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Generalization of Fixed-Point Results in Complex-Valued Bipolar Metric Space with Applications

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Abstract: We undertake this study with the objective of introducing certain control functions in the contractive condition to prove fixed-point theorems in the framework of complex-valued bipolar metric spaces. The incorporation of control functions broadens the applicability of the contractive condition. This approach yields key results consistent with previous studies. In support of our results, we offer two insightful examples that demonstrate the concepts discussed. Additionally, we present the notion of interpolative contraction in this new and generalized metric space and prove fixed-point theorems for non-self mappings. To demonstrate the application of our approach, we reproduce key findings from several established studies in the field.

Keywords: control functions; complex-valued bipolar metric space; interpolative contraction; fixed point

MSC: 47H10, 54H25

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1. Introduction

Fixed-point (FP) theory, a cornerstone of mathematical analysis, delves into the fascinating world of self-mapping functions. It seeks to identify elements within a set that remain unchanged when the function is applied. This seemingly simple concept has profound implications across various branches of mathematics, including differential equations, optimization, and functional analysis. Antón-Sancho [1–3] investigated the existence and properties of FPs within the space of principal E_6-bundles defined over a compact algebraic curve. FPs in the space of vector bundles defined over a compact Riemann surface subjected to various automorphisms are also identified and analyzed. The foundation of FP theory is arguably laid by the Banach contraction principle (BCP) [4]. This powerful theorem, established by Stefan Banach in 1922, guarantees the existence of a unique FP for a contraction mapping within a complete metric space (CMS). It provides a clear and elegant method to analyze iterative processes and has become a cornerstone for numerous extensions and applications. Building upon the BCP, Kannan [5] introduced his influential FP theorem in 1968. This theorem broadened the applicability of FP theory by relaxing the contractive condition, allowing for a wider range of functions to possess fixed points. Further enriching the landscape, Reich [6] presented his FP theorem. This theorem ingeniously integrated the BCP and Kannan's result, offering a more general framework for establishing FPs under specific conditions. The quest for more versatile FP theorems continued with Fisher's contribution [7]. His FP theorem incorporated rational expressions within the contractive condition, leading to a wider range of functions satisfying the necessary criteria for fixed points. Subsequently, Khan [8] introduced another rational expression within the contractive condition, further expanding the applicability of FP theory. Following the above two results, Jaggi [9] incorporated a new rational expression into the contractive condition of the

Banach principle. This refinement, known as Jaggi's FP theorem, demonstrates the continuous evolution of FP theory with ever-expanding tools for analyzing self-mapping functions.

The fruitful exploration of FPs in these theorems is facilitated by the rich theory of CMSs, offering a robust framework for analyzing distances. Metric spaces have numerous applications across various scientific fields, like mathematics, computer science, physics, chemistry, and biology. This wide applicability has led researchers to explore extensions and generalizations of the concept. Azam et al. [10] began the idea of complex-valued metric spaces (CVMSs) by replacing the real number range with complex numbers. Their work established the existence of common FPs for certain mappings using Fisher's contractions. Subsequently, Rouzkard et al. [11] extended this result by incorporating a rational term into the contraction condition. Sintunavarat et al. [12] further generalized these findings by introducing variable-dependent control functions. Sitthikul et al. [13] contributed by proving common FP theorems within the framework of CVMSs. Ahmad et al. [14] demonstrated the appropriateness of their findings by establishing the existence of solutions to a second-order differential equation. In classical MSs and CVMSs, we typically consider the distance between elements within a single set. This naturally leads to the question of how to define and compute the distance between elements in two different sets. Such issues arise in various fields of mathematics. To address these issues, Mutlu et al. [15] unveiled the concept of bipolar metric spaces (bipMSs). This new notion has facilitated the advancement of FP theory, particularly in developing Banach's and Kannan's FP results. Later on, Paul et al. [16] established FP results for contractions involving Jaggi's and Khan's rational expressions in the framework of bipMSs. Kishore et al. [17] presented applications of their outcomes in the domains of integral equations and homotopy theory. Extensive research has been conducted on the existence of FPs within the framework of bipMSs (see [18–20] and references therein). Recently, Siva [21] put forward the notion of complex-valued bipolar metric spaces (CVbipMSs) and proved FP theorems for contravariant mappings under rational contractions.

This research leverages the concept of CVbipMSs to obtain FP results for rational contractions containing certain control functions. Our approach successfully recovers several established findings documented in the literature. Some non-trivial examples to support our findings are also furnished. Additionally, we present the notion of interpolative contraction in CVbipMSs and prove FP theorems for non-self mappings.

2. Preliminaries

To establish a solid groundwork for the ensuing discussions, we present essential definitions, notations, and preliminary results in this section. Pioneering the field, the Banach contraction principle [4] is formulated as follows:

Theorem 1 ([4]). *Let* $(\Omega, \mathfrak{o})$ *be a CMS and* $\mathcal{A} : \Omega \to \Omega$, *satisfying*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa \mathfrak{o}(\mathfrak{r}, \mathfrak{j})$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, *and for some* $0 \leq \kappa < 1$; *then,* $\mathcal{A}$ *has a unique fixed point (UFP).*

In a significant contribution to the field, Kannan [5] established the subsequent theorem:

Theorem 2 ([5]). *Let* $(\Omega, \mathfrak{o})$ *be a CMS and* $\mathcal{A} : \Omega \to \Omega$, *satisfying*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa(\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}))$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, *and for some* $0 \leq \kappa < \frac{1}{2}$; *then,* $\mathcal{A}$ *has a UFP.*

Reich [6] subsequently unified the Banach contraction principle and Kannan's fixed-point result through the following approach.

