axioms

MDPI

Article

Limit Property of an L^2-Normalized Solution for an L^2-Subcritical Kirchhoff-Type Equation with a Variable Exponent

Xincai Zhu * and Hanxiao Wu

School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China; wuhanxiao19991228@163.com
* Correspondence: zhuxc68@163.com

Abstract: This paper is concerned with the following L^2-subcritical Kirchhoff-type equation $-\Big(a+b\big(\int_{\mathbb{R}^2}|\nabla u|^2dx\big)^s\Big)\Delta u+V(x)u=\mu u+\beta|u|^2u,\ x\in\mathbb{R}^2$, with $\int_{\mathbb{R}^2}|u|^2dx=1$. We give a detailed analysis of the limit property of the L^2-normalized solution when exponent s tends toward 0 from the right (i.e., $s\searrow 0$). Our research extends previous works, in which the authors have displayed the limit behavior of L^2-normalized solutions when $s=1$ as $a\searrow 0$ or $b\searrow 0$.

Keywords: L^2-subcritical Kirchhoff-type equation; variable exponent; L^2-normalized solution; limit property

MSC: 32J20; 35J60; 35Q40; 46N50

1. Introduction

In this paper, we study the following Kirchhoff-type equation with $\int_{\mathbb{R}^2}|u|^2dx=1$:

$$-\Big(a+b\big(\int_{\mathbb{R}^2}|\nabla u|^2dx\big)^s\Big)\Delta u+V(x)u=\mu u+\beta|u|^2u,\ x\in\mathbb{R}^2, \tag{1}$$

where $a,b>0$ are constants, exponent $s>0$, parameter $\beta>0$, and μ denotes a Lagrange multiplier. The above $\big(\int_{\mathbb{R}^2}|\nabla u|^2dx\big)^s\Delta u$ in (1) is called a variable nonlocal term, and $\beta|u|^2u$ is an L^2-subcritical term.

In recent decades, many different types of elliptic equations involved in (1) have been studied extensively through variational methods and functional analysis techniques. For instance, when $s=0$ and $\beta>0$ in (1), Equation (1) comes from the well-known Bose-Einstein condensates [1–3], which can be described by a Gross–Pitaevskii (G-P) energy functional [4,5]. Especially for $V(x)$ in ring-shaped, multi-well, sinusoidal, and periodic forms, the authors in [6–9] presented the existence, non-existence, and limit properties of L^2-normalized solutions when β tends to a threshold value β^*. Furthermore, if $V(x)$ is a logarithmic or homogeneous function [10,11], the local uniqueness was also analyzed as β approaches β^*.

For $s=1$, Equation (1) is regarded as a classic Kirchhoff-type equation. When $V(x)=0$, Ye [12,13] gave a quantitative classification of existence and nonexistence for the L^2-normalized solution. Meng and Zeng [14] have displayed a detailed limit behavior of the L^2-normalized solution when $V(x)$ is a periodic function. Somewhat similarly, there are many works [15–18] involved in the existence, non-existence, and limit properties of L^2-normalized solutions when (1) possesses an L^2-subcritical term. If $V(x)$ is a polynomial potential, Tang, Zhou, and their co-workers in [15,19] obtained some results on the refined limit behavior of L^2-normalized solutions as $b\searrow 0$ or as $a\searrow 0$.

Inspired by previous works, in the present paper, we shall study L^2-normalized solutions for the L^2-subcritical Kirchhoff-type Equation (1) with a variable nonlocal term.

check for updates

Citation: Zhu, X.; Wu, H. Limit Property of an L^2-Normalized Solution for an L^2-Subcritical Kirchhoff-Type Equation with a Variable Exponent. *Axioms* **2024**, *13*, 571. https://doi.org/10.3390/axioms13090571

Academic Editors: Andrey Zahariev and Hristo Kiskinov

Received: 31 July 2024
Revised: 20 August 2024
Accepted: 21 August 2024
Published: 23 August 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

A well-known result shows that the L^2-normalized solution of (1) can be solved by dealing with the following constrained minimization problem

$$e(s,\beta) := \inf_{u\in\mathcal{K}} E_{s,\beta}(u), \tag{2}$$

where $E_{s,\beta}(u)$ fulfills

$$E_{s,\beta}(u) = a\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)^{s+1} + \int_{\mathbb{R}^2}V(x)u^2dx - \frac{\beta}{2}\int_{\mathbb{R}^2}|u|^4dx. \tag{3}$$

The above $\mathcal{K}$ in (2) satisfies

$$\mathcal{K} := \Big\{u\in\mathcal{H} : \int_{\mathbb{R}^2}|u|^2 = 1\Big\} \tag{4}$$

and $\mathcal{H}$ such as

$$\mathcal{H} := \Big\{u\in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2}V(x)|u|^2dx < \infty\Big\}$$

with norm $\|u\|_{\mathcal{H}} := \Big(\int_{\mathbb{R}^2}|\nabla u|^2dx + \int_{\mathbb{R}^2}\big(1+V(x)|u|^2\big)dx\Big)^{\frac{1}{2}}$.

Because many physicists are often interested in L^2-normalized solutions, we study minimization problem (1.2) with $\int_{\mathbb{R}^n}|u|^2 = 1$. In terms of physics, if $\int_{\mathbb{R}^n}|u|^2 = c$, the constant c denotes the number of particles, mass, density, etc. The readers are encouraged to refer to the papers [12,13,20] for more details on physical aspects. Furthermore, we mention that, for $\int_{\mathbb{R}^2}|u|^2 = c$, the techniques to deal with these problems are essentially the same. Without loss of generality, we only consider $\int_{\mathbb{R}^n}|u|^2 = 1$.

For convenience, the structure of our paper is arranged as follows. The main three theorems of this paper are introduced in Section 2. In Section 3, we shall give a detailed analysis of the limit properties of least energy $e(s,\beta)$ and minimizer $u_{s,\beta}$ as $s\searrow 0$ when $0<\beta<\beta^*$. In Section 4, for any $\beta>\beta^*$ and positive sequence $\{s_k\}$ with $s_k\searrow 0$ as $k\to\infty$, we are devoted not only to establishing a refined energy estimation of $e(s_k,\beta)$, but also analyzing the concrete limit behavior of constraint minimizer $u_{s_k,\beta}$.

2. Main Results

To obtain the detailed limit behavior of the L^2-normalized solution, some results on the existence and non-existence of constraint minimizers for (2) are necessary. For this reason, the potential $V(x)$ is restricted to satisfying

$$V(x)\in L^\infty_{loc}(\mathbb{R}^2) \text{ and } \lim_{|x|\to\infty}V(x) = +\infty \quad (V_1).$$

Next, we introduce a nonlinear scalar field equation [21],

$$-\Delta u + u - u^3 = 0,\ \ x\in\mathbb{R}^2,\ \ u\in H^1(\mathbb{R}^2), \tag{5}$$

where (5) admits a unique (up to translation) positive radially symmetric solution $Q\in H^1(\mathbb{R}^2)$. At the same time, one can obtain from (5) that

$$\|\nabla Q\|^2_{L^2} = \|Q\|^2_{L^2} = \frac{1}{2}\|Q\|^4_{L^4}. \tag{6}$$

In view of [22] (Proposition 4.1), the $Q(x)$ satisfies

$$|\nabla Q(x)|, Q(|x|) = O(|x|^{-\frac{1}{2}}e^{-|x|}) \ \text{ as } \ |x|\to\infty. \tag{7}$$

Furthermore, the following Gagliardo–Nirenberg (G-N) inequality [23] is necessary

$$\int_{\mathbb{R}^2} |u|^4 dx \leq \frac{2}{\|Q\|_{L^2}^2} \int_{\mathbb{R}^2} |\nabla u|^2 dx \int_{\mathbb{R}^2} |u|^2 dx, \ u \in H^1(\mathbb{R}^2), \tag{8}$$

where $Q(x)$ is given by (5).

