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Fuzzy Evaluation Model for Critical Components of Machine Tools

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Abstract: The rapid progression of emerging technologies like the Internet of Things (IoT) and Big Data analytics for manufacturing has driven innovation across various industries worldwide. Production data are utilized to construct a model for quality evaluation and analysis applicable to components processed by machine tools, ensuring process quality for critical components and final product quality for the machine tools. Machine tool parts often encompass several quality characteristics concurrently, categorized into three types: smaller-the-better, larger-the-better, and nominal-the-better. In this paper, an evaluation index for the nominal-the-better quality characteristic was segmented into two single-sided Six Sigma quality indexes. Furthermore, the process quality of the entire component product was assessed by n single-sided Six Sigma quality indexes. According to numerous studies, machine tool manufacturers conventionally base their decisions on small sample sizes (n), considering timeliness and costs. However, this often leads to inconsistent evaluation results due to significant sampling errors. Therefore, this paper established fuzzy testing rules using the confidence intervals of the q single-sided Six Sigma quality indices, serving as the fuzzy quality evaluation model for components of machine tools.

Keywords: machine tools; quality characteristics; single-sided Six Sigma quality indices; confidence intervals; fuzzy evaluation model

MSC: 62C05; 62C86



Citation: Chen, K.-S.; Yao, K.-C.; Cheng, C.-H.; Yu, C.-M.; Chang, C.-H. Fuzzy Evaluation Model for Critical Components of Machine Tools. *Axioms* **2024**, *13*, 555. <https://doi.org/10.3390/axioms13080555>

Academic Editor: Darjan Karabašević

Received: 5 July 2024

Revised: 2 August 2024

Accepted: 12 August 2024

Published: 14 August 2024



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1. Introduction

Numerous studies have indicated that an increasing number of machine tool manufacturers in the industry chain are responding to global warming and intensifying global competition. Accordingly, they prioritize the processing of competitive core parts in which they excel, while gradually outsourcing the production of components in which they are less proficient [1–3]. Many researchers have also highlighted that the in-process quality of each component can greatly influence the final product quality for machine tools. Consequently, a growing number of component-processing factories are striving to secure outsourcing orders from machine tool manufacturers by enhancing their process capabilities [4–6]. Obviously, as new technologies such as the Internet of Things (IoT) and Big Data analytics in manufacturing continue to swiftly advance, they are driving innovation across industries worldwide. A model for quality evaluation and analysis is built using production data, being suitable for all critical parts processed by machine tools. This model can verify the process quality for the key components and ensure the final product quality for the machine tools.

Chen et al. [7] claimed that several quality characteristics (QCs), composed of the smaller-the-better (STB), larger-the-better (LTB), and nominal-the-better (NTB) types, typically coexist in machine tool components. The tolerance for the NTB quality characteristic is given by

$T \pm d$, where T denotes the target value, $d = USL - T = T - LSL$, and USL and LSL represent the upper and lower specification limits, respectively. For example, the inner and outer diameters of an axis are considered *NBT* quality characteristics, while the axis itself is classified as an *STB* quality characteristic. Since these values are typically greater than zero and cannot be negative, therefore, the target value is denoted as $T = 0$ (roundness), and only the USL value is defined. Additionally, some characteristics, such as axial roundness and runout, are classified as *STB* quality characteristics. Furthermore, the normal operating time (lifetime) of the axis, where a longer duration is preferable, is classified as a functional *LTB* quality characteristic, with only the LSL value defined. According to Wu et al. [8] and Yin et al. [9], all quality characteristics must satisfy the required process capabilities to ensure that the final component products achieve the quality levels expected by customers. Chen et al. [10] modified both the unilateral specification index and bilateral specification index proposed by Kane [11] to introduce two unilateral Six Sigma quality indices, Q_{pu} and Q_{pl} , and one bilateral Six Sigma quality index, Q_{pk} . If the process mean m derived from the target value does not exceed 1.5 Sigma and the process standard deviation is equal to $1/6$ of the tolerance, then the process quality can be said to have reached the Six Sigma quality level [12,13]. Following this concept, when the process quality reaches k Sigma quality level, the process mean m from the target value does not exceed 1.5 Sigma, and the process standard deviation is equal to $1/k$ of the tolerance. Therefore, when a process reaches the k Sigma quality level, the value of Q_{pk} is at least k , and the relationship between Q_{pk} and process yield ($Yield\%$) is denoted as $Yield\% \geq 2\Phi(Q_{pk} - 1.5) - 1$. In other words, when the process reaches the 6 Sigma level, the value of Q_{pk} is greater than or equal to 6, and the process yield is given as $Yield\% \geq 2\Phi(4.5) - 1 = 0.9999932$. Evidently, the Six Sigma quality index can completely demonstrate both the process yield and process quality level.