Theorem 3 ([6]). *Let* $(\Omega, \mathfrak{o})$ *be a CMS and* $\mathcal{A} : \Omega \to \Omega$, *satisfying*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1 \mathfrak{o}(\mathfrak{r}, \mathfrak{j}) + \kappa_2(\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}))$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, *and for some* $0 \leq \kappa_1, \kappa_2 < 1$, *with* $\kappa_1 + 2\kappa_2 < 1$; *then,* $\mathcal{A}$ *has a UFP.*

Leveraging the Banach contraction principle, Fisher [7] formulated a refined contractive condition incorporating a novel rational expression. This approach yields the next result.

Theorem 4 ([7]). *Let* $\mathcal{A}$ *be a self-mapping on the CMS* $(\Omega, \mathfrak{o})$. *We investigate the following contractive condition for* $\mathcal{A}$:

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1 \mathfrak{o}(\mathfrak{r}, \mathfrak{j}) + \kappa_2 \frac{\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})}{1 + \mathfrak{o}(\mathfrak{r}, \mathfrak{j})}$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, *and for some* $0 \leq \kappa_1, \kappa_2 < 1$, *with* $\kappa_1 + \kappa_2 < 1$; *then,* $\mathcal{A}$ *has a UFP.*

In an extension of the Banach contraction principle, Jaggi [9] incorporated a new rational expression into the contractive condition, achieving the following result.

Theorem 5 ([9]). *Consider a continuous self-map* $\mathcal{A}$ *defined on a CMS* $(\Omega, \mathfrak{o})$. *Suppose* $\mathcal{A}$ *fulfills the following contractive condition:*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1 \mathfrak{o}(\mathfrak{r}, \mathfrak{j}) + \kappa_2 \frac{\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})}{\mathfrak{o}(\mathfrak{r}, \mathfrak{j})}$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, $\mathfrak{r} \neq \mathfrak{j}$, *and for some* $0 \leq \kappa_1, \kappa_2 < 1$, *with* $\kappa_1 + \kappa_2 < 1$; *then,* $\mathcal{A}$ *has a UFP.*

In 1976, Khan [8] introduced a novel rational expression within the contractive condition, leading to the following result.

Theorem 6 ([8]). *Let* $(\Omega, \mathfrak{o})$ *be a CMS and* $\mathcal{A} : \Omega \to \Omega$, *satisfying*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa \frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{j})}{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{j})}$$

for all $\mathfrak{r}, \mathfrak{j} \in \Omega$, *with* $0 \leq \kappa < 1$; *then,* $\mathcal{A}$ *has a UFP.*

Azam et al. [10] put forward the idea of the CVMS in such a fashion.

Definition 1 ([10]). *If* $z = x + iy$ *is a complex number, then*

- x *represents the real part of* z;
- y *represents the imaginary part of* z;
- i *is the imaginary unit, defined as the square root of* -1 *(i.e.,* $i^2 = -1$*).*

Now, let $z_1, z_2 \in \mathbb{C}$. *The set* $\mathbb{C}$ *is equipped with a partial order* $\precsim$ *defined by*

$$z_1 \precsim z_2 \Leftrightarrow Re(z_1) \leqslant Re(z_2),\ Im(z_1) \leqslant Im(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of these assertions is satisfied:

$$\begin{aligned}
&\text{(a) } \mathrm{Re}(z_1) = \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) < \mathrm{Im}(z_2),\\
&\text{(b) } \mathrm{Re}(z_1) < \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) = \mathrm{Im}(z_2),\\
&\text{(c) } \mathrm{Re}(z_1) < \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) < \mathrm{Im}(z_2),\\
&\text{(d) } \mathrm{Re}(z_1) = \mathrm{Re}(z_2),\ \mathrm{Im}(z_1) = \mathrm{Im}(z_2).
\end{aligned}$$

Definition 2 ([10]). *Let $\Omega \neq \emptyset$ and $\mathfrak{o} : \Omega \times \Omega \to \mathbb{C}$ be a mapping satisfying*

(i) $0 \precsim \mathfrak{o}(\mathfrak{r}, \mathfrak{j})$, *and* $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = 0 \Leftrightarrow \mathfrak{r} = \mathfrak{j}$,
(ii) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = \mathfrak{o}(\mathfrak{j}, \mathfrak{r})$,
(iii) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) \precsim \mathfrak{o}(\mathfrak{r}, z) + \mathfrak{o}(z, \mathfrak{j})$,

$\forall\ \mathfrak{r}, \mathfrak{j}, z \in \Omega$. *Then,* $(\Omega, \mathfrak{o})$ *is said to be a CVMS.*

Example 1 ([10]). *Let $\Omega = [0, 1]$ and $\mathfrak{r}, \mathfrak{j} \in \Omega$. Define $\mathfrak{o} : \Omega \times \Omega \to \mathbb{C}$ by*

$$\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = \begin{cases} 0, & \text{if } \mathfrak{r} = \mathfrak{j}, \\ \frac{i}{2}, & \text{if } \mathfrak{r} \neq \mathfrak{j}. \end{cases}$$

Then, $(\Omega, \mathfrak{o})$ *is a CVMS.*

Mutlu et al. [15] originated the thought of bipMSs in such a manner.