Based on the assumption (V_1), we recall from [20,24,25] that the following existence and non-existence results of the constraint minimizer for $e(s,\beta)$ hold.

Theorem 1. *Assume that (V_1) holds and $s = 0$ in (1); then, there exists a critical value $\beta^* := (a+b)\|Q\|_{L^2}^2$ such that $e(0,\beta)$ has at least one minimizer if $0 < \beta < \beta^*$, and $e(0,\beta)$ has no minimizer for $\beta \geq \beta^*$. Furthermore, for any $s > 0$ and $\beta > 0$, $e(s,\beta)$ has at least one minimizer.*

Notice that $e(0,\beta)$ is a constrained minimization problem related to the classic G-P functional, and the conclusion of Theorem 1 comes from [20] (Theorem 1). Also for $s > 0$, $e(s,\beta)$ is associated with a Kirchhoff-type equation, and the related results can be obtained from [24] (Theorem 1.2) or [25] (Theorem 1). We, however, state the results of Theorem 1 here for the reader's convenience, and the detailed proof process of Theorem 1 can be found in [20,24,25].

The above Theorem 1 gives the fact that $e(0,\beta)$ has no minimizer for $\beta \geq \beta^*$, but for any $s > 0$ and $\beta > 0$, $e(s,\beta)$ has at least one minimizer. An interesting question is what happens on the constraint minimizer of $e(s,\beta)$ when s tends to 0 from the right. In truth, a priori analysis shows that the constant

$$\beta^* = (a+b)\|Q\|_{L^2}^2 \tag{9}$$

is a critical threshold, which is the criterion used to judge the constraint minimizer of $e(s,\beta)$ blowing up or converging as $s \searrow 0$. According to Theorem 1, we set $u_{s,\beta}$ be a constraint minimizer of $e(s,\beta)$ for any $s > 0$; then we always say that $|u_{s,\beta}|$ is also a nonnegative minimizer of $e(s,\beta)$ by the fact that $E(u_{s,\beta}) \geq E(|u_{s,\beta}|)$.

Next, we firstly establish some results on the limit properties of the least energy $e(s,\beta)$ and nonnegative minimizer $u_{s,\beta}$ as $s \searrow 0$ when $0 < \beta < \beta^*$.

Theorem 2. *Assume that (V_1) holds and that for any $s > 0$, $0 < \beta < \beta^*$, let $u_{s,\beta}$ be a nonnegative minimizer of $e(s,\beta)$. Then, we have*

$$\lim_{s \searrow 0} e(s,\beta) = e(0,\beta). \tag{10}$$

Furthermore, for any $s > 0$, it admits a $u_0 \in \mathcal{H}$ such that as $s \searrow 0$

$$u_{s,\beta} \to u_0 \ \text{ strongly in } \mathcal{H}, \tag{11}$$

where u_0 is a nonnegative minimizer of $e(0,\beta)$.

Theorem 1 shows that for any $s > 0$ and $\beta > \beta^*$, $e(s,\beta^*)$ has at least one minimizer. However, for $s = 0$ and $\beta > \beta^*$, $e(s,\beta^*)$ has no minimizer. We next are concerned with the limit property of the minimizer as $s \searrow 0$ when $\beta > \beta^*$. For this goal, a more appropriate assumption about the potential $V(x)$ is given as follows:

$$V(x) \in C_{loc}^{\alpha}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2), \ \alpha \in (0,1) \text{ and } \min_{\mathbb{R}^2} V(x) = 0 \quad (V_2).$$

Further, we denote the class of minima for $V(x)$ by

$$\mathcal{Z} = \{x_i : V(x_i) = 0, \ i = 1,2,3\cdots\}. \tag{12}$$

Theorem 3. *Suppose that* (V_1) *and* (V_2) *hold. For any* $\beta > \beta^*$ *and positive sequence* $\{s_k\}$ *with* $s_k \searrow 0$ *as* $k \to \infty$, *set* $u_{s_k,\beta}$ *to be the nonnegative minimizer of* $e(s_k, \beta)$, *and then* $u_{s_k,\beta}$ *exists as a unique local maximum point* $z_{s_k,\beta} \in \mathbb{R}^2$ *such that* $\{s_k\}$ *has a subsequence (still denoted by* $\{s_k\}$*) satisfying, as* $k \to \infty$,

$$\lim_{k\to\infty} z_{s_k,\beta} = x_0 \in \mathcal{Z} \text{ and } V(x_0) = 0 \tag{13}$$

and

$$\epsilon_{s_k,\beta} u_{s_k,\beta}(\epsilon_{s_k,\beta} x + z_{s_k,\beta}) \to \frac{Q(|x|)}{\|Q\|_{L^2}} \quad \text{strongly in } H^1(\mathbb{R}^2),$$

where Q *is given by* (5) *and* $\epsilon_{s_k,\beta} := \left(\int_{\mathbb{R}^2} |\nabla u_{s_k,\beta}|^2 dx\right)^{-\frac{1}{2}}$. *Moreover, the* $\epsilon_{s_k,\beta}$ *and* $e(s_k, \beta)$ *satisfy* $k \to \infty$

$$\epsilon_{s_k,\beta}^{-2s_k} \to \frac{b\beta + a(\beta - \beta^*)}{b\beta^*} \tag{14}$$

and

$$e(s_k, \beta) \approx -\frac{s_k b}{s_k + 1}\left[\frac{b\beta + a(\beta - \beta^*)}{b\beta^*}\right]^{\frac{s_k+1}{s_k}}, \tag{15}$$

where the function $f(s_k) \approx g(s_k)$ *means* $\lim_{k\to\infty} \frac{f(s_k)}{g(s_k)} = 1$.

