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Lagrange Duality and Saddle-Point Optimality Conditions for Nonsmooth Interval-Valued Multiobjective Semi-Infinite Programming Problems with Vanishing Constraints

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Abstract: This article deals with a class of nonsmooth interval-valued multiobjective semi-infinite programming problems with vanishing constraints (NIMSIPVC). We introduce the VC-Abadie constraint qualification (VC-ACQ) for NIMSIPVC and employ it to establish Karush–Kuhn–Tucker (KKT)-type necessary optimality conditions for the considered problem. Regarding NIMSIPVC, we formulate interval-valued weak vector, as well as interval-valued vector Lagrange-type dual and scalarized Lagrange-type dual problems. Subsequently, we establish the weak, strong, and converse duality results relating the primal problem NIMSIPVC and the corresponding dual problems. Moreover, we introduce the notion of saddle points for the interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC. Furthermore, we derive the saddle-point optimality criteria for NIMSIPVC by establishing the relationships between the solutions of NIMSIPVC and the saddle points of the corresponding Lagrangians of NIMSIPVC, under convexity assumptions. Non-trivial illustrative examples are provided to demonstrate the validity of the established results. The results presented in this paper extend the corresponding results derived in the existing literature from smooth to nonsmooth optimization problems, and we further generalize them for interval-valued multiobjective semi-infinite programming problems with vanishing constraints.

Keywords: interval-valued multiobjective programming; vanishing constraints; Clarke subdifferential; Lagrange duality; saddle point

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1. Introduction

In recent times, mathematical programming problems with vanishing constraints (MPVC) have emerged as a very important and interesting optimization problem (see, for instance, [1–3]). The initial formulation and investigation of MPVC are attributed to the work of Achtziger and Kanzow [1]. The term vanishing constraints is derived from the fact that in various applications of MPVC, some of the constraints are often seen to vanish or become redundant at some points within the feasible set. One of the primary challenges encountered in the investigation of MPVC is that the feasible set may be non-convex and non-connected, despite the presence of convex constraint functions (see, for instance, [1,4]). Furthermore, in general, the majority of standard constraint qualifications, such as the Mangasarian–Fromovitz constraint qualification (MFCQ) and the linear independence constraint qualification (LICQ), are violated at an arbitrary feasible point of MPVC (see [1,4]). For a more comprehensive study of MPVC in various settings, we refer the readers to [2,3,5–7] and the references cited therein. If the feasible set of MPVC is defined by an infinite number of inequality constraints, then MPVC is termed a semi-infinite

programming problem with vanishing constraints (SIPVC). Tung [8] studied the optimality conditions and duality results for SIPVC involving continuously differentiable functions.

Multiobjective optimization problems involve the optimization of two or more conflicting objectives simultaneously. Due to their diverse applications in various real-world problems, including science and engineering (see, for instance, [9,10]), multiobjective optimization problems have been extensively studied by numerous researchers in various settings (see [7,11–13] and the references cited therein). Maeda [14] studied constraint qualifications for multiobjective optimization problems involving differentiable functions. Further, Li [15] established KKT-type necessary optimality conditions for nonsmooth multiobjective optimization problems. Guu et al. [16] derived strong KKT-type sufficient optimality criteria for multiobjective SIPVC under generalized convexity assumptions. Antczak [17] established both necessary and sufficient optimality conditions for multiobjective SIPVC involving invex functions. Further, Tung [18,19] derived KKT-type necessary optimality conditions as well as duality results for multiobjective SIPVC involving smooth and nonsmooth functions under convexity assumptions.

In general, optimization problems involve deterministic values for the coefficients of objective and constraint functions, leading to precise solutions. However, it is significant that many real-life optimization problems involve uncertain or imprecise data due to measurement errors or variations due to market fluctuations (see [20,21]). Therefore, to address uncertain optimization problems, researchers have developed several optimization methodologies, such as stochastic optimization, fuzzy optimization, and interval-valued optimization (see [20–26] and the references cited therein). Interval-valued optimization problems have gained significant attention from several researchers in the past few years (see, for instance, [27–35] and the references cited therein). Wu [36] studied KKT-type optimality conditions for multiobjective interval-valued optimization problems. Further, Singh et al. [37] derived KKT-type optimality conditions for interval-valued multiobjective programming problems involving generalized differentiable functions. Tung [38] established KKT-type optimality conditions for semi-infinite programming problems involving multiple interval-valued objective functions under convexity assumptions. Further, optimality conditions and duality results for interval-valued SIPVC were developed by Su and Dinh [39]. Recently, Yadav and Gupta [40] investigated optimality criteria as well as duality results for multiobjective interval-valued semi-infinite programming problems with vanishing constraints.

Over the past few decades, Lagrange duality and saddle-point optimality criteria have gained significant attention (see, for instance, [41,42] and the references cited therein). Sawaragi et al. [43] studied Lagrange duality theory for multiobjective optimization problems under convexity and regularity assumptions. Further, Luc [11] discussed Lagrange duality and saddle-point optimality conditions for multiobjective optimization problems involving set-valued data. Wang et al. [44] further extended the results obtained by Sawargi et al. [43] to cone-subconvexlike functions. Jayswal et al. [45] studied saddle-point optimality conditions for interval-valued optimization problems involving nonsmooth functions. Further, Dar et al. [46] studied optimality and saddle-point optimality conditions for interval-valued nondifferentiable multiobjective fractional programming problems. Recently, Tung et al. [47] studied Lagrange duality and saddle-point optimality conditions for multiobjective SIPVC.

It is worth noting that several researchers have studied constraint qualifications and necessary optimality conditions for both single-objective and multiobjective optimization problems (see, for instance, [2,6,14,16,18,19,27] and the references cited therein). Moreover, Lagrange duality and saddle-point optimality criteria for various nonlinear programming problems have been investigated by numerous researchers (see, for instance, [42,44,47,48] and the references cited therein). However, KKT-type necessary optimality conditions for NIMSIPVC have not been investigated yet via Clarke subdifferentials. Furthermore, Lagrange duality and saddle-point optimality criteria for interval-valued multiobjective SIPVC involving nonsmooth locally Lipschitz functions have not been explored before.

In this paper, our primary objective is to address these research gaps by introducing VC-ACQ and establishing optimality conditions, Lagrange duality results, and saddle-point optimality conditions for NIMSIPVC in terms of Clarke subdifferentials.

Motivated by the results established in [14,16,19,44,47], in this paper, we consider a class of interval-valued multiobjective semi-infinite programming problems with vanishing constraints involving nonsmooth locally Lipschitz functions. We introduce the notions of VC-stationary points and VC-ACQ for NIMSIPVC. Subsequently, we derive KKT-type necessary optimality conditions for NIMSIPVC by employing VC-ACQ. We formulate several Lagrange-type dual problems corresponding to NIMSIPVC, including interval-valued weak vector, interval-valued vector, and scalarized Lagrange-type dual problems. We derive the weak, converse, and strong duality results relating NIMSIPVC to the corresponding dual problems. In addition, we introduce the notions of saddle points for the interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC. Further, we establish the saddle-point optimality criteria for NIMSIPVC by establishing the relationships between saddle points and LU-efficient solutions of the Lagrangians of NIMSIPVC and the primal problem NIMSIPVC, respectively.

The novelty and contributions of this paper are as follows. Firstly, to establish the KKT-type necessary optimality conditions for NIMSIPVC, we extend and generalize the corresponding results derived by Achtziger and Kanzow [1] from smooth MPVC to nonsmooth multiobjective SIPVC involving interval-valued objective functions. Secondly, the KKT-type necessary optimality criteria investigated in this article extend and generalize the corresponding results established by Tung [18,19] to a broader class of optimization problems, namely NIMSIPVC. Thirdly, the Lagrange duality results and saddle-point optimality conditions for NIMSIPVC studied in this article extend and generalize the corresponding results derived in [47] from smooth multiobjective SIPVC to interval-valued multiobjective SIPVC involving nonsmooth locally Lipschitz functions. Moreover, it is significant to note that nonsmoothness is a common phenomenon that often appears in several real-world problems in the fields of science, engineering, and technology (see, for instance, [49–51]). Furthermore, interval-valued optimization is preferred over stochastic and fuzzy optimization for addressing uncertain optimization problems, as it can handle data uncertainty in situations where it is difficult to determine an exact probability distribution or fuzzy membership function (see [31,41]). In light of the above discussion and the fact that NIMSIPVC belongs to a broader category of optimization problems, the results derived in this article can be applied to study a more general class of mathematical programming problems compared to the existing results in the available literature.

The rest of this article is organized as follows. Some basic mathematical preliminaries and fundamental concepts used are discussed in Section 2. We establish KKT-type necessary optimality conditions for NIMSIPVC by employing VC-ACQ in Section 3. In Section 4, we formulate interval-valued vector Lagrange-type dual problems corresponding to NIMSIPVC, followed by the weak, converse, and strong duality results. Further, we establish saddle-point optimality criteria for NIMSIPVC by utilizing the saddle points of an interval-valued vector Lagrangian. In addition, we formulate the scalarized Lagrange-type dual problems corresponding to NIMSIPVC and derive weak, converse, and strong duality results in Section 5. We also derive the saddle-point optimality criteria for NIMSIPVC in this section. Section 6 concludes the work presented in this paper and provides various future research avenues.

2. Mathematical Preliminaries and Definitions

In this article, the notations $\mathbb{N}$ and $\mathbb{R}^n$ denote the set of natural numbers and the Euclidean space of dimension n, respectively. The notation $\mathbb{R}^n_+$ represents the non-negative orthant of $\mathbb{R}^n$. The standard inner product is denoted by the symbol $\langle \cdot, \cdot \rangle$. Let $\mathcal{L} \subseteq \mathbb{R}$ be an infinite set. Then, $\mathbb{R}^{|\mathcal{L}|}$ signifies the linear space, which is defined as follows:

$$\mathbb{R}^{|\mathcal{L}|} := \{u = (u_k)_{k \in \mathcal{L}} \mid u_k = 0, \ \forall k \in \mathcal{L}, \text{ except } u_k \neq 0, \text{ for finitely many } k \in \mathcal{L}\}.$$

The symbol $\mathbb{R}_+^{|\mathcal{L}|}$ denotes the positive cone of $\mathbb{R}^{|\mathcal{L}|}$, which is defined as follows:

$$\mathbb{R}_+^{|\mathcal{L}|} := \{u = (u_k)_{k\in\mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|} \mid u_k \geq 0,\ \forall k \in \mathcal{L}\}.$$

Let $\mathcal{C} \subseteq \mathbb{R}^n$. The symbols $\mathrm{cl}(\mathcal{C})$, $\mathrm{span}(\mathcal{C})$, $\mathrm{co}(\mathcal{C})$, and $\mathrm{pos}(\mathcal{C})$ denote the closure, span, convex hull, and positive conic hull of $\mathcal{C}$, respectively. The following sets are used in the subsequent parts of this article:

$$\begin{aligned}\mathcal{C}^- &:= \{z \in \mathbb{R}^n : \langle z, u\rangle \leq 0, \forall u \in \mathcal{C}\},\\ \mathcal{C}^{\ominus} &:= \{z \in \mathbb{R}^n : \langle z, u\rangle < 0, \forall u \in \mathcal{C}\},\\ \mathcal{C}^0 &:= \{z \in \mathbb{R}^n : \langle z, u\rangle = 0, \forall u \in \mathcal{C}\}.\end{aligned}$$

Let us consider $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$. The following relations are from [18,19]:

$$\begin{aligned}\mathrm{pos}(C_1 \cup C_2) &= \mathrm{pos}(C_1) + \mathrm{pos}(C_2),\\ \mathrm{span}(C_1 \cup C_2) &= \mathrm{span}(C_1) + \mathrm{span}(C_2).\end{aligned}$$

Let $u, v \in \mathbb{R}^n$. We define the following notations that are used in the remainder of this paper:

$$\begin{aligned}u \prec v &\Leftrightarrow u_j < v_j,\ \forall j = 1, 2, \ldots, l.\\ u \preceq v &\Leftrightarrow \begin{cases} u_j \leq v_j, & \forall j = 1, 2, \ldots, l,\\ u_k < v_k, & \text{for at least one } k \in \{1, 2, \ldots, l\}.\end{cases}\\ u \preccurlyeq v &\Leftrightarrow u_j \leq v_j,\ \forall j \in \{1, 2, \ldots, l\}.\end{aligned}$$

A function $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be a locally Lipschitz function around $\overline{u} \in H$ if there exists a neighborhood V of $\overline{u}$ and a constant $K > 0$ such that $|\Psi(u) - \Psi(v)| \leq K||u - v||$, $\forall u, v \in V$.

In the following definition, we recall the notion of a contingent cone of a non-empty subset of $\mathbb{R}^n$ (see [49]).

Definition 1. *Let $\emptyset \neq H \subseteq \mathbb{R}^n$ and $\overline{u} \in \mathrm{cl}(H)$. The contingent cone to H at $\overline{u}$ is denoted by $\mathcal{T}(\overline{u}, H)$ and is given by*

$$\mathcal{T}(\overline{u}, H) := \{d \in \mathbb{R}^n |\ d_k \in \mathbb{R}^n,\ t_k \downarrow 0,\ \overline{u} + t_k d_k \in H,\ \forall k \in \mathbb{N}\}.$$

The notion of a convex subset of $\mathbb{R}^n$ is recalled in the following definition (see, for instance, [52]).

Definition 2. *Let $H \subseteq \mathbb{R}^n$ and u, w be any two arbitrary distinct elements of H. H is said to be a convex set if the following condition holds:*

$$(1 - t)u + tw \in H,\ \forall t \in [0, 1].$$

In the following definition, we recall the notions of the Clarke directional derivative and the Clarke subdifferential for a real-valued locally Lipschitz function (see [49]).

Definition 3. *Let $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function around $u \in H$. For such a function, we have the following definitions:*

(i) The Clarke directional derivative of Ψ at u in direction v is defined as follows:

$$\Psi^\circ(u; v) := \limsup_{y \to u, t \downarrow 0} \frac{\Psi(y + tv) - \Psi(u)}{t}.$$

(ii) The Clarke subdifferential of Ψ *at* u *is given by*

$$\partial_c \Psi(u) := \{\eta \in \mathbb{R}^n | \Psi^\circ(u; v) \geq \langle \eta, v \rangle, \ \forall v \in \mathbb{R}^n\}.$$

The following lemma from [49] presents the various properties of the Clarke directional derivative and the Clarke subdifferential of a real-valued locally Lipschitz function, which are used in this paper.

Lemma 1. *Let* $\Psi, \phi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ *be two locally Lipschitz functions around* $u \in H$*. Then, the following statements hold:*

(i) $\partial_c \Psi(u)$ *is a non-empty, convex, and compact subset of* $\mathbb{R}^n$*.*

(ii) For every $v \in \mathbb{R}^n$*, the Clarke directional derivative of* Ψ *at* u *satisfies the following:*

$$\Psi^\circ(u; v) := max\{\langle \eta, v \rangle | \ \eta \in \partial_c \Psi(u)\}.$$

(iii) The set-valued map $u \rightrightarrows \partial_c \Psi(u)$ *is an upper semicontinuous set-valued function, provided* Ψ *is locally Lipschitz on* $\mathbb{R}^n$*.*

(iv) For any $\sigma \in \mathbb{R}$*, we have* $\partial_c(\sigma \Psi(u)) = \sigma \partial_c \Psi(u)$*. Furthermore,* $\partial_c(\Psi + \phi)(u) \subseteq \partial_c \Psi(u) + \partial_c \phi(u)$*.*

(v) If Ψ *is locally Lipschitz on an open set containing* $[u, v]$*, then*

$$\Psi(u) - \Psi(v) = \langle \xi^*, v - u \rangle,$$

for some $z \in [u, v)$ *and* $\xi^* \in \partial_c \Psi(z)$*.*

Now, we recall the definition of a convex function defined on a convex set $H \subseteq \mathbb{R}^n$ (see, for instance, [52,53]).

Definition 4. *Let* $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz function. We define the following properties for* Ψ*:*

(i) Convexity at $\overline{u} \in H$ *if the following condition holds:*

$$\Psi(u) - \Psi(\overline{u}) \geq \langle \xi, u - \overline{u} \rangle, \ \forall \xi \in \partial_c \Psi(u), \forall u \in H.$$

(ii) Strict convexity at $\overline{u} \in H$ *if the following condition holds:*

$$\Psi(u) - \Psi(\overline{u}) > \langle \xi, u - \overline{u} \rangle, \ \forall \xi \in \partial_c \Psi(u), \ \forall u \in H \setminus \{\overline{u}\}.$$

Now, we discuss some important concepts and definitions related to interval analysis (see [54]).

Let $\mathcal{I}$ be the collection of all closed intervals in $\mathbb{R}$, defined as follows:

$$\mathcal{I} := \{[m^L, m^U] | m^L \leq m^U, \ m^L, \ m^U \in \mathbb{R}\}.$$

For any two intervals $\mathcal{M} = [m^L, m^U]$ and $\mathcal{N} = [n^L, n^U] \in \mathcal{I}$, we define the following:

(a_1) $\mathcal{M} + \mathcal{N} := \{m + n : m \in \mathcal{M} \text{ and } n \in \mathcal{N}\} = [m^L + n^L, m^U + n^U]$.

(b_1) $-\mathcal{M} := \{-m : m \in \mathcal{M}\} = [-m^U, -m^L]$.

It is worth noting that any real number m can be represented as a closed interval, since $\mathcal{M}_m = [m, m]$: Let $\mathcal{M} = [m^L, m^U]$ and $\mathcal{N} = [n^L, n^U] \in \mathcal{I}$. We define the following relations:

(a_2) $\mathcal{M} \preceq_{LU} \mathcal{N} \iff m^L \leq n^L$ and $m^U \leq n^U$.

(b_2) $\mathcal{M} \prec_{LU} D \iff \mathcal{M} \preceq_{LU} \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$, that is, one of the following conditions is satisfied:

$$m^L < n^L \text{ and } m^U < n^U \text{ or } m^L \leq n^L \text{ and } m^U < n^U, \text{ or } m^L < n^L \text{ and } m^U \leq n^U.$$

(c_2) $\mathcal{M} \prec^s_{LU} \mathcal{N} \iff m^L < n^L$ and $m^U < n^U$.

Let $\mathcal{I}^l$ ($l \in \mathbb{N}$) be the collection of all interval-valued vectors, where each element $\mathbf{M} \in \mathcal{I}^l$ can be defined as follows:

$\mathbf{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_l)$ such that for every $i = 1, 2, \ldots, l$, $\mathcal{M}_i = \left[m_i^L, m_i^U\right]$ is a closed interval. Given two arbitrary interval-valued vectors, $\mathbf{M}$ and $\mathbf{N}$, the following relationships hold:

(a_3) $\mathbf{M} \preceq_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \preceq_{LU} \mathcal{N}_i$, $\forall i = 1, 2, \ldots, l$.
(b_3) $\mathbf{M} \prec_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \preceq_{LU} \mathcal{N}_i$, $\forall i = 1, 2, \ldots, l$, and $\mathcal{M}_k \prec_{LU} \mathcal{N}_k$ for some $i \neq k$.
(c_3) $\mathbf{M} \prec^s_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \prec^s_{LU} \mathcal{N}_i$, $\forall i = 1, 2, \ldots, l$.

Remark 1. *1. If* $\mathbf{M} \not\prec_{LU} \mathbf{N}$, *then from* ($a_3$), ($b_3$), *and* ($c_3$), *we have*

$$\begin{aligned} &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+ \setminus \{0\},\ (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+ \setminus \{0\}, \\ \text{or, } &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+ \setminus \{0\}, (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+, \\ \text{or, } &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+,\ (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+ \setminus \{0\}, \end{aligned}$$

where $\mathcal{M}^L = (\mathcal{M}_1^L, \mathcal{M}_2^L, \ldots, \mathcal{M}_l^L) \in \mathbb{R}^l$, $\mathcal{M}^U = (\mathcal{M}_1^U, \mathcal{M}_2^U, \ldots, \mathcal{M}_l^U) \in \mathbb{R}^l$.

2. If $\mathbf{M} \not\prec^s_{LU} \mathbf{N}$, *then from* ($a_3$), ($b_3$), *and* ($c_3$), *we have*

$$(\mathcal{M}^L - \mathcal{N}^L) \notin -\operatorname{int}\mathbb{R}^l_+,\ (\mathcal{M}^U - \mathcal{N}^U) \notin -\operatorname{int}\mathbb{R}^l_+.$$

A function $\Psi : \mathbb{R}^n \to \mathcal{I}$ is termed an interval-valued function if $\Psi(u) = \left[\Psi^L(u), \Psi^U(u)\right]$, where $\Psi^L, \Psi^U : \mathbb{R}^n \to \mathbb{R}$ are real-valued functions such that $\Psi^L(u) \leq \Psi^U(u), \forall u \in \mathbb{R}^n$. An interval-valued function $\Psi : H \to \mathcal{I}$ is known as a locally Lipschitz function on H if Ψ^L, Ψ^U are locally Lipschitz on H.