In fact, Q_{pl} is suitable for the processes with *LTB* quality characteristics, whereas Q_{pu} is applicable to processes with *STB* quality characteristics. In addition, the Six Sigma quality index, $Q_{pk} = \min\{Q_{pu}, Q_{pl}\}$, is applicable to the processes with *NTB* quality characteristics. This paper assumes, without loss of generality, that components have a total of q quality characteristics, consisting of u *STB*, l *LTB*, and b *NTB* quality characteristics. In many engineering projects, Q_{pu} , Q_{pl} , and Q_{pk} are used to establish quality assessment models for key parts. However, due to the complexity of the Q_{pk} estimator, practical application can be challenging. According to numerous studies, machine tool manufacturers frequently make decisions based on small sample sizes (n), considering both timeliness and costs. Nevertheless, this practice often results in inconsistent evaluation results due to significant sampling errors [14–16]. To address the above problems, this paper applied the properties of Q_{pk} that can be decomposed into Q_{pu} and Q_{pl} . Then, Q_{pk} was replaced with Q_{pu} and Q_{pl} , and the confidence intervals of these q one-sided Six Sigma quality indexes for machine tool components were adopted. Subsequently, a fuzzy quality evaluation model was developed for machine tool components, aimed at facilitating industrial application. The fuzzy evaluation method proposed in this paper is not a traditional fuzzy method but is based on statistical confidence intervals [17,18]. This model can reduce the risk of misjudgments caused by sampling errors. At the same time, because it integrates expert knowledge and data accumulated from past experiences, it compensates for the issue of poor evaluation accuracy caused by small sample sizes (n). Additionally, the evaluation model established in this paper contributes to seizing improvement opportunities for all key parts and achieving the goal of enhancing quality levels. As engineers improve the process quality of all key parts, the quality levels of machine tools will also be boosted. This will not only increase the product value of all key parts and machine tools but also reduce the carbon footprint associated with rework and scrap after the product is sold. Finally, it can also lessen carbon emissions and cost losses resulting from maintenance.

The structure of the remaining sections is described as follows. Section 2 demonstrates how to establish the $100(1-\alpha)\%$ upper confidence limits for Six Sigma quality indices. Section 3 develops a fuzzy hypothesis testing method based on upper confidence limits and establishes a fuzzy quality evaluation model for machine tool components using these

fuzzy testing rules. Section 4 illustrates the application of the proposed fuzzy quality evaluation model with a data case study. Finally, Section 5 presents the conclusions.

2. Upper Confidence Limits for Six Sigma Quality Indexes

As mentioned in the Introduction, Six Sigma quality indices can fully reflect process yields and process quality levels. Additionally, this section illustrates how to utilize this feature, along with Boole's inequality and DeMorgan's rules, to derive confidence intervals. Subsequently, the background information related to these confidence intervals, presented in the Introduction, is incorporated into this section to help readers better understand the upper confidence limits for Six Sigma quality indices.

The evaluation index Q_{pk} containing *NTB* quality characteristics is divided into Q_{pu} and Q_{pl} . The total number of quality characteristics is $q = q_u + q_l$ where $q_u = u + b$ represents the total number of *STB* quality characteristics, and $q_l = l + b$ represents the total number of *LTB* quality characteristics. Additionally, q single-sided Six Sigma quality indexes can be employed to assess the overall process quality of entire component product. Under the assumption of normality, supposing the random variable X_h denotes a process distribution for the quality characteristic h , then X_h will follow a normal distribution with a mean μ_h and a standard deviation σ_h , where $h = 1, 2, \dots, q$. In an attempt to standardize tolerances for different quality characteristics, this paper denotes a random variable as $Y_h = (X_h - T_h)/d_h$. Then, Y_h follows a normal distribution with a mean δ_h and a standard deviation γ_h , where $\delta_h = (\mu_h - T_h)/d_h$, $\gamma_h = \sigma_h/d_h$, and

$$(T_h, d_h) = \begin{cases} (0, USL_h), & QC\ h \in STB\ type \\ (2LSL_h, LSL_h), & QC\ h \in LTB\ type \\ ((USL_h + LSL_h)/2, (USL_h - LSL_h)/2), & QC\ h \in NTB\ type \end{cases} \quad (1)$$