Definition 3 ([15]). *Let $\Omega \neq \emptyset$ and $\mho \neq \emptyset$ and let $\mathfrak{o} : \Omega \times \mho \to [0, +\infty)$, satisfying*

(bi_1) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = 0 \Longleftrightarrow \mathfrak{r} = \mathfrak{j}$,
(bi_2) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = \mathfrak{o}(\mathfrak{j}, \mathfrak{r})$, *if* $\mathfrak{r}, \mathfrak{j} \in \Omega \cap \mho$,
(bi_3) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) \leq \mathfrak{o}(\mathfrak{r}, \mathfrak{j}') + \mathfrak{o}(\mathfrak{r}', \mathfrak{j}') + \mathfrak{o}(\mathfrak{r}', \mathfrak{j})$,

for all $(\mathfrak{r}, \mathfrak{j}), (\mathfrak{r}', \mathfrak{j}') \in \Omega \times \mho$.
Then, $(\Omega, \mho, \mathfrak{o})$ *is called a bipMS.*

Example 2 ([15]). *Let Ω and $\mho$ be two classes of all singleton and compact subsets of $\mathbb{R}$. Define $\mathfrak{o} : \Omega \times \mho \to [0, +\infty)$ by*

$$\mathfrak{o}(\mathfrak{r}, \Xi) = |\mathfrak{r} - \inf(\Xi)| + |\mathfrak{r} - \sup(\Xi)|$$

for $\{\mathfrak{r}\} \subseteq \Omega$ and $\Xi \subseteq \mho$; then, $(\Omega, \mho, \mathfrak{o})$ *is a complete bipMS.*

Definition 4 ([15]). *Let $(\Omega_1, \mho_1, \mathfrak{o}_1)$ and $(\Omega_2, \mho_2, \mathfrak{o}_2)$ be two bipMSs. A mapping $\mathcal{A} : \Omega_1 \cup \mho_1 \rightrightarrows \Omega_2 \cup \mho_2$ is said to be a covariant mapping if $\mathcal{A}(\Omega_1) \subseteq \Omega_2$ and $\mathcal{A}(\mho_1) \subseteq \mho_2$. Similarly, a mapping $\mathcal{A} : \Omega_1 \cup \mho_1 \rightrightarrows \Omega_2 \cup \mho_2$ is called a contravariant mapping if $\mathcal{A}(\Omega_1) \subseteq \mho_2$ and $\mathcal{A}(\Omega_2) \subseteq \mho_1$.*

The covariant mapping will be represented by the symbol $\mathcal{A} : (\Omega_1, \mho_1) \rightrightarrows (\Omega_2, \mho_2)$ and the contravariant mapping by $\mathcal{A} : (\Omega_1, \mho_1) \rightleftarrows (\Omega_2, \mho_2)$.

3. Complex-Valued Bipolar Metric Space

This section establishes the foundation for our research by presenting the idea of a CVbipMS. It includes the definitions of covariant and contravariant mappings, as well as the relevant lemmas. Very recently, Siva [21] defined the notion of the CVbipMS in the following way.

Definition 5 ([21]). *Let $\Omega \neq \emptyset$ and $\mho \neq \emptyset$ and $\mathfrak{o} : \Omega \times \mho \to \mathbb{C}$ be a function fulfilling the following conditions:*

(cvb1) $0 \precsim \mathfrak{o}(\mathfrak{r}, \mathfrak{j})$, *and* $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = 0 \Longleftrightarrow \mathfrak{r} = \mathfrak{j}$ *for* $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$,
(cvb2) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = \mathfrak{o}(\mathfrak{j}, \mathfrak{r})$, *for all* $\mathfrak{r}, \mathfrak{j} \in \Omega \cap \mho$

(cvb3) $\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) \precsim \mathfrak{o}(\mathfrak{r}, \mathfrak{j}^{/}) + \mathfrak{o}(\mathfrak{r}^{/}, \mathfrak{j}^{/}) + \mathfrak{o}(\mathfrak{r}^{/}, \mathfrak{j})$,

for all $\mathfrak{r}, \mathfrak{r}^{/} \in \Omega$ *and* $\mathfrak{j}, \mathfrak{j}^{/} \in \mho$.

Then, the triple $(\Omega, \mho, \mathfrak{o})$ is called a CVbipMS.

Remark 1 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS. Then, the space* $(\Omega, \mho, \mathfrak{o})$ *is said to be disjoint if* $\Omega \cap \mho = \emptyset$. *If* $\Omega \cap \mho \neq \emptyset$, *the space* $(\Omega, \mho, \mathfrak{o})$ *is referred to as a joint. The sets* Ω *and* $\mho$ *are referred to as the left and right poles of* $(\Omega, \mho, \mathfrak{o})$, *respectively.*

Example 3 ([21]). *Let* Ω *be the set of functions such that* $h : \mathbb{C} \to \{z : 1 \leq Re(z) \leq 3, Im(z) = 0\}$ *and* $\mho$ *be the set of all complex numbers* $\mathbb{C}$. *Now, we define* $\mathfrak{o} : \Omega \times \mho \to \mathbb{C}$ *by*

$$\mathfrak{o}(h, \mathfrak{j}) = h(\mathfrak{j}),$$

where $(h, \mathfrak{j}) \in \Omega \times \mho$. *Then,* $(\Omega, \mho, \mathfrak{o})$ *is a CVbipMS.*

Lemma 1 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS. A left sequence* $\{\mathfrak{r}_s\}_{s=1}^{\infty}$ *converges to a right point* $\mathfrak{j}$ *iff* $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j})| \to 0$, *and also, a right sequence* $\{\mathfrak{j}_s\}_{s=1}^{\infty}$ *converges to a left point* $\mathfrak{r}$ *if and only if* $|\mathfrak{o}(\mathfrak{r}, \mathfrak{j}_s)| \to 0$ *as* $s \to \infty$.

Lemma 2 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS. When a central point serves as a sequence's limit, the central point acts as the sequence's unique limit.*

Lemma 3 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS. If a left sequence* $\{\mathfrak{r}_s\}_{s=1}^{\infty}$ *converges to* $\mathfrak{j}$ *and a right sequence* $\{\mathfrak{j}_s\}_{s=1}^{\infty}$ *converges to* $\mathfrak{r}$, *then* $\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) \to \mathfrak{o}(\mathfrak{r}, \mathfrak{j})$ *as* $s \to \infty$.