Let us define the following sets for a non-empty subset $\mathcal{A} \subset \mathcal{I}^l$:

$$\begin{aligned} \operatorname{WMin}\mathcal{A} := \{M \in \mathcal{A} | &(I^L - M^L) \notin -\operatorname{int}\mathbb{R}^l_+,\ (I^U - M^U) \notin -\operatorname{int}\mathbb{R}^l_+, \forall I \in \mathcal{A}\}, \\ \operatorname{Min}\mathcal{A} := \{M \in \mathcal{A} | &(I^L - M^L) \notin -\mathbb{R}^l_+ \setminus \{0\},\ (I^U - M^U) \notin -\mathbb{R}^l_+ \setminus \{0\},\ \forall I \in \mathcal{A}, \\ &\text{or } (I^L - M^L) \notin -\mathbb{R}^l_+,\ (I^U - M^U) \notin -\mathbb{R}^l_+ \setminus \{0\},\ \forall I \in \mathcal{A}, \\ &\text{or } (I^L - M^L) \notin -\mathbb{R}^l_+ \setminus \{0\},\ (I^U - M^U) \notin -\mathbb{R}^l_+,\ \forall I \in \mathcal{A}\}. \end{aligned}$$

The notion of LU-convexity of an interval-valued function defined on a convex subset is presented in the following definition (see, for instance, [36,41]).

Definition 5. *Let* $\Psi : H \subseteq \mathbb{R}^n \to \mathcal{I}$ *be any interval-valued function on a convex set* H. Ψ *is said to be LU-convex at* $u \in H$ *if* Ψ^L *and* Ψ^U *are convex at* u.

The following lemmas from [52] are instrumental in establishing KKT-type necessary optimality conditions for NIMSIPVC.

Lemma 2. *Let* $\{\mathcal{D}_i | i \in \mathcal{L}\}$ *be any arbitrary collection of non-empty convex sets in* $\mathbb{R}^n$. *Further, let*

$$\mathcal{B} = \operatorname{pos}\left(\bigcup_{i \in \mathcal{L}} \mathcal{D}_i\right).$$

Then, any non-zero vector lying in set $\mathcal{B}$ *can be expressed as a non-negative combination of at most* n *linearly independent vectors, each belonging to some different set* $\mathcal{D}_i$.

Lemma 3. *Let $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{P}$ be any arbitrary (not necessarily finite) index sets. Consider the maps $d_i : \mathcal{D} \to \mathbb{R}^n$, $e_j : \mathcal{E} \to \mathbb{R}^n$, and $f_m : \mathcal{P} \to \mathbb{R}^n$, as follows:*

$$\begin{aligned} d_i &= d(i) = (d_1(i), \ldots, d_n(i)), \\ e_j &= e(j) = (e_1(j), \ldots, e_n(j)), \\ f_m &= f(m) = (f_1(m), \ldots, f_n(m)). \end{aligned}$$

Further, suppose that the set $\text{co}\{d_i | i \in \mathcal{D}\} + \text{pos}\{e_j | j \in \mathcal{E}\} + \text{span}\{f_m | m \in \mathcal{P}\}$ is a closed set. Then, the following statements are equivalent:

Statement I. The following system of inequalities

$$\begin{aligned} \langle d_i, v \rangle &< 0, i \in \mathcal{D}, \mathcal{D} \neq \emptyset, \\ \langle e_j, v \rangle &\leq 0, j \in \mathcal{E}, \\ \langle f_m, v \rangle &= 0, m \in \mathcal{P}, \end{aligned}$$

has no solution $v \in \mathbb{R}^n$.

Statement II. The following relation holds:

$$0 \in \text{co}\{d_i | i \in \mathcal{D}\} + \text{pos}\{e_j | j \in \mathcal{E}\} + \text{span}\{f_m | m \in \mathcal{P}\}.$$

Lemma 4. *Suppose that $\mathcal{D}$ is any non-empty and compact subset of $\mathbb{R}^n$. Then, the following statements hold:*

(a) The convex hull of $\mathcal{D}$ is a compact set.
(b) The $\text{pos}(\mathcal{D})$ is a closed cone, provided $0 \notin \text{co}(\mathcal{D})$.

3. Optimality Conditions for NIMSIPVC

In this section, we consider a multiobjective interval-valued semi-infinite programming problem with vanishing constraints involving nonsmooth locally Lipschitz functions. We introduce the notions of a VC-stationary point and a VC-linearized cone for NIMSIPVC. Further, we present VC-ACQ for the considered problem and employ it to derive KKT-type necessary optimality conditions for NIMSIPVC in terms of Clarke subdifferentials. In addition, we demonstrate that the standard constraint qualifications, namely the MFCQ and LICQ, are not satisfied at the LU-efficient solution of NIMSIPVC.

Consider the following nonsmooth multiobjective interval-valued semi-infinite programming problem with vanishing constraints on $\mathbb{R}^n$:

$$\begin{aligned} \text{(NIMSIPVC) Minimize } \mathcal{F}(u) &= (\mathcal{F}_1(u), \mathcal{F}_2(u), \ldots, \mathcal{F}_l(u)), \\ &= \Big([\mathcal{F}_1^L(u), \mathcal{F}_1^U(u)], [\mathcal{F}_2^L(u), \mathcal{F}_2^U(u)] \ldots, [\mathcal{F}_l^L(u), \mathcal{F}_l^U(u)]\Big), \\ \text{subject to } \Psi_k(u) &\leq 0, \ \forall k \in \mathcal{L}, \\ \zeta_i(u) &= 0, \ \forall i \in \mathcal{B} = \{1, 2, \ldots, r\}, \\ \mathcal{Q}_i(u) &\geq 0, \ \forall i \in \mathcal{C} = \{1, 2, \ldots, s\}, \\ \mathcal{Q}_i(u)\mathcal{R}_i(u) &\leq 0, \ \forall i \in \mathcal{C} = \{1, 2, \ldots, s\}, \end{aligned}$$

where $\mathcal{F}_i^L$, $\mathcal{F}_i^U : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{J}^{\mathcal{F}} = \{1, 2, \ldots, l\})$, $\Psi_k : \mathbb{R}^n \to \mathbb{R}$ $(k \in \mathcal{L})$, $\zeta_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{B})$, $\mathcal{Q}_i$, $\mathcal{R}_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{C})$ are locally Lipschitz functions on $\mathbb{R}^n$. The index set $\mathcal{L}$ is considered to be arbitrary (not necessarily finite).

Remark 2. *Under the following conditions, NIMSIPVC reduces to well-known problems:*

1. *If $\mathcal{L}$ is a finite set, $\mathcal{J}^{\mathcal{F}} = \{1\}$, $\mathcal{F}_1^L(u) = \mathcal{F}_1^U(u)$, $\forall u \in \mathbb{R}^n$, then NIMSIPVC reduces to the problem MPVC, as considered by Achtziger and Kanzow [1].*
2. *If $\mathcal{J}^{\mathcal{F}} = \{1\}$, then NIMSIPVC reduces to a semi-infinite interval-valued optimization problem with vanishing constraints, as considered by Joshi et al. [27].*

3. *If* $\mathcal{J}^{\mathcal{F}} = \{1\}$, $\mathcal{F}_1^L(u) = \mathcal{F}_1^U(u)$, $\forall u \in \mathbb{R}^n$, *and* $\mathcal{B} = \emptyset = \mathcal{C}$, *then NIMSIPVC reduces to the semi-infinite programming problem, as considered by Kanzi [55].*
4. *If* $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$, *and if* $\mathcal{L}$ *is a finite set with* $\mathcal{B} = \emptyset = \mathcal{C}$, *then NIMSIPVC reduces to the multiobjective constrained optimization problem (P), as considered by Maeda [14].*
5. *If* $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, *and* $\forall u \in \mathbb{R}^n$, *then NIMSIPVC reduces to the problem (P), as considered by Tung [19].*
6. *If for every* $i \in \mathcal{J}^{\mathcal{F}}$, $\mathcal{F}_i$ *is a continuously differentiable real-valued function, that is, for every* $i \in \mathcal{J}^{\mathcal{F}}$, $u \in \mathbb{R}^n$, $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall u \in \mathbb{R}^n$, *and* $\partial_c \mathcal{F}_i(u) = \{\nabla \mathcal{F}_i(u)\}$. *Moreover, if* Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ *are continuously differentiable functions, then NIMSIPVC reduces to the problem (P), as considered by Tung [18] and Tung et al. [47].*

The feasible set for NIMSIPVC is given by

$$\mathcal{G} := \{u \in \mathbb{R}^n | \Psi_k(u) \leq 0,\ k \in \mathcal{L},\ \zeta_i(u) = 0,\ i \in \mathcal{B},\ \mathcal{Q}_i(u) \geq 0,\ i \in \mathcal{C},\ \mathcal{Q}_i(u)\mathcal{R}_i(u) \leq 0,\ i \in \mathcal{C}\}.$$

Let $\overline{u}$ be any feasible element of NIMSIPVC. Then, the following sets are used in this paper:

$$\mathbb{P}(\overline{u}) := \{k \in \mathcal{L} | \Psi_k(\overline{u}) = 0\},\ \mathbb{P}^{\Psi}(\overline{u}) := \{\sigma^{\Psi} \in \mathbb{R}_+^{|\mathcal{L}|} | \sigma_k^{\Psi} \Psi_k(\overline{u}) = 0,\ \forall k \in \mathcal{L}\},$$

where $\mathbb{P}(\overline{u})$ signifies the index set of all active inequality constraints and $\mathbb{P}^{\Psi}(\overline{u})$ contains all active constraint multipliers at $\overline{u}$.

In the following definition, we recall the notions of LU-efficient solutions for NIMSIPVC (see [19,47]).

Definition 6. *Let* $\overline{u}$ *be an arbitrary feasible element of NIMSIPVC. Then* $\overline{u}$ *is referred to as the following:*

(i) A locally LU-efficient solution of NIMSIPVC if there exists a neighborhood V *of* $\overline{u}$ *such that for every* $u \in V \cap \mathcal{G}$, *the following conditions hold:*

$$\mathcal{F}_i(u) \not\leq_{LU} \mathcal{F}_i(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$
$$\mathcal{F}_j(u) \not< _{LU} \mathcal{F}_j(\overline{u}),\ \text{for at least one } j \in \mathcal{J}^{\mathcal{F}}.$$

The symbol Eff_{loc} *denotes the set of all locally LU-efficient solutions of NIMSIPVC.*

(ii) A locally weakly LU-efficient solution of NIMSIPVC if there exists a neighborhood V *of* $\overline{u}$ *such that for any* $u \in \mathcal{G} \cap V$, *the following condition holds:*

$$\mathcal{F}_i(u) \not<^s_{LU} \mathcal{F}_i(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

The set of all locally weakly LU-efficient solutions of NIMSIPVC is denoted by $WEff_{loc}$.

Remark 3. *The following observations are noteworthy:*

1. *If* $V = \mathbb{R}^n$ *in Definition 6, then* $\overline{u} \in \mathcal{G}$ *is known as an LU-efficient and a weakly LU-efficient solution of NIMSIPVC.*
2. *The symbols Eff and WEff denote the sets of all LU-efficient and weakly LU-efficient solutions of NIMSIPVC, respectively.*

Consider an arbitrary feasible element $\overline{u}$. The following index sets are employed in the remainder of this article:

$$\begin{aligned}
\mathcal{H}_{+}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) > 0\},\\
\mathcal{H}_{0}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) = 0\},\\
\mathcal{H}_{+0}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) > 0,\ \mathcal{R}_i(\overline{u}) = 0\},\\
\mathcal{H}_{+-}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) > 0,\ \mathcal{R}_i(\overline{u}) < 0\},\\
\mathcal{H}_{0+}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) = 0,\ \mathcal{R}_i(\overline{u}) > 0\},\\
\mathcal{H}_{00}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) = 0,\ \mathcal{R}_i(\overline{u}) = 0\},\\
\mathcal{H}_{0-}(\overline{u}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{u}) = 0,\ \mathcal{R}_i(\overline{u}) < 0\}.
\end{aligned}$$

In the following definition, we extend the notion of a VC-stationary point from multiobjective SIPVC to NIMSIPVC (see [18]).

Definition 7. *Let $\overline{u}$ be an arbitrary feasible element of NIMSIPVC. Then, $\overline{u}$ is known as a VC-stationary point of NIMSIPVC if there exists $(\lambda^L, \lambda^U, \sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:*

$$\begin{aligned}
0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\lambda_i^L \partial_c \mathcal{F}_i^L(\overline{u}) + \lambda_i^U \partial_c \mathcal{F}_i^U(\overline{u})\right) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \partial_c \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \partial_c \zeta_i(\overline{u}) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{u})\\
+ \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{u}),
\end{aligned} \tag{1}$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} (\lambda_i^L + \lambda_i^U) = 1$, $\sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\overline{u})} = 0$, $\sigma^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u})} \geq 0$, $\sigma^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u})} \geq 0$, *and* $\sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}) \cup \mathcal{H}_{0+}(\overline{u})} = 0$.

Remark 4. *The following observations are noteworthy:*

1. *If $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$, and $\mathcal{L}$ is a finite set, then Definition 7 reduces to Definition 2.2, as presented by Hoheisel and Kanzow [4].*
2. *The symbol VC_{SP} denotes the set of all VC-stationary points of NIMSIPVC.*

For any element $\overline{u} \in \mathcal{G}$, we define the following sets:

$$\begin{aligned}
\mathbb{P}_{+}^{\Psi}(\overline{u}) &:= \{k \in \mathbb{P}(\overline{u}) | \sigma_k^{\Psi} > 0\},\\
\mathcal{B}_{+}^{\zeta}(\overline{u}) &:= \{i \in \mathcal{B} | \sigma_i^{\zeta} > 0\}, & \mathcal{B}_{-}^{\zeta}(\overline{u}) &:= \{i \in \mathcal{B} | \sigma_i^{\zeta} < 0\},\\
\overline{\mathbb{H}}_{0}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_0(\overline{u}) | \sigma_i^{\mathcal{Q}} > 0\}, & \overline{\mathbb{H}}_{0}^{-}(\overline{u}) &:= \{i \in \mathcal{H}_0(\overline{u}) | \sigma_i^{\mathcal{Q}} < 0\},\\
\overline{\mathbb{H}}_{0+}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{0+}(\overline{u}) | \sigma_i^{\mathcal{Q}} > 0\}, & \overline{\mathbb{H}}_{0+}^{-}(\overline{u}) &:= \{i \in \mathcal{H}_{0+}(\overline{u}) | \sigma_i^{\mathcal{Q}} < 0\},\\
\overline{\mathbb{H}}_{0-}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_0(\overline{u}) | \sigma_i^{\mathcal{Q}} > 0\},\\
\mathbb{H}_{+0}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{+0}(\overline{u}) | \sigma_i^{\mathcal{R}} > 0\}, & \mathbb{H}_{+0}^{-}(\overline{u}) &:= \{i \in \mathcal{H}_{+0}(\overline{u}) | \sigma_i^{\mathcal{R}} < 0\},\\
\mathbb{H}_{+-}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{+-}(\overline{u}) | \sigma_i^{\mathcal{R}} < 0\}, & \mathbb{H}_{0+}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{0+}(\overline{u}) | \sigma_i^{\mathcal{R}} > 0\},\\
\mathbb{H}_{0+}^{-}(\overline{u}) &:= \{i \in \mathcal{H}_{0+}(\overline{u}) | \sigma_i^{\mathcal{R}} < 0\}, & \mathbb{H}_{00}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{00}(\overline{u}) | \sigma_i^{\mathcal{R}} > 0\},\\
\mathbb{H}_{00}^{-}(\overline{u}) &:= \{i \in \mathcal{H}_{00}(\overline{u}) | \sigma_i^{\mathcal{R}} < 0\}, & \mathbb{H}_{0-}^{+}(\overline{u}) &:= \{i \in \mathcal{H}_{0-}(\overline{u}) | \sigma_i^{\mathcal{R}} > 0\}.
\end{aligned}$$

Now, we extend the definition of a VC-linearized cone given by Tung [38] from smooth multiobjective semi-infinite programming problems to a broader class of optimization problems, namely NIMSIPVC.

Definition 8. *For an arbitrary feasible element $\overline{u}$ of NIMSIPVC, the VC-linearized cone at $\overline{u}$ is given by*

$$\begin{aligned}\mathcal{L}_{VC}(\overline{u}) := \{\nu \in \mathbb{R}^n | \langle \eta_k^{\Psi}, \nu\rangle \leq 0,\ \eta_k^{\Psi} \in \partial_c \Psi_k(\overline{u}),\ \forall k \in \mathbb{P}(\overline{u}),\\ \langle \eta_i^{\zeta}, \nu\rangle = 0,\ \eta_i^{\zeta} \in \partial_c \zeta_i(\overline{u}),\ \forall i \in \mathcal{B},\\ \langle \eta_i^{\mathcal{Q}}, \nu\rangle = 0,\ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u}),\ \forall i \in \mathcal{H}_{0+}(\overline{u}),\\ \langle \eta_i^{\mathcal{Q}}, \nu\rangle \geq 0,\ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u}),\ \forall i \in \mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}),\\ \langle \eta_i^{\mathcal{R}}, \nu\rangle \leq 0,\ \eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{u}),\ \forall i \in \mathcal{H}_{+0}(\overline{u}),\\ \langle \eta_i^{\mathcal{R}}, \nu\rangle \leq 0,\ \eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{u}),\ \forall i \in \mathcal{H}_{00}(\overline{u})\}.\end{aligned}$$

For an arbitrary element $\overline{u} \in \mathcal{G}$, we define the following sets that are used in the subsequent sections of this article:

$$\begin{aligned}&\mathcal{E}_{\Psi} := \bigcup_{k\in\mathbb{P}(\overline{u})} \partial_c \Psi_k(\overline{u}), \quad \mathcal{E}_{\zeta} := \bigcup_{i\in\mathcal{B}} \partial_c \zeta_i(\overline{u}), \quad \mathcal{E}_{\mathcal{Q}_1} := \bigcup_{i\in\mathcal{H}_{0+}(\overline{u})} \partial_c \mathcal{Q}_i(\overline{u}),\\ &\mathcal{E}_{\mathcal{Q}_2} := \bigcup_{i\in\mathcal{H}_{00}(\overline{u})\cup\mathcal{H}_{0-}(\overline{u})} -\partial_c \mathcal{Q}_i(\overline{u}), \quad \mathcal{E}_{\mathcal{R}_1} := \bigcup_{i\in\mathcal{H}_{+0}(\overline{u})} \partial_c \mathcal{R}_i(\overline{u}),\\ &\mathcal{E}_{\mathcal{R}_2} := \bigcup_{i\in\mathcal{H}_{+0}(\overline{u})\cup\mathcal{H}_{00}(\overline{u})} \partial_c \mathcal{R}_i(\overline{u}),\end{aligned}$$

Remark 5. *In view of Definition 8, it is worth noting that*

$$\mathcal{L}_{VC}(\overline{u}) = (\mathcal{E}_{\Psi})^- \cap (\mathcal{E}_{\zeta})^0 \cap (\mathcal{E}_{\mathcal{Q}_1})^0 \cap (\mathcal{E}_{\mathcal{Q}_2})^- \cap (\mathcal{E}_{\mathcal{R}_2})^-.$$

In the following definition, we introduce VC-ACQ for NIMSIPVC.

Definition 9. *Let $\overline{u}$ be an arbitrary feasible element of NIMSIPVC. Then, VC-ACQ for NIMSIPVC is satisfied at $\overline{u}$ if*

$$\mathcal{L}_{VC}(\overline{u}) \subseteq \mathcal{T}(\overline{u}, \mathcal{G}).$$

In other words, VC-ACQ is satisfied at $\overline{u}$, provided that the VC-linearized cone at $\overline{u}$ is a subset of the contingent cone to the feasible set $\mathcal{G}$ at $\overline{u}$.

Remark 6. *The following observation is noteworthy:*

1. *In view of Remark 1, Definition 9 extends the definitions of VC-ACQ presented by Tung (see [18,19,38]) to a broader class of optimization problems, namely NIMSIPVC.*

In the next theorem, we derive the KKT-type necessary optimality conditions for NIMSIPVC using VC-ACQ.

Theorem 1. *Let $\overline{u}$ be a locally weakly LU-efficient solution of NIMSIPVC and $\mathcal{K}_2 := \text{pos}(\mathcal{E}_{\Psi} \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) + \text{span}(\mathcal{E}_{\zeta} \cup \mathcal{E}_{\mathcal{Q}_1})$ be a closed set. Further, suppose that VC-ACQ is satisfied at $\overline{u}$. Then, $\overline{u}$ is a VC-stationary point of NIMSIPVC.*

Proof. From the given hypotheses, $\overline{u}$ is a locally weakly LU-efficient solution of NIMSIPVC. This implies that there exists a neighborhood V of $\overline{u}$ such that no point $u \in V \cap \mathcal{G}$ satisfies

$$\mathcal{F}_i(u) \prec^s_{LU} \mathcal{F}_i(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}} = \{1, 2, \ldots, l\}. \tag{2}$$

In other words, there is no feasible element of NIMSIPVC in the neighborhood V of $\overline{u}$ for which every component of the objective function is strictly better than the corresponding component of the objective function at $\overline{u}$.

