$.

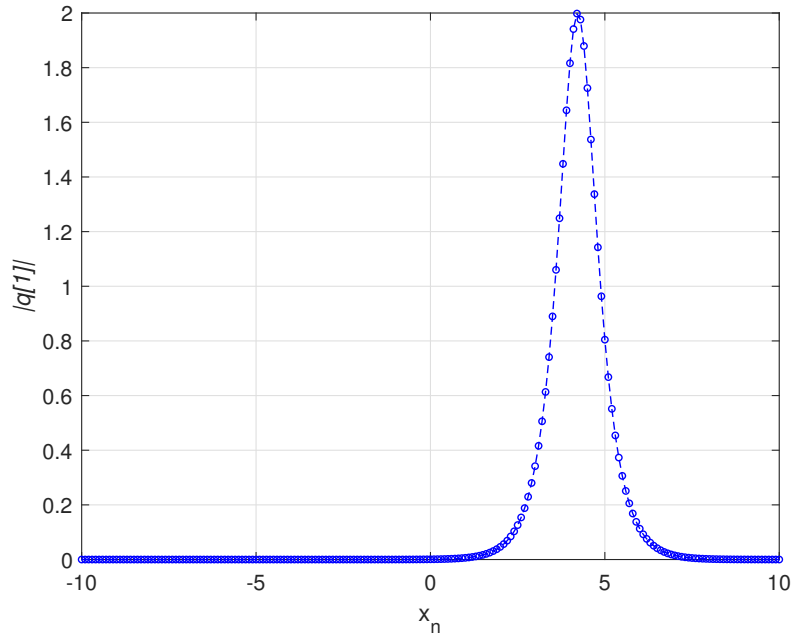



**Figure 3.** Single-soliton solution of (33) with $\eta = 1, h = 0.1$.

**Case2: Single-soliton solution of the mKdV equation on the $q$-discrete time scale.**
Taking $\mathbb{T} = \{t_0, t_0q, t_0q^2, t_0q^3, \ldots\}, \mathbb{X} = \{x_0, x_0q, x_0q^2, x_0q^3, \ldots\}$, we have

$$\mu_k = q^{k-1}t_0 - q^k t_0, \quad \nu_k = q^{k-1}x_0 - q^k x_0, \quad k = 1, 2\cdots,$$

where $x_0, t_0, q$ are all constants. Assuming $x_0 > 0, t_0 > 0, 0 < q < 1$ and using the properties of the Cayley exponential function, we have

$$E_\alpha^{\rho_2}(x) \equiv E_\alpha(\rho_2(x)) = \frac{1 + \frac{1}{2}\alpha v(x)}{1 - \frac{1}{2}\alpha v(x)} E_\alpha(x),$$

Thus, the exponential function on the $q$-discrete time scale is obtained.

$$E_\alpha(q^n x_0) \equiv \varepsilon_q^{\alpha x} = \prod_{j=0}^{\infty} \frac{1+\frac{1}{2}\alpha x(q-1)q^j}{1-\frac{1}{2}\alpha x(q-1)q^j}, x_0 > 0, 0 < q < 1.$$

By combining this with the Cayley $q$-exponential function, we have

$$\varepsilon_q^z = \prod_{j=0}^{\infty} \frac{1+\frac{1}{2}z(q-1)q^j}{1-\frac{1}{2}z(q-1)q^j}, 0 < q < 1,$$

Thus, the single-soliton solution on the $q$-discrete time scale can be obtained, as follows:

$$q[1]_{q-dis\times q-dis} = -\frac{4\eta}{M_2 + M_2^{-1}}, \tag{34}$$

where $M_2 = \varepsilon_q^{2\eta x}\varepsilon_q^{-8\eta^3 t}, 0 < q < 1$.

**Case3: Single-soliton solution of the mKdV equation on the continuous time scale.**

When we take the continuous time scale, we have

$$\mu(t) = \nu(x) = 0.$$

Subsequently, the Cayley exponential function is reduced to

$$E_\beta(x) = \exp\left(\int_0^x \beta(\tau)d\tau\right).$$

Thus, we can obtain the single-soliton solution on the continuous time scale of the classical mKdV equation, as follows:

$$aq[1]_{\mathbb{R}\times\mathbb{R}} = -2\eta \operatorname{sech}\left(2\eta x - 8\eta^3 t\right), \tag{35}$$

where $\eta$ is a parameter, commonly known as wave number or wave velocity parameter. Figure 4 is the single-soliton solution with $\eta = 0.3$.