Definition 6 ([21]). *Let* $(\Omega_1, \mho_1, \mathfrak{o}_1)$ *and* $(\Omega_2, \mho_2, \mathfrak{o}_2)$ *be two CVbipMSs. The mapping* $\mathcal{A} : \Omega_1 \cup \mho_1 \to \Omega_2 \cup \mho_2$ *is a mapping.*

(i) If $\mathcal{A}(\Omega_1) \subseteq \Omega_2$ *and* $\mathcal{A}(\mho_1) \subseteq \mho_2$, *then* $\mathcal{A}$ *is said to be a covariant mapping from* $(\Omega_1, \mho_1, \mathfrak{o}_1)$ *to* $(\Omega_2, \mho_2, \mathfrak{o}_2)$. *We use* $\mathcal{A} : \Omega_1 \cup \mho_1 \rightrightarrows \Omega_2 \cup \mho_2$ *to represent covariant mappings.*

(ii) If $\mathcal{A}(\Omega_1) \subseteq \mho_2$ *and* $\mathcal{A}(\mho_1) \subseteq \Omega_2$, *then* $\mathcal{A}$ *is said to be contravariant mapping from* $(\Omega_1, \mho_1, \mathfrak{o}_1)$ *to* $(\Omega_2, \mho_2, \mathfrak{o}_2)$. *We express the contravariant mappings as* $\mathcal{A} : \Omega_1 \cup \mho_1 \rightleftarrows \Omega_2 \cup \mho_2$.

Definition 7 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS.*

(i) A sequence $(\mathfrak{r}_s, \mathfrak{j}_s)$ *on* $\Omega \times \mho$ *is reputed to be bisequence on* $(\Omega, \mho, \mathfrak{o})$.

(ii) Assuming that sequences $(\mathfrak{r}_s)$ *and* $(\mathfrak{j}_s)$ *converge, their corresponding bisequence* $(\mathfrak{r}_s, \mathfrak{j}_s)$ *also converges. If both* $(\mathfrak{r}_s)$ *and* $(\mathfrak{j}_s)$ *converge to the same limit, the bisequence* $(\mathfrak{r}_s, \mathfrak{j}_s)$ *is termed biconvergent.*

(iii) If for each $z \in \mathbb{C}$, *there exists* $s_0 \in \mathbb{N}$ *such that a bisequence* $(\mathfrak{r}_s, \mathfrak{j}_s)$ *in a CVbipMS* $(\Omega, \mho, \mathfrak{o})$ *is often described as a Cauchy, if for every* $\epsilon > 0$, $\exists\ s_0 \in \mathbb{N}$, *such that* $\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s+t}) \preceq z$, *for* $s \geq s_0$.

Lemma 4 ([21]). *Let* $(\Omega, \mho, \mathfrak{o})$ *be a CVbipMS and let* $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ *be a bisequence. Then,* $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ *is a Cauchy iff* $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s+t})| \to 0$ *as* $s \to \infty$, *where* $t \in \mathbb{N}$.

4. Main Results

The primary goal of this section is to establish the core findings of our research. We begin by proving a foundational proposition that serves as a cornerstone for subsequent results.

Proposition 1. *Let* $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ *and* $\mathfrak{r}_0 \in \Omega$, $\mathfrak{j}_0 = \mathcal{A}\mathfrak{r}_0 \in \mho$, *and* $\mathfrak{r}_1 = \mathcal{A}\mathfrak{j}_0$. *Define the bisequence* $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ *on* $(\Omega, \mho, \mathfrak{o})$ *as*

$$\mathfrak{j}_s = \mathcal{A}\mathfrak{r}_s \text{ and } \mathfrak{r}_{s+1} = \mathcal{A}\mathfrak{j}_s.$$

for all $s = 0, 1, 2, ...$

Assume that there exists $\kappa : \Omega \times \mho \to [0,1)$ *satisfying*

$$\kappa(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa(\mathfrak{r}, \mathfrak{j}) \text{ and } \kappa(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa(\mathfrak{r}, \mathfrak{j})$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$. *Then,*

$$\kappa(\mathfrak{r}_s, \mathfrak{j}) \leq \kappa(\mathfrak{r}_0, \mathfrak{j}) \text{ and } \kappa(\mathfrak{r}, \mathfrak{j}_s) \leq \kappa(\mathfrak{r}, \mathfrak{j}_0)$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$ *and* $s = 0, 1, 2, ...$

Proof. Let $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$ and $s = 0, 1, 2, ...$ Then, we have

$$\kappa(\mathfrak{r}_s, \mathfrak{j}) = \kappa(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}) \leq \kappa(\mathfrak{j}_{s-1}, \mathfrak{j}) = \kappa(\mathcal{A}\mathfrak{r}_{s-1}, \mathfrak{j}) \leq \kappa(\mathfrak{r}_{s-1}, \mathfrak{j}) \leq ... \leq \kappa(\mathfrak{r}_0, \mathfrak{j}).$$

Similarly, we have

$$\kappa(\mathfrak{r}, \mathfrak{j}_s) = \kappa(\mathfrak{r}, \mathcal{A}\mathfrak{r}_s) \leq \kappa(\mathfrak{r}, \mathfrak{r}_s) = \kappa(\mathfrak{r}, \mathcal{A}\mathfrak{j}_{s-1}) \leq \kappa(\mathfrak{r}, \mathfrak{j}_{s-1}) \leq ... \leq \kappa(\mathfrak{r}, \mathfrak{j}_0).$$

□

Theorem 7. *Let* $(\Omega, \mho, \mathfrak{o})$ *be a complete CVbipMS and let* $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$. *If there exist mappings* $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0,1)$ *such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$
$\kappa_2(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_2(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$,
$\kappa_3(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_3(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$,
(b) $\kappa_1(\mathfrak{r}, \mathfrak{j}) + 2\kappa_2(\mathfrak{r}, \mathfrak{j}) + \kappa_3(\mathfrak{r}, \mathfrak{j}) < 1$
(c)

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \preceq\ & \kappa_1(\mathfrak{j}, \mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) + \kappa_2(\mathfrak{j}, \mathfrak{r})(\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})) \\ & + \kappa_3(\mathfrak{j}, \mathfrak{r}) \frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{1 + \mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned} \tag{1}$$

for all $\mathfrak{r} \in \Omega$ *and* $\mathfrak{j} \in \mho$, *then* $\mathcal{A}$ *has a UFP in* $\Omega \cup \mho$.