To begin with, let us verify the following condition:

$$\left(\bigcup_{i\in\mathcal{J}^{\mathcal{F}}}\partial_c\mathcal{F}_i^L(\overline{u})\cup\partial_c\mathcal{F}_i^U(\overline{u})\right)^{\ominus}\bigcap\mathcal{T}(\overline{u},\mathcal{G})=\emptyset. \tag{3}$$

The following cases arise:
Case I. If $0\in\partial_c\mathcal{F}_i^L(\overline{u})$ or $0\in\partial_c\mathcal{F}_i^U(\overline{u})$ for at least one $i\in\mathcal{J}^{\mathcal{F}}$, then there does not exist any $\nu\in\mathbb{R}^n$ such that the following inequalities hold:

$$\begin{aligned}&\langle\xi_i^L,\nu\rangle<0,\forall\xi_i^L\in\partial_c\mathcal{F}_i^L(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}}\\&\langle\xi_i^U,\nu\rangle<0,\forall\xi_i^U\in\partial_c\mathcal{F}_i^U(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}}.\end{aligned} \tag{4}$$

That is,

$$\left(\bigcup_{i\in\mathcal{J}^{\mathcal{F}}}\partial_c\mathcal{F}_i^L(\overline{u})\cup\partial_c\mathcal{F}_i^U(\overline{u})\right)^{\ominus}=\emptyset. \tag{5}$$

In view of the fact that the intersection of an empty set with any subset of $\mathbb{R}^n$ is always an empty set, we have

$$\left(\bigcup_{i\in\mathcal{J}^{\mathcal{F}}}\partial_c\mathcal{F}_i^L(\overline{u})\cup\partial_c\mathcal{F}_i^U(\overline{u})\right)^{\ominus}\cap\mathcal{T}(\overline{u},\mathcal{G})=\emptyset. \tag{6}$$

Case II. Assume $0\notin\partial_c\mathcal{F}_i^L(\overline{u})$ and $0\notin\partial_c\mathcal{F}_i^U(\overline{u})$ for any $i\in\mathcal{J}^{\mathcal{F}}$. On the contrary, suppose that there exists $\nu\in\mathbb{R}^n$ such that $\nu\in\left(\bigcup_{i\in\mathcal{J}^{\mathcal{F}}}\partial_c\mathcal{F}_i^L(\overline{u})\cup\partial_c\mathcal{F}_i^U(\overline{u})\right)^{\ominus}\bigcap\mathcal{T}(\overline{u},\mathcal{G})$. It follows that

$$\begin{aligned}&\langle\xi_i^L,\nu\rangle<0,\ \forall\xi_i^L\in\partial_c\mathcal{F}_i^L(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}},\\&\langle\xi_i^U,\nu\rangle<0,\ \forall\xi_i^U\in\partial_c\mathcal{F}_i^U(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}}.\end{aligned} \tag{7}$$

Moreover, $\nu\in\mathcal{T}(\overline{u},\mathcal{G})$. This implies that there exist real sequences $t_m\downarrow 0,\nu_m\to\nu$ as $m\to\infty$ and $\nu_m\in\mathbb{R}^n$ such that $\overline{u}+t_m\nu_m\in\mathcal{G}$ for all $m\in\mathbb{N}$. In light of the mean value theorem from Lemma 1(v), for every $m\in\mathbb{N}$, there exists $y_m\in(\overline{u},\overline{u}+t_m\nu_m)$ and $\xi_m^L\in\partial_c\mathcal{F}_1^L(y_m)$, satisfying the following condition:

$$\mathcal{F}_1^L(\overline{u}+t_m\nu_m)-\mathcal{F}_1^L(\overline{u})=t_m\langle\xi_m^L,\nu_m\rangle, \tag{8}$$

In view of the fact that $\partial_c\Psi_1^L(y_m)$ is a compact set in $\mathbb{R}^n$, this implies that $\{\xi_m^L\}_{m=1}^{\infty}\subset\partial_c\Psi_1^L(y_m)$ is a bounded sequence in $\mathbb{R}^n$. By utilizing the upper semicontinuity of map $u\mapsto\partial_c\mathcal{F}_1^L(u)$, we obtain some subsequence $\xi_{m_k}^L$ of sequence ξ_m^L such that $\xi_{m_k}^L\to\overline{\xi}_1^L\in\partial_c\mathcal{F}_1^L(\overline{u})$. In view of (7), we infer that

$$\langle\overline{\xi}_1^L,\nu\rangle<0. \tag{9}$$

From (8), we have

$$\frac{\mathcal{F}_1^L(\overline{u}+t_{m_k}\nu_{m_k})-\mathcal{F}_1^L(\overline{u})}{t_{m_k}}=\langle\xi_{m_k}^L,\nu_{m_k}\rangle\to\langle\overline{\xi}_1^L,\nu\rangle<0. \tag{10}$$

Therefore, there exists a natural number M_1 such that

$$\mathcal{F}_1^L(\overline{u}+t_{m_k}\nu_{m_k}))<\mathcal{F}_1^L(\overline{u}),\ \forall k>M_1. \tag{11}$$

Hence, there exists a subsequence $\{\overline{u}+t_m^1\nu_m^1\}_{m=1}^{\infty}$ of the sequence $\{\overline{u}+t_m\nu_m\}_{m=1}^{\infty}$ such that

$$\mathcal{F}_1^L(\overline{u}+t_m^1\nu_m^1))<\mathcal{F}_1^L(\overline{u}). \tag{12}$$

By following similar steps, there exists a subsequence $\{\overline{u} + t_m^2 \nu_m^2\}_{m=1}^{\infty}$ of the sequence $\{\overline{u} + t_m^1 \nu_m^1\}_{m=1}^{\infty}$ such that

$$\begin{aligned} \mathcal{F}_1^L(\overline{u} + t_m^2 \nu_m^2)) < \mathcal{F}_1^L(\overline{u}), \\ \mathcal{F}_2^L(\overline{u} + t_m^2 \nu_m^2)) < \mathcal{F}_2^L(\overline{u}). \end{aligned} \tag{13}$$

Similarly, we can obtain a subsequence $\{\overline{u} + t_m^l \nu_m^l\}_{m=1}^{\infty}$ of the sequence $\{\overline{u} + t_m \nu_m\}_{m=1}^{\infty}$ such that

$$\begin{aligned} &\mathcal{F}_1^L(\overline{u} + t_m^l \nu_m^l)) < \mathcal{F}_1^L(\overline{u}), \\ &\mathcal{F}_2^L(\overline{u} + t_m^l \nu_m^l)) < \mathcal{F}_2^L(\overline{u}), \\ &\cdots \ \cdots \ \cdots \ \cdots \ \cdots\cdots \\ &\mathcal{F}_l^L(\overline{u} + t_m^l \nu_m^l) < \mathcal{F}_l^L(\overline{u}). \end{aligned} \tag{14}$$

In a similar manner, we obtain the following inequalities:

$$\begin{aligned} &\mathcal{F}_1^U(\overline{u} + t_m^l \nu_m^l)) < \mathcal{F}_1^U(\overline{u}), \\ &\mathcal{F}_2^U(\overline{u} + t_m^l \nu_m^l)) < \mathcal{F}_2^U(\overline{u}), \\ &\cdots \ \cdots \ \cdots \ \cdots \ \cdots\cdots \\ &\mathcal{F}_l^U(\overline{u} + t_m^l \nu_m^l) < \mathcal{F}_l^U(\overline{u}). \end{aligned} \tag{15}$$

In view of the definition of the contingent cone (see Definition 1), we infer that $\overline{u} + t_m^l \nu_m^l$ is a feasible element of NIMSIPVC for sufficiently large $m \in \mathbb{N}$ such that $\overline{u} + t_m^l \nu_m^l \in V$. This contradicts the fact that $\overline{u}$ is a locally weakly LU-efficient solution of NIMSIPVC. Hence,

$$\left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{u}) \cup \partial_c \mathcal{F}_i^U(\overline{u}) \right)^{\ominus} \bigcap \mathcal{T}(\overline{u}, \mathcal{G}) = \emptyset. \tag{16}$$

From the given hypothesis, VC-ACQ holds at $\overline{u}$. This implies that there does not exist any $\nu \in \mathbb{R}^n$ such that the following system of inequalities has any solution. That is,

$$\begin{aligned} &\langle \xi_i^L, \nu \rangle < 0, \ \xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{u}), \ \forall i \in \mathcal{J}^{\mathcal{F}}, \\ &\langle \xi_i^U, \nu \rangle < 0, \ \xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{u}), \ \forall i \in \mathcal{J}^{\mathcal{F}}, \\ &\langle \eta_k^{\Psi}, \ \nu \rangle \leq 0, \ \eta_k^{\Psi} \in \partial_c \Psi_k(\overline{u}), \ \forall k \in \mathbb{P}(\overline{u}), \\ &\langle \eta_i^{\zeta}, \nu \rangle = 0, \ \eta_i^{\zeta} \in \partial_c \zeta_i(\overline{u}), \ \forall i \in \mathcal{B}, \\ &\langle \eta_i^{\mathcal{Q}}, \nu \rangle = 0, \ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u}), \ \forall i \in \mathcal{H}_{0+}(\overline{u}), \\ &\langle \eta_i^{\mathcal{Q}}, \nu \rangle \geq 0, \ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u}), \ \forall i \in \mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}), \\ &\langle \eta_i^{\mathcal{R}}, \nu \rangle \leq 0, \ \eta_i^{\mathcal{H}} \in \partial_c \mathcal{R}_i(\overline{u}), \ \forall i \in \mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u}). \end{aligned} \tag{17}$$

Moreover, from Lemma 1, $\operatorname{co}\{\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{u}) \cup \partial_c \mathcal{F}_i^U(\overline{u}))\}$ is a compact set. This implies that $\operatorname{co}\{\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{u}) \cup \partial_c \mathcal{F}_i^U(\overline{u}))\} + \mathcal{K}_2$ is a closed set. From Lemma 3, it follows that

$$0 \in \operatorname{co}\left\{ \bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{u}) \cup \partial_c \mathcal{F}_i^U(\overline{u})) \right\} + \operatorname{pos}(\mathcal{E}_{\Psi} \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) + \operatorname{span}(\mathcal{E}_{\zeta} \cup \mathcal{E}_{\mathcal{Q}_1}). \tag{18}$$

Equivalently,

$$\begin{aligned} 0 \in \operatorname{co}\left\{ \bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{u}) \cup \partial_c \mathcal{F}_i^U(\overline{u})) \right\} + \operatorname{pos}(\mathcal{E}_{\Psi}) + \operatorname{pos}(\mathcal{E}_{\mathcal{Q}_2}) + \operatorname{pos}(\mathcal{E}_{\mathcal{R}_2}) + \operatorname{span}(\mathcal{E}_{\zeta}) \\ + \operatorname{span}(\mathcal{E}_{\mathcal{Q}_1}). \end{aligned} \tag{19}$$

If we set $\sigma_k^{\Psi} = 0$ $(k \in \mathcal{L} \setminus \mathbb{P}(\overline{u}))$, then from (19), there exists $(\lambda^L, \lambda^U, \sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$\begin{aligned} 0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \Big(\lambda_i^L \partial_c \mathcal{F}_i^L(\overline{u}) + \lambda_i^U \partial_c \mathcal{F}_i^U(\overline{u})\Big) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \partial_c \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \partial_c \zeta_i(\overline{u}) \\ - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{u}), \end{aligned} \tag{20}$$

with $\sum_{i \in \mathcal{J}^{\mathcal{F}}} (\lambda_i^L + \lambda_i^U) = 1, \sigma^{\mathcal{Q}}_{\mathcal{H}_+(\overline{u})} = 0, \sigma^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u})} \geq 0, \sigma^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u})} \geq 0$, and $\sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}) \cup \mathcal{H}_{0+}(\overline{u})} = 0$. □

Remark 7. *The following observations are noteworthy:*

1. *If $\mathcal{J}^{\mathcal{F}} = \{1\}$, $\mathcal{L}$ is a finite set, and $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Then, in view of Remark 2, Theorem 1 reduces to Theorem 1, as derived by Achtziger and Kanzow [1].*
2. *If $\mathcal{F}_i$ is a real-valued function for every $i \in \mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $\forall u \in \mathbb{R}^n$. Then, in view of Remark 2, Theorem 1 reduces to Proposition 3.1(ii), as derived by Tung [19].*
3. *In view of Remark 2, Theorem 1 generalizes Proposition 1(ii), as deduced by Tung [18], from smooth multiobjective SIPVC to a broader class of mathematical programming problems, specifically NIMSIPVC.*

In the following example, we illustrate the significance of KKT-type necessary optimality conditions for NIMSIPVC, as stated in Theorem 1.

Example 1. *Consider the problem $(\mathcal{P}_1)$ as follows:*

$$\begin{aligned} (\mathcal{P}_1)\ \text{Minimize } \mathcal{F}(u) &= (\mathcal{F}_1(u), \mathcal{F}_2(u)), \\ &= \left(\Big[|u_1 - 1|, |u_1 - 1| + u_2^2\Big], \Big[\frac{1}{2}(u_1 - 1)^2, \frac{1}{2}\Big((u_1 - 1)^2 + (u_2 - 1)^2\Big)\Big]\right), \\ \text{subject to } \Psi_k(u) &= (k-1)(u_2 - 1) \leq 0, k \in \mathcal{L} = [0, 1], \\ \mathcal{Q}_1(u) &= (u_1 - 1) \geq 0, \\ \mathcal{Q}_1(u)\mathcal{R}_1(u) &= (u_1 - 1)(u_2 - 1) \leq 0. \end{aligned}$$

The feasible set of the considered problem is given as follows:

$$\begin{aligned} \mathcal{G} = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 > 1,\ u_2 = 1\} \cup \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 = 1\} \\ \cup \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 > 1\}. \end{aligned}$$

Evidently, $\overline{u} = (1, 1)$ is an LU-efficient solution of $(\mathcal{P}_1)$. In particular, $\overline{u}$ is a weakly LU-efficient solution of $(\mathcal{P}_1)$.

The contingent cone to the set $\mathcal{G}$ at $\overline{u}$ is given by

$$\begin{aligned} \mathcal{T}(\overline{u}, \mathcal{G}) = \{(d_1, d_2) \in \mathbb{R}^2 | d_1 > 0,\ d_2 = 0\} \cup \{(d_1, d_2) \in \mathbb{R}^2 | d_1 = 0,\ d_2 = 0\} \\ \cup \{(d_1, d_2) \in \mathbb{R}^2 | d_1 = 0,\ d_2 > 0\}. \end{aligned}$$

The VC-linearized cone at $\overline{u}$ is given by

$$\mathcal{L}_{VC}(\overline{u}) = \{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 0,\ d_2 = 0\} \subseteq \mathcal{T}(\overline{u}, \mathcal{G}).$$

This implies that VC-ACQ is satisfied at $\overline{u}$*. Now, we have*

$$\begin{aligned}
&\partial_c \mathcal{F}_1^L(\overline{u}) = \text{co}\{(-1,0),(1,0)\}, \quad && \partial_c \mathcal{F}_1^U(\overline{u}) = \text{co}\{(-1,2),(1,2)\},\\
&\partial_c \mathcal{F}_2^L(\overline{u}) = \{(2,0)\}, && \partial_c \mathcal{F}_2^U(\overline{u}) = \{(2,2)\},\\
&\partial_c \Psi_k(\overline{u}) = \{(0,(k-1))\}, && \partial_c \mathcal{Q}_1(\overline{u}) = \{(1,0)\},\\
&\partial_c \mathcal{R}_1(\overline{u}) = \{(0,1)\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\partial_c \Psi_k(\overline{u}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_2 \geq 0\},\\
(-\partial_c \mathcal{Q}_1(\overline{u}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_1 \geq 0\},\\
(-\partial_c \mathcal{R}_1(\overline{u}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_2 \leq 0\}.
\end{aligned}$$

Therefore,

$$\mathcal{K}_2 = \text{pos}(\mathcal{E}_\Psi \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) = \mathbb{R}_+ \times \mathbb{R},$$

is a closed set. All the hypotheses stated in Theorem 1 are satisfied at $\overline{u}$*, which implies that* $\overline{u}$ *is a VC-stationary point of* $(\mathcal{P}_1)$*. That is, there exist* $\lambda_1^L = \frac{1}{4} = \lambda_2^L = \lambda_1^U = \lambda_2^U$ *and* $\overline{\sigma} = (\overline{\sigma}_k^\Psi, \overline{\sigma}_1^\mathcal{Q}, \overline{\sigma}_1^\mathcal{R})$ *such that*

$$\overline{\sigma}_k^\Psi = \begin{cases} 1, & k = 0,\\ 0, & \textit{otherwise}, \end{cases}$$

$\overline{\sigma}_1^\mathcal{Q} = \frac{1}{2}$, $\overline{\sigma}_1^\mathcal{R} = 0$*. If we choose* $\xi_1^L = (-1,0)$, $\xi_1^U = (-1,2)$, $\xi_2^L = (2,0)$, $\xi_2^U = (2,2)$, $\eta_k^\Psi = (0,k-1)$, $\eta_1^\mathcal{Q} = (1,0)$, $\eta_1^\mathcal{R} = (0,1)$*, then the following condition holds:*

$$0 = \lambda_1^L \xi_1^L + \lambda_2^L \xi_2^L + \lambda_1^U \xi_1^U + \lambda_2^U \xi_2^U + \sum_{k \in [0,1]} \overline{\sigma}_k^\Psi \eta_k - \overline{\sigma}_1^\mathcal{Q} \eta^\mathcal{Q} + \overline{\sigma}_1^\mathcal{R} \eta^\mathcal{R}.$$

Remark 8. *Here, we provide some non-trivial examples to demonstrate that the standard constraint qualifications, namely the linear independent constraint qualification (LICQ) and the Mangasarian–Fromovitz constraint qualification (MFCQ), are not satisfied at locally weakly LU-efficient solutions of NIMSIPVC:*

1. *It is worth noting that the LICQ is not satisfied for NIMSIPVC at* $\overline{u} = (1,1)$ *in Example 1. Let* $h_1(u) = (u_1 - 1)(u_2 - 1)$ *in the aforementioned example (1). Then, the Clarke subdifferentials of* h_1, Ψ_k $(k \in [0,1])$*, and* $\mathcal{Q}_1$ *at* $\overline{u}$ *are given as follows:*

$$\partial_c h_1(\overline{u}) = \{(0,0)\}, \quad \partial_c \Psi_k(\overline{u}) = \{(0,k-1)\},\ \forall k \in [0,1],\ \partial_c \mathcal{Q}_1(\overline{u}) = (1,0).$$

   Notably, $\eta_1^h \in \partial_c h_1(\overline{u})$, $\eta_1^\Psi \in \partial_c \Psi_k(\overline{u})$ $(k \in [0,1])$, $\eta_1^\mathcal{Q} \in \partial_c \mathcal{Q}_1(\overline{u})$ *are not linearly independent vectors. Hence, the LICQ is not satisfied at* $\overline{u}$*.*
2. *It is worth noting that the MFCQ is also not satisfied for NIMSIPVC at LU-efficient or weakly LU-efficient solutions. Let us consider the following example:*

$$\begin{aligned}
(\mathcal{P}_2)\ &\textit{Minimize } \mathcal{F}(u) = (\mathcal{F}_1(u), \mathcal{F}_2(u)),\\
&= \Big(\Big[|u_2 - 1|, |u_2 - 1| + (u_1 - 1)^2\Big], \Big[(u_1 - 1)^2, (u_1 - 1)^2 + (u_2 - 1)^2\Big]\Big),\\
&\textit{subject to } \Psi_k(u) = -k(u_2 - 1) \leq 0,\ k \in [0,1],\\
&\zeta_1(u) = u_1 - 1 = 0,\\
&\mathcal{Q}_1(u) = (u_1 - u_2) \geq 0,\\
&h_1(u) = \mathcal{Q}_1(u)\mathcal{R}_1(u) = (u_1 - u_2)(u_2 - 1) \leq 0.
\end{aligned}$$

   The feasible set of the considered problem is given as follows:

$$\mathcal{G} = \{(u_1,u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 = 1\}.$$

Evidently, $\overline{u} = (1,1)$ is an LU-efficient solution of $(\mathcal{P}_2)$. In particular, $\overline{u}$ is a weakly LU-efficient solution of $(\mathcal{P}_2)$.
Now, the Clarke subdifferentials of each constraint function at $\overline{u}$ are given as follows:

$$\begin{aligned} \partial_c\zeta_1(\overline{u}) &= \{(1,0)\}, \qquad \partial_c\Psi_k(\overline{u}) = \{(0,-k)\},\ \forall k \in [0,1], \\ \partial_c\mathcal{Q}_1(\overline{u}) &= \{(1,-1)\}, \qquad \partial_c h_1(\overline{u}) = \{(0,1)\}. \end{aligned}$$

Evidently, $\partial_c\zeta_1(\overline{u})$ is a linearly independent set. Suppose that there exists a vector $d = (d_1, d_2) \in \mathbb{R}^2$ such that

$$\begin{aligned} &\langle \eta_1^{\zeta}, (d_1,d_2)\rangle = 0,\ \eta_1^{\zeta} \in \partial_c\zeta_1(\overline{u}), \qquad \langle \eta_k^{\Psi}, (d_1,d_2)\rangle < 0,\ \eta_k^{\Psi} \in \partial_c\Psi_k(\overline{u}),\ k \in [0,1], \\ &\langle \eta_1^{\mathcal{Q}}, (d_1,d_2)\rangle < 0,\ \eta_1^{\mathcal{Q}} \in \partial_c\mathcal{Q}_1(\overline{u}), \qquad \langle \eta_1^{h}, (d_1,d_2)\rangle < 0,\ \eta_1^{h} \in \partial_c h_1(\overline{u}). \end{aligned} \tag{21}$$

It follows that

$$\begin{aligned} &\langle \eta_1^{\zeta}, (d_1,d_2)\rangle = d_1 = 0,\ \eta_1^{\zeta} \in \partial_c\zeta_1(\overline{u}), \\ &\langle \eta_k^{\Psi}, (d_1,d_2)\rangle = -kd_2 < 0,\ \eta_k^{\Psi} \in \partial_c\Psi_k(\overline{u}),\ k \in [0,1], \\ &\langle \eta_1^{\mathcal{Q}}, (d_1,d_2)\rangle = d_1 - d_2 < 0,\ \eta_1^{\mathcal{Q}} \in \partial_c\mathcal{Q}_1(\overline{u}), \\ &\langle \eta_1^{h}, (d_1,d_2)\rangle = d_2 < 0,\ \eta_1^{h} \in \partial_c h_1(\overline{u}). \end{aligned} \tag{22}$$

It is evident from (22) that the system of inequalities in (21) does not have any solution $d \in \mathbb{R}^2$. This demonstrates that the MFCQ is not satisfied at $\overline{u}$.