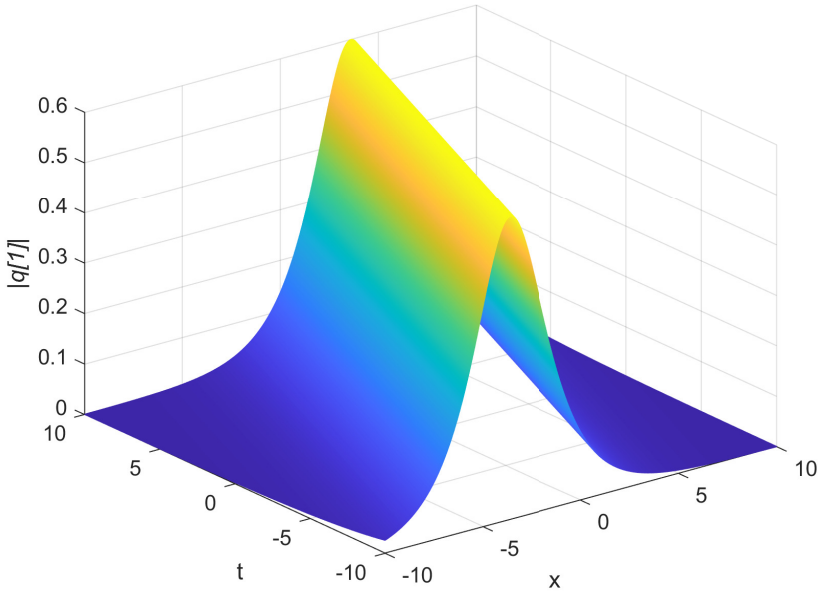



**Figure 4.** Single-soliton solution of (35) with $\eta = 0.3$.

As can be seen from Figures 4–6, with the increase in $\eta$, the amplitude of solitons increases, the width of solitons narrows, and the wave crest becomes sharper. In addition, the time term of (35) is $-8\eta^3 t$, which means that the propagation speed of the soliton is

proportional to $\eta$. Therefore, as $\eta$ increases, the propagation speed of solitons increases significantly.