Proof. Let $\mathfrak{r}_0 \in \Omega$, $\mathfrak{j}_0 = \mathcal{A}\mathfrak{r}_0 \in \mho$, and $\mathfrak{r}_1 = \mathcal{A}\mathfrak{j}_0$. Define the bisequence $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ on $(\Omega, \mho, \mathfrak{o})$ as

$$\mathfrak{j}_s = \mathcal{A}\mathfrak{r}_s \text{ and } \mathfrak{r}_{s+1} = \mathcal{A}\mathfrak{j}_s. \tag{2}$$

Now, by (1), we have

$$\begin{aligned} \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_s,) \\ &\preceq \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\quad + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \frac{\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})} \\ &= \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}. \end{aligned}$$

By proposition (1), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) &\preceq \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})} \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})} \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| &\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|\frac{|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|}{|1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|} \\
&\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \\
&= \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|
\end{aligned}$$

which further implies that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0) + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}{1 - \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0) - \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|. \tag{3}$$

Similarly, by (1), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1},) \\
&\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1}) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})} \\
&= \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})}.
\end{aligned}$$

By proposition (1), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})} \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})} \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})},
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| &\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\frac{|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|}{|1 + \mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \\
&\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|
\end{aligned}$$

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}{1 - \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0) - \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|. \tag{4}$$

Set $\mathfrak{b} = \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}{1-\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)-\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)} < 1$. Then, by (3) and (4), it is easy to see that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \mathfrak{b}^{2s}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|. \tag{5}$$

Similarly, we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \mathfrak{b}^{2s-1}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| \tag{6}$$

for all $s = 1, 2, ...$ For $t, s \in \mathbb{N}$ (if $t > s$), we have

$$\begin{aligned}
|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| &\leq |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_t)| \\
&\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_t)| \\
&\leq \cdots \\
&\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1} + ... + \mathfrak{b}^{2s-1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| + |\mathfrak{o}(\mathfrak{r}_t, \mathfrak{j}_t)| \\
&\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1} + ... + \mathfrak{b}^{2t-1} + \mathfrak{b}^{2t}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|.
\end{aligned}$$

Similarly, if $s > t$, then we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| \leq \left(\mathfrak{b}^{2t+1} + \mathfrak{b}^{2t+2} + ... + \mathfrak{b}^{2s+1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|.$$

Now, taking the limit as $t, s \to \infty$, we obtain $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| \to 0$. Thus, $(\mathfrak{r}_s, \mathfrak{j}_s)$ is a Cauchy in $(\Omega, \mho, \mathfrak{o})$. By using the completeness of the CVbipMS $(\Omega, \mho, \mathfrak{o})$, we find that $(\mathfrak{r}_s, \mathfrak{j}_s)$ biconverges to a limit point $\mathfrak{k} \in \Omega \cap \mho$. Therefore, $(\mathfrak{r}_s) \to \mathfrak{k}$, $(\mathfrak{j}_s) \to \mathfrak{k}$. Hence, $\mathcal{A}\mathfrak{r}_s = \mathfrak{j}_s \to$

$\mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$ implies $\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s) \to \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})$ as $s \to \infty$ by using Lemma (3). Now, by (1), we have

$$\begin{aligned}\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s) &\preceq \kappa_1(\mathfrak{r}_s, \mathfrak{k})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k}) + \kappa_2(\mathfrak{r}_s, \mathfrak{k})(\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s) + \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{k})\frac{\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})} \\ &= \kappa_1(\mathfrak{r}_s, \mathfrak{k})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k}) + \kappa_2(\mathfrak{r}_s, \mathfrak{k})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{k})\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})}.\end{aligned}$$

By proposition (1), we have

$$\begin{aligned}\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s) &\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{k})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k}) + \kappa_2(\mathfrak{r}_0, \mathfrak{k})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})) \\ &\quad + \kappa_3(\mathfrak{r}_0, \mathfrak{k})\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})},\end{aligned}$$

which implies

$$\begin{aligned}|\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s)| &\leq \kappa_1(\mathfrak{r}_0, \mathfrak{k})|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})| + \kappa_2(\mathfrak{r}_0, \mathfrak{k})(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})|) \\ &\quad + \kappa_3(\mathfrak{r}_0, \mathfrak{k})\frac{|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)||\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})|}{|1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})|}.\end{aligned}$$