3. Limit Properties of $0 < \beta < \beta^*$

As stated in Theorem 1, $e(s, \beta)$ has at least one minimizer for any $0 < \beta < \beta^*$ and $s \geq 0$. In the following section, we shall give the proof of Theorem 2, which is related to the limit properties of least energy $e(s, \beta)$ and nonnegative minimizer $u_{s,\beta}$ as $s \searrow 0$. Before this, we introduce a well-known compact embedding theorem [26] (Theorem 2.1) such that if $V(x)$ satisfies (V_1), then

$$\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2)(2 \leq q < \infty) \text{ is compact.} \tag{16}$$

Proof of Theorem 2. For $0 < \beta < \beta^*$ and $s > 0$, we define the constrained minimization problem without nonlinear terms such as

$$d(s) := \inf_{u\in\mathcal{K}} E_s(u) \tag{17}$$

and

$$E_s(u) = a\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{b}{s+1}\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{s+1} + \int_{\mathbb{R}^2} V(x)u^2 dx. \tag{18}$$

We first claim that (17) has at least one minimizer. In truth, it is easy to know from (18) that for any $s > 0$, $E_s(u)$ is bounded from below on $\mathcal{K}$. Let $\{u_n\}$ be a minimizing sequence of $d(s)$; one can obtain from (17) and (18) that $\{u_n\}$ is bounded in $\mathcal{H}$. Furthermore, the (16) yields the fact that there exists a $u_s \in \mathcal{K}$ and $\{u_n\}$ has a subsequence (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u_s \text{ weakly in } \mathcal{K},\ u_n \to u_s \text{ strongly in } L^q(2 \leq q < \infty),$$

which together with

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^2} |\nabla u_s|^2 dx$$

gives

$$d(s) = \liminf_{n\to\infty} E_s(u_n) \geq E_s(u_s) \geq d(s). \tag{19}$$

Therefore, we conclude that u_s is a minimizer of $d(s)$ for any $s > 0$.

For any $s > 0$, set u_s be a minimizer of $d(s)$, then one can obtain from (17) that

$$d(s) = E_s(u_s). \tag{20}$$

In fact, from a fixed function $\eta(x) \in C_0^\infty(\mathbb{R}^2) \cap \mathcal{K}$, we obtain from (18) and (20) that there exists a constant $M > 0$ satisfying

$$0 \leq \lim_{s \searrow 0} d(s) = \lim_{s \searrow 0} E_s(u_s) \leq \lim_{s \searrow 0} E_s(\eta) \leq M. \tag{21}$$

For $0 < \beta < \beta^*$, applying the definitions of $d(s)$ and $e(s,\beta)$, we then deduce from (20) and (21) that as $s \searrow 0$,

$$e(s,\beta) \leq E_{s,\beta}(u_s) \leq E_s(u_s) = d(s) \leq E_s(\eta) \leq M. \tag{22}$$

In view of Theorem 1, set $u_{s,\beta}$ be a nonnegative minimizer of $e(s,\beta)$; then, we have

$$\begin{aligned} e(s,\beta) = E_{s,\beta}(u_{s,\beta}) =& a\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx\Big)^{s+1} \\ &+ \int_{\mathbb{R}^2} V(x) u_{s,\beta}^2 dx - \frac{\beta}{2}\int_{\mathbb{R}^2} |u_{s,\beta}|^4 dx. \end{aligned} \tag{23}$$

Next, we claim that there exists a constant $D > 0$ such that

$$\lim_{s \searrow 0} \int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx \leq D. \tag{24}$$

If (24) is false, that is, $\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx \to \infty$ as $s \searrow 0$. Since $0 < \beta < \beta^*$, applying the G-N inequality (8), we then deduce from (9) and (23) that there exists an arbitrarily large constant $\mathcal{P} > 0$ satisfying

$$\begin{aligned} e(s,\beta) = E_{s,\beta}(u_{s,\beta}) =& a\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx\Big)^{s+1} \\ &+ \int_{\mathbb{R}^2} V(x) u_{s,\beta}^2 dx - \frac{\beta(a+b)}{\beta^*}\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx \\ \geq& (a+b)\Big(\frac{\beta^* - \beta}{\beta^*}\Big)\int_{\mathbb{R}^2} |\nabla u_{s,\beta}|^2 dx \geq \mathcal{P}, \end{aligned} \tag{25}$$

which is a contradiction with (22). Therefore, (24) holds, and then $\{u_{s,\beta}\}$ is bounded in $\mathcal{H}$.

Using again (16), it implies that there exists a $u_0 \in \mathcal{K}$, and $\{u_{s,\beta}\}$ has a subsequence (still denoted by $\{u_{s,\beta}\}$) such that as $s \searrow 0$,

$$u_{s,\beta} \rightharpoonup u_0 \text{ weakly in } \mathcal{K}, u_{s,\beta} \to u_0 \text{ strongly in } L^q (2 \leq q < \infty). \tag{26}$$

Combining (23) and (26), one can obtain

$$\lim_{s \searrow 0} e(s,\beta) = \lim_{s \searrow 0} E_{s,\beta}(u_{s,\beta}) \geq \liminf_{s \searrow 0} E_{s,\beta}(u_{s,\beta}) \geq E_{0,\beta}(u_0) \geq e(0,\beta). \tag{27}$$

Additionally, by the definition of $e(0,\beta)$ and (22), for any $\epsilon > 0$, there exists a $u_\epsilon \in \mathcal{K}$ satisfying

$$E_{0,\beta}(u_\epsilon) \leq e(0,\beta) + \epsilon \leq M + \epsilon. \tag{28}$$

Similar to the proof of (24), we deduce from (28) that u_ϵ is bounded in $\mathcal{H}$. By this fact, we have

$$\lim_{s \searrow 0} \Big[\frac{b}{s+1}\Big(\int_{\mathbb{R}^2} |\nabla u_\epsilon|^2 dx\Big)^{s+1} - b\int_{\mathbb{R}^2} |\nabla u_\epsilon|^2 dx\Big] = 0, \tag{29}$$

which, together with (28) and (29), gives that as $s \searrow 0$,

$$\begin{aligned} e(s,\beta) \leq E_{s,\beta}(u_\epsilon) =& E_{0,\beta}(u_\epsilon) + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2} |\nabla u_\epsilon|^2 dx\Big)^{s+1} \\ &- b\int_{\mathbb{R}^2} |\nabla u_\epsilon|^2 dx \leq e(0,\beta) + \epsilon. \end{aligned} \tag{30}$$

Combining with (27)–(30), we have

$$e(0,\beta) \le E_{0,\beta}(u_0) \le \lim_{s\searrow 0} e(s,\beta) \le e(0,\beta)+\epsilon. \tag{31}$$

Letting $\epsilon \to 0$, we then conclude from (31) that u_0 is a minimizer of $e(0,\beta)$ and $u_{s,\beta} \to u_0$ strongly in $\mathcal{H}$ as $s \searrow 0$. We thereby complete the proof of Theorem 2. □

4. Limit Behavior Analysis of $\beta > \beta^*$

In this section, we are concerned with the limit property of minimizer as $s \searrow 0$ when $\beta > \beta^*$. Before this, we define a constrained minimization problem without potential such as

$$\bar{e}(s,\beta) := \inf_{u\in\mathcal{S}} \bar{E}_{s,\beta}(u), \tag{32}$$

where $\bar{E}_{s,\beta}(u)$ and S satisfy

$$\bar{E}_{s,\beta}(u) = a\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2} |\nabla u|^2 dx\Big)^{s+1} - \frac{\beta}{2}\int_{\mathbb{R}^2} |u|^4 dx,$$

$$\mathcal{S} := \Big\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = 1\Big\}.$$

In order to attain our goal, the following existence result of constrained minimizer for (32) is established for any $s > 0$ when $\beta > \beta^*$. Note from [24] (Theorem 1.1) that for $s = 0$, a similar result can also be found.