4. Interval-Valued Vector Lagrange-Type Duality Models and Saddle Points for NIMSIPVC

In this section, we formulate interval-valued vector Lagrange-type dual problems for NIMSIPVC, namely the interval-valued weak vector and interval-valued vector Lagrange-type dual problems. Further, we establish various weak, strong, and converse duality results to elucidate the relationship between the primal problem (NIMSIPVC) and its associated Lagrange-type dual problems. Moreover, this section addresses the notion of saddle points for the interval-valued vector Lagrangian of NIMSIPVC, particularly weakly LU-saddle points and LU-saddle points.

Let us formulate the interval-valued weak vector Lagrange-type dual problem for NIMSIPVC. Consider $\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$, and $e = (1,1,\ldots,1) \in \mathbb{R}^l$. Then, the interval-valued vector Lagrangian $\mathbb{L} : \mathbb{R}^n \times \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \to \mathcal{I}^l$ is defined as follows:

$$\mathbb{L}(u,\sigma) = (\mathbb{L}_1(u,\sigma), \mathbb{L}_2(u,\sigma), \ldots, \mathbb{L}_l(u,\sigma)),$$

where $\mathbb{L}_i(u,\sigma) = [\mathbb{L}_i^L(u,\sigma), \mathbb{L}_i^U(u,\sigma)]$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, and

$$\begin{aligned} \mathbb{L}_i^L(u,\sigma) &:= \mathcal{F}_i^L(u) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(u) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(u) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(u),\ \forall i \in \mathcal{J}^{\mathcal{F}}, \\ \mathbb{L}_i^U(u,\sigma) &:= \mathcal{F}_i^U(u) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(u) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(u) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(u),\ \forall i \in \mathcal{J}^{\mathcal{F}}. \end{aligned} \tag{23}$$

In other words, for every $u \in \mathbb{R}^n$ and $\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$, we have

$$\mathbb{L}(u,\sigma) = \mathcal{F}(u) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(u) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(u) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(u) \right) e, \tag{24}$$

where $e = (1,1,\ldots,1) \in \mathbb{R}^l$.

4.1. Interval-Valued Weak Vector Lagrange-Type Duality

We define an interval-valued weak vector Lagrangian dual function $\psi : \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \rightrightarrows \mathcal{I}^l$ as follows:

$$\psi(\sigma) = \text{WMin}\{\mathbb{L}(u,\sigma)|u \in \mathcal{G}\}.$$

Let $u \in \mathcal{G}$. Then, the interval-valued weak vector Lagrange-type dual problem of NIMSIPVC is formulated as follows:

$$\begin{aligned}(\text{VCD}^{\text{WVL}}(u))\ \text{WMax}\ \psi(\sigma)\\ \text{subject to}\ \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)} \geq 0,\\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)} \leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)} \geq 0.\end{aligned}$$

The feasible set of $\text{VCD}^{\text{WVL}}(u)$ is denoted by $\mathcal{G}_{\text{WVL}(u)}$ and is defined as follows:

$$\begin{aligned}\mathcal{G}_{\text{WVL}(u)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s | \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)} \geq 0,\\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)} \leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)} \geq 0\}.\end{aligned}$$

The notion of a weakly LU-efficient point of $\text{VCD}^{\text{WVL}}(u)$ is presented in the following definition by extending the corresponding definition presented by Tung et al. [47] from smooth multiobjective SIPVC to a broader class of optimization problems, namely NIMSIPVC. For more details, we refer readers to [44].

Definition 10. *An element $\overline{I} \in \bigcup_{\sigma\in\mathcal{G}_{WVL(u)}} \psi(\sigma)$ is said to be a weakly LU-efficient point of $VCD^{WVL}(u)$ if and only if*

$$\overline{I} \in WMax \bigcup_{\sigma\in\mathcal{G}_{WVL(u)}} \psi(\sigma).$$

Equivalently, there does not exist any $I \in \bigcup_{\sigma\in\mathcal{G}_{WVL(u)}} \psi(\sigma)$ such that

$$\overline{I} \prec^{s}_{LU} I.$$

Remark 9. *It is worth noting that $VCD^{WVL}(u)$ depends on the feasible point u.*

Now, we formulate the interval-valued weak vector Lagrange-type dual problem of NIMSIPVC, independent of any feasible point, as follows:

$$\begin{aligned}(\text{VCD}^{\text{WVL}})\ \text{WMax}\ \psi(\sigma)\\ \text{subject to}\ \sigma \in \mathcal{G}_{\text{WVL}} = \bigcap_{u\in\mathcal{G}} \mathcal{G}_{\text{WVL}(u)}.\end{aligned}$$

Remark 10. *One can easily see that the feasible set of VCD^{WVL} is a non-empty set. That is, $\mathcal{G}_{WVL} = \bigcap_{u\in\mathcal{G}} \mathcal{G}_{WVL(u)} \neq \emptyset$.*

In the following theorem, we derive weak duality results that relate NIMSIPVC to its corresponding Lagrange-type dual problem VCD^{WVL}.

Theorem 2. *Let u be an arbitrary element of $\mathcal{G}$ and $I \in \bigcup_{\sigma\in\mathcal{G}_{WVL(u)}} \psi(\sigma)$. Then,*

$$\mathcal{F}(u) \not\prec^{s}_{LU} I.$$

Proof. From the given hypothesis, there exists $\sigma \in \mathcal{G}_{\mathrm{WVL}(u)}$ such that $I \in \psi(\sigma)$. Therefore, we have

$$\mathcal{F}(u) + \left(\sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \right) e \not\prec_{LU}^{s} I. \tag{25}$$

On the contrary, suppose that

$$\mathcal{F}(u) \prec_{LU}^{s} I, \tag{26}$$

which implies that

$$\mathcal{F}_i^L(u) < I_i^L,\ \mathcal{F}_i^U(u) < I_i^U, \forall i \in \mathcal{J}^{\mathcal{F}}. \tag{27}$$

In view of the fact that $u \in \mathcal{G}$, we infer that

$$\Psi_k(u) \leq 0\ (k \in \mathcal{L}),\ \zeta_i(u) = 0\ (i \in \mathcal{B}),\ \mathcal{Q}_i(u) \geq 0\ (i \in \mathcal{C}).$$

Moreover, $\sigma \in \mathcal{G}_{\mathrm{WVL}(u)}$ implies that

$$\begin{aligned} \sum_{k \in \mathbb{P}(u)} \sigma_k^{\Psi} \Psi_k(u) = 0, \quad \sum_{k \in \mathcal{L} \setminus \mathbb{P}(u)} \sigma_k^{\Psi} \Psi_k(u) \leq 0, \\ \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) = 0, \\ \sum_{i \in \mathcal{H}_0(u)} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) = 0, \quad \sum_{i \in \mathcal{H}_+(u)} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) \geq 0, \\ \sum_{i \in \mathcal{H}_{0+}(u)} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \leq 0, \quad \sum_{i \in \mathcal{H}_{+-}(u) \cup \mathcal{H}_{0-}(u)} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \leq 0. \end{aligned}$$

Hence, it follows that

$$\sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \leq 0, \tag{28}$$

and hence

$$\mathcal{F}_i^L(u) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \leq \mathcal{F}_i^L(u),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}_i^U(u) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) \leq \mathcal{F}_i^U(u),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

From (27), we have

$$\mathcal{F}_i^L(u) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) < I_i^L,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}_i^U(u) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(u) < I_i^U,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

which contradicts (25). Hence, the proof of the theorem is complete. □

Remark 11. *If for every $i \in \mathcal{J}^{\mathcal{F}}$, $\mathcal{F}_i$ is a real-valued function, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ are continuously differentiable functions, then $\partial_c \mathcal{F}_i(u) = \{\nabla \mathcal{F}_i(u)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c \Psi_k(u) = \{\nabla \Psi_k(u)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(u) = \{\nabla \zeta_i(u)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(u) = \{\nabla \mathcal{Q}_i(u)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(u) = \{\nabla \mathcal{R}_i(u)\}$ $(i \in \mathcal{C})$. In this case, Theorem 2 reduces to Proposition 3.1, as derived by Tung et al. [47].*

The relationship between a weakly LU-efficient solution and a weakly LU-efficient point of NIMSIPVC and VCD^{WVL} is established in the following theorem.

Theorem 3. *Consider an arbitrary* $\overline{u} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{u})}$, *and* $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. *Then,* $\mathcal{F}(\overline{u})$ *is a weakly LU-efficient point of VCD*$^{WVL}(\overline{u})$.

Proof. On the contrary, suppose that $\mathcal{F}(\overline{u})$ is not a weakly LU-efficient point of VCD$^{\mathrm{WVL}}(\overline{u})$. This implies that there exists $I \in \psi(\overline{\sigma})$ for some $\overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{u})}$ such that

$$\mathcal{F}(\overline{u}) \prec^{s}_{LU} I. \tag{29}$$

In view of the fact that $\overline{u} \in \mathcal{G}, \overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{u})}$, and from the proof of Theorem 2, we have

$$\begin{aligned}
&\mathcal{F}^{L}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(\overline{u}) \leq \mathcal{F}^{L}_{i}(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}},\\
&\mathcal{F}^{U}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(\overline{u}) \leq \mathcal{F}^{U}_{i}(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.
\end{aligned} \tag{30}$$

From (29), we have

$$\begin{aligned}
&\mathcal{F}^{L}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(\overline{u}) < I^{L}_{i},\ \forall i \in \mathcal{J}^{\mathcal{F}},\\
&\mathcal{F}^{U}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(\overline{u}) < I^{U}_{i},\ \forall i \in \mathcal{J}^{\mathcal{F}},
\end{aligned} \tag{31}$$

which contradicts the fact that $I \in \psi(\overline{\sigma})$. Therefore, $\mathcal{F}(\overline{u})$ is a weakly LU-efficient point of VCD$^{\mathrm{WVL}}(\overline{u})$. □

In the following theorem, we derive a converse duality result that relates the primal problem (NIMSIPVC) to the corresponding interval-valued weak vector Lagrange-type dual problem VCD^{WVL}.

Theorem 4. *Let* $\overline{u} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{WVL}$, *and* $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. *Then,* $\overline{u} \in WEff$.

Proof. On the contrary, suppose that $\overline{u} \notin$ WEff. This implies that there exists $u \in \mathcal{G}$ such that

$$\mathcal{F}(u) \prec^{s}_{LU} \mathcal{F}(\overline{u}). \tag{32}$$

By following similar steps as in Theorem 2 and in view of the fact that $u \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}} \subset \mathcal{G}_{\mathrm{WVL}(u)}$, it follows that

$$\sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(u) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(u) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(u) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(u) \leq 0. \tag{33}$$

From (32) and (33), for every $i \in \mathcal{J}^{\mathcal{F}}$, we have

$$\begin{aligned}
&\mathbb{L}^{L}_{i}(u,\overline{\sigma}) := \mathcal{F}^{L}_{i}(u) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(u) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta(u) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(u) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(u) < \mathcal{F}^{L}_{i}(\overline{u}),\\
&\mathbb{L}^{U}_{i}(u,\overline{\sigma}) := \mathcal{F}^{U}_{i}(u) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_{k}(u) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_{i}(u) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_{i}(u) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_{i}(u) < \mathcal{F}^{U}_{i}(\overline{u}),
\end{aligned}$$

which contradicts the fact that $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. Therefore, $\overline{u} \in$ WEff. □

Remark 12. *Theorems 3 and 4 generalize Proposition 3.2 established by Tung et al. [47] from smooth multiobjective semi-infinite programming problems with vanishing constraints to a broader class of optimization problems, specifically NIMSIPVC.*

In the following example, we illustrate the significance of Theorems 2–4.

Example 2. *Consider the problem* $(\mathcal{P}_1)$ *from Example 1.*

The feasible set of the considered problem is given as follows:

$$\mathcal{G} = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 > 1,\ u_2 = 1\} \cup \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 = 1\} \cup \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 > 1\}.$$

For the sake of convenience, we break the feasible set into three disjoint sets as follows:

$$\mathcal{G}^1 = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 > 1,\ u_2 = 1\},$$
$$\mathcal{G}^2 = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 = 1\},$$
$$\mathcal{G}^3 = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 = 1,\ u_2 > 1\}.$$

Formulate the interval-valued vector Lagrangian for $(\mathcal{P}_1)$ *as follows:*

$$\mathbb{L}(u, \sigma) = \mathcal{F}(u) + \left(\sum_{k \in [0,1]} \sigma_k^{\Psi} \Psi_k(u) - \sigma_1^{\mathcal{Q}} \mathcal{Q}(u) + \sigma_1^{\mathcal{R}} \mathbb{R}(u) \right) e.$$

Then,

$$\mathbb{L}^L(u, \sigma) = \begin{pmatrix} |u_1 - 1| + \sum\limits_{k \in [0,1]} \sigma_k^{\Psi}(k-1)(u_2 - 1) - \sigma_1^{\mathcal{Q}}(u_2 - 1) + \sigma_1^{\mathcal{R}}(u_1 - 1) \\ \frac{1}{2}(u_1 - 1)^2 + \sum\limits_{k \in [0,1]} \sigma_k^{\Psi}(k-1)(u_2 - 1) - \sigma_1^{\mathcal{Q}}(u_2 - 1) + \sigma_1^{\mathcal{R}}(u_1 - 1) \end{pmatrix},$$

and

$$\mathbb{L}^U(u, \sigma) = \begin{pmatrix} |u_1 - 1| + u_2^2 + \sum\limits_{k \in [0,1]} \sigma_k^{\Psi}(k-1)(u_2 - 1) - \sigma_1^{\mathcal{Q}}(u_2 - 1) + \sigma_1^{\mathcal{R}}(u_1 - 1) \\ \frac{1}{2}((u_1 - 1)^2 + (u_2 - 1)^2) + \sum\limits_{k \in [0,1]} \sigma_k^{\Psi}(k-1)(u_2 - 1) - \sigma_1^{\mathcal{Q}}(u_2 - 1) + \sigma_1^{\mathcal{R}}(u_1 - 1) \end{pmatrix}.$$

Now, we define $\psi : \mathbb{R}_+^{|[0,1]|} \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^2$ *as follows:*

$$\psi(\sigma) = WMin\{\mathbb{L}(u, \sigma) | u \in \mathcal{G}\}.$$

Consider an arbitrary point $u \in \mathcal{G}^1$. *Then,* $\mathbb{P}(u) = [0, 1]$, $\mathcal{H}_{0+}(u) = \{1\}$. *The interval-valued weak vector Lagrange-type dual problem* $VCD^{WVL1}(u)$ *of* $(\mathcal{P}_1)$ *is formulated as follows:*

$$(VCD^{WVL\,1}(u))\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^{\Psi} \in \mathbb{R}_+^{|[0,1]|},\ \sigma_1^{\mathcal{R}} \in \mathbb{R},\ \sigma_1^{\mathcal{Q}} \geq 0.$$

Similarly, for $u \in \mathcal{G}^2$, *we formulate the following interval-valued weak vector Lagrange-type dual problem corresponding to* $(\mathcal{P}_1)$*:*

$$(VCD^{WVL\,2}(u))\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^{\Psi} \in \mathbb{R}_+^{|\mathcal{L}|},\ \sigma_1^{\mathcal{R}} \in \mathbb{R},\ \sigma_1^{\mathcal{Q}} \in \mathbb{R},$$

and for $u \in \mathcal{G}^3$, *the interval-valued weak vector Lagrange-type dual problem of* $(\mathcal{P}_1)$ *is given by*

$$(VCD^{WVL\,3}(u))\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^{\Psi} \in \mathbb{R}_+^{|\mathcal{L}|},\ \sigma_1^{\mathcal{R}} \leq 0,\ \sigma_1^{\mathcal{Q}} \in \mathbb{R}.$$

The interval-valued weak vector Lagrange-type dual problem, which is independent of a feasible point, is defined as follows:

$$(VCD^{WVL})\ WMax\ \psi(\sigma)$$
$$subject\ to\ \sigma^{\Psi} \in \mathbb{R}_{+}^{|[0,1]|},\ \sigma_1^{\mathcal{R}} \leq 0,\ \sigma_1^{\mathcal{Q}} \geq 0.$$

Let $\overline{u} = (1,1)$. *Then,* $\mathcal{F}(\overline{u}) = ([0,1],[0,0])$. *Let* $\overline{\sigma} = (\overline{\sigma}_k^{\Psi}, \overline{\sigma}_1^{\mathcal{Q}}, \overline{\sigma}_1^{\mathcal{R}})$ *such that*

$$\overline{\sigma}_k^{\Psi} = \begin{cases} 1, & k = 1, \\ 0, & otherwise, \end{cases}$$

$\overline{\sigma}_1^{\mathcal{Q}} = 0$, $\overline{\sigma}_1^{\mathcal{R}} = 1$.

Now,

$$\psi(\overline{\sigma}) = WMin\left\{\begin{pmatrix} \left[|u_1-1| + (u_1-1), |u_1-1| + (u_1-1) + u_2^2\right] \\ \left[\frac{1}{2}(u_1-1)^2 + (u_1-1), \frac{1}{2}((u_1-1)^2 + (u_2-2)^2) + (u_1-1)\right] \end{pmatrix} | u \in \mathcal{G}\right\}.$$

Then, one can verify that $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. *Therefore, from Theorem 3,* $\mathcal{F}(\overline{u})$ *is a weakly LU-efficient point of* $VCD^{WVL}(\overline{u})$.

Let $\overline{\sigma} = (\overline{\sigma}_k^{\Psi}, \overline{\sigma}_1^{\mathcal{Q}}, \overline{\sigma}_1^{\mathcal{R}})$ *such that* $\overline{\sigma}_k^{\Psi} = 0$, $\forall k \in [0,1]$, $\overline{\sigma}_1^{\mathcal{Q}} = 0$, $\overline{\sigma}_1^{\mathcal{R}} = 0$. *Then, one can verify that* $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. *Therefore, from Theorem 4, we conclude that* $(1,1)$ *is a weakly LU-efficient solution of problem* $(\mathcal{P}_1)$.

In the next theorem, we derive the strong duality result, which elucidates the relationship between NIMSIPVC and the interval-valued weak vector Lagrange-type dual problem.

Theorem 5. *Let* $\overline{u} \in WEff_{loc}$ *such that VC-ACQ is satisfied at* $\overline{u}$, *and let* $\mathcal{K}_2$ *be a closed set. Further, assume that* $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ *and* Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{u}) \cup \mathbb{H}_{00}^{+}(\overline{u}))$ *are LU-convex and convex at* $\overline{u}$, *respectively. Then, there exists* $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{u})}$ *such that* $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$. *Furthermore,* $\mathcal{F}(\overline{u})$ *is a weakly LU-efficient point of* $VCD^{WVL}(\overline{u})$.