Taking the limit as $s \to \infty$, we obtain

$$(1 - \kappa_2(\mathfrak{r}_0, \mathfrak{k}))|\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})|$$

which is possible only if $\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k}) = 0$. Hence, $\mathcal{A}\mathfrak{k} = \mathfrak{k}$. Thus, $\mathcal{A}$ has a fixed point. Now, if $\mathfrak{v}$ is another fixed point of $\mathcal{A}$, then $\mathcal{A}\mathfrak{v} = \mathfrak{v}$ implies that $\mathfrak{v} \in \Omega \cap \mho$. Then,

$$\begin{aligned}\mathfrak{o}(\mathfrak{k}, \mathfrak{v}) &= \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{v}) \preceq \kappa_1(\mathfrak{v}, \mathfrak{k})\mathfrak{o}(\mathfrak{v}, \mathfrak{k}) \\ &\quad + \kappa_2(\mathfrak{v}, \mathfrak{k})(\mathfrak{o}(\mathfrak{v}, \mathcal{A}\mathfrak{v}) + \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})) \\ &\quad + \kappa_3(\mathfrak{v}, \mathfrak{k})\frac{\mathfrak{o}(\mathfrak{v}, \mathcal{A}\mathfrak{v}).\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{v}, \mathfrak{k})} \\ &= \kappa_1(\mathfrak{v}, \mathfrak{k})\mathfrak{o}(\mathfrak{v}, \mathfrak{k})\end{aligned}$$

which is a contradiction, except $\mathfrak{k} = \mathfrak{v}$. □

Example 4. *Let* $\Omega = \left\{0, \frac{1}{2}, 2\right\}$ *and* $\mho = \left\{0, \frac{1}{2}\right\}$. *Define* $\mathfrak{o} : \Omega \times \mho \to \mathbb{C}$ *by*

$$\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = |\mathfrak{r} - \mathfrak{j}| + i|\mathfrak{r} - \mathfrak{j}|$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$. *Then,* $(\Omega, \mho, \mathfrak{o})$ *is a complete CVbipMS. Define* $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ *by* $\mathcal{A}(0) = 0$, $\mathcal{A}(\frac{1}{2}) = 0$ *and* $\mathcal{A}(2) = \frac{1}{2}$. *Then,* $\mathcal{A}$ *is a contravariant mapping. Define* $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0, 1)$ *by*

$$\kappa_1(\mathfrak{r}, \mathfrak{j}) = \frac{\mathfrak{r}^2\mathfrak{j}^2}{3}$$

$$\kappa_2(\mathfrak{r}, \mathfrak{j}) = \frac{\mathfrak{r}^2}{25} + \frac{\mathfrak{j}^2}{36}$$

$$\kappa_3(\mathfrak{r}, \mathfrak{j}) = \frac{\mathfrak{r}^2}{64} + \frac{\mathfrak{j}^2}{81}.$$

Consequently, hypotheses (a) and (b) of Theorem 7 are satisfied by these control functions, and the mapping $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ *fulfills inequality 1. Thus, there exists a UFP* 0 *of the mapping* $\mathcal{A}$.

Corollary 1. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$. If there exist mappings $\kappa_1, \kappa_2 : \Omega \times \mho \to [0, 1)$ such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$
$\kappa_2(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_2(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$,
(b) $\kappa_1(\mathfrak{r}, \mathfrak{j}) + 2\kappa_2(\mathfrak{r}, \mathfrak{j}) < 1$
(c)

$$\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \preceq \kappa_1(\mathfrak{j}, \mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) + \kappa_2(\mathfrak{j}, \mathfrak{r})(\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r}))$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, then $\mathcal{A}$ has a UFP.

Proof. Define $\kappa_3 : \Omega \times \mho \to [0, 1)$ as $\kappa_3(\mathfrak{j}, \mathfrak{r}) = 0$ in Theorem 7. □

Corollary 2. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$. If there exist mappings $\kappa_1, \kappa_3 : \Omega \times \mho \to [0, 1)$ such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$
$\kappa_3(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_3(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$,
(b) $\kappa_1(\mathfrak{r}, \mathfrak{j}) + \kappa_3(\mathfrak{r}, \mathfrak{j}) < 1$
(c)

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \quad \preceq \quad & \kappa_1(\mathfrak{j}, \mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) \\ & + \kappa_3(\mathfrak{j}, \mathfrak{r}) \frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{1 + \mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, then $\mathcal{A}$ has a UFP.

Proof. Define $\kappa_2 : \Omega \times \mho \to [0, 1)$ as $\kappa_2(\mathfrak{j}, \mathfrak{r}) = 0$ in Theorem 7. □

Remark 2. *If $\Omega = \mho$ in the above corollary, then CVbipMS is reduced to a CVMS. This allows us to directly recover a result established by Sitthikul et al.* [13].

Remark 3. *If $\Omega = \mho$ and we define $\kappa_1, \kappa_3 : \Omega \times \mho \to [0, 1)$ by $\kappa_1(\mathfrak{r}, \mathfrak{j}) = \kappa_1(\mathfrak{r})$ and $\kappa_3(\mathfrak{r}, \mathfrak{j}) = \kappa_3(\mathfrak{r})$ in the above corollary, we directly arrive at the central finding established by Sintunavarat et al.* [12].

Remark 4. *If $\Omega = \mho$ and we define $\kappa_1, \kappa_3 : \Omega \times \mho \to [0, 1)$ by $\kappa_1(\mathfrak{r}, \mathfrak{j}) = \kappa_1$ and $\kappa_3(\mathfrak{r}, \mathfrak{j}) = \kappa_3$ in the above corollary, we essentially achieve the conditions of Azam et al.* [10], *leading directly to their main result.*

Corollary 3 ([21]). *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$. If there exist the constants $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ with $\kappa_1 + 2\kappa_2 + \kappa_3 < 1$ such that*

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \quad \preceq \quad & \kappa_1\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) + \kappa_2(\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})) \\ & + \kappa_3 \frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{1 + \mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned} \tag{7}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, then $\mathcal{A}$ has a UFP.