Lemma 1. *For any $s > 0$ and $\beta > \beta^*$, $\bar{e}(s,\beta)$ has at least one nonnegative minimizer.*

Proof. We firstly claim that for any $s > 0$ and $\beta > \beta^*$,

$$\bar{e}(s,\beta) < 0. \tag{33}$$

In truth, we choose a test function such as

$$u_\theta(x) = \frac{\theta Q(\theta x)}{\|Q\|_{L^2}} (\theta > 0) \text{ and } Q(x) \text{ is given by (8).} \tag{34}$$

It is derived from (6), (34) that $u_\theta \in \mathcal{S}$ and

$$\int_{\mathbb{R}^2} |\nabla u_\theta|^2 dx = \theta^2,\ \Big(\int_{\mathbb{R}^2} |\nabla u_\theta|^2 dx\Big)^{s+1} = \theta^{2(s+1)},\ \frac{\beta}{2}\int_{\mathbb{R}^2} |u_\theta|^4 dx = \frac{\beta}{\|Q\|_{L^2}^2}\theta^2. \tag{35}$$

Hence, it follows from (9) and (35) that

$$\begin{aligned} \bar{E}_{s,\beta}(u_\theta) &= a\theta^2 + \frac{b}{s+1}\theta^{2(s+1)} - \frac{\beta}{\|Q\|_{L^2}^2}\theta^2 \\ &= a\theta^2 + \frac{b}{s+1}\theta^{2(s+1)} - \frac{(a+b)\beta}{\beta^*}\theta^2. \end{aligned} \tag{36}$$

For $s > 0$ and $\beta > \beta^*$, choosing $\theta = (\frac{\beta}{\beta^*})^{\frac{1}{2s}}$ and putting it into (36), we have

$$\bar{e}(s,\beta) \le \bar{E}_{s,\beta}(u_\theta) = -a\Big(\frac{\beta}{\beta^*}\Big)^{\frac{1}{s}}\Big(\frac{\beta-\beta^*}{\beta}\Big) - \frac{sb}{s+1}\Big(\frac{\beta}{\beta^*}\Big)^{\frac{s+1}{s}} < 0. \tag{37}$$

Therefore, (33) holds.

By applying (8), one obtains

$$\bar{E}_{s,\beta}(u) \geq a\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)^{s+1} - \frac{(a+b)\beta}{\beta^*}\int_{\mathbb{R}^2}|\nabla u|^2dx \tag{38}$$

which yields that for any fixed $s > 0$, $\bar{E}_{s,\beta}(u)$ is bounded from below on $\mathcal{S}$. According to (33) and (38), and choosing $\{\bar{u}_n\}$ as a minimizing sequence of $\bar{e}(s,\beta)$, it then follows from (33) and (38) that $\{\bar{u}_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Recall from [27] (Appendix A.III) and [24] (Lemma 2.3), we see that there exists a nonnegative and non-increasing sequence $\{\bar{u}_n^*\} \subseteq \mathcal{S} \cap H_r^1(\mathbb{R}^2)$ such that $\{\bar{u}_n^*\}$ is also a minimizing sequence of $\bar{e}(s,\beta)$. At the same time, $\{\bar{u}_n^*\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Applying [28] (Proposition 1.7.1), there exists a subsequence of $\{\bar{u}_n^*\}$ (still denoted by $\{\bar{u}_n^*\}$) and a function $\bar{u}_0 \in H_r^1(\mathbb{R}^2)$ satisfying

$$\bar{u}_n^* \rightharpoonup \bar{u}_0 \text{ weakly in } H_r^1(\mathbb{R}^2),\ \bar{u}_n^* \to \bar{u}_0 \text{ strongly in } L^q(\mathbb{R}^2)\ (2 < q < \infty). \tag{39}$$

In the following, we prove that $\int_{\mathbb{R}^2}|\bar{u}_0|^2dx = 1$. Firstly, one can derive from (39) that $\int_{\mathbb{R}^2}|\bar{u}_0|^2dx \neq 0$. If not, we have, as $n \to \infty$,

$$\int_{\mathbb{R}^2}|\bar{u}_n^*|^4dx \to 0,$$

which, together with (32) and (33), yields that

$$0 > \bar{e}(s,\beta) = \lim_{n\to\infty}\Big[a\int_{\mathbb{R}^2}|\nabla\bar{u}_n^*|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla\bar{u}_n^*|^2dx\Big)^{s+1}\Big] \geq 0. \tag{40}$$

However, this is a contradiction, and hence, $\int_{\mathbb{R}^2}|\bar{u}_0|^2dx \neq 0$. Using the fact, we obtain from (33) and (39) that

$$0 > \bar{e}(s,\beta) = \lim_{n\to\infty}\bar{E}_{s,\beta}(\bar{u}_n^*) \geq \bar{E}_{s,\beta}(\bar{u}_0). \tag{41}$$

Denote $l := \int_{\mathbb{R}^2}|\bar{u}_0|^2dx$; then the above results show that $l \in (0,1]$. Setting $\bar{u}_l(x) := \bar{u}_0(l^{\frac{1}{2}}x) \in \mathcal{S}$, we obtain from (32) and (41) that

$$\begin{aligned}
\bar{e}(s,\beta) &\leq \bar{E}_{s,\beta}(\bar{u}_l)\\
&= a\int_{\mathbb{R}^2}|\nabla\bar{u}_0|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla\bar{u}_0|^2dx\Big)^{s+1} - \frac{\beta}{2l}\int_{\mathbb{R}^2}|\bar{u}_0|^4dx\\
&= \frac{1}{l}\Big[al\int_{\mathbb{R}^2}|\nabla\bar{u}_0|^2dx + \frac{bl}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla\bar{u}_0|^2dx\Big)^{s+1} - \frac{\beta}{2}\int_{\mathbb{R}^2}|\bar{u}_0|^4dx\Big]\\
&\leq \frac{1}{l}\bar{E}_{s,\beta}(\bar{u}_0) \leq \frac{1}{l}\bar{e}(s,\beta),
\end{aligned} \tag{42}$$

this yields from (33) that $l \geq 1$, that is, $\int_{\mathbb{R}^2}|\bar{u}_0|^2dx \geq 1$. Since $l = \int_{\mathbb{R}^2}|\bar{u}_0|^2dx \in (0,1]$, one has $\int_{\mathbb{R}^2}|\bar{u}_0|^2dx = 1$, which, together with (41) and (42), implies that $\bar{u}_0$ is a minimizer of $\bar{e}(s,\beta)$. By the definition of $\bar{e}(s,\beta)$, we see that $|\bar{u}_0|$ is a minimizer, too. Therefore, we always say that $\bar{e}(s,\beta)$ has at least one nonnegative minimizer. □

Lemma 2. *For any fixed $\beta > \beta^*$, set $\bar{u}_{s,\beta}$ as a nonnegative minimizer of $\bar{e}(s,\beta)$; then, $\bar{e}(s,\beta)$ satisfies, as $s \searrow 0$,*

$$\bar{e}(s,\beta) \approx -\frac{sb}{s+1}\Big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\Big]^{\frac{s+1}{s}}, \tag{43}$$

where $f(s) \approx g(s)$ means $\lim_{s\to 0}\frac{f(s)}{g(s)} = 1$. As $s \searrow 0$, the minimizer $\bar{u}_{s,\beta}$ behaves like

$$\bar{u}_{s,\beta} = \frac{\theta Q(\theta x)}{\|Q\|_{L^2}} \text{ and } \theta = \Big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\Big]^{\frac{1}{2s}}, \tag{44}$$

where Q satisfies Equation (5).