In Equation (1), USL_h and LSL_h represent the upper specification limit and the lower specification limit of the quality characteristic h , respectively. Regarding Six Sigma quality indices, they are defined as follows:

$$Q_{ph} = \begin{cases} Q_{puh} = \frac{1 - \delta_h}{\gamma_h}, & h = 1, 2, \dots, b, b + 1, \dots, q_u \\ Q_{plh} = \frac{1 + \delta_h}{\gamma_h}, & h = 1, 2, \dots, b, b + 1, \dots, q_l \end{cases} \quad (2)$$

where $q_u = b + u$ represents the total number of *STB* quality characteristics, and $q_l = b + l$ represents the total number of *LTB* quality characteristics. For the *STB* quality characteristics, the process yield is denoted as follows:

$$p_h = p(Y_h \leq 1) = p\left(Z \leq \frac{1 - \delta_h}{\gamma_h}\right) = \Phi(Q_{puh}) \quad (3)$$

Similarly, for the *LTB* quality characteristics, the process yield is expressed in the following equation:

$$p_h = p(Y_h \geq -1) = p\left(Z \geq -\frac{1 + \delta_h}{\gamma_h}\right) = p\left(Z \leq \frac{1 + \delta_h}{\gamma_h}\right) = \Phi(Q_{plh}) \quad (4)$$

where $\Phi(\bullet)$ denotes the cumulative function for the standard normal distribution. Based on Equations (3)–(5), $p_h = \Phi(Q_{ph})$ is derived, and the product yield p_T of the component is defined as follows:

$$p_T = 1 - \sum_{i \in \{u, l\}} \sum_{h=1}^{q_i} (1 - \Phi(Q_{pih})) \quad (5)$$

As mentioned previously, when the process quality for each quality characteristic attains to the required quality level expected by the customer, the product can be considered

qualified. Let Event E_h represent the level of process quality for the h^{th} quality characteristic satisfying the requirements of the specification, as shown below:

$$E_h = \begin{cases} E_{uh} = \{Y_h \leq 1\}, h = 1, 2, \dots, b, b+1, \dots, q_u \\ E_{lh} = \{-1 \leq Y_h\}, h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (6)$$

According to Equation (6), in this paper, we let Event $E_T = \cap_{i \in \{u,l\}} \cap_{h=1}^{q_i} E_{ih}$, where Event E_T denotes the product quality level of a critical component satisfying customer requirements. Based on Boole's inequality and DeMorgan's rules [19], the following expression holds:

$$\begin{aligned} p(E_T) &= p\left(\cap_{i \in \{u,l\}} \cap_{h=1}^{q_i} E_{ih}\right) \\ &= 1 - p\left(\cup_{i \in \{u,l\}} \cup_{h=1}^{q_i} E_{ih}^C\right) \geq 1 - \sum_{i \in \{u,l\}} \sum_{h=1}^{q_i} (1 - \Phi(Q_{pih})) \\ &= p_T \end{aligned} \quad (7)$$

Let the Six Sigma quality product index for critical components of machine tools be

$$Q_{pT} = \Phi^{-1} \left\{ 1 - \sum_{i \in \{u,l\}} \sum_{h=1}^{q_i} (1 - \Phi(Q_{pih})) \right\} \quad (8)$$

Then, the product yield is written as $p_T = \Phi(Q_{pT})$. If the desired value of the Six Sigma quality product index is k , then the required value of the Six Sigma quality index for the quality characteristic h can be expressed as follows:

$$k' = \Phi^{-1} \left(1 - \frac{1 - \Phi(k)}{q_u + q_l} \right) \quad (9)$$

k' is the desired value of the quality level for each quality characteristic. Clearly, the value of k' exceeds the required value k of the product quality level. This result is consistent with the perspective presented by Chen et al. [10].

Let $(Y_{h,1}, \dots, Y_{h,j}, \dots, Y_{h,n})$ denote a random sample of the quality characteristic h with the sample sizes of n . Then, the maximum likelihood estimation (MLE) of δ_h and γ_h can be expressed, respectively, as follows:

$$\delta_h^* = \frac{1}{n} \sum_{j=1}^n Y_{h,j} \quad (10)$$

and

$$\gamma_h^* = \sqrt{\frac{1}{n} \sum_{j=1}^n (Y_{h,j} - \delta_h^*)^2} \quad (11)$$

Therefore, the estimators of the Six Sigma quality indices for the quality characteristic h can be defined as follows:

$$Q_{ph}^* = \begin{cases} Q_{puh}^* = \frac{1 - \delta_h^*}{\gamma_h^*}, h = 1, 2, \dots, b, b+1, \dots, q_u \\ Q_{plh}^* = \frac{1 + \delta_h^*}{\gamma_h^*}, h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (12)$$

Let the random variable be $K = n\gamma_h^{*2} / \gamma_h^2$. Then, K follows a chi-squared distribution with $n - 1$ degrees of freedom (i.e., χ_{n-1}^2). Thus,

$$1 - \alpha/2 = p\{K \leq \chi_{1-\alpha/2; n-1}^2\} = p\left\{\frac{\gamma_h^*}{\gamma_h} \leq \sqrt{\frac{\chi_{1-\alpha/2; n-1}^2}{n}}\right\} \quad (13)$$

where $\chi^2_{1-\alpha/2;n-1}$ represents the lower $1 - \alpha/2$ quantile of the chi-squared distribution with $n - 1$ degrees of freedom.