Proof. Define $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0, 1)$ by $\kappa_1(\mathfrak{r}, \mathfrak{j}) = \kappa_1$, $\kappa_2(\mathfrak{r}, \mathfrak{j}) = \kappa_2$ and $\kappa_3(\mathfrak{r}, \mathfrak{j}) = \kappa_3$ in Theorem 7. □

Theorem 8. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist mappings $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0, 1)$ such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$,
$\kappa_2(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_2(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$,
$\kappa_3(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_3(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$,

(b) $\kappa_1(\mathfrak{r},\mathfrak{j})+2\kappa_2(\mathfrak{r},\mathfrak{j})+\kappa_3(\mathfrak{r},\mathfrak{j})<1$
(c)

$$\begin{aligned}\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathcal{A}\mathfrak{j}) &\preceq \kappa_1(\mathfrak{j},\mathfrak{r})\mathfrak{o}(\mathfrak{j},\mathfrak{r})+\kappa_2(\mathfrak{j},\mathfrak{r})(\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{j})+\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathfrak{r}))\\ &\quad+\kappa_3(\mathfrak{j},\mathfrak{r})\frac{\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathfrak{r})}{\mathfrak{o}(\mathfrak{j},\mathfrak{r})}\end{aligned} \tag{8}$$

for all $(\mathfrak{r},\mathfrak{j})\in\Omega\times\mho$, *with* $\mathfrak{r}\neq\mathfrak{j}$, *then* $\mathcal{A}$ *has a UFP in* $\Omega\cup\mho$.

Proof. Let $\mathfrak{r}_0\in\Omega$, $\mathfrak{j}_0=\mathcal{A}\mathfrak{r}_0\in\mho$, and $\mathfrak{r}_1=\mathcal{A}\mathfrak{j}_0$. Define the bisequence $\{\mathfrak{r}_s,\mathfrak{j}_s\}$ on $(\Omega,\mho,\mathfrak{o})$ as

$$\mathfrak{j}_s=\mathcal{A}\mathfrak{r}_s \text{ and } \mathfrak{r}_{s+1}=\mathcal{A}\mathfrak{j}_s. \tag{9}$$

Now, by (1), we have

$$\begin{aligned}\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1},\mathcal{A}\mathfrak{r}_s,)\\ &\preceq \kappa_1(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})\\ &\quad+\kappa_2(\mathfrak{r}_s,\mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s,\mathcal{A}\mathfrak{r}_s)+\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1},\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_s,\mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_s,\mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1},\mathfrak{j}_{s-1})}{\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}\\ &= \kappa_1(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})+\kappa_2(\mathfrak{r}_s,\mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_s,\mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}{\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}\\ &= \kappa_1(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})+\kappa_2(\mathfrak{r}_s,\mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s).\end{aligned}$$

By proposition (1), we have

$$\begin{aligned}\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s) &\preceq \kappa_1(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})\\ &\quad+\kappa_2(\mathfrak{r}_s,\mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_s,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)\\ &\preceq \kappa_1(\mathfrak{r}_0,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})\\ &\quad+\kappa_2(\mathfrak{r}_0,\mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_0,\mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)\\ &\preceq \kappa_1(\mathfrak{r}_0,\mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})\\ &\quad+\kappa_2(\mathfrak{r}_0,\mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))\\ &\quad+\kappa_3(\mathfrak{r}_0,\mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s).\end{aligned}$$

This implies that

$$\begin{aligned}|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)| &\leq \kappa_1(\mathfrak{r}_0,\mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})|+\kappa_2(\mathfrak{r}_0,\mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)|+|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})|)\\ &\quad+\kappa_3(\mathfrak{r}_0,\mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)|\\ &\leq \kappa_1(\mathfrak{r}_0,\mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})|+\kappa_2(\mathfrak{r}_0,\mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)|+|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})|)\\ &\quad+\kappa_3(\mathfrak{r}_0,\mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)|\end{aligned}$$

which further implies that

$$|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)|\leq\frac{\kappa_1(\mathfrak{r}_0,\mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0,\mathfrak{j}_0)}{1-\kappa_2(\mathfrak{r}_0,\mathfrak{j}_0)-\kappa_3(\mathfrak{r}_0,\mathfrak{j}_0)}|\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})|. \tag{10}$$

Similarly, by (8), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1},) \\
&\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1}) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})}{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})} \\
&= \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})} \\
&= \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}).
\end{aligned}$$

By proposition (1), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}),
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| &\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|,
\end{aligned}$$

which implies

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}{1-\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)-\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|. \tag{11}$$

Set $\mathfrak{b} = \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}{1-\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)-\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)} < 1$. Then, by (10) and (11), it is easy to see that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \mathfrak{b}^{2s}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|. \tag{12}$$

Similarly, we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \mathfrak{b}^{2s-1}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| \tag{13}$$

for all $s = 1, 2, ...$ By following the same procedure as outlined in Theorem 7, we can demonstrate that $(\mathfrak{r}_s, \mathfrak{j}_s)$ forms a Cauchy in $(\Omega, \mho, \mathfrak{o})$. By using the completeness of $(\Omega, \mho, \mathfrak{o})$, we find that $(\mathfrak{r}_s, \mathfrak{j}_s)$ biconverges to a point $\mathfrak{k} \in \Omega \cap \mho$. So, $(\mathfrak{r}_s) \to \mathfrak{k}, (\mathfrak{j}_s) \to \mathfrak{k}$ as $s \to \infty$. Hence, $\mathcal{A}\mathfrak{r}_s = \mathfrak{j}_s \to \mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$. Also, $\mathcal{A}\mathfrak{j}_s = \mathfrak{r}_{s+1} \to \mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$. Since $\mathcal{A}$ is continuous,

$$\mathcal{A}\mathfrak{k} = \lim_{s\to\infty} \mathcal{A}\mathfrak{r}_s = \lim_{s\to\infty} \mathfrak{j}_s = \mathfrak{k}.$$