Proof. Choosing the same test function as (34), similar to the estimations (35) and (36), we have

$$\bar{E}_{s,\beta}(u_\theta) = a\theta^2 + \frac{b}{s+1}\theta^{2(s+1)} - (a+b)\frac{\beta}{\beta^*}\theta^2. \tag{45}$$

For any fixed $\beta > \beta^*$, we choose $\theta^2 = \big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\big]^{\frac{1}{s}} \to \infty$ as $s \searrow 0$. Taking θ into (45), it then yields that as $s \searrow 0$

$$\bar{e}(s,\beta) \le \bar{E}_{s,\beta}(u_\theta) \le -\frac{sb}{s+1}\big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\big]^{\frac{s+1}{s}}. \tag{46}$$

We next give the lower energy estimation of $\bar{e}(s,\beta)$ when $\beta > \beta^*$ as $s \searrow 0$. In view of Lemma 1, we assume that $\bar{u}_{s,\beta} \in \mathcal{S}$ is a nonnegative minimizer of $\bar{e}(s,\beta)$. Applying the G-N inequality (8) and (9), it is deduced that

$$\begin{aligned}\bar{e}(s,\beta) &= a\int_{\mathbb{R}^2}|\nabla \bar{u}_{s,\beta}|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla \bar{u}_{s,\beta}|^2dx\Big)^{s+1} - \frac{\beta}{2}\int_{\mathbb{R}^2}|\bar{u}_{s,\beta}|^4dx \\ &\ge \big[a - (a+b)\frac{\beta}{\beta^*}\big]\int_{\mathbb{R}^2}|\nabla \bar{u}_{s,\beta}|^2dx + \frac{b}{s+1}\Big(\int_{\mathbb{R}^2}|\nabla \bar{u}_{s,\beta}|^2dx\Big)^{s+1}.\end{aligned} \tag{47}$$

Defining a function $f(t) := \big[a - (a+b)\frac{\beta}{\beta^*}\big]t + \frac{b}{s+1}t^{s+1}$ and $t = \int_{\mathbb{R}^2}|\nabla \bar{u}_{s,\beta}|^2dx \in (0,+\infty)$, a simple calculation shows that $f(t)$ attains its unique minimum at

$$t = t_s = \big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\big]^{\frac{1}{s}}. \tag{48}$$

It then follows from (47) that as $s \searrow 0$

$$\bar{e}(s,\beta) = f(t) \ge f(t_s) \ge -\frac{sb}{s+1}\big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\big]^{\frac{s+1}{s}}. \tag{49}$$

Moreover, the procedures of (46) and (49) yield that $\bar{e}(s,\beta)$ attains its minimum at $\bar{u}_{s,\beta} = \frac{\theta Q(\theta x)}{\|Q\|_{L^2}}$ as $s \searrow 0$, where $\theta = \big[\frac{b\beta + a(\beta-\beta^*)}{b\beta^*}\big]^{\frac{1}{2s}}$. □

To obtain more detailed information about the minimizer and $e(s,\beta)$ as $s \searrow 0$, we always assume that $u_{s,\beta}$ is a nonnegative minimizer of $e(s,\beta)$. It then follows that $u_{s,\beta}$ satisfies the elliptic equation

$$-\Big(a + b\Big(\int_{\mathbb{R}^2}|\nabla u_{s,\beta}|^2dx\Big)^s\Big)\Delta u_{s,\beta} + V(x)u_{s,\beta} = \mu_{s,\beta}u_{s,\beta} + \beta|u_{s,\beta}|^2u_{s,\beta},\ x \in \mathbb{R}^2, \tag{50}$$

where $\mu_{s,\beta} \in \mathbb{R}$ denotes a Lagrange multiplier. Furthermore, we set

$$\epsilon_{s,\beta} := \Big(\int_{\mathbb{R}^2}|\nabla u_{s,\beta}|^2dx\Big)^{-\frac{1}{2}} > 0. \tag{51}$$

Lemma 3. *For $\beta > \beta^*$, the $\epsilon_{s,\beta}$, $e(s,\beta)$ and potential energy satisfy $s \searrow 0$*

$$\epsilon_{s,\beta} \to 0, \quad e(s,\beta) - \bar{e}(s,\beta) \to 0 \tag{52}$$

and

$$\int_{\mathbb{R}^2} V(x)u_{s,\beta}^2dx \to 0. \tag{53}$$

Moreover, as $s \searrow 0$,

$$\epsilon_{s,\beta}^{-2s} \to \frac{b\beta + a(\beta-\beta^*)}{b\beta^*}. \tag{54}$$

Proof. If $\epsilon_{s,\beta} \nrightarrow 0$, then $\{u_{s,\beta}\}$ is bounded in $\mathcal{H}$. Applying (16), there admits a $\hat{u} \in \mathcal{K}$ and $\{u_{s,\beta}\}$ exists as a subsequence (still denoted by $\{u_{s,\beta}\}$) satisfying, as $s \searrow 0$,

$$u_{s,\beta} \rightharpoonup \hat{u} \text{ weakly in } \mathcal{K}, u_{s,\beta} \to \hat{u} \text{ strongly in } L^q (2 \le q < \infty). \tag{55}$$

Similar to (27) and (31), it is deduced that

$$e(0,\beta) \le E_{0,\beta}(\hat{u}) \le \lim_{s \searrow 0} e(s,\beta) \le e(0,\beta), \tag{56}$$

which yields that $\hat{u}$ is a minimizer of $e(0,\beta)$. But this is impossible because Theorem 1 shows that $e(0,\beta)$ has no minimizer for $\beta > \beta^*$. Hence, we have $\epsilon_{s,\beta} \to 0$ as $s \searrow 0$.

On the one hand, by the definitions of $e(s,\beta)$ and $\bar{e}(s,\beta)$, one directly derives that

$$e(s,\beta) - \bar{e}(s,\beta) \ge 0. \tag{57}$$

On the other hand, we turn to estimate an upper bound of $e(s,\beta) - \bar{e}(s,\beta)$ as $s \searrow 0$. Toward this goal, we choose a cut-off function $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$ such as $0 \le \varphi(x) \le 1$, and $\varphi(x) = 1$ for $|x| \le 1$, $\varphi(x) = 0$ for $|x| \ge 2$, $|\nabla \varphi(x)| \le 2$ for $x \in \mathbb{R}^2$. Define

$$\hat{u}_{s,\beta} := A_{s,\beta} \varphi(x - x_0) \bar{u}_{s,\beta}(x - x_0), \tag{58}$$

where x_0 satisfies $V(x_0) = 0$ and $\bar{u}_{s,\beta}$ is given by (44). The above $A_{s,\beta}$ is chosen so that $\int_{\mathbb{R}^2} |\hat{u}_{s,\beta}|^2 dx = 1$. Applying (7), one can calculate from (58) that

$$1 \le A_{s,\beta}^2 \le 1 + Ce^{-2\theta} \text{ as } s \searrow 0. \tag{59}$$

Based on this fact, we then have

$$\int_{\mathbb{R}^2} |\nabla \hat{u}_{s,\beta}|^2 dx \le \int_{\mathbb{R}^2} |\nabla \bar{u}_{s,\beta}|^2 dx + Ce^{-\theta} \text{ as } s \searrow 0, \tag{60}$$