Proof. Since $\overline{u} \in \mathrm{WEff}_{loc}$ and VC-ACQ is satisfied at $\overline{u}$, it follows from Theorem 1 that $\overline{u} \in \mathrm{VC}_{SP}$. Therefore, there exist $\overline{\lambda}^L, \overline{\lambda}^U \in \mathbb{R}_+^l \times \mathbb{R}_+^l$, $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$\begin{aligned} 0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \partial_c \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \partial_c \mathcal{F}_i^U(\overline{u})\right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \partial_c \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \partial_c \zeta_i(\overline{u}) \\ - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{u}), \end{aligned} \tag{34}$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L + \overline{\lambda}_i^U\right) = 1$, $\overline{\sigma}_{\mathcal{H}_+(\overline{u})}^{\mathcal{Q}} = 0$, $\overline{\sigma}_{\mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u})}^{\mathcal{Q}} \geq 0$, $\overline{\sigma}_{\mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u})}^{\mathcal{R}} \geq 0$, and $\overline{\sigma}_{\mathcal{H}_{+-}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}) \cup \mathcal{H}_{0+}(\overline{u})}^{\mathcal{R}} = 0$. This implies that there exist $\hat{\xi}_i^L \in \partial_c \mathcal{F}_i^L(\overline{u})$, $\hat{\xi}_i^U \in \partial_c \mathcal{F}_i^U(\overline{u})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\hat{\eta}_k^{\Psi} \in \partial_c \Psi_k(\overline{u})$ $(k \in \mathcal{L})$, $\hat{\eta}_i^{\zeta} \in \partial_c \zeta_i(\overline{u})$ $(i \in \mathcal{B})$, $\hat{\eta}_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u})$ $(i \in \mathcal{C})$, $\hat{\eta}_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{u})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \hat{\xi}_i^L + \overline{\lambda}_i^U \hat{\xi}_i^U\right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \hat{\eta}_k^{\Psi} + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \hat{\eta}_i^{\zeta} - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \hat{\eta}_i^{\mathcal{Q}} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \hat{\eta}_i^{\mathcal{R}} = 0. \tag{35}$$

Further, one can obtain the following:

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})=0. \tag{36}$$

Hence,

$$\mathbb{L}_i^L(\overline{u},\overline{\sigma})=\mathcal{F}_i^L(\overline{u}),\ \mathbb{L}_i^U(\overline{u},\overline{\sigma})=\mathcal{F}_i^U(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}}. \tag{37}$$

Let us assume that there exists $\tilde{u}\in\mathcal{G}$ such that

$$\mathbb{L}(\tilde{u},\overline{\sigma})\prec_{LU}^{s}\mathbb{L}(\overline{u},\overline{\sigma})=\mathcal{F}(\overline{u}). \tag{38}$$

That is,

$$\mathbb{L}_i^L(\tilde{u},\overline{\sigma})<\mathbb{L}_i^L(\overline{u},\overline{\sigma}),\quad \mathbb{L}_i^U(\tilde{u},\overline{\sigma})<\mathbb{L}_i^U(\overline{u},\overline{\sigma}),\ \forall i\in\mathcal{J}^{\mathcal{F}}. \tag{39}$$

Equivalently, for every $i\in\mathcal{J}^{\mathcal{F}}$, the following inequalities hold:

$$\begin{aligned}
&\mathcal{F}_i^L(\tilde{u})+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_i^L(\overline{u})+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),\\
&\mathcal{F}_i^U(\tilde{u})+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_i^U(\overline{u})+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}).
\end{aligned} \tag{40}$$

Multiplying both equations by $\overline{\lambda}_i^L$ and $\overline{\lambda}_i^U$, and then adding them, we obtain

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{u})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u})\\
&+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\lambda_i^U\mathcal{F}_i^U(\overline{u})-\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})-\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})\\
&+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})<0.
\end{aligned} \tag{41}$$

Now, from the LU-convexity of $\mathcal{F}_i$ $(i\in\mathcal{J}^{\mathcal{F}})$ at $\overline{u}$, we have

$$\begin{aligned}
\mathcal{F}_i^L(\tilde{u})-\mathcal{F}_i^L(\overline{u})&\geq\langle\xi_i^L,\tilde{u}-\overline{u}\rangle,\ \forall\xi_i^L\in\partial_c\mathcal{F}_i^L(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}},\\
\mathcal{F}_i^U(\tilde{u})-\mathcal{F}_i^U(\overline{u})&\geq\langle\xi_i^U,\tilde{u}-\overline{u}\rangle,\ \forall\xi_i^U\in\partial_c\mathcal{F}_i^U(\overline{u}),\ \forall i\in\mathcal{J}^{\mathcal{F}}.
\end{aligned} \tag{42}$$

Moreover, from the convexity assumptions of all the constraint functions at $\overline{u}$, we have the following inequalities:

$$\begin{aligned}
\langle\eta_k^{\Psi},\tilde{u}-\overline{u}\rangle&\leq\Psi_k(\tilde{u})-\Psi_k(\overline{u}),\ \forall\eta_k^{\Psi}\in\partial_c\Psi_k(\overline{u}),\ k\in\mathbb{P}^{\Psi}(\overline{u}),\\
\langle\eta_i^{\zeta},\tilde{u}-\overline{u}\rangle&\leq\zeta_i(\tilde{u})-\zeta_i(\overline{u})=0,\ \forall\eta_i^{\zeta}\in\partial_c\zeta_i(\overline{u}),\ i\in\mathcal{B}_+^{\zeta}(\overline{u}),\\
\langle-\eta_i^{\zeta},\tilde{u}-\overline{u}\rangle&\leq(-\zeta_i)(\tilde{u})-(-\zeta_i)(\overline{u})=0,\ \forall-\eta_i^{\zeta}\in\partial_c(-\zeta_i(\overline{u})),\ i\in\mathcal{B}_-^{\zeta}(\overline{u}),\\
\langle\eta_i^{\mathcal{Q}},\tilde{u}-\overline{u}\rangle&\leq\mathcal{Q}_i(\tilde{u})-\mathcal{Q}_i(\overline{u}),\ \forall\eta_i^{\mathcal{Q}}\in\partial_c\mathcal{Q}_i(\overline{u}),\ i\in\overline{\mathbb{H}}_{0+}^{-}(\overline{u}),\\
\langle-\eta_i^{\mathcal{Q}},\tilde{u}-\overline{u}\rangle&\leq-\mathcal{Q}_i(\tilde{u})-(-\mathcal{Q}_i(\overline{u})),\ \forall-\eta_i^{\mathcal{Q}}\in\partial_c(-\mathcal{Q}_i(u)),\ i\in\overline{\mathbb{H}}_{0+}^{+}(\overline{u})\cup\overline{\mathbb{H}}_{0-}^{+}(\overline{u})\cup\overline{\mathbb{H}}_{00}^{+}(\overline{u}),\\
\langle\eta_i^{\mathcal{R}},\tilde{u}-\overline{u}\rangle&\leq\mathcal{R}_i(\tilde{u})-\mathcal{R}_i(\overline{u}),\ \forall\eta_i^{\mathcal{R}}\in\partial_c\mathcal{R}_i(\overline{u}),\ i\in\mathbb{H}_{+0}^{+}(\overline{u})\cup\mathbb{H}_{+-}^{+}(\overline{u})\cup\mathbb{H}_{00}^{+}(\overline{u}).
\end{aligned} \tag{43}$$

On multiplying the above inequalities by $\overline{\sigma}_k^{\Psi}>0$ $(k\in\mathbb{P}^{\Psi}(\overline{u}))$, $\overline{\sigma}_i^{\zeta}>0$ $(i\in\mathcal{B}_+^{\zeta}(\overline{u}))$, $\overline{\sigma}_i^{\zeta}<0$ $(i\in\mathcal{B}_-^{\zeta}(\overline{u}))$, $\overline{\sigma}_i^{\mathcal{Q}}<0$ $(i\in\overline{\mathbb{H}}_0^{-}(\overline{u}))$, $\overline{\sigma}_i^{\mathcal{Q}}>0$ $(i\in\overline{\mathbb{H}}_{0+}^{+}(\overline{u})\cup\overline{\mathbb{H}}_{0-}^{+}(\overline{u})\cup\overline{\mathbb{H}}_{00}^{+}(\overline{u}))$, $\overline{\sigma}_i^{\mathcal{R}}>0$ $(i\in\mathbb{H}_{+0}^{+}(\overline{u})\cup\mathbb{H}_{+-}^{+}(\overline{u})\cup\mathbb{H}_{00}^{+}(\overline{u}))$, respectively, and then adding them, we obtain

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{u})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u})\\
&-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right)\\
&-\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})-\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\\
&\geq\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^U\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{u}-\overline{u}\right\rangle
\end{aligned}\tag{44}$$

From (41), we have that for every $\xi_i^L\in\partial_c\mathcal{F}_i^L(\overline{u})$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\xi_i^U\in\partial_c\mathcal{F}_i^U(\overline{u})$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\eta_k^{\Psi}\in\partial_c\Psi_k(\overline{u})$ $(k\in\mathcal{L})$, $\eta_i^{\zeta}\in\partial_c\zeta_i(\overline{u})$ $(i\in\mathcal{B})$, $\eta_i^{\mathcal{Q}}\in\partial_c\mathcal{Q}_i(\overline{u})$ $(i\in\mathcal{C})$, $\eta_i^{\mathcal{R}}\in\partial_c\mathcal{R}_i(\overline{u})$ $(i\in\mathcal{C})$,

$$\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^L\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{u}-\overline{u}\right\rangle<0,\tag{45}$$

which contradicts (35). Therefore, there does not exist any $u\in\mathcal{G}$ such that $\mathbb{L}(u,\overline{\sigma})\prec^s_{LU}\mathbb{L}(\overline{u},\overline{\sigma})$. From (37), we have that $\mathcal{F}(\overline{u})=\mathbb{L}(\overline{u},\overline{\sigma})\in\psi(\overline{\sigma})$. Furthermore, from Theorem 3, $\mathcal{F}(\overline{u})$ is a weakly LU-efficient point of $\mathrm{VCD}^{\mathrm{WVL}}(\overline{u})$. □

Remark 13. *If $\mathcal{F}_i$ is a real-valued function for every $i\in\mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u)=\mathcal{F}_i^U(u)$, $\forall i\in\mathcal{J}^{\mathcal{F}}$, $\forall u\in\mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i\in\mathcal{J}^{\mathcal{F}})$, Ψ_k $(k\in\mathcal{L})$, ζ_i $(i\in\mathcal{B})$, $\mathcal{Q}_i$ $(i\in\mathcal{C})$, $\mathcal{R}_i$ $(i\in\mathcal{C})$ are continuously differentiable functions, then Theorem 5 reduces to Proposition 3.4, as deduced by Tung et al. [47].*

Now, we provide an example to demonstrate the significance of Theorem 5.

Example 3. *Consider Example 1 and let $\overline{u}=(1,1)$. From Example 1, $\overline{u}$ is a VC-stationary point of $(\mathcal{P}_1)$. Moreover, $\mathcal{F}(\overline{u})=([0,1],[0,0])$ and $\mathcal{F}_i$ $(i=1,2)$, Ψ_k, $\mathcal{Q}_1$, $\mathcal{R}_1$ are LU-convex and convex at $\overline{u}$, respectively. Therefore, all the hypotheses in Theorem 5 are satisfied. Hence, from Theorem 5, there exists $\overline{\sigma}=(\overline{\sigma}_k^{\Psi},\overline{\sigma}_1^{\mathcal{Q}},\overline{\sigma}_1^{\mathcal{R}})$ such that*

$$\overline{\sigma}_k^{\Psi}=\begin{cases}1, & k=0,\\ 0, & \textit{otherwise},\end{cases}$$

$\overline{\sigma}_1^{\mathcal{Q}}=0$, $\overline{\sigma}_1^{\mathcal{R}}=1$ such that $\mathcal{F}(\overline{u})\in\psi(\overline{\sigma})$ and $\mathcal{F}(\overline{u})$ is a weakly LU-efficient point of $(VCD)^{WVL2}(\overline{u})$.

4.2. Interval-Valued Vector Lagrange-Type Duality

In this subsection, we formulate an interval-valued vector Lagrange-type dual problem corresponding to NIMSIPVC and further elucidate the weak and strong duality results.

Let us define a set-valued function $\psi^V:\mathbb{R}_+^{|\mathcal{L}|}\times\mathbb{R}^r\times\mathbb{R}^s\times\mathbb{R}^s\rightrightarrows\mathcal{I}^l$ as follows:

$$\psi^V(\sigma):=\mathrm{Min}\{\mathbb{L}(u,\sigma)|u\in\mathcal{G}\}.$$

Let us formulate the interval-valued vector Lagrange-type dual problem of NIMSIPVC for an arbitrary $u\in\mathcal{G}$ as follows:

$$\begin{aligned}
(\mathrm{VCD}^{\mathrm{VL}}(u))\ &\mathrm{Max}\ \psi^V(\sigma)\\
&\text{subject to } \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)}\geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)}\leq 0,\\
&\qquad\sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)}\geq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)}\geq 0.
\end{aligned}$$

The feasible set of VCD^{VL} is denoted by $\mathcal{G}_{\mathrm{VL}(u)}$ and is given by

$$\mathcal{G}_{\mathrm{VL}(u)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_{+}^{|\mathcal{L}|} \times \mathbb{R}^{r} \times \mathbb{R}^{s} \times \mathbb{R}^{s}) | \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)} \leq 0,$$
$$\sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)} \geq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)} \geq 0\}.$$

In the next definition, we extend the definition of a weakly LU-efficient point of VCD$^{VL}(u)$ from Tung et al. [47]. For further details, we refer readers to [11,44].

Definition 11. *An interval-valued vector $\overline{I} \in \bigcup_{\sigma\in\mathcal{G}_{VL(u)}} \psi^{V}(\sigma)$ is said to be an LU-efficient point of VCD$^{VL}(u)$ if*

$$\overline{I} \in Max \bigcup_{\sigma\in\mathcal{G}_{VL(u)}} \psi^{V}(\sigma).$$

Equivalently, there does not exist any $I \in \bigcup_{\sigma\in\mathcal{G}_{VL(u)}} \psi^{V}(\sigma)$ such that

$$\overline{I} \prec_{LU} I.$$

Remark 14. *It is worth noting that VCD$^{VL}(u)$ depends on the feasible point u.*

Now, we propose the interval-valued vector Lagrange-type dual problem for NIMSIPVC, which is independent of the choice of a feasible element, as follows:

$$(\mathrm{VCD}^{\mathrm{VL}})\ \mathrm{Max}\ \psi^{V}(\sigma)$$
$$\text{subject to } \sigma \in \mathcal{G}_{\mathrm{VL}} = \bigcap_{u\in\mathcal{G}} \mathcal{G}_{\mathrm{VL}(u)}.$$

Remark 15. *One can easily see that the feasible region of VCD^{VL} is always non-empty i.e., $\mathcal{G}_{VL} = \bigcap_{u\in\mathcal{G}} \mathcal{G}_{VL(u)} \neq \emptyset$.*

In the following theorem, we establish the weak duality result that elucidates the relationship between NIMSIPVC and VCD^{VL}. The proof is analogous to the proof of Theorem 2 so we omit it.

Theorem 6. *Let u be an arbitrary element of $\mathcal{G}$ and $I \in \bigcup_{\sigma\in\mathcal{G}_{VL(u)}} \psi^{V}(\sigma)$. Then,*

$$\mathcal{F}(u) \not\prec_{LU} I.$$

Remark 16. *Theorem 6 generalizes Proposition 3.6, as derived by Tung et al. [47], from smooth multiobjective SIPVC to NIMSIPVC, which belongs to a broader category of optimization problems.*

In the following theorem, we derive the relationship between a feasible point of NIMSIPVC and an LU-efficient point of VCD^{VL}. The proof is analogous to the proof of Theorem 3 so we omit it.

Theorem 7. *Consider an arbitrary $\overline{u} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{VL(\overline{u})}$ and $\mathcal{F}(\overline{u}) \in \psi^{V}(\overline{\sigma})$. Then, $\mathcal{F}(\overline{u})$ is an LU-efficient point of VCD$^{VL}(\overline{u})$.*

In the following theorem, we derive the converse duality result that relates our primal problem (NIMSIPVC) to the corresponding interval-valued vector Lagrange-type dual problem VCD^{VL}. The proof is analogous to the proof of Theorem 4 so we omit it.

Theorem 8. *Let $\overline{u} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{VL}$, and $\mathcal{F}(\overline{u}) \in \psi^{V}(\overline{\sigma})$. Then, $\overline{u} \in Eff$.*

Remark 17. *Theorems 7 and 8 generalize Proposition 3.7 deduced by Tung et al. [47] from smooth multiobjective SIPVC to NIMSIPVC, which belongs to a more general category of optimization problems.*

In the following theorem, we derive the strong duality result relating NIMSIPVC to the interval-valued vector Lagrange-type dual problem of NIMSIPVC.

Theorem 9. *Let $\overline{u} \in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{u}$ and let $\mathcal{K}_2$ be a closed set. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ and Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}^{\zeta}_{+})$, $-\zeta_i$ $(i \in \mathcal{B}^{\zeta}_{-})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{u}))$,$-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}^{+}_{+0}(\overline{u}) \cup \mathbb{H}^{+}_{00}(\overline{u}))$ are strictly LU-convex and convex at $\overline{u}$, respectively. Then, there exists $\overline{\sigma} \in \mathcal{G}_{VL(\overline{u})}$ such that $\mathcal{F}(\overline{u}) \in \psi^{V}(\overline{\sigma})$. Furthermore, $\mathcal{F}(\overline{u})$ is an LU-efficient point of $VCD^{VL}(\overline{u})$.*

Proof. Following similar lines as in the proof of Theorem 5, we obtain

$$\sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}) = 0. \tag{46}$$

Hence, we have

$$\mathbb{L}^{L}_{i}(\overline{u},\overline{\sigma}) = \mathcal{F}^{L}_{i}(\overline{u}), \quad \mathbb{L}^{U}_{i}(\overline{u},\overline{\sigma}) = \mathcal{F}^{U}_{i}(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}}. \tag{47}$$

Let us assume that there exists some $\tilde{u} \in \mathcal{G}$ such that

$$\mathbb{L}(\tilde{u},\overline{\sigma}) \prec_{LU} \mathbb{L}(\overline{u},\overline{\sigma}). \tag{48}$$

This implies that

$$\mathbb{L}^{L}_{i}(\tilde{u},\overline{\sigma}) \leq \mathbb{L}^{L}_{i}(\overline{u},\overline{\sigma}),\ \mathbb{L}^{U}_{i}(\tilde{u},\overline{\sigma}) \leq \mathbb{L}^{U}_{i}(\overline{u},\overline{\sigma}),\ \forall i \in \mathcal{J}^{\mathcal{F}}, \tag{49}$$

and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, exactly one of the following relations holds:

$$\begin{cases} \mathbb{L}^{L}_{p}(\tilde{u},\overline{\sigma}) < \mathbb{L}^{L}_{p}(\overline{u},\overline{\sigma}) \\ \mathbb{L}^{U}_{p}(\tilde{u},\overline{\sigma}) \leq \mathbb{L}^{U}_{p}(\overline{u},\overline{\sigma}) \end{cases} \text{or} \begin{cases} \mathbb{L}^{L}_{p}(\tilde{u},\overline{\sigma}) \leq \mathbb{L}^{L}_{p}(\overline{u},\overline{\sigma}) \\ \mathbb{L}^{U}_{p}(\tilde{u},\overline{\sigma}) < \mathcal{L}^{U}_{p}(\overline{u},\overline{\sigma}) \end{cases} \text{or} \quad \begin{cases} \mathbb{L}^{L}_{p}(\tilde{u},\overline{\sigma}) < \mathbb{L}^{L}_{p}(\overline{u},\overline{\sigma}) \\ \mathbb{L}^{U}_{p}(\tilde{u},\overline{\sigma}) < \mathbb{L}^{U}_{p}(\overline{u},\overline{\sigma}). \end{cases}$$

Equivalently, for every $i \in \mathcal{J}^{\mathcal{F}}$, the following inequalities hold:

$$\begin{aligned} &\mathcal{F}^{L}_{i}(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\tilde{u}) \\ \leq &\mathcal{F}^{L}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}), \\ &\mathcal{F}^{U}_{i}(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\tilde{u}) \\ \leq &\mathcal{F}^{U}_{i}(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}), \end{aligned} \tag{50}$$

and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, exactly one of the following conditions hold:

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),\\
&\mathcal{F}_p^U(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
\leq&\mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),
\end{aligned} \tag{51}$$

or

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
\leq&\mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),\\
&\mathcal{F}_p^U(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),
\end{aligned} \tag{52}$$

or

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),\\
&\mathcal{F}_p^U(\tilde{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
<&\mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}).
\end{aligned} \tag{53}$$

On multiplying by $\overline{\lambda}_i^L,\ \overline{\lambda}_i^U \geq 0\ (i \in \mathcal{J}^{\mathcal{F}})$ such that $\sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L + \overline{\lambda}_i^U\right) = 1$, we have

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\tilde{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\tilde{u})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
&- \sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{u})\right) - \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) - \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\\
&\leq 0.
\end{aligned} \tag{54}$$

In view of the fact that $\mathcal{F}_i^L,\ \mathcal{F}_i^U, \forall i \in \mathcal{J}^{\mathcal{F}}$ are strictly LU-convex at $\overline{u}$, we have

$$\begin{aligned}
\mathcal{F}_i^L(\tilde{u}) - \mathcal{F}_i^L(\overline{u}) &> \langle \xi_i^L, \tilde{u} - \overline{u}\rangle,\ \forall \xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}}\\
\mathcal{F}_i^U(\tilde{u}) - \mathcal{F}_i^U(\overline{u}) &> \langle \xi_i^U, \tilde{u} - \overline{u}\rangle,\ \forall \xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{u}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.
\end{aligned} \tag{55}$$

Following similar steps as in Theorem 5, along with the convexity assumptions of all the constraint functions, we obtain

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\tilde{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\tilde{u})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})\\
&- \sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{u})\right) - \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) - \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\\
&> \left\langle \sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^U \xi_i^U\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\eta_k^{\Psi} + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\eta_i^{\zeta} - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}} + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}}, \tilde{u} - \overline{u} \right\rangle.
\end{aligned} \tag{56}$$

From (54) and (56), we obtain that for every $\xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{u})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{u})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\eta_k^{\Psi} \in \partial_c \Psi_k(\overline{u})$ $(k \in \mathcal{L})$, $\eta_i^{\zeta} \in \partial_c \zeta_i(\overline{u})$ $(i \in \mathcal{B})$, $\eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{u})$ $(i \in \mathcal{C})$, $\eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{u})$ $(i \in \mathcal{C})$,

$$\left\langle \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^U \xi_i^U \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \eta_k^{\Psi} + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \eta_i^{\zeta} - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \eta_i^{\mathcal{Q}} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \eta_i^{\mathcal{R}}, \tilde{u} - \overline{u} \right\rangle < 0, \quad (57)$$

which contradicts the fact that $\overline{u} \in \mathrm{VC}_{SP}$. From Theorem 7, $\mathcal{F}(\overline{u})$ is an LU-efficient point of $\mathrm{VCD}^{\mathrm{VL}}(\overline{u})$. Furthermore, from Theorem 8, we conclude that $\overline{u} \in \mathrm{Eff}$. This completes the proof. □

Remark 18. *Theorem 9 generalizes Proposition 3.8, as deduced by Tung et al. [47], from the smooth case of multiobjective SIPVC to nonsmooth SIPVC, including multiple interval-valued objective functions.*

4.3. Interval-Valued Vector Saddle-Point Optimality Criteria

In this subsection, we introduce the notion of LU-saddle points for the interval-valued vector Lagrangian of NIMSIPVC, specifically weakly LU-saddle points and LU-saddle points. Further, we establish several relationships between the efficient solutions of NIMSIPVC and the saddle points for the interval-valued vector Lagrangian of NIMSIPVC.