For *STB* quality characteristics, this paper defines the random variable Z as

$$Z = \sqrt{n} \left[Q_{puh} - Q_{puh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) \right] \quad (14)$$

Z is distributed as a standardized normal distribution. Thus, we have

$$1 - \alpha/2 = p\{Z \leq Z_{\alpha/2}\} = p\left\{Q_{puh} \leq Q_{puh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}}\right\} \quad (15)$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standardized normal distribution. Then, this paper defines Event Z_{uh} and Event K_h as follows:

$$Z_{uh} = \left\{ Q_{puh} \leq Q_{puh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}} \right\} \text{ and } K_h = \left\{ \frac{S_h}{\gamma_h} \leq \sqrt{\frac{\chi^2_{1-\alpha/2;n-1}}{n}} \right\} \quad (16)$$

Obviously, the probabilities of both Event Z_{uh} and Event K_h are equal to $1 - \alpha/2$. Drawing from Boole's inequality and DeMorgan's theorem, this paper obtains

$$p(Z_{uh} \cap K_h) \geq 1 - p(Z_{uh}^c) - p(K_h^c) = 1 - \alpha \quad (17)$$

where Z_{uh}^c represents the complement for Event Z_{uh} and K_h^c represents the complement for Event K_h . Thus,

$$p\left\{Q_{puh} \leq Q_{puh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}}, \frac{\gamma_h^*}{\gamma_h} \leq \sqrt{\frac{\chi^2_{1-\alpha/2;n-1}}{n}}\right\} \geq 1 - \alpha \quad (18)$$

Equivalently,

$$p\left\{Q_{puh} \leq Q_{puh}^* \times \sqrt{\frac{\chi^2_{1-\alpha/2;n-1}}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}}\right\} \geq 1 - \alpha \quad (19)$$

For *LTB* quality characteristics, this paper defines the random variable Z as follows:

$$Z = \sqrt{n} \left[Q_{plh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) - Q_{plh} \right] \quad (20)$$

Z is distributed as a standardized normal distribution. Accordingly, this paper obtains

$$1 - \alpha/2 = p\{Z \geq -Z_{\alpha/2}\} = p\left\{Q_{plh} \leq Q_{plh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}}\right\} \quad (21)$$

where $Z_{\alpha/2}$ indicates the upper $\alpha/2$ quantile of the standardized normal distribution. Then, this paper defines Event Z_{lh} as follows:

$$Z_{lh} = \left\{ Q_{plh} \leq Q_{plh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}} \right\} \quad (22)$$

Obviously, the probability of Event Z_{lh} equals $1 - \alpha/2$. By means of Boole's inequality and DeMorgan's theorem, this paper obtains

$$p(Z_{lh} \cap K_h) \geq 1 - p(Z_{lh}^c) - p(K_h^c) = 1 - \alpha \quad (23)$$

where Z_{lh}^c is the compliment of Event Z_{lh} . Therefore,

$$p \left\{ Q_{plh} \leq Q_{plh}^* \times \left(\frac{\gamma_h^*}{\gamma_h} \right) + \frac{Z_{\alpha/2}}{\sqrt{n}}, \frac{\gamma_h^*}{\gamma_h} \leq \sqrt{\frac{\chi_{1-\alpha/2;n-1}^2}{n}} \right\} \geq 1 - \alpha \quad (24)$$

Equivalently,

$$p \left\{ Q_{Plh} \leq Q_{plh}^* \times \sqrt{\frac{\chi_{1-\alpha/2;n-1}^2}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}} \right\} \geq 1 - \alpha \quad (25)$$

Next, according to the needs of practical operations, let $(y_{h,1}, \dots, y_{h,j}, \dots, y_{h,n})$ denote the observed values of $(Y_{h,1}, \dots, Y_{h,j}, \dots, Y_{h,n})$ in this paper. Then, the observed values of δ_h^* and γ_h^* are expressed individually as follows:

$$\delta_{h0}^* = \frac{1}{n} \sum_{j=1}^n y_{h,j} \text{ and } \gamma_{h0}^* = \sqrt{\frac{1}{n} \sum_{j=1}^n (y_{h,j} - \delta_{h0}^*)^2} \quad (26)$$