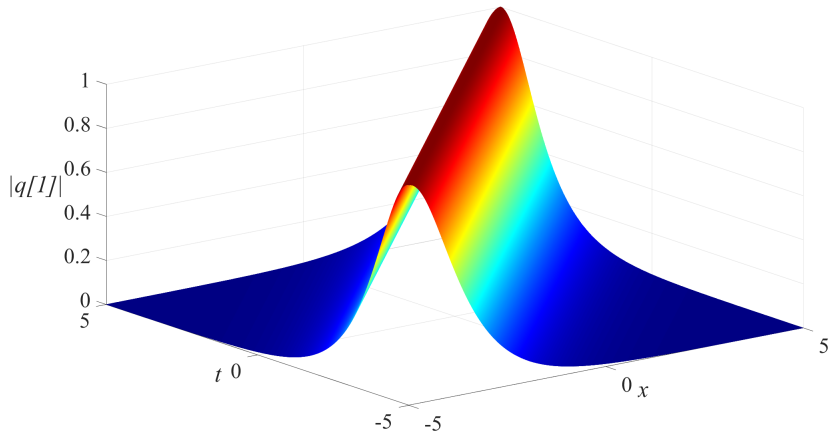



**Figure 5.** Single-soliton solution of (35) with $\eta = 0.5$.

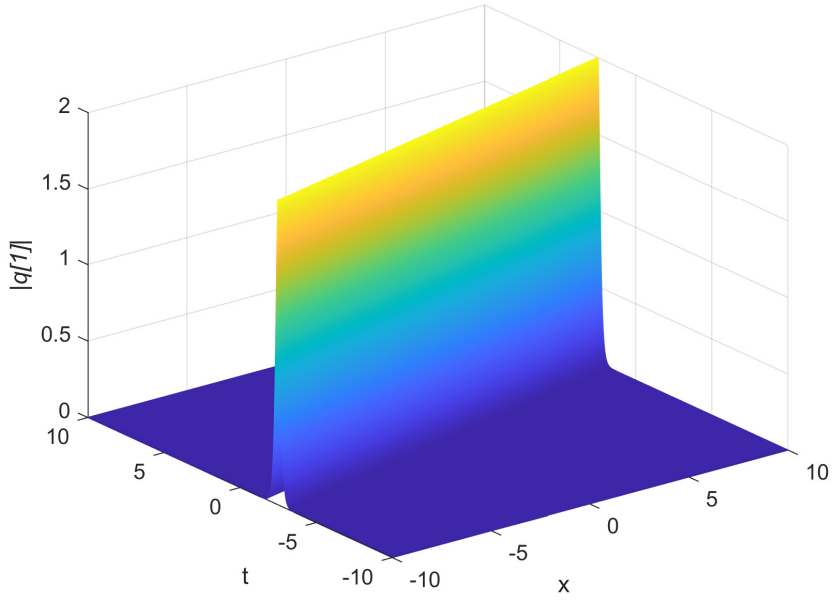



**Figure 6.** Single-soliton solution of (35) with $\eta = 1$.

**Case4: Single-soliton solution of the mKdV equation on the Cantor set.**

When we consider the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{C}$, we can find

$$\begin{aligned} &\mu(t) = 0, \\ &\nu(x) = \begin{cases} \frac{1}{3^{l+1}}, & x \in \mathbb{L}, \\ 0, & x \in \mathbb{C} \backslash \mathbb{L}, \end{cases} \end{aligned} \tag{36}$$

where $\mathbb{C}$ is a Cantor set and $\mathbb{L}$ contains the left discrete elements of the Cantor set, as follows:

$$\mathbb{L} = \left\{ \sum_{j=1}^{m} \frac{b_j}{3^j} + \frac{1}{3^{n+1}} : n \in N, b_j \in \{0, 2\}, 1 \leq j \leq n \right\}. \tag{37}$$

Based on Definition 3, we have

$$E_{\pm\eta}(x,0)=\left[\frac{1\pm\frac{\eta}{2\times3^{m+1}}}{1\mp\frac{\eta}{2\times3^{m+1}}}\right]^{\frac{x}{3^{m+1}}}. \tag{38}$$

Therefore, Equation (30) can be reduced to

$$\begin{aligned}\phi&=\left[\frac{1+\frac{\eta}{2\times3^{n+1}}}{1-\frac{\eta}{2\times3^{n+1}}}\right]^{\frac{x}{3^{n+1}}}e^{-4\eta^3t},\\ \psi&=\left[\frac{1-\frac{\eta}{2\times3^{n+1}}}{1+\frac{\eta}{2\times3^{n+1}}}\right]^{\frac{x}{3^{n+1}}}e^{4\eta^3t}.\end{aligned} \tag{39}$$

By substituting the above equation into Equation (27), the single-soliton solution of the mKdV equation is derived on the Cantor set, as follows:

$$q[1]_{\mathbb{R}\times\mathbb{C}}=\begin{cases}-\frac{4\eta}{M_4+M_4^{-1}}, x\in\mathbb{L}, t\in\mathbb{R},\\ -2\eta\,\mathrm{sech}\left(2\eta x-8\eta^3t\right), x\in\mathbb{C}\backslash\mathbb{L}, t\in\mathbb{R},\end{cases} \tag{40}$$

where $M_4=\left[\frac{1+\frac{\eta}{2\times3^{n+1}}}{1-\frac{\eta}{2\times3^{n+1}}}\right]^{\frac{2x}{3^{n+1}}}e^{-8\eta^3t}$.

## 4. Conclusions

We directly constructed the spectral problem for the mKdV equation on time scales—combined with the zero-curvature equation on time scales—to obtain the mKdV equation on the time scale, and we verified its feasibility. Subsequently, the mKdV equation on time scales was degenerated into specific forms on discrete and continuous time scales, respectively. Furthermore, utilizing the DT, we derived the soliton solutions of the mKdV equation on time scales with a zero background. Specifically, we discussed the explicit forms of single-soliton solutions on four special time scales ($\mathbb{R}$, $\mathbb{Z}$, and $q$-discrete, $\mathbb{C}$). The research in this paper was carried out in (1 + 1) dimensions; however, in our future work, we will extend this method to (1 + 2) and even (1 + 3) dimensions. However, this paper only studies soliton solutions on time scales, and other complex solutions are not studied, which is also a possible direction for future research.

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