In the next definition, we extend the notion of saddle points for the interval-valued vector Lagrangian of NIMSIPVC, as presented by Tung et al. [47], to the vector Lagrangian of smooth multiobjective SIPVC.

Definition 12. *Let $\overline{u} \in \mathcal{G}$ and $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{u})}$ be arbitrary elements. Then, $(\overline{u}, \overline{\sigma})$ is known as follows:*

(i) A weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC if the following condition holds:

$$\mathbb{L}(u, \overline{\sigma}) \not\prec_{LU}^{s} \mathbb{L}(\overline{u}, \overline{\sigma}) \not\prec_{LU}^{s} \mathbb{L}(\overline{u}, \sigma), \ \forall u \in \mathcal{G}, \forall \sigma \in \mathcal{G}_{WVL(\overline{u})}.$$

(ii) An LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC if the following condition holds:

$$\mathbb{L}(u, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u}, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u}, \sigma), \ \forall u \in \mathcal{G}, \forall \sigma \in \mathcal{G}_{VL(\overline{u})}.$$

The symbols $WS^{\mathbb{L}}$ and $S^{\mathbb{L}}$ denote the sets of all weakly LU-saddle points and LU-saddle points for the interval-valued vector Lagrangian of NIMSIPVC, respectively.

Remark 19. *It is worth noting that*

$$S^{\mathbb{L}} \subseteq WS^{\mathbb{L}}.$$

In the following theorem, we derive the relationship between a weakly LU-efficient solution and a weakly LU-saddle point of NIMSIPVC and its interval-valued vector Lagrangian of NIMSIPVC, respectively.

Theorem 10. *Let $\overline{u} \in WEff$ such that VC-ACQ is satisfied at $\overline{u}$, and let $\mathcal{K}_2$ be a closed set. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ and Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{u}) \cup \mathbb{H}_{00}^{+}(\overline{u}))$ are LU-convex and convex at $\overline{u}$, respectively. Then, there exists $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{u})}$ such that $(\overline{u}, \overline{\sigma}) \in WS^{\mathbb{L}}$.*

Proof. From Theorem 5, there exists $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{u})}$ such that

$$\mathcal{F}(\overline{u}) = \mathbb{L}(\overline{u}, \overline{\sigma}), \quad (58)$$

and

$$\mathbb{L}(u,\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{u},\overline{\sigma}),\ \forall u \in \mathcal{G}. \tag{59}$$

We claim that

$$\mathbb{L}(\overline{u},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{u},\sigma),\ \forall \sigma \in \mathcal{G}_{WVL(\overline{u})}. \tag{60}$$

On the contrary, suppose that there exists $\sigma \in \mathcal{G}_{WVL(\overline{u})}$ such that

$$\mathbb{L}(\overline{u},\overline{\sigma}) \prec^{s}_{LU} \mathbb{L}(\overline{u},\sigma),\ \forall \sigma \in \mathcal{G}_{WVL(\overline{u})}. \tag{61}$$

From (58), we obtain

$$\mathcal{F}(\overline{u}) \prec^{s}_{LU} \mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e. \tag{62}$$

Furthermore, we can rewrite the above inequality for every $i \in \mathcal{J}^{\mathcal{F}}$ as follows:

$$\begin{aligned}
\mathcal{F}_i^{L}(\overline{u}) &< \mathcal{F}_i^{L}(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}),\\
\mathcal{F}_i^{U}(\overline{u}) &< \mathcal{F}_i^{U}(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}).
\end{aligned} \tag{63}$$

It follows that

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) > 0. \tag{64}$$

In view of the fact that $\overline{u} \in \mathcal{G}$ and $\sigma \in \mathcal{G}_{WVL(\overline{u})}$, we have

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) \leq 0, \tag{65}$$

which contradicts (64). Therefore, there does not exist any $\sigma \in \mathcal{G}_{WVL(\overline{u})}$ such that

$$\mathbb{L}(\overline{u},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{u},\sigma), \forall \sigma \in \mathcal{G}_{WVL(\overline{u})}. \tag{66}$$

Therefore, from (59) and (66), $(\overline{u},\overline{\sigma}) \in \mathrm{WS}^{\mathbb{L}}$. This completes the proof. □

Remark 20. *Theorem 10 generalizes Proposition 3.10(i), as derived by Tung et al. [47], from smooth multiobjective semi-infinite programming problems with vanishing constraints to a broader class of optimization problems, specifically NIMSIPVC.*

In the following theorem, we establish the relationship between a weakly LU-saddle point and a weakly LU-efficient point of the interval-valued vector Lagrangian for NIMSIPVC and $\mathrm{VCD}^{\mathrm{WVL}}(\overline{u})$, respectively.

Theorem 11. *Let $(\overline{u},\overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{WVL(\overline{u})}$ be a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. Then, $\mathcal{F}(\overline{u}) \in \psi(\overline{\sigma})$, where $\mathcal{F}(\overline{u})$ is a weakly LU-efficient point of $VCD^{WVL}(\overline{u})$.*

Proof. From the given hypothesis, $(\overline{u},\overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{\mathrm{WVL}(\overline{u})}$ is a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. It follows that

$$\mathbb{L}(\overline{u},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{u},\sigma),\ \forall \sigma \in \mathcal{G}_{\mathrm{WVL}(\overline{u})}. \tag{67}$$

Equivalently,

$$\begin{aligned}\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \\ \not<^s_{LU} \mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e.\end{aligned} \tag{68}$$

If we assume that $\sigma = 0$, then (68) can be rewritten as follows:

$$\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \not<^s_{LU} \mathcal{F}(\overline{u}). \tag{69}$$

Moreover, by following similar steps as in Theorem 2, we deduce that

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) \leq 0. \tag{70}$$

If

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) < 0, \tag{71}$$

then

$$\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \prec^s_{LU} \mathcal{F}(\overline{u}), \tag{72}$$

which contradicts (69). Therefore, we have

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) = 0. \tag{73}$$

This implies that $\mathbb{L}(\overline{u},\overline{\sigma}) = \mathcal{F}(\overline{u})$. In view of the given hypothesis, $(\overline{u},\overline{\sigma})$ is a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. It follows that

$$\mathbb{L}(u,\overline{\sigma}) \not<^s_{LU} \mathbb{L}(\overline{u},\overline{\sigma}),\ \forall u \in \mathcal{G}. \tag{74}$$

This demonstrates that $\mathcal{F}(\overline{u}) = \mathbb{L}(\overline{u},\overline{\sigma}) \in \mathrm{WMin}\{\mathbb{L}(u,\overline{\sigma})|u \in \mathcal{G}\} = \psi(\overline{\sigma})$. In view of Theorem 3, $\mathcal{F}(\overline{u})$ is a weakly LU-efficient point of $\mathrm{VCD}^{\mathrm{WVL}}(\overline{u})$. □

Remark 21. *If $\mathcal{F}_i$ is a real-valued function for every $i \in \mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ are continuously differentiable functions, then $\partial_c\mathcal{F}_i(u) = \{\nabla\mathcal{F}_i(u)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(u) = \{\nabla\Psi_k(u)\}$ $(k \in \mathcal{L})$, $\partial_c\zeta_i(u) = \{\nabla\zeta_i(u)\}$ $(i \in \mathcal{B})$, $\partial_c\mathcal{Q}_i(u) = \{\nabla\mathcal{Q}_i(u)\}$ $(i \in \mathcal{C})$, $\partial_c\mathcal{R}_i(u) = \{\nabla\mathcal{R}_i(u)\}$ $(i \in \mathcal{C})$. In this case, Theorem 11 reduces to Proposition 3.10(ii), as derived by [47].*

In the following theorem, we establish the relationship between a VC-stationary point and a weakly LU-saddle point of NIMSIPVC and the interval-valued vector Lagrangian of NIMSIPVC, respectively.

Theorem 12. *Let $\overline{u} \in VC_{SP}$. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ and Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{u}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{u}) \cup \mathbb{H}_{00}^{+}(\overline{u}))$ are LU-convex and convex at $\overline{u}$. Then, there exists $\overline{\sigma} = (\overline{\sigma}^{\Psi},\overline{\sigma}^{\zeta},\overline{\sigma}^{\mathcal{Q}},\overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that $(\overline{u},\overline{\sigma}) \in WS^{\mathbb{L}}$.*

Proof. In view of the fact that $\overline{u} \in \mathrm{VC}_{\mathrm{SP}}$, there exist $(\overline{\lambda}^L, \overline{\lambda}^U) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+$, $\overline{\sigma} = (\overline{\sigma}^\Psi, \overline{\sigma}^\zeta, \overline{\sigma}^\mathcal{Q}, \overline{\sigma}^\mathcal{R}) \in \mathbb{P}^\Psi(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$\begin{aligned} 0 \in \sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}^L_i \partial_c \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \partial_c \mathcal{F}^U_i(\overline{u}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \partial_c \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \partial_c \zeta_i(\overline{u}) \\ - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \partial_c \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \partial_c \mathcal{R}_i(\overline{u}), \end{aligned} \tag{75}$$

where $\sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}^L_i + \overline{\lambda}^U_i \right) = 1$, $\overline{\sigma}^\mathcal{Q}_{\mathcal{H}_+(\overline{u})} = 0$, $\overline{\sigma}^\mathcal{Q}_{\mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u})} \geq 0$, $\overline{\sigma}^\mathcal{R}_{\mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u})} \geq 0$, and $\overline{\sigma}^\mathcal{R}_{\mathcal{H}_{+-}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}) \cup \mathcal{H}_{0+}(\overline{u})} = 0$. This implies that there exist $\hat{\xi}^L_i \in \partial_c \mathcal{F}^L_i(\overline{u})$, $\hat{\xi}^U_i \in \partial_c \mathcal{F}^U_i(\overline{u})$ $(i \in \mathcal{J}^\mathcal{F})$, $\hat{\eta}^\Psi_k \in \partial_c \Psi_k(\overline{u})$ $(k \in \mathcal{L})$, $\hat{\eta}^\zeta_i \in \partial_c \zeta_i(\overline{u})$ $(i \in \mathcal{B})$, $\hat{\eta}^\mathcal{Q}_i \in \partial_c \mathcal{Q}_i(\overline{u})$ $(i \in \mathcal{C})$, $\hat{\eta}^\mathcal{R}_i \in \partial_c \mathcal{R}_i(\overline{u})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}^L_i \hat{\xi}^L_i + \overline{\lambda}^U_i \hat{\xi}^U_i \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \hat{\eta}^\Psi_k + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \hat{\eta}^\zeta_i - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \hat{\eta}^\mathcal{Q}_i + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \hat{\eta}^\mathcal{R}_i = 0. \tag{76}$$

We divide the main proof into two parts:

(a) We claim that

$$\mathbb{L}(u, \overline{\sigma}) \not\prec^s_{LU} \mathbb{L}(\overline{u}, \overline{\sigma}),\ \forall u \in \mathcal{G},\ \overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{u})}. \tag{77}$$

On the contrary, suppose that there exists $\tilde{u} \in \mathcal{G}$ such that

$$\mathbb{L}(\tilde{u}, \overline{\sigma}) \prec^s_{LU} \mathbb{L}(\overline{u}, \overline{\sigma}). \tag{78}$$

This implies that for every $i \in \mathcal{J}^\mathcal{F}$, the following inequalities hold:

$$\begin{aligned} &\mathcal{F}^L_i(\tilde{u}) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\tilde{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\tilde{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\tilde{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\tilde{u}) \\ <&\mathcal{F}^L_i(\overline{u}) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\overline{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\overline{u}), \\ &\mathcal{F}^U_i(\tilde{u}) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\tilde{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\tilde{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\tilde{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\tilde{u}) \\ <&\mathcal{F}^U_i(\overline{u}) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\overline{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\overline{u}). \end{aligned} \tag{79}$$

Let us multiply the first and second inequalities by $\overline{\lambda}^L_i$ and $\overline{\lambda}^U_i$, respectively. On adding them, we obtain

$$\begin{aligned} &\sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\tilde{u}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\tilde{u}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\tilde{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\tilde{u}) \\ &- \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\tilde{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\tilde{u}) - \sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{u}) \right) \\ &- \sum_{k \in \mathcal{L}} \overline{\sigma}^\Psi_k \Psi_k(\overline{u}) - \sum_{i \in \mathcal{B}} \overline{\sigma}^\zeta_i \zeta_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{Q}_i \mathcal{Q}_i(\overline{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^\mathcal{R}_i \mathcal{R}_i(\overline{u}) \\ &< 0. \end{aligned} \tag{80}$$

Following similar steps as in the proof of Theorem 5, we have

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{u})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{u})\\
&-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{u})-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right)\\
&-\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})-\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\\
\geq&\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^U\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{u}-\overline{u}\right\rangle.
\end{aligned}\tag{81}$$

From (80) and (81), we have that for every $\xi_i^L\in\partial_c\mathcal{F}_i^L(\overline{u})$, $\xi_i^U\in\partial_c\mathcal{F}_i^U(\overline{u})$, $\eta_k^{\Psi}\in\partial_c\Psi_k(\overline{u})$,$\eta_i^{\zeta}\in\partial_c\zeta_i(\overline{u})$, $\eta_i^{\mathcal{Q}}\in\partial_c\mathcal{Q}_i(\overline{u})$, $\eta_i^{\mathcal{R}}\in\partial_c\mathcal{R}_i(\overline{u})$,

$$\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^L\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{u}-\overline{u}\right\rangle<0,\tag{82}$$

which contradicts the fact that $\overline{u}\in\mathrm{VC}_{SP}$. Therefore,

$$\mathbb{L}(u,\overline{\sigma})\not\prec_{LU}^{s}\mathbb{L}(\overline{u},\overline{\sigma}),\ \forall u\in\mathcal{G}.\tag{83}$$

(b) In this part, we claim that

$$\mathbb{L}(\overline{u},\overline{\sigma})\not\prec_{LU}^{s}\mathbb{L}(\overline{u},\sigma),\ \forall\sigma\in\mathcal{G}_{WVL(\overline{u})}.\tag{84}$$

On the contrary, suppose that there exists $\sigma\in\mathcal{G}_{\mathrm{WVL}(\overline{u})}$ such that

$$\mathbb{L}(\overline{u},\overline{\sigma})\prec_{LU}^{s}\mathbb{L}(\overline{u},\sigma).\tag{85}$$

From (85), it follows that

$$\begin{aligned}
&\mathcal{F}(\overline{u})+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e\\
\prec_{LU}^{s}&\mathcal{F}(\overline{u})+\left(\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e.
\end{aligned}\tag{86}$$

In view of the fact that $\overline{u}\in\mathrm{VC}_{SP}$, we obtain

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})=0.\tag{87}$$

Hence, from (86) and (87), we deduce that

$$\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})>0.\tag{88}$$

Since $\overline{u}\in\mathcal{G}$ and $\sigma\in\mathcal{G}_{WVL(\overline{u})}$, we infer that

$$\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\leq0,\tag{89}$$

which contradicts (88). Therefore,

$$\mathbb{L}(\overline{u},\overline{\sigma})\not\prec_{LU}^{s}\mathbb{L}(\overline{u},\sigma),\ \forall\sigma\in\mathcal{G}_{WVL(\overline{u})}.\tag{90}$$

From (83) and (90), we can conclude that $(\overline{u},\overline{\sigma})\in\mathrm{WS}^{\mathbb{L}}$.

□

Remark 22. *Theorem 12 generalizes Proposition 3.11, as derived by [47], to a general category of nonsmooth multiobjective optimization problems, specifically NIMSIPVC.*

Now, the following example illustrates the significance of Theorem 12.

Example 4. *Consider the Problem* $(\mathcal{P}_1)$ *in Example 1.*

In Example 1, $\overline{u}$ *is a VC-stationary point of* $(\mathcal{P}_1)$*. Furthermore, one can observe that* $\mathcal{F}_i$ $(i = 1, 2)$, Ψ_k $(k \in [0, 1])$, $\mathcal{Q}$, *and* $\mathcal{R}$ *are LU-convex and convex at* $\overline{u}$*, respectively. Therefore, all the hypotheses in Theorem 12 are satisfied at* $\overline{u}$*, which implies that* $(\overline{u}, \overline{\sigma})$ *is a weakly LU-saddle point for the interval-valued vector Lagrangian of* $(\mathcal{P}_1)$*.*

In the following theorem, we establish a relationship between an LU-weakly local efficient solution of NIMSIPVC and an LU-saddle point of interval-valued vector Lagrangian of NIMSIPVC.

Theorem 13. *Let* $\overline{u} \in WEff_{loc}$ *such that VC-ACQ is satisfied at* $\overline{u}$*, and let* $\mathcal{K}_2$ *be a closed set. Further, assume that* $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ *and* Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u})), \zeta_i$ $(i \in \mathcal{B}^{\zeta}_{+}), -\zeta_i$ $(i \in \mathcal{B}^{\zeta}_{-})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}^{+}_{+0}(\overline{u}) \cup \mathbb{H}^{+}_{00}(\overline{u}))$ *are LU-convex and convex at* $\overline{u}$*, respectively. Then, there exists* $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ *such that* $(\overline{u}, \overline{\sigma}) \in S^{\mathbb{L}}$.

Proof. From the given hypotheses, $\overline{u} \in \mathrm{WEff}_{loc}$ such that VC-ACQ is satisfied at $\overline{u}$, and $\mathcal{K}_2$ is a closed set. Therefore, from Theorem 1, $\overline{u} \in VC_{SP}$. This implies that there exists $\overline{\sigma} = (\overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \partial_c \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \partial_c \mathcal{F}^U_i(\overline{u}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \partial_c \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \partial_c \zeta_i(\overline{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \partial_c \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \partial_c \mathcal{R}_i(\overline{u}), \tag{91}$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i + \overline{\lambda}^U_i \right) = 1$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_+(\overline{u})} = 0$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u})} \geq 0$, $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{u}) \cup \mathcal{H}_{00}(\overline{u})} \geq 0$, and $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{u}) \cup \mathcal{H}_{0-}(\overline{u}) \cup \mathcal{H}_{0+}(\overline{u})} = 0$. Therefore, $\overline{\sigma} \in \mathcal{G}_{VL(\overline{u})}$.