Furthermore, the observed values of Q_{ph}^* are defined as

$$Q_{ph0}^* = \begin{cases} Q_{puh0}^* = \frac{1-\delta_{h0}^*}{\gamma_{h0}^*}, & h = 1, 2, \dots, b, b+1, \dots, q_u \\ Q_{plh0}^* = \frac{1+\delta_{h0}^*}{\gamma_{h0}^*}, & h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (27)$$

As noted above, the observed values for the $100(1 - \alpha)\%$ upper confidence limits of Q_{ph} are written as

$$UQ_{ph0} = \begin{cases} UQ_{puh0} = Q_{puh0}^* \times \sqrt{\frac{\chi_{1-\alpha/2;n-1}^2}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_u \\ UQ_{plh0} = Q_{plh0}^* \times \sqrt{\frac{\chi_{1-\alpha/2;n-1}^2}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (28)$$

Based on the above observed values for the $100(1 - \alpha)\%$ upper confidence limits of Q_{ph} , this paper constructs a fuzzy membership function in the next section and develops upper confidence limits for the fuzzy testing method.

3. A Fuzzy Hypothesis Testing Method Based on Upper Confidence Limits

Many studies have highlighted that industries often need to make quick decisions with small sample sizes due to cost and time constraints. Confidence intervals are one of the tools used for conducting statistical tests in such situations. In the case of small sample sizes, the length of their confidence intervals is relatively wide, which can lead to substantial errors and increase the likelihood of misjudgments. Some studies have suggested that fuzzy tests based on confidence intervals can be effectively utilized when the sample sizes are not large. Since past data and expert experience can be incorporated, the precision of the tests can be maintained [20,21]. Therefore, in this paper, we develop a fuzzy testing approach for index Q_{ph} using the upper limits of the index derived from the previous section. Additionally, the fuzzy evaluation points for the quality characteristic h , guided by the fuzzy testing rules, can lay the groundwork for the subsequent creation of the fuzzy radar evaluation chart.

As stated earlier, the required value of the Six Sigma quality index of the quality characteristic h is k' , assuming the required value of the Six Sigma quality product index is k . Also, the null hypothesis is written as $H_0: Q_{ph} \geq k'$, while the alternative hypothesis is expressed as $H_1: Q_{ph} < k'$. If the observed value of the upper confidence limit, denoted as UQ_{ph0} , is adopted for statistical testing, then the testing rules are established as follows:

- (1) Suppose $UQ_{ph0} \geq k'$, H_0 is retained, leading to the conclusion that $Q_{ph} \geq k'$.
- (2) Suppose $UQ_{ph0} < k'$, H_0 is not retained, leading to the conclusion $Q_{ph} < k'$.

Based on the method introduced by Chen et al. [22] and the aforementioned rules for statistical testing, this paper develops a fuzzy hypothesis testing method utilizing the upper confidence limits. As described by Chen et al. [22], the α -cuts of the triangular fuzzy number $\tilde{U}Q_{ph0}$ is obtained as

$$\tilde{U}Q_{ph0}[\alpha] = \begin{cases} [UQ_{ph0}(1), UQ_{ph0}(\alpha)], & \text{for } 0.01 \leq \alpha \leq 1 \\ [UQ_{ph0}(1), UQ_{ph0}(0.01)], & \text{for } 0 \leq \alpha \leq 0.01 \end{cases} \quad (29)$$

where

$$UQ_{ph0}(\alpha) = \begin{cases} UQ_{puh0}(\alpha) = Q_{puh0}^* \times \sqrt{\frac{\chi_{1-\alpha/2, n-1}^2}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_u \\ UQ_{plh0}(\alpha) = Q_{plh0}^* \times \sqrt{\frac{\chi_{1-\alpha/2, n-1}^2}{n}} + \frac{Z_{\alpha/2}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (30)$$

Accordingly, the fuzzy number with a half-triangular distribution is introduced as $\tilde{U}Q_{ph0} = (UQ_{hM}, UQ_{hR})$, where $UQ_{hM} = UQ_{ph0}(1)$ and $UQ_{hR} = UQ_{ph0}(0.01)$ can be shown as follows:

$$UQ_{hR} = \begin{cases} UQ_{puh0}(0.01) = Q_{puh0}^* \times \sqrt{\frac{\chi_{0.995, n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_u \\ UQ_{plh0}(0.01) = Q_{plh0}^* \times \sqrt{\frac{\chi_{0.995, n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}}, & h = 1, 2, \dots, b, b+1, \dots, q_l \end{cases} \quad (31)$$

The membership function of the fuzzy number $\tilde{U}Q_{ph0}$ with a half-triangular distribution is defined as

$$\zeta_h(x) = \begin{cases} 0 & \text{if } x < UQ_{hM} \\ 1 & \text{if } x = UQ_{hM} \\ \alpha & \text{if } UQ_{hM} < x < UQ_{hR} \\ 0 & \text{if } UQ_{hR} \leq x \end{cases} \quad (32)$$

where α is directed by $UQ_{ph0}(\alpha) = x$. The membership function $\zeta_h(x)$, represented with the vertical line $x = k'$, is depicted in Figure 1.