$$\int_{\mathbb{R}^2} |\hat{u}_{s,\beta}|^4 dx \ge \int_{\mathbb{R}^2} |\bar{u}_{s,\beta}|^4 dx - Ce^{-2\theta} \text{ as } s \searrow 0 \tag{61}$$

and

$$\int_{\mathbb{R}^2} V(x) |\hat{u}_{s,\beta}|^2 dx = V(x_0) + o(1) = o(1), \tag{62}$$

where $o(1) \to 0$ as $s \searrow 0$. Equations (60)–(62) together with (57) yield that

$$\begin{aligned} 0 \le e(s,\beta) - \bar{e}(s,\beta) \le & E_{s,\beta}(\hat{u}_{s,\beta}) - \bar{E}_{s,\beta}(\bar{u}_{s,\beta}) + \int_{\mathbb{R}^2} V(x) |\hat{u}_{s,\beta}|^2 dx \\ \le & Ce^{-\frac{1}{2}\theta} + o(1) \to 0 \text{ as } s \searrow 0 \end{aligned}$$

which then completes the proof of (52). Furthermore, we obtain from the definitions of $e(s,\beta)$ and $\bar{e}(s,\beta)$ that

$$\int_{\mathbb{R}^2} V(x) u_{s,\beta}^2 dx = E_{s,\beta}(u_{s,\beta}) - \bar{E}_{s,\beta}(u_{s,\beta}) \le e(s,\beta) - \bar{e}(s,\beta) \to 0 \text{ as } s \searrow 0,$$

and hence (53) holds. Furthermore, the estimation of (54) is deduced directly from Lemma 2 and (52) and (53). □

Lemma 4. *For $\beta > \beta^*$ and any positive sequence $\{s_k\}$ with $s_k \searrow 0$ as $k \to \infty$, the nonnegative minimizer $u_{s_k,\beta}$ has at least one local maximum point $z_{s_k,\beta}$. Define an L^2-normalized function*

$$w_{s_k,\beta}(x) := \epsilon_{s_k,\beta} u_{s_k,\beta}(\epsilon_{s_k,\beta} x + z_{s_k,\beta}), \ \epsilon_{s_k,\beta} \tag{63}$$

as given by (51)*; then, there exists a finite ball* $B_2(0)$ *and a constant* $\eta > 0$ *such that*

$$\int_{B_2(0)} w^2_{s_k,\beta}(x)dx \geq \eta > 0. \tag{64}$$

Furthermore, $z_{s_k,\beta}$ *admits a convergent subsequence (still denoted by* $z_{s_k,\beta}$*) such that*

$$z_{s_k,\beta} \to z_0 \text{ as } k \to \infty,$$

where z_0 *satisfies* $V(z_0) = 0$*, that is, where* z_0 *is a global minimum point of* $V(x)$*.*

Proof. Since $u_{s_k,\beta}$ is a nonnegative minimizer of $e(s_k, \beta)$, it satisfies (50). Using the results of Lemma 2 and (52), we have, as $k \to \infty$,

$$e(s_k, \beta) \approx -\frac{s_k b}{s_k + 1}\Big[\frac{b\beta + a(\beta - \beta^*)}{b\beta^*}\Big]^{\frac{s_k+1}{s_k}}$$

which then yields from (54) that, as $k \to \infty$,

$$\epsilon^{2(s_k+1)}_{s_k,\beta} e(s_k, \beta) \approx \frac{s_k b}{s_k + 1} \to 0. \tag{65}$$

Define an L^2-normalized function

$$\bar{w}_{s_k,\beta}(x) := \epsilon_{s_k,\beta} u_{s_k,\beta}(\epsilon_{s_k,\beta} x),\ \epsilon_{s_k,\beta} \text{ is given by (51).} \tag{66}$$

One then obtains from (53) and (65) that, as $k \to \infty$,

$$\epsilon^{2(s_k+1)}_{s_k,\beta} e(s_k, \beta) = a\epsilon^{2s_k}_{s_k,\beta} + \frac{b}{s_k + 1} - \frac{\beta}{2}\epsilon^{2s_k}_{s_k,\beta}\int_{\mathbb{R}^2} |\bar{w}_{s_k,\beta}|^4 dx \to 0$$

which derives from (54) that

$$\int_{\mathbb{R}^2} |\bar{w}_{s_k,\beta}|^4 dx \to \frac{2(a + b)}{\beta^*} \text{ as } k \to \infty. \tag{67}$$

Because $u_{s_k,\beta}$ satisfies (50), by the definition of $e(s_k, \beta)$, it follows that $\mu_{s_k,\beta}$ fulfills

$$\mu_{s_k,\beta} = e(s_k, \beta) + \frac{s_k b}{s_k + 1}\Big(\int_{\mathbb{R}^2} |\nabla u_{s_k,\beta}|^2 dx\Big)^{s_k+1} - \frac{\beta}{2}\int_{\mathbb{R}^2} |u_{s_k,\beta}|^4 dx,$$

which, together with (54), (65) and (67), yields that as $k \to \infty$,

$$\begin{aligned}\mu_{s_k,\beta}\epsilon^{2(s_k+1)}_{s_k,\beta} =& \epsilon^{2(s_k+1)}_{s_k,\beta} e(s_k, \beta) + \frac{s_k b}{s_k + 1} \\ &- \frac{\beta}{2}\epsilon^{2s_k}_{s_k,\beta}\int_{\mathbb{R}^2} |\bar{w}_{s_k,\beta}|^4 dx \to -\frac{b\beta(a + b)}{b\beta + a(\beta - \beta^*)}.\end{aligned} \tag{68}$$

Based on (V_1), (V_2) and (68), one can use the method of [29] (Lemma 3.5) to acquire that $u_{s_k,\beta}$ has at least one local maximum point by applying the standard regularity theory and comparison principle to elliptic Equation (50). Set $z_{s_k,\beta}$ as the local maximum point and define the L^2-normalized function $w_{s_k,\beta}$ such as (63); then, it follows from (50), (54) and (67) that

$$\int_{\mathbb{R}^2} |w_{s_k,\beta}|^4 dx \to \frac{2(a + b)}{\beta^*} > 0 \text{ as } k \to \infty \tag{69}$$

and $w_{s_k,\beta}$ satisfies

$$\begin{aligned} -\Delta w_{s_k,\beta} + A_{s_k}\epsilon_{s_k,\beta}^{2(s_k+1)} V(\epsilon_{s_k,\beta}x + z_{s_k,\beta}) w_{s_k,\beta} \\ = A_{s_k}\mu_{s_k,\beta}\epsilon_{s_k,\beta}^{2(s_k+1)} w_{s_k,\beta} + A_{s_k}\beta\epsilon_{s_k,\beta}^{2s_k} w_{s_k,\beta}^3,\ x \in \mathbb{R}^2 \end{aligned} \tag{70}$$

where $\lim\limits_{k\to\infty} A_{s_k} = \frac{b\beta + a(\beta-\beta^*)}{b\beta(a+b)} > 0$. Thus, for s_k small enough, we have

$$-\Delta w_{s_k,\beta} - c(x) w_{s_k,\beta} \le 0 \text{ in } \mathbb{R}^2,$$

where $c(x) = A_{s_k}\beta\epsilon_{s_k,\beta}^{2s_k} w_{s_k,\beta}^2$. Applying the De Giorgi–Nash–Moser theory [30] (Theorem 4.1), one obtains that

$$\max_{B_1(0)} w_{s_k,\beta} \le C\Big(\int_{B_2(0)} |w_{s_k,\beta}|^2 dx\Big)^{\frac{1}{2}}, \tag{71}$$

where $C > 0$ is a constant dependent on the upper bound of $\|w_{s_k,\beta}\|_{L^4(B_2(0))}$.