Moreover, there exist $\hat{\xi}^L_i \in \partial_c \mathcal{F}^L_i(\overline{u})$, $\hat{\xi}^U_i \in \partial_c \mathcal{F}^U_i(\overline{u})$ $(i \in \mathcal{J}^{\mathcal{F}}), \hat{\eta}^{\Psi}_k \in \partial_c \Psi_k(\overline{u})$ $(k \in \mathcal{L})$, $\hat{\eta}^{\zeta}_i \in \partial_c \zeta_i(\overline{u})$ $(i \in \mathcal{B})$, $\hat{\eta}^{\mathcal{Q}}_i \in \partial_c \mathcal{Q}_i(\overline{u})$ $(i \in \mathcal{C})$, $\hat{\eta}^{\mathcal{R}}_i \in \partial_c \mathcal{R}_i(\overline{u})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \hat{\xi}^L_i + \overline{\lambda}^U_i \hat{\xi}^U_i \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \hat{\eta}^{\Psi}_k + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \hat{\eta}^{\zeta}_i - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \hat{\eta}^{\mathcal{Q}}_i + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \hat{\eta}^{\mathcal{R}}_i = 0. \tag{92}$$

Evidently, in view of the fact that $(\overline{u}, \overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{u})}$, we have

$$\mathcal{L}(\overline{u}, \overline{\sigma}) = \mathcal{F}(\overline{u}). \tag{93}$$

From Theorem 9, $\mathcal{F}(\overline{u}) \in \psi^V(\overline{\sigma})$, which yields the following equation:

$$\mathbb{L}(u, \overline{\sigma}) \not\prec_{LU} \mathcal{F}(\overline{u}) = \mathbb{L}(\overline{u}, \overline{\sigma}), \ \forall u \in \mathcal{G}. \tag{94}$$

We now claim that

$$\mathbb{L}(\overline{u}, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u}, \sigma), \ \forall \sigma \in \mathcal{G}_{VL(\overline{u})}. \tag{95}$$

On the contrary, suppose that there exists $\sigma \in \mathcal{G}_{\mathrm{VL}(\overline{u})}$ such that

$$\mathbb{L}(\overline{u}, \overline{\sigma}) \prec_{LU} \mathbb{L}(\overline{u}, \sigma). \tag{96}$$

This implies that for every $i \in \mathcal{J}^{\mathcal{F}}$, we have

$$\begin{aligned} \mathcal{F}_i^L(\overline{u}) &\leq \mathcal{F}_i^L(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sigma_i^{\mathcal{R}} \sum_{i\in\mathcal{C}} \mathcal{R}_i(\overline{u}), \\ \mathcal{F}_i^U(\overline{u}) &\leq \mathcal{F}_i^U(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \end{aligned} \tag{97}$$

and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, exactly one of the following relations holds:

$$\begin{aligned} \mathcal{F}_p^L(\overline{u}) &< \mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \\ \mathcal{F}_p^U(\overline{u}) &\leq \mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \end{aligned} \tag{98}$$

or

$$\begin{aligned} \mathcal{F}_p^L(\overline{u}) &\leq \mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \\ \mathcal{F}_p^U(\overline{u}) &< \mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \end{aligned} \tag{99}$$

or

$$\begin{aligned} \mathcal{F}_p^L(\overline{u}) &< \mathcal{F}_p^L(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}), \\ \mathcal{F}_p^U(\overline{u}) &< \mathcal{F}_p^U(\overline{u}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}). \end{aligned} \tag{100}$$

Therefore, from (97), (98), (99), and (100), we have

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) > 0. \tag{101}$$

However, if $(\overline{u}, \sigma) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{u})}$, it follows that

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) \leq 0, \tag{102}$$

which contradicts (101). Therefore,

$$\mathbb{L}(\overline{u}, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u}, \sigma),\ \forall \sigma \in \mathcal{G}_{VL(\overline{u})}. \tag{103}$$

From (94) and (103), we conclude that $(\overline{u}, \overline{\sigma}) \in S^{\mathbb{L}}$. This completes the proof. □

Remark 23. *If $\mathcal{F}_i$ is a real-valued function for every $i \in \mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ are continuously differentiable functions, then $\partial_c \mathcal{F}_i(u) = \{\nabla \mathcal{F}_i(u)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c \Psi_k(u) = \{\nabla \Psi_k(u)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(u) = \{\nabla \zeta_i(u)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(u) = \{\nabla \mathcal{Q}_i(u)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(u) = \{\nabla \mathcal{R}_i(u)\}$ $(i \in \mathcal{C})$. In this case, Theorem 13 reduces to Proposition 3.13(i), as derived by [47].*

In the following theorem, we derive the necessary condition for a saddle point of the interval-valued vector Lagrangian of NIMSIPVC.

Theorem 14. *If $(\overline{u}, \overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{u})}$ is an LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC, then $\mathcal{F}(\overline{u}) \in \psi^V(\overline{u})$ such that $\mathcal{F}(\overline{u})$ is an LU-efficient point of $VCD^{VL}(\overline{u})$.*

Proof. Since $(\overline{u},\overline{\sigma}) \in S^{\mathbb{L}}$, we have

$$\mathbb{L}(\overline{u},\overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u},\sigma),\ \forall \sigma \in \mathcal{G}_{\mathrm{WVL}(\overline{u})}. \tag{104}$$

It follows that

$$\begin{aligned} &\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \\ \not\prec_{LU}\ &\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e. \end{aligned} \tag{105}$$

If we assume that $\sigma = 0$, then (105) can be rewritten as follows:

$$\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \not\prec_{LU} \mathcal{F}(\overline{u}) \tag{106}$$

Moreover, by following similar steps as in Theorem 2, we deduce that

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) \leq 0. \tag{107}$$

If

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) < 0, \tag{108}$$

then

$$\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \prec^{s}_{LU} \mathcal{F}(\overline{u}). \tag{109}$$

It follows that

$$\mathcal{F}(\overline{u}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u})\right)e \prec_{LU} \mathcal{F}(\overline{u}), \tag{110}$$

which contradicts (106). Therefore, we have

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) = 0, \tag{111}$$

which implies that $\mathbb{L}(\overline{u},\overline{\sigma}) = \mathcal{F}(\overline{u})$. In view of the given hypothesis, $(\overline{u},\overline{\sigma}) \in \mathrm{WS}^{\mathbb{L}}$. It follows that

$$\mathbb{L}(u,\overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{u},\overline{\sigma}), \forall u \in \mathcal{G}. \tag{112}$$

This demonstrates that $\mathcal{F}(\overline{u}) = \mathbb{L}(\overline{u},\overline{\sigma}) \in \mathrm{Min}\{\mathbb{L}(u,\overline{\sigma})|u \in \mathcal{G}\} = \psi^{V}(\overline{\sigma})$. From Theorem 9, we conclude that $\mathcal{F}(\overline{u})$ is an LU-efficient point of $\mathrm{VCD}^{\mathrm{VL}}(\overline{u})$. □

Remark 24. *Theorem 14 generalizes Proposition 3.13(ii), as derived by Tung et al. [47], from smooth semi-infinite programming problems with vanishing constraints to a broader class of optimization problems, specifically NIMSIPVC.*

In the following theorem, we establish a relationship between a VC-stationary point of NIMSIPVC and an LU-saddle point of the interval-valued vector Lagrangian of NIMSIPVC. The proof is analogous to the proof of Theorem 12 so we omit it.

Theorem 15. *Let* $\overline{u} \in VC_{SP}$. *Further, assume that* $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ *and* Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}^{\zeta}_{+})$, $-\zeta_i$ $(i \in \mathcal{B}^{\zeta}_{-})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{u}))$,

$\mathcal{R}_i$ $(i \in \mathbb{H}^+_{+0}(\overline{u}) \cup \mathbb{H}^+_{00}(\overline{u}))$ *are strictly LU-convex and convex at* $\overline{u}$*, respectively. Then there exists* $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ *such that* $(\overline{u}, \overline{\sigma}) \in S^{\mathbb{L}}$.

Remark 25. *Theorem 15 generalizes Proposition 3.14, as deduced by Tung et al. [47], from smooth multiobjective SIPVC to NIMSIPVC.*

5. Scalarized Lagrange-Type Duality and Saddle-Point Optimality Criteria for NIMSIPVC

In this section, we investigate scalarized Lagrange-type dual problems corresponding to NIMSIPVC. Further, we establish both weak and strong duality results that relate the primal problem (NIMSIPVC) to its corresponding scalarized Lagrange-type dual problem. In addition, we introduce the notion of a saddle point for the scalarized Lagrangian of NIMSIPVC and present the saddle-point optimality criteria for NIMSIPVC.

5.1. Scalarized Lagrange-Type Duality

In this subsection, we formulate the scalarized Lagrange-type dual problem associated with NIMSIPVC. We derive various weak and strong duality results that elucidate the relationship between the scalarized Lagrange-type dual problem and the primal problem (NIMSIPVC).

Let $\overline{\lambda}^L_i, \overline{\lambda}^U_i \geq 0$, $(i \in \mathcal{J}^{\mathcal{F}})$ be fixed elements, and $\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^{|\mathcal{L}|}_+ \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$. The scalarized Lagrangian of NIMSIPVC is a function $\mathbb{L}^s : \mathbb{R}^n \times \mathbb{R}^{|\mathcal{L}|}_+ \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ defined as follows:

$$\begin{aligned}\mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) := \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(u) + \overline{\lambda}^U_i \mathcal{F}^U_i(u)\right) + \sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(u) \\ - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(u).\end{aligned} \tag{113}$$

We define the scalarized Lagrangian dual map $\Psi_0 : \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{R}^{|\mathcal{L}|}_+ \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ as follows:

$$\Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) := \text{Minimize}_{u \in \mathcal{G}} \mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}).$$

The scalarized Lagrange-type dual problem for NIMSIPVC is given as follows:

$$\begin{aligned}(\text{VCD}^{\text{SL}}(u, \overline{\lambda}^L, \overline{\lambda}^U)) \text{ Maximize } & \Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \\ \text{subject to } & \sigma^{\Psi}_{\mathcal{L} \setminus \mathbb{P}(u)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u) \cup \mathcal{H}_{0-}(u)} \geq 0, \\ & \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)} \leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)} \geq 0.\end{aligned}$$

The feasible set of $(\text{VCD})^{\text{SL}}(u, \overline{\lambda}^L, \overline{\lambda}^U)$ is denoted by $\mathcal{G}_{SL(u, \overline{\lambda}^L, \overline{\lambda}^U)}$ and is defined as follows:

$$\begin{aligned}\mathcal{G}_{SL(u, \overline{\lambda}^L, \overline{\lambda}^U)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^{|\mathcal{L}|}_+ \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s | \sigma^{\Psi}_{\mathcal{L} \setminus \mathbb{P}(u)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)} \leq 0, \\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u) \cup \mathcal{H}_{0-}(u)} \geq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)} \geq 0\}.\end{aligned}$$

Remark 26. *It is worth noting that* $VCD^{SL}(u, \overline{\lambda}^L, \overline{\lambda}^U)$ *depends on the feasible point* u.

The scalarized Lagrange-type dual problem, independent of an element's choice from the feasible set $\mathcal{G}$, is defined as follows:

$$\begin{aligned}(\text{VCD}^{\text{SL}}(\overline{\lambda}^L, \overline{\lambda}^U)) \text{ Maximize } & \Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \\ \text{subject to } & \sigma \in \mathcal{G}_{\text{SL}(\overline{\lambda}^L, \overline{\lambda}^U)} = \bigcap_{u \in \mathcal{G}} \mathcal{G}_{\text{SL}(u, \overline{\lambda}^L, \overline{\lambda}^U)} \neq \emptyset.\end{aligned}$$

In the following theorem, we establish weak duality results that demonstrate the relationship between NIMSIPVC and VCD$^{SL}(u, \overline{\lambda}^L, \overline{\lambda}^U)$.

Theorem 16. *Let u and σ be any elements of $\mathcal{G}$ and $\mathcal{G}_{SL(u,\overline{\lambda}^L,\overline{\lambda}^U)}$, respectively. Then,*

$$\Psi_0\Big(\overline{\lambda}^L, \overline{\lambda}^U, \sigma\Big) \leq \sum_{i\in\mathcal{J}^{\mathcal{F}}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(u) + \overline{\lambda}_i^U \mathcal{F}_i^U(u)\Big). \tag{114}$$

Proof. From the definition of Ψ_0 and the given hypothesis that $(u, \sigma) \in \mathcal{G} \times \mathcal{G}_{\mathrm{SL}(u,\overline{\lambda}^L,\overline{\lambda}^U)}$, we have

$$\begin{aligned}\Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) &= \mathrm{Minimize}_{u\in\mathcal{G}} \mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \sigma)\\ &\leq \sum_{i\in\mathcal{J}^{\mathcal{F}}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(u) + \overline{\lambda}_i^U \mathcal{F}_i^U(u)\Big) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(u) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(u) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(u)\\ &\quad + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(u).\end{aligned} \tag{115}$$

On utilizing the feasibility of u and σ, we have

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(u) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(u) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(u) \leq 0. \tag{116}$$

From (115) and (116), we infer that

$$\Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \sum_{i\in\mathcal{J}^{\mathcal{F}}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(u) + \overline{\lambda}_i^U \mathcal{F}_i^U(u)\Big). \tag{117}$$

This completes the proof. □

Remark 27. *If $\mathcal{F}_i$ is a real-valued function for every $i \in \mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ are continuously differentiable functions, then $\partial_c\mathcal{F}_i(u) = \{\nabla\mathcal{F}_i(u)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(u) = \{\nabla\Psi_k(u)\}$ $(k \in \mathcal{L})$, $\partial_c\zeta_i(u) = \{\nabla\zeta_i(u)\}$ $(i \in \mathcal{B})$, $\partial_c\mathcal{Q}_i(u) = \{\nabla\mathcal{Q}_i(u)\}$ $(i \in \mathcal{C})$, $\partial_c\mathcal{R}_i(u) = \{\nabla\mathcal{R}_i(u)\}$ $(i \in \mathcal{C})$. In this case, Theorem 16 reduces to Proposition 4.1, as derived by Tung et al. [47].*

In the next corollary, we derive the weak duality result that relates NIMSIPVC to VCD^{SL}.

Corollary 1. *Let u and σ be any arbitrary elements of $\mathcal{G}$ and $\mathcal{G}_{SL}$, respectively. Then,*

$$\Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \sum_{i\in\mathcal{J}^{\mathcal{F}}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(u) + \overline{\lambda}_i^U \mathcal{F}_i^U(u)\Big). \tag{118}$$

Remark 28. *Corollary 1 generalizes Corollary 4.2, as derived by Tung et al. [47], from smooth multiobjective SIPVC to nonsmooth multiobjective SIPVC involving interval-valued objective functions.*

In the following theorem, we establish the strong duality result that relates NIMSIPVC to VCD$^{SL}(u, \overline{\lambda}^L, \overline{\lambda}^U)$.

Theorem 17. *Let $\overline{u} \in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{u}$, and let $\mathcal{K}_2$ be a closed set. Furthermore, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ and Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta}(\overline{u}))$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta}(\overline{u}))$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^-(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^+(\overline{u}) \cup \overline{\mathbb{H}}_{00}^+(\overline{u}) \cup \overline{\mathbb{H}}_{0-}^+(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^+(\overline{u}) \cup$*

$\mathbb{H}^{+}_{00}(\overline{u})$) *are LU-convex and convex at* $\overline{u}$*, respectively. Then, there exists* $(\overline{\lambda}^{L}, \overline{\lambda}^{U}) \in \mathbb{R}^{l}_{+} \times \mathbb{R}^{l}_{+}$, $\sum\limits_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i} + \overline{\lambda}^{U}_{i}\right) = 1$ *such that* $\overline{\sigma} \in \mathcal{G}_{SL(\overline{u},\overline{\lambda}^{l},\overline{\lambda}^{U})}$ *is an optimal solution of* $VCD^{SL}(\overline{u}, \overline{\lambda}^{L}, \overline{\lambda}^{U})$*, and*

$$\Psi_0(\overline{\lambda}^{L}, \overline{\lambda}^{U}, \overline{\sigma}) = \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(\overline{u}) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(\overline{u})\right). \tag{119}$$

Proof. From the given hypothesis, if $\overline{u} \in \mathrm{WEff}_{loc}$ and VC-ACQ is satisfied at $\overline{u}$, then from Theorem 1, $\overline{u} \in \mathrm{VC}_{SP}$, which implies that there exist $(\overline{\lambda}^{L}, \overline{\lambda}^{U}) \in \mathbb{R}^{l}_{+} \times \mathbb{R}^{l}_{+}$ and $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^{r} \times \mathbb{R}^{s} \times \mathbb{R}^{s}$ satisfying:

$$\begin{aligned} 0 \in \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\partial_c\mathcal{F}^{L}_{i}(\overline{u}) + \overline{\lambda}^{U}_{i}\partial_c\mathcal{F}^{U}_{i}(\overline{u})\right) + \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\partial_c\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\partial_c\zeta_i(\overline{u}) \\ - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\partial_c\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\partial_c\mathcal{R}_i(\overline{u}), \end{aligned} \tag{120}$$

where $\sum\limits_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i} + \overline{\lambda}^{U}_{i}\right) = 1$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_{+}(\overline{u})} = 0$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{u})\cup\mathcal{H}_{0-}(\overline{u})} \geq 0$, $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{u})\cup\mathcal{H}_{00}(\overline{u})} \geq 0$, and $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{u})\cup\mathcal{H}_{0-}(\overline{u})\cup\mathcal{H}_{0+}(\overline{u})} = 0$. This implies that there exist $\hat{\xi}^{L}_{i} \in \partial_c\mathcal{F}^{L}_{i}(\overline{u})$, $\hat{\xi}^{U}_{i} \in \partial_c\mathcal{F}^{U}_{i}(\overline{u})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\hat{\eta}^{\Psi}_{k} \in \partial_c\Psi_k(\overline{u})$ $(k \in \mathcal{L})$, $\hat{\eta}^{\zeta}_{i} \in \partial_c\zeta_i(\overline{u})$, $\hat{\eta}^{\mathcal{Q}}_{i} \in \partial_c\mathcal{Q}_i(\overline{u})$ $(i \in \mathcal{C})$, $\hat{\eta}^{\mathcal{R}}_{i} \in \partial_c\mathcal{R}_i(\overline{u})$ $(i \in \mathcal{C})$ such that

$$\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\hat{\xi}^{L}_{i} + \overline{\lambda}^{U}_{i}\hat{\xi}^{U}_{i}\right) + \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\hat{\eta}^{\Psi}_{k} + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\hat{\eta}^{\zeta}_{i} - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\hat{\eta}^{\mathcal{Q}}_{i} + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\hat{\eta}^{\mathcal{R}}_{i} = 0. \tag{121}$$

Moreover, in view of the fact that $\overline{u} \in \mathrm{VC}_{SP}$ and properties of $\overline{\sigma}$, one has

$$\sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}) = 0. \tag{122}$$

It follows that

$$\mathbb{L}^{s}(\overline{u}, \overline{\lambda}^{L}, \overline{\lambda}^{U}, \overline{\sigma}) = \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(\overline{u}) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(\overline{u})\right). \tag{123}$$

On following similar steps as in the proof of Theorem 5, we have

$$\begin{aligned} &\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(u) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(u)\right) + \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\Psi_k(u) + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\zeta_i(u) - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(u) \\ &- \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(\overline{u}) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(\overline{u})\right) - \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) - \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}) \\ &\geq \left\langle \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\xi^{L}_{i} + \overline{\lambda}^{U}_{i}\xi^{U}_{i}\right) + \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\eta^{\Psi}_{k} + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\eta^{\zeta}_{i} - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\eta^{\mathcal{Q}}_{i} + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\eta^{\mathcal{R}}_{i}, u - \overline{u}\right\rangle. \end{aligned} \tag{124}$$

From (121) and (124), we obtain the following inequality:

$$\begin{aligned} &\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(u) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(u)\right) + \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\Psi_k(u) + \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\zeta_i(u) - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(u) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(u) \\ &- \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}^{L}_{i}\mathcal{F}^{L}_{i}(\overline{u}) + \overline{\lambda}^{U}_{i}\mathcal{F}^{U}_{i}(\overline{u})\right) - \sum_{k\in\mathcal{L}}\overline{\sigma}^{\Psi}_{k}\Psi_k(\overline{u}) - \sum_{i\in\mathcal{B}}\overline{\sigma}^{\zeta}_{i}\zeta_i(\overline{u}) + \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{Q}}_{i}\mathcal{Q}_i(\overline{u}) - \sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}_{i}\mathcal{R}_i(\overline{u}) \\ &\geq 0. \end{aligned} \tag{125}$$

This implies that

$$\begin{aligned}&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(u)+\overline{\lambda}_i^U\mathcal{F}_i^U(u)\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(u)+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(u)-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(u)+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(u)\\ \geq&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{u})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{u})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}).\end{aligned}\tag{126}$$

Hence,

$$\mathbb{L}^s(u,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})\geq\mathbb{L}^s(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})=\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right),\ \forall u\in\mathcal{G}.\tag{127}$$

Now,

$$\begin{aligned}\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})&=\text{Minimize}_{u\in\mathcal{G}}\mathbb{L}^s(u,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})\\&=\mathbb{L}^s(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})\\&=\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right).\end{aligned}\tag{128}$$

Therefore, from Theorem 16,

$$\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\sigma)\leq\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}),\ \forall\sigma\in\mathcal{G}_{SL(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U)}.\tag{129}$$

Therefore, $\overline{\sigma}$ is an optimal solution of $\text{VCD}^{\text{SL}}(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U)$. This completes the proof. □

Remark 29. *If $\mathcal{F}_i$ is a real-valued function for every $i\in\mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u)=\mathcal{F}_i^U(u)$, $i\in\mathcal{J}^{\mathcal{F}}$, $\forall u\in\mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i\in\mathcal{J}^{\mathcal{F}})$, Ψ_k $(k\in\mathcal{L})$, ζ_i $(i\in\mathcal{B})$, $\mathcal{Q}_i$ $(i\in\mathcal{C})$, $\mathcal{R}_i$ $(i\in\mathcal{C})$ are continuously differentiable functions, then $\partial_c\mathcal{F}_i(u)=\{\nabla\mathcal{F}_i(u)\}$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(u)=\{\nabla\Psi_k(u)\}$ $(k\in\mathcal{L})$, $\partial_c\zeta_i(u)=\{\nabla\zeta_i(u)\}$ $(i\in\mathcal{B})$, $\partial_c\mathcal{Q}_i(u)=\{\nabla\mathcal{Q}_i(u)\}$ $(i\in\mathcal{C})$, $\partial_c\mathcal{R}_i(u)=\{\nabla\mathcal{R}_i(u)\}$ $(i\in\mathcal{C})$. In this case, Theorem 17 reduces to Proposition 4.4, as derived by Tung et al. [47].*

The following example demonstrates the significance of Theorems 16 and 17.