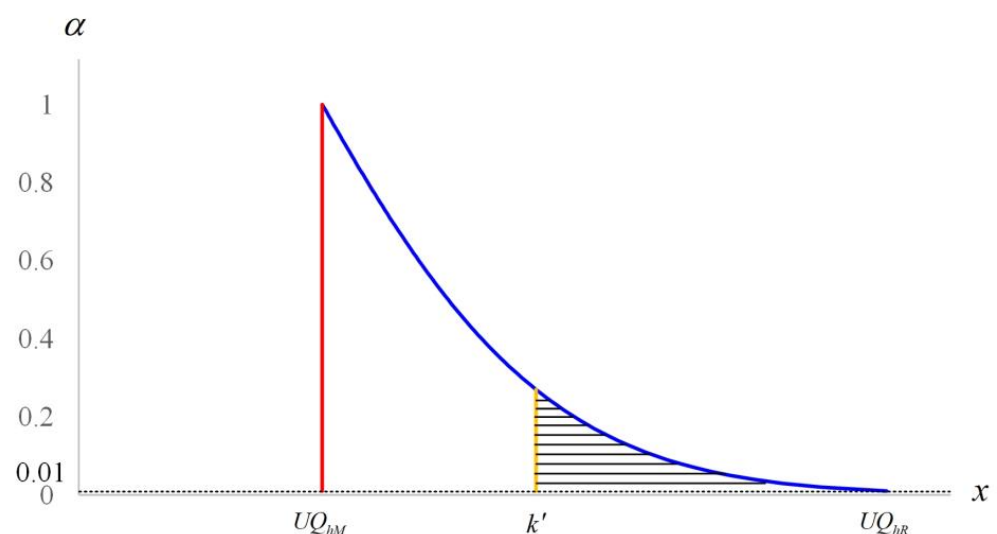


Figure 1. The membership function of the fuzzy number with a half-triangular distribution, represented with the vertical line $x = k'$.

As suggested by Chen et al. [22], let Set A_{hT} denote the area in the graph for the membership function of the fuzzy number with a half-triangular distribution. In Figure 1, let Set A_{hT} represent the area between the membership function and the x-axis. Then,

$$A_{hT} = \{(x, \alpha) | UQ_{hM} \leq x \leq UQ_{ph0}(\alpha)\} \quad (33)$$

Similarly, let Set A_{hR} denote the area from the vertical line $x = k'$ to the left of the membership function $\zeta_h(x)$. Then,

$$A_{hR} = \{(x, \alpha) | k' \leq x \leq UQ_{ph0}(\alpha)\} \quad (34)$$

According to Equations (33) and (34), $d_{hT} = 2(UQ_{hR} - UQ_{hM})$ and $d_{hR} = (UQ_{hR} - k')$ are expressed, respectively, as follows:

$$d_{hT} = 2 \left(Q_{ph0}^* \sqrt{\frac{\chi_{0.995;n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}} \right) - 2Q_{ph0}^* \sqrt{\frac{\chi_{0.5;n-1}^2}{n}} \quad (35)$$

and

$$d_{hR} = Q_{ph0}^* \times \sqrt{\frac{\chi_{0.995;n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}} - k' \quad (36)$$

Then, $d_{hR}/d_{hT} = (UQ_{hR} - k')/2(UQ_{hR} - UQ_{hM})$ can be written as follows:

$$d_{hR}/d_{hT} = \frac{Q_{ph0}^* \times \sqrt{\frac{\chi_{0.995;n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}} - k'}{2Q_{ph0}^* \times \left(\sqrt{\frac{\chi_{0.995;n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}} - \sqrt{\frac{\chi_{0.5;n-1}^2}{n}} \right)} \quad (37)$$

As noted by some studies, this paper defines $0 < \phi \leq 0.5$ [22,23]. The fuzzy decision rules are listed as follows:

- (1) If $d_{hR}/d_{hT} \leq \phi$, then reject H_0 and conclude that $Q_{ph} < k'$.
- (2) If $d_{hR}/d_{hT} > \phi$, then do not reject H_0 and conclude that $Q_{ph} \geq k'$.