In fact, we see that 0 is a local maximum point of $w_{s_k,\beta}$ due to $z_{s_k,\beta}$, which is a local maximum point of $u_{s_k,\beta}$. We next argue that there exists a constant $\eta > 0$ satisfying

$$w_{s_k,\beta}(0) \ge \eta > 0 \text{ as } k \to \infty. \tag{72}$$

If (72) is not true, then for any $R > 0$, one obtain

$$\sup_{y\in\mathbb{R}^2} \int_{B_R(y)} |w_{s_k,\beta}|^2 dx \to 0 \text{ as } k \to \infty$$

which, together with the vanishing lemma [31] (Lemma 1.1), yields that $\int_{\mathbb{R}^2} |w_{s_k,\beta}|^4 dx \to 0$ as $k \to \infty$. However, this is a contradiction with (69). Hence, (72) holds, and then (64) follows from (71) and (72).

Next, we need to prove that $z_{s_k,\beta}$ is bounded uniformly as $k \to \infty$. If this is false, then there exists a subsequence $\{s_k\}$ (still denoted by $\{s_k\}$) with $s_k \searrow 0$ as $k \to \infty$ such that $z_{s_k,\beta} \to \infty$; one then obtains from (63) and (64) that

$$\begin{aligned} \int_{\mathbb{R}^2} V(x) u_{s_k,\beta}^2 dx &= \int_{\mathbb{R}^2} V(\epsilon_{s_k,\beta}x + z_{s_k,\beta}) |w_{s_k,\beta}|^2 dx \\ &\ge \int_{B_2(0)} V(\epsilon_{s_k,\beta}x + z_{s_k,\beta}) |w_{s_k,\beta}|^2 dx \ge \mathcal{D}\eta \end{aligned} \tag{73}$$

where $\mathcal{D}$ is an arbitrarily large constant. However, this contradicts (53). Hence, $\{z_{s_k,\beta}\}$ is a bounded sequence in $\mathbb{R}^2$.

Passing to a subsequence if necessary, there is a $z_0 \in \mathbb{R}^2$ such that $z_{s_k,\beta} \to z_0$. In truth, we can prove that $V(z_0) = 0$. If not, one denotes $K := V(z_0) > 0$; then, (64), together with Fatou's Lemma, gives

$$\begin{aligned} &\liminf_{k\to\infty} \int_{\mathbb{R}^2} V(\epsilon_{s_k,\beta}x + z_{s_k,\beta}) |w_{s_k,\beta}|^2 dx \\ &\ge \int_{B_2(0)} \liminf_{k\to\infty} V(\epsilon_{s_k,\beta}x + z_{s_k,\beta}) |w_{s_k,\beta}|^2 dx \ge V(z_0)\eta = K\eta > 0 \end{aligned} \tag{74}$$

which is a contradiction with (53). We thereby complete the proof of Lemma 4. □

In the final part, we shall give the proof of Theorem 3.

Proof of Theorem 3. Let $u_{s_k,\beta}$ be a nonnegative minimizer of $e(s_k,\beta)$ and $z_{s_k,\beta}$ be is its local maximum point. Define a function the same as (63), then, for any positive sequence $\{s_k\}$ with $s_k \searrow 0$ as $k \to \infty$. Using the definitions of (51) and (63), it follows that

$$\int_{\mathbb{R}^2} |w_{s_k,\beta}|^2 = 1 \text{ and } \int_{\mathbb{R}^2} |\nabla w_{s_k,\beta}|^2 = 1. \tag{75}$$

It then shows that $\{w_{s_k,\beta}\}$ is a bounded sequence in $H^1(\mathbb{R}^2)$. Passing to a subsequence if necessary (still denoted by $\{w_{s_k,\beta}\}$), there exists a $w_0 \in H^1(\mathbb{R}^2)$ satisfying, as $k \to \infty$,

$$w_{s_k,\beta} \rightharpoonup w_0 \text{ weakly in } H^1(\mathbb{R}^2). \tag{76}$$

Since $w_{s_k,\beta}$ satisfies the elliptic equation (70), passing to the weak limit, it is deduced from (9), (54), (68) and (76) that w_0 satisfies (in the weak sense)

$$-\Delta w_0 + w_0 = \|Q\|_{L^2}^2 w_0^3,\ x \in \mathbb{R}^2. \tag{77}$$

In fact, since (72) holds, we always say that $w_0 > 0$ applyies the strong maximum principle to (77). Moreover, because (5) has a unique (up to translation) positive radially symmetric solution $Q(x)$, it is restricted to the fact that there exists a $y_0 \in \mathbb{R}^2$ such that w_0 fulfills

$$w_0 = \frac{Q(|x - y_0|)}{\|Q\|_{L^2}} \tag{78}$$

which, together with (6), gives

$$\int_{\mathbb{R}^2} |w_0|^2 = 1,\ \int_{\mathbb{R}^2} |\nabla w_0|^2 = 1.$$

The above results show that $w_{s_k,\beta} \to w_0$ strongly in $L^2(\mathbb{R}^2)$ as $k \to \infty$. By applying the Hölder and Sobolev inequalities, one further derives that $\|u\|_{L^4(\mathbb{R}^2)} \le C\|u\|_{L^2(\mathbb{R}^2)}^{\gamma}\|u\|_{H^1(\mathbb{R}^2)}^{1-\gamma}$ for any $u \in H^1(\mathbb{R}^2)$ with $\gamma \in (0,1)$. This indicates that $w_{s_k,\beta} \to w_0$ strongly in $L^4(\mathbb{R}^2)$ as $k \to \infty$. One then obtains from (75)–(77) that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} |\nabla w_{s_k,\beta}|^2 dx = \int_{\mathbb{R}^2} |\nabla w_0|^2 dx,$$

that is,

$$w_{s_k,\beta} \to w_0 \text{ strongly in } H^1(\mathbb{R}^2) \text{ as } k \to \infty. \tag{79}$$

Under the assumption of (V_2), using the technique of proving Theorem 1.2 in [7], one infers from (70) that $w_{s_k,\beta} \in C_{loc}^{2,\alpha_1}(\mathbb{R}^2)$, $\alpha_1 \in (0,1)$. Hence, we have $w_0 \in C_{loc}^2(\mathbb{R}^2)$ and

$$w_{s_k,\beta} \to w_0 \text{ in } C_{loc}^2(\mathbb{R}^2) \text{ as } k \to \infty. \tag{80}$$