Example 5. *Consider the problem $(\mathcal{P}_s)$ as follows:*

$$\begin{aligned}(\mathcal{P}_s)\ \textit{Minimize}\ \mathcal{F}(u)&=(\mathcal{F}_1(u),\mathcal{F}_2(u))\\&=\left([|u_1-1|,|u_1-1|+(u_1-1)^2],\left[(u_2-1)^2,(u_1-1)^2+(u_2-1)^2\right]\right),\\ \textit{subject to}\ \Psi_k(u)&=(k-1)(u_2-1)\leq 0,k\in\mathcal{L}=[0,1],\\ \mathcal{Q}_1(u)&=(u_2-u_1)\geq 0,\\ \mathcal{R}_1(u)\mathcal{Q}_1(u)&=(u_2-1)(u_2-u_1)\leq 0.\end{aligned}$$

The feasible set of $(\mathcal{P}_s)$ is given as follows:

$$\mathcal{G}=\cup_{i=1}^3\mathcal{G}^i,$$

where $\mathcal{G}_i, i=1,2,3$ are defined as follows:

$$\begin{aligned}\mathcal{G}^1&=\{(u_1,u_2)\in\mathbb{R}^2|u_2-1=0,\ u_2-u_1>0\}=\{(u_1,u_2)\in\mathbb{R}^2|u_2=1,\ u_1<1\},\\ \mathcal{G}^2&=\{(u_1,u_2)\in\mathbb{R}^2|u_2-1>0,\ u_2-u_1=0\}=\{(u_1,u_2)\in\mathbb{R}^2|u_2>1,\ u_1>1\},\\ \mathcal{G}^3&=\{(u_1,u_2)\in\mathbb{R}^2|u_2-1=0,\ u_2-u_1=0\}=\{(u_1,u_2)\in\mathbb{R}^2|u_2=1,\ u_1=1\}.\end{aligned}$$

Now, we formulate the scalarized Lagrangian for $(\mathcal{P}_s)$ for some fixed $\overline{\lambda}=(\overline{\lambda}_1^L,\overline{\lambda}_1^U,\overline{\lambda}_2^L,\overline{\lambda}_2^U)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+,\sum_{i=1}^2\left(\overline{\lambda}_i^L+\overline{\lambda}_i^U\right)=1$ as follows:

$$\mathbb{L}^s(u,\overline{\lambda},\sigma)=\sum_{i=1}^{2}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(u)+\overline{\lambda}_i^U\mathcal{F}_i^U(u)\Big)+\sum_{k\in[0,1]}\sigma_k^{\Psi}(k-1)(u_2-1)-\sigma^{\mathcal{Q}}\mathcal{Q}_1(u)+\sigma^{\mathcal{R}}\mathcal{R}_1(u). \tag{130}$$

Moreover, $\Psi_0 := Minimize_{u\in\mathcal{G}}\mathbb{L}^s(u,\overline{\lambda},\sigma)$.

Formulate the scalarized Lagrange-type dual problem corresponding to $(\mathcal{P}_s)$ *as follows:*

$$\begin{aligned}(VCD_1^{SL}(u,\overline{\lambda}))\ \text{Maximize}\ &\Psi_0(\overline{\lambda},\sigma)\\ &\text{subject to}\ \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)}\geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)}\geq 0,\\ &\qquad\sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)}\leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)}\geq 0.\end{aligned}$$

The feasible region of $VCD_1^{SL}(u,\overline{\lambda})$ *corresponding to* $\mathcal{G}^1$ *is given by*

$$\mathcal{G}^1_{SL(u,\overline{\lambda})}=\{(\sigma^{\Psi},\sigma^{\mathcal{Q}},\sigma^{\mathcal{R}})|\sigma_k^{\Psi}\in\mathbb{R}_+^{|\mathcal{L}|},\ \sigma^{\mathcal{Q}}\geq 0,\ \sigma^{\mathcal{R}}\in\mathbb{R}\}.$$

Formulate the scalarized Lagrange-type dual problem corresponding to $\mathcal{G}^2$ *of* $(\mathcal{P}_s)$ *as follows:*

$$\begin{aligned}(VCD_2^{SL}(u,\overline{\lambda}))\ \text{Maximize}\ &\Psi_0(\overline{\lambda},\sigma)\\ &\text{subject to}\ \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)}\geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)}\geq 0,\\ &\qquad\sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)}\leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)}\geq 0.\end{aligned}$$

The feasible set of $VCD_2^{SL}(u,\overline{\lambda})$ *corresponding to* $\mathcal{G}^2$ *is given by*

$$\mathcal{G}^2_{SL(u,\overline{\lambda})}=\{(\sigma^{\Psi},\sigma^{\mathcal{Q}},\sigma^{\mathcal{R}})|\sigma_k^{\Psi}\in\mathbb{R}_+^{|\mathcal{L}|},\ \sigma^{\mathcal{Q}}\in\mathbb{R},\ \sigma^{\mathcal{R}}\leq 0\}.$$

Formulate the scalarized Lagrange-type dual problem corresponding to $\mathcal{G}^3$ *as follows:*

$$\begin{aligned}(VCD_3^{SL}(u,\overline{\lambda}))\ \text{Maximize}\ &\Psi_0(\overline{\lambda},\sigma)\\ &\text{subject to}\ \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(u)}\geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(u)\cup\mathcal{H}_{0-}(u)}\geq 0,\\ &\qquad\sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(u)}\leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(u)}\geq 0.\end{aligned}$$

The feasible set of $VCD_3^{SL}(u,\overline{\lambda})$ *corresponding to* $\mathcal{G}^3$ *is given by*

$$\mathcal{G}^3_{SL(u,\overline{\lambda})}=\{(\sigma^{\Psi},\sigma^{\mathcal{Q}},\sigma^{\mathcal{R}})|\sigma_k^{\Psi}\in\mathbb{R}_+^{|\mathcal{L}|},\ \sigma^{\mathcal{Q}}\in\mathbb{R},\ \sigma^{\mathcal{R}}\in\mathbb{R}\}.$$

Moreover, the scalarized Lagrange-type dual problem, independent of the choice of a feasible element of $(\mathcal{P}_s)$*, is formulated as*

$$\begin{aligned}(VCD^{SL}(\overline{\lambda}))\ \text{Maximize}\ &\Psi_0(\overline{\lambda},\sigma)\\ &\text{subject to}\ (\sigma^{\Psi},\sigma^{\mathcal{Q}},\sigma^{\mathcal{R}})\in\mathcal{G}_{SL(\overline{\lambda})}=\bigcap_{i=1,2,3}\mathcal{G}^i_{(SL)(\overline{\lambda})}.\end{aligned}$$

Therefore, for any $(\sigma^{\Psi},\sigma^{\mathcal{Q}},\sigma^{\mathcal{R}})\in\mathcal{G}_{SL(\overline{\lambda})},\ u\in\mathcal{G}$*, the following inequality holds:*

$$\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}(k-1)(u_2-1)-\sigma^{\mathcal{Q}}\mathcal{Q}_1(u)+\sigma^{\mathcal{R}}\mathcal{R}_1(u)\leq 0. \tag{131}$$

This implies that

$$\Psi_0(\overline{\lambda},\sigma)\leq\mathbb{L}^s(u,\overline{\lambda},\sigma)\leq\sum_{i=1}^{2}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(u)+\overline{\lambda}_i^U\mathcal{F}_i^U(u)\Big). \tag{132}$$

Therefore, Theorem 16 and Corollary 1 are satisfied.

It is worth noting that $\overline{u} = (1,1)$ *is a locally weakly LU-efficient solution of* $(\mathcal{P}_s)$ *such that VC-ACQ holds at* $\overline{u}$. *Let* $\overline{u} = (1,1)$. *Then,* $\mathcal{F}(\overline{u}) = ([0,1],[0,0])$. *Choose* $\overline{\sigma} = (\overline{\sigma}_k^{\Psi}, \overline{\sigma}_1^{\mathcal{Q}}, \overline{\sigma}_1^{\mathcal{R}})$ *such that*

$$\overline{\sigma}_k^{\Psi} = \begin{cases} 1, & k = 0, \\ 0, & otherwise, \end{cases}$$

$\overline{\sigma}_1^{\mathcal{Q}} = \overline{\lambda}_1^L + 1$, $\overline{\sigma}_1^{\mathcal{R}} = 0$. *Moreover,* $\mathcal{F}_1$, $\mathcal{F}_2$, Ψ_k $(k \in \mathcal{L})$, $\mathcal{Q}$, *and* $\mathcal{R}$ *are LU-convex and convex functions at* $\overline{u}$, *which implies that all hypotheses in Theorem 17 are satisfied at* $\overline{u}$. *Therefore, Theorem 17 holds, that is,*

$$\Psi_0(\overline{\lambda}, \overline{\sigma}) = \sum_{i=1}^{2} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{u}) \right). \tag{133}$$

5.2. Saddle-Point Optimality Criteria

In this subsection, we introduce the notion of a saddle point for the scalarized Lagrangian corresponding to NIMSIPVC and further explore saddle-point optimality criteria for NIMSIPVC.

Definition 13. *Let* $\left(\overline{\lambda}^L, \overline{\lambda}^U\right) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+$ *be a fixed element such that* $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L + \overline{\lambda}_i^U\right) = 1$. *Further, we assume that* $\overline{u} \in \mathcal{G}$, *and* $\overline{\sigma} \in \mathcal{G}_{SL(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U)}$. *Then,* $(\overline{u}, \overline{\sigma})$ *is known as a saddle point for the scalarized Lagrangian of NIMSIPVC if the following condition holds:*

$$\mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \leq \mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}),\ \forall u \in \mathcal{G},\ \forall \sigma \in \mathcal{G}_{SL(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U)}.$$

The relationship between a locally LU-weakly efficient solution of NIMSIPVC and a saddle point of scalarized Lagrangian of NIMSIPVC is established in the following theorem.

Theorem 18. *Let* $\overline{u} \in WEff_{loc}$. *Further, assume that all the hypotheses in Theorem 17 are satisfied at* $\overline{u}$. *Then, there exists* $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ *such that* $(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma})$ *is a saddle point for the scalarized Lagrangian of NIMSIPVC.*

Proof. From the proof of Theorem 17, we have

$$\mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \geq \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}),\ \forall u \in \mathcal{G}. \tag{134}$$

We are left to prove that

$$\mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}),\ \forall \sigma \in \mathcal{G}_{SL(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U)}. \tag{135}$$

Following similar steps as in the proof of Theorem 17 and Corollary 1, we have the following condition:

$$\begin{aligned} \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) &= \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{u}) \right) \\ &\geq \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{u}) \right) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{u}) \\ &\qquad - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{u}) \\ &= \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma). \end{aligned} \tag{136}$$

This completes the proof. □

Remark 30. *Theorem 18 generalizes Proposition 4.7, as deduced by Tung et al. [47], from smooth multiobjective SIPVC to nonsmooth multiobjective interval-valued semi-infinite programming problems with vanishing constraints.*

In the following theorem, we establish a relationship between the saddle point of the scalarized Lagrangian of NIMSIPVC and the VC-stationary point of the primal problem (NIMSIPVC).

Theorem 19. *Let $\overline{u} \in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{u}$, and let $\mathcal{K}_2$ be a closed set. Furthermore, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ and Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{u}))$, ζ_i $(i \in \mathcal{B}^{\zeta}_{+}(\overline{u}))$, $-\zeta_i$ $(i \in \mathcal{B}^{\zeta}_{-}(\overline{u}))$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{u}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{u}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{u}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}^{+}_{+0}(\overline{u}) \cup \mathbb{H}^{+}_{00}(\overline{u}))$ are LU-convex and convex at $\overline{u}$, respectively. Then, there exists $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{u}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that $(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma})$ is a saddle point for the scalarized Lagrangian of NIMSIPVC.*

Proof. In view of the definition of the scalarized Lagrangian of NIMSIPVC, we have

$$\begin{aligned} &\mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) - \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) = \\ &\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(u) + \overline{\lambda}^U_i \mathcal{F}^U_i(u) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(u) + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(u) - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(u) \\ &- \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{u}) \right) - \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\overline{u}) - \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\overline{u}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{u}) - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\overline{u}). \end{aligned} \tag{137}$$

Furthermore, by employing the convexity assumptions on the objective functions and constraint functions, we obtain the following condition by following the analogous steps in the proofs of Theorems 5 and 17 as follows:

$$\begin{aligned} \mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) - \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) &\geq \left\langle \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \xi^L_i + \overline{\lambda}^U_i \xi^U_i \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \eta^{\Psi}_k + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \eta^{\zeta}_i \right. \\ &\quad \left. - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \eta^{\mathcal{Q}}_i + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \eta^{\mathcal{R}}_i, u - \overline{u} \right\rangle \\ &\geq 0, \end{aligned} \tag{138}$$

due to the fact that $\overline{u} \in \mathrm{VC}_{SP}$. This implies that

$$\mathbb{L}^s(u, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \geq \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}), \ \forall u \in \mathcal{G}. \tag{139}$$

Since $\overline{u} \in \mathcal{G}$ and $\sigma \in \mathcal{G}_{SL(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U)}$, it follows that

$$\sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(u) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(u) - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(u) + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(u) \leq 0. \tag{140}$$

Now,

$$\begin{aligned} \mathbb{L}^s(\overline{u}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) &= \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{u}) \right) + \sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(\overline{u}) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(\overline{u}) \\ &\quad - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{u}) + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\overline{u}) \\ &\leq \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{u}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{u}) \right). \end{aligned} \tag{141}$$

Since $\overline{u} \in \mathrm{VC}_{SP}$, the following condition holds:

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) = 0. \tag{142}$$

Therefore, from (141) and (142), we have

$$\begin{aligned}\mathbb{L}^s(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U,\sigma) \leq \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U\mathcal{F}_i^U(\overline{u})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{u}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{u}) \\ - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{u}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{u}) \\ = \mathbb{L}^s(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}).\end{aligned} \tag{143}$$

Hence, $(\overline{u},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})$ is a saddle point for the scalarized Lagrangian of NIMSIPVC. □

Remark 31. *If $\mathcal{F}_i$ is a real-valued function for every $i \in \mathcal{J}^{\mathcal{F}}$, then $\mathcal{F}_i^L(u) = \mathcal{F}_i^U(u)$, $i \in \mathcal{J}^{\mathcal{F}}$, $\forall u \in \mathbb{R}^n$. Moreover, if $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathcal{L})$, ζ_i $(i \in \mathcal{B})$, $\mathcal{Q}_i$ $(i \in \mathcal{C})$, $\mathcal{R}_i$ $(i \in \mathcal{C})$ are continuously differentiable functions, then $\partial_c\mathcal{F}_i(u) = \{\nabla\mathcal{F}_i(u)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(u) = \{\nabla\Psi_k(u)\}$ $(k \in \mathcal{L})$, $\partial_c\zeta_i(u) = \{\nabla\zeta_i(u)\}$ $(i \in \mathcal{B})$, $\partial_c\mathcal{Q}_i(u) = \{\nabla\mathcal{Q}_i(u)\}(i \in \mathcal{C})$, $\partial_c\mathcal{R}_i(u) = \{\nabla\mathcal{R}_i(u)\}$ $(i \in \mathcal{C})$. In this case, Theorem 19 reduces to Proposition 4.8, as established by Tung et al. [47].*

Now, we provide a non-trivial example to demonstrate the validity of Theorem 19.

Example 6. *Consider the problem $(\mathcal{P}_s)$ from Example 5.*

It can be verified that $\overline{u} = (1,1)$ is a VC-stationary point of $(\mathcal{P}_s)$. Therefore, there exist $\overline{\lambda}_1^L = \overline{\lambda}_2^L = \frac{1}{4} = \overline{\lambda}_1^U = \overline{\lambda}_2^U$, $\overline{\sigma}^{\mathcal{Q}} = \frac{1}{2}$, $\overline{\sigma}^{\mathcal{R}} = 1$, and

$$\overline{\sigma}^{\Psi} = \begin{cases} \frac{1}{2}, & k = 0, \\ 0, & \text{otherwise}, \end{cases}$$

such that

$$0 \in \sum_{i=1}^{2}\left(\overline{\lambda}_i^L\partial_c\mathcal{F}_i^L(\overline{u}) + \overline{\lambda}_i^U\partial_c\mathcal{F}_i^U(\overline{u})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\partial_c\Psi_k(\overline{u}) - \overline{\sigma}^{\mathcal{Q}}\partial_c\mathcal{Q}_1(\overline{u}) + \overline{\sigma}^{\mathcal{R}}\partial_c\mathcal{R}_1(\overline{u}). \tag{144}$$

Furthermore, $\mathcal{F}_i$ $(i = 1,2)$ and Ψ_k $(k \in \mathcal{L})$, $\mathcal{Q}_1$, and $\mathcal{R}_1$ are LU-convex and convex at $\overline{u}$, respectively. Therefore, from Theorem 19, $(\overline{u},\overline{\lambda},\overline{\sigma})$ is a saddle point for the scalarized Lagrangian of $(\mathcal{P}_s)$.

6. Conclusions and Future Research Directions

In this article, we have investigated KKT-type necessary optimality conditions, Lagrange-type duality, and saddle-point optimality conditions for NIMSIPVC. We have presented the VC-ACQ for NIMSIPVC and employed it to derive the KKT-type necessary optimality conditions. We have formulated interval-valued weak vector, interval-valued vector, and scalarized Lagrange-type dual problems corresponding to NIMSIPVC. Subsequently, we have derived the weak, strong, and converse duality results relating NIMSIPVC to its corresponding Lagrange-type dual problems. We have introduced the notion of saddle points for the interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC and derived the saddle-point optimality conditions for NIMSIPVC.

The results derived in this paper extend and generalize several well-known results existing in the literature. In particular, we have extended the corresponding results presented in [1] from smooth to nonsmooth MPVC and generalized them for interval-valued multiobjective SIPVC. Moreover, the KKT-type necessary optimality conditions established in this paper extend and generalize the corresponding results derived in [18,19] from mul-

tiobjective SIPVC to interval-valued multiobjective SIPVC involving nonsmooth locally Lipschitz functions. Furthermore, we have extended and generalized the corresponding results developed by Tung et al. [47] to a broader class of optimization problems, namely NIMSIPVC. Several non-trivial examples have been provided to illustrate the significance of the established results.

The results derived in this paper are applicable to various real-life problems, such as portfolio optimization [32], machine learning [31], and energy systems [56]. Moreover, it is worth noting that topology optimization of truss structures relies on deterministic scenarios, such as fixed external loads and structural parameters (see [1]). However, several engineering problems involve uncertainty in loads and manufacturing errors, which makes it essential to incorporate uncertainty into truss topology design problems. Consequently, in view of the work of Achtziger and Kanzow [1], the truss topology design problem can be modeled as NIMSIPVC to address uncertainty in external loads. Due to unmodeled or unpredictable external forces, as well as noisy or incomplete sensor measurements, uncertainty may arise in robot motion-planning problems (see [57]). Therefore, to address uncertainty in robot motion-planning problems, we can formulate these problems as NIMSIPVC.

It is significant to observe that all the functions involved in the various results derived in this paper are assumed to be convex functions. However, in several real-life problems, non-convex functions are often encountered (see, for instance, [25,58]). In view of this, several results related to Lagrange duality and saddle-point optimality conditions for NIMSIPVC may not be applicable to nonsmooth interval-valued multiobjective semi-infinite programming problems involving non-convex functions. Moreover, the Clarke subdifferential of a locally Lipschitz function may be too large for many significant applications, especially for necessary optimality conditions. For instance, minimizing $-|x|$ over $\mathbb{R}$ illustrates that the Clarke subdifferential of this function is a very large set that does not contain the minimizer of the function (see [50]). We intend to address these limitations in our future studies.

In future work, investigating Lagrange duality for nonsmooth interval-valued multiobjective semi-infinite programming problems with switching constraints would be an interesting problem. In view of the fact that the limiting subdifferential is the smallest among all robust subdifferentials and is contained in the Clarke subdifferential (see, for instance, [50,51]), following the research work of Kanzi [59], the results of this paper could be further refined by employing the limiting subdifferential rather than the Clarke subdifferential. Furthermore, it would be interesting to establish the weak, strong, and converse duality results for NIMSIPVC using the limiting subdifferential. Saddle-point optimality criteria for NIMSIPVC could be further extended by establishing interrelations between the solutions of NIMSIPVC and the saddle points of corresponding Lagrangians in terms of limiting subdifferentials. In addition, the findings established in the present article can be generalized from the setting of Euclidean spaces to more general frameworks, particularly Riemannian manifolds.

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