Attempting to facilitate practical application, we propose the decision value ev_h of the Six Sigma quality index for the quality characteristic h as follows:

$$ev_h = d_{hR} - \phi d_{hT} = (1 - 2\phi) \left(Q_{ph0}^* \times \sqrt{\frac{\chi_{0.995;n-1}^2}{n}} + \frac{Z_{0.005}}{\sqrt{n}} \right) + 2\phi Q_{ph0}^* \sqrt{\frac{\chi_{0.5;n-1}^2}{n}} \quad (38)$$

According to Equation(38), the decision value ev_h of the Six Sigma quality index for the quality characteristic h will replace d_{hR}/d_{hT} for decision making. The decision can be made simply by directly comparing the ev_h value with the required value k' , thus facilitating the work of the field engineer. It is also conducive to the promotion of this method in practice. Then, the fuzzy decision rules are reaffirmed below:

- (1) Supposing $ev_h \leq k'$, H_0 is rejected, resulting in the conclusion that $Q_{ph} < k'$.
- (2) Supposing $ev_h > k'$, H_0 is not rejected, resulting in the conclusion that $Q_{ph} \geq k'$.

4. An Example for Practical Application

Several studies have indicated that Central Taiwan is the hub of the machine tool industry. The industry chain of machine tools consists of machine tool manufacturers, crucial component suppliers, and various downstream machining manufacturers (i.e., customers) who purchase machine tools [24–26]. Components processed by machine tools typically exhibit several key quality characteristics (QCs), such as gears, shafts, and bearings. This study used a specific axis as an example, including outer diameter 1, outer diameter 2, cylindrical surface roundness, and concentricity. Additionally, it features a total

of four quality characteristics, with two categorized under the *STB* type and two under the *LTB* type, as illustrated in Table 1 below.

Table 1. The tolerances of four important QCs presented in the axis machining process.

No.	QC	Type	Target	d
1	outer diameter 1	NTB	12	0.02
2	outer diameter 2	NTB	12	0.02
3	roundness	STB	0	0.03
4	concentricity	STB	0	0.04

Customers require that the process quality of the axis reach a minimum 5 Sigma quality level. According to Equation (10), the required process quality level for each quality characteristic is computed as

$$k' = \Phi^{-1} \left(1 - \frac{1 - \Phi(5)}{4 + 2} \right) = 5.33. \quad (39)$$

Thus, both the null and the alternative hypotheses are defined as

$$H_0: Q_{ph} \geq 5.33 \text{ versus } H_1: Q_{ph} < 5.33,$$

where $h = 1, 2, 3, 4, 5, 6$. Then, this study randomly selected 30 samples, each containing 4 quality characteristics and a total of 6 evaluation indices. According to Equations (26), (27) and (38), the relevant sample statistics of these four quality characteristics were computed using EXCEL 2016 software as follows:

$$QC1: \quad \delta_{10}^* = 0.23, \gamma_{10}^* = 0.181, \begin{cases} Q_{p10}^* = Q_{pu10}^* = 4.254 \\ Q_{p20}^* = Q_{pl20}^* = 6.796 \end{cases}$$

$$\begin{cases} UQ_{p10} = UQ_{pu10} = 6.089 \\ UQ_{p20} = UQ_{pl20} = 9.446 \end{cases} \begin{cases} ev_1 = 4.646 \\ ev_2 = 7.253 \end{cases}$$

$$QC2: \quad \delta_{20}^* = 0.14, \gamma_{20}^* = 0.162, \begin{cases} Q_{p30}^* = Q_{pu30}^* = 5.309 \\ Q_{p40}^* = Q_{pl40}^* = 7.037 \end{cases}$$

$$\begin{cases} UQ_{p30} = UQ_{pu30} = 7.842 \\ UQ_{p40} = UQ_{pl40} = 9.765 \end{cases} \begin{cases} ev_3 = 5.727 \\ ev_4 = 7.500 \end{cases}$$

$$QC3: \quad \delta_{30}^* = 0.65, \gamma_{30}^* = 0.092, Q_{p50}^* = Q_{pu50}^* = 3.804$$

$$UQ_{p50} = UQ_{pu50} = 5.495 \quad ev_5 = 4.184$$

$$QC4: \quad \delta_{40}^* = 0.59, \gamma_{40}^* = 0.081, Q_{p60}^* = Q_{pu60}^* = 5.062$$

$$UQ_{p60} = UQ_{pu60} = 7.156 \quad ev_6 = 5.474$$

Quality engineers set $\phi = 0.2$ based on past accumulated data analysis results and professional experience. Then, according to the fuzzy testing rules set for Six Sigma quality indexes, the decision values, $ev_1 = 4.646$ and $ev_5 = 4.184$, are less than 5.33. Therefore, improvements are needed for Quality Characteristic 1 and Quality Characteristic 3. In fact, $UQ_{p10} = 6.089$ and $UQ_{p50} = 5.495$. These two confidence upper limits both exceed the required index value of 5.33. According to statistical testing rules, H_0 cannot be rejected. However, the point estimates of the index are 4.254 and 3.804, which are both significantly lower than the required value of 5.33. Clearly, this discrepancy results from the excessively long confidence intervals caused by small sample sizes of n . Subsequently, taking the fifth index as an example, we compared the differences between the statistical testing method and the fuzzy testing method based on the sizes of the observed values, as shown in Table 2.