Since the origin is a unique critical point (up to translation) of $Q(x)$, one then concludes from (78) that the origin is the unique critical point of w_0. Therefore, one obtains

$$w_0(x) = \frac{1}{\|Q\|_{L^2}} Q(|x|). \tag{81}$$

At last, set y_k as the local maximum point of $w_{s_k,\beta}$; then, the (72) shows that $w_{s_k,\beta}(y_k) \ge \eta > 0$. Taking γ small enough, one infers from [32] (Lemma 4.2) that the origin is the unique local maximum point of $w_{s_k,\beta}$ as $k \to \infty$. It then yields that $z_{s_k,\beta}$ is the unique maximum point of $u_{s_k,\beta}$. The proof of Theorem 3 is thereby completed. □

Author Contributions: Formal analysis, H.W.; Data curation, H.W.; Writing—original draft, X.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported by National Nature Science Foundation of China (NSFC), grant number 11901500; Nanhu Scholars Program for Young Scholars of XYNU.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Anderson, M.H.; Ensher, J.R.; Matthews, M.R.; Wieman, C.E.; Cornell, E.A. Observation of Bose-Einstein condensation in a dilute atomic vapor. *Science* **1995**, *269*, 198–201. [CrossRef] [PubMed]
2. Dalfovo, F.; Giorgini, S.; Pitaevskii, L.P.; Stringari, S. Theory of Bose-Einstein condensation in trapped gases. *Rev. Mod. Phys.* **1999**, *71*, 463–512. [CrossRef]
3. Fetter, A.L. Rotating trapped Bose-Einstein condensates. *Rev. Mod. Phys.* **2009**, *81*, 647. [CrossRef]
4. Gross, E.P. Hydrodynamics of a superfluid condensate. *J. Math. Phys.* **1963**, *4*, 195–207. [CrossRef]
5. Pitaevskii, L.P. Vortex lines in an imperfect Bose gas. *Sov. Phys. JETP* **1961**, *13*, 451–454.
6. Guo, Y.J.; Wang, Z.Q.; Zeng, X.Y.; Zhou, H.S. Properties of ground states of attractive Gross–Pitaevskii equations with multi-well potentials. *Nonlinearity* **2018**, *31*, 957–979. [CrossRef]
7. Guo, Y.J.; Zeng, X.Y.; Zhou, H.S. Energy estimates and symmetry breaking in attactive Bose-Einstein condensates with ring-shaped potentials. *Ann. I'nst. H. Poincaré Anal. Non Linéaire* **2016**, *33*, 809–828.
8. Wang, Q.X.; Zhao, D. Existence and mass concentration of 2D attractive Bose-Einstein condensates with periodic potentials. *J. Differ. Equ.* **2017**, *262*, 2684–2704. [CrossRef]
9. Zhu, X.C.; Wang, C.J. Mass concentration behavior of attractive Bose-Einstein condensates with sinusoidal potential in a circular region. *Mediterr. J. Math.* **2024**, *21*, 12. [CrossRef]
10. Guo, Y.J.; Liang, W.N.; Li, Y. Existence and uniqueness of constraint minimizers for the planar Schrödinger-Poisson system with logarithmic potentials. *J. Differ. Equ.* **2023**, *369*, 299–352. [CrossRef]
11. Guo, Y.J.; Lin, C.S.; Wei, J.C. Local uniqueness and refined spike profiles of ground states for two-dimensional attractive Bose-Einstein condensates. *SIAM J. Math. Anal.* **2017**, *49*, 3671–3715. [CrossRef]
12. Ye, H.Y. The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations. *Z. Angew. Math. Phys.* **2015**, *66*, 1483–1497. [CrossRef]
13. Ye, H.Y. The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations. *Math. Methods Appl. Sci.* **2015**, *38*, 2663–2679. [CrossRef]
14. Meng, X.Y.; Zeng, X.Y. Existence and asymptotic behavior of minimizers for the Kirchhoff functional with periodic potentials. *J. Math. Anal. Appl.* **2022**, *507*, 125727. [CrossRef]
15. Guo, H.L.; Zhou, H.S. Properties of the minimizers for a constrained minimization problem arising in Kirchhoff equation. *Discrete Contin. Dyn. Syst.* **2021**, *41*, 1023–1050. [CrossRef]
16. Li, G.B.; Ye, H.Y. On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials. *J. Differ. Equ.* **2019**, *266*, 7101–7123. [CrossRef]
17. Li, Y.H.; Hao, X.C.; Shi, J.P. The existence of constrained minimizers for a class of nonlinear Kirchhoff-Schrödinger equations with doubly critical exponents in dimension four. *Nonlinear Anal.* **2019**, *186*, 99–112. [CrossRef]
18. Zhu, X.C.; Wang, C.J.; Xue, Y.F. Constraint minimizers of Kirchhoff-Schrödinger energy functionals with L^2-subcritical perturbation. *Mediterr. J. Math.* **2021**, *18*, 224. [CrossRef]
19. Hu, T.X.; Tang, C.L. Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations. *Calc. Var.* **2021**, *60*, 210. [CrossRef]
20. Guo, Y.J.; Seiringer, R. On the mass concentration for Bose-Einstein condensates with attactive interactions. *Lett. Math. Phys.* **2014**, *104*, 141–156. [CrossRef]
21. Kwong, M.K. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.* **1989**, *105*, 243–266. [CrossRef]
22. Gidas, B.; Ni, W.M.; Nirenberg, L. Symmetry of Positive Solutions of Nonlinear Elliptic Equations in $\mathbb{R}^n$. In *Mathematical Analysis and Applications*; Part A; Academic Press: New York, NY, USA, 1981; Volume 7, pp. 369–402.
23. Weinstein, M.I. Nonlinear Schrödinger equations and sharp interpolations estimates. *Comm. Math. Phys.* **1983**, *87*, 567–576. [CrossRef]
24. Guo, H.L.; Zhang, Y.M.; Zhou, H.S. Blow-up solutions for a Kirchhoff type elliptic equation with trapping potential. *Commun. Pur. Appl. Anal.* **2018**, *17*, 1875–1897. [CrossRef]
25. Zhu, X.C.; Wu, H.X. Existence and limit behavior of constraint minimizers for a varying non-local Kirchhoff-type energy functional. *Mathematics* **2024**, *12*, 661. [CrossRef]
26. Bartsch, T.; Wang, Z.Q. Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^n$. *Comm. Partial Differ. Equ.* **1995**, *20*, 1725–1741. [CrossRef]

27. Berestycki, H.; Lions, P.L. Nonlinear scalar field equations, I. Existence of a ground state. *Arch. Ration. Mech. Anal.* **1983**, *82*, 313–346. [CrossRef]
28. Cazenave, T. *Semilinear Schrödinger Equations*; Courant Institute of Mathematical Sciences: New York, NY, USA, 2003. [CrossRef]
29. Zeng, X.Y. Asymptotic properties of standing waves for mass subcritical nonlinear Schrödinger equations. *Discrete Contin. Dyn. Syst.* **2017**, *37*, 1749–1762. [CrossRef]
30. Han, Q.; Lin, F.H. *Elliptic Partial Differential Equations*; Courant Lecture Note in Mathematics 1; Courant Institute of Mathematical Science/AMS: New York, NY, USA, 2011.
31. Lions, P.L. The concentration-compactness principle in the caclulus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1984**, *1*, 223–283. [CrossRef]
32. Ni, W.M.; Takagi, I. On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **1991**, *44*, 819–851. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.