Table 2. Comparisons between the statistical testing and fuzzy testing methods based on index observations.

Q_{p50}^*	UQ_{p50}	ev_5	Statistical Test	Fuzzy Test
3.60	5.225	3.975	reject H_0	reject H_0
3.70	5.357	4.077	do not reject H_0	reject H_0
3.80	5.495	4.184	do not reject H_0	reject H_0
3.90	5.621	4.283	do not reject H_0	reject H_0
4.00	5.753	4.385	do not reject H_0	reject H_0
4.10	5.886	4.488	do not reject H_0	reject H_0
4.20	6.018	4.590	do not reject H_0	reject H_0
4.30	6.150	4.693	do not reject H_0	reject H_0
4.40	6.282	4.795	do not reject H_0	reject H_0
4.50	6.414	4.898	do not reject H_0	reject H_0
4.60	6.546	5.001	do not reject H_0	reject H_0
4.70	6.678	5.103	do not reject H_0	reject H_0
4.80	6.810	5.206	do not reject H_0	reject H_0
4.90	6.942	5.308	do not reject H_0	reject H_0
5.00	7.074	5.411*	do not reject H_0	do not reject H_0

According to Table 2, since small sample sizes of n result in relatively long confidence intervals, the statistical testing method does not reject H_0 when $Q_{p50}^* = 3.7$, because the upper limit of its confidence interval exceeds the required value of 5.33. This might lose opportunities for process improvements. In contrast, if the fuzzy testing method is applied, then H_0 is rejected, indicating that process improvements are necessary. As $Q_{p50}^* = 5.0$, the fuzzy testing method does not reject H_0 for $ev_5 = 5.41$, exceeding the required value of 5.33. Therefore, it is evident that in cases of smaller sample sizes of n , the method proposed in this paper is more reasonable than the statistical testing method, since it can incorporate past data and professional expertise [27].

5. Conclusions

Machine tool components typically encompass multiple quality characteristics simultaneously, composing *STB*, *LTB*, and *NTB* types. The index for evaluation with *NTB* quality characteristics is segmented into two single-sided Six Sigma quality indexes. Thus, the entire component product's process quality can be assessed by q single-sided Six Sigma quality indexes to assure customers of all critical components' process quality, as well as the final product quality of machine tools. Considering timeliness and costs, machine tool manufacturers usually make decisions using small sample sizes, denoted as n . Consequently, inconsistent evaluation results are often caused by substantial sampling errors. Therefore, in this paper, we proposed a $100(1-\alpha)\%$ confidence interval for the single-sided Six Sigma quality index, based on recommendations from studies that account for small sample sizes. First, we derived a triangular fuzzy number as well as determined its membership function. Subsequently, we utilized the membership function to build a fuzzy testing model incorporating the confidence intervals. This model can serve as a fuzzy quality evaluation model for machine tool components. Finally, we presented a case study to demonstrate how to apply this fuzzy evaluation model to machine tool components. In this paper, we adopted a method that can incorporate historical data and professional expertise. It is suggested that future research can involve the collection of data from practical cases to establish models for calculating historical data, thereby exploring the values of ϕ more scientifically to make objective decisions.

Author Contributions: Conceptualization, K.-S.C., K.-C.Y. and C.-M.Y.; methodology, K.-S.C. and C.-M.Y.; software, C.-H.C. (Chien-Hsin Cheng); validation, C.-H.C. (Chien-Hsin Cheng) and C.-H.C. (Chen-Hsu Chang); formal analysis, K.-S.C. and C.-M.Y.; resources, C.-H.C. (Chien-Hsin Cheng); data curation, C.-H.C. (Chen-Hsu Chang); writing—original draft preparation, K.-S.C., K.-C.Y., C.-H.C. (Chien-Hsin Cheng), C.-M.Y. and C.-H.C. (Chen-Hsu Chang); writing—review and editing, K.-S.C., K.-C.Y. and C.-M.Y.; visualization, C.-H.C. (Chien-Hsin Cheng); supervision, K.-S.C.; project administration, C.-M.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

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