Thus, $\mathcal{A}$ has an FP. Now, if $\mathfrak{v}$ is another FP of $\mathcal{A}$, then $\mathcal{A}\mathfrak{v} = \mathfrak{v}$ implies that $\mathfrak{v} \in \Omega \cap \mho$. Then,

$$\begin{aligned} \mathfrak{o}(\mathfrak{k}, \mathfrak{v}) &= \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{v}) \preceq \kappa_1(\mathfrak{v}, \mathfrak{k})\mathfrak{o}(\mathfrak{v}, \mathfrak{k}) \\ &\quad +\kappa_2(\mathfrak{v}, \mathfrak{k})(\mathfrak{o}(\mathfrak{v}, \mathcal{A}\mathfrak{v}) + \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})) \\ &\quad +\kappa_3(\mathfrak{v}, \mathfrak{k})\frac{\mathfrak{o}(\mathfrak{v}, \mathcal{A}\mathfrak{v}).\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{\mathfrak{o}(\mathfrak{v}, \mathfrak{k})} \\ &= \kappa_1(\mathfrak{v}, \mathfrak{k})\mathfrak{o}(\mathfrak{v}, \mathfrak{k}), \end{aligned}$$

which leads to a contradiction. Hence, $\mathfrak{k} = \mathfrak{v}$. □

Corollary 4. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist some constants $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $\kappa_1 + 2\kappa_2 + \kappa_3 < 1$ and*

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) &\preceq \kappa_1\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) + \kappa_2(\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})) \\ &\quad +\kappa_3\frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{\mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, with $\mathfrak{r} \neq \mathfrak{j}$, then $\mathcal{A}$ has a UFP in $\Omega \cup \mho$.

Proof. Define $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0, 1)$ as $\kappa_1(\mathfrak{r}, \mathfrak{j}) = \kappa_1, \kappa_2(\mathfrak{r}, \mathfrak{j}) = \kappa_2$ and $\kappa_3(\mathfrak{r}, \mathfrak{j}) = \kappa_3$ in Theorem 8. □

Remark 5. *If we set $Im(z) = 0$ in the given Definition 5, then the CVbipMS is reduced to a bip-MS, directly yielding a key result established by Paul et al.* [16].

Corollary 5. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist mappings $\kappa_1, \kappa_3 : \Omega \times \mho \to [0, 1)$ such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ and $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$,
$\kappa_3(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$ and $\kappa_3(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$,
(b) $\kappa_1(\mathfrak{r}, \mathfrak{j}) + \kappa_3(\mathfrak{r}, \mathfrak{j}) < 1$,
(c)

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) &\preceq \kappa_1(\mathfrak{j}, \mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) \\ &\quad +\kappa_3(\mathfrak{j}, \mathfrak{r})\frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{\mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, with $\mathfrak{r} \neq \mathfrak{j}$, then $\mathcal{A}$ has a UFP in $\Omega \cup \mho$.

Proof. Define $\kappa_2 : \Omega \times \mho \to [0, 1)$ as $\kappa_2(\mathfrak{r}, \mathfrak{j}) = 0$ in Theorem 8. □

Corollary 6. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist some constants $\kappa_1, \kappa_3 \in [0, 1)$ such that $\kappa_1 + \kappa_3 < 1$ and*

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) &\preceq \kappa_1\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) \\ &\quad +\kappa_3\frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})}{\mathfrak{o}(\mathfrak{j}, \mathfrak{r})} \end{aligned}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, with $\mathfrak{r} \neq \mathfrak{j}$, then $\mathcal{A}$ has a UFP in $\Omega \cup \mho$.

Remark 6. *Setting the imaginary component of z to zero in Definition 5 reduces the CVbipMS to a bipMS, yielding the result of Paul et al.* [16] *as a corollary.*

Theorem 9. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist mappings $\kappa_1, \kappa_2, \kappa_3 : \Omega \times \mho \to [0, 1)$ such that*

(a) $\kappa_1(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_1(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_1(\mathfrak{r}, \mathfrak{j})$
$\kappa_2(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_2(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_2(\mathfrak{r}, \mathfrak{j})$,
$\kappa_3(\mathcal{A}\mathfrak{r}, \mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$ *and* $\kappa_3(\mathfrak{r}, \mathcal{A}\mathfrak{j}) \leq \kappa_3(\mathfrak{r}, \mathfrak{j})$,
(b) $\kappa_1(\mathfrak{r}, \mathfrak{j}) + 2\kappa_2(\mathfrak{r}, \mathfrak{j}) + \kappa_3(\mathfrak{r}, \mathfrak{j}) < 1$
(c)

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathcal{A}\mathfrak{j}) \preceq\ & \kappa_1(\mathfrak{j}, \mathfrak{r})\mathfrak{o}(\mathfrak{j}, \mathfrak{r}) + \kappa_2(\mathfrak{j}, \mathfrak{r})(\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})) \\ & + \kappa_3(\mathfrak{j}, \mathfrak{r}) \frac{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{j})\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathcal{A}\mathfrak{r}, \mathfrak{r})\mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{j})}{\mathfrak{o}(\mathfrak{j}, \mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathfrak{r}, \mathcal{A}\mathfrak{j})} \end{aligned} \tag{14}$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$, then $\mathcal{A}$ has a UFP in $\Omega \cup \mho$.

Proof. Let $\mathfrak{r}_0 \in \Omega$, $\mathfrak{j}_0 = \mathcal{A}\mathfrak{r}_0 \in \mho$, and $\mathfrak{r}_1 = \mathcal{A}\mathfrak{j}_0$. Define the bisequence $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ on $(\Omega, \mho, \mathfrak{o})$ as

$$\mathfrak{j}_s = \mathcal{A}\mathfrak{r}_s \text{ and } \mathfrak{r}_{s+1} = \mathcal{A}\mathfrak{j}_s. \tag{15}$$

Now, by (14), we have

$$\begin{aligned} \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_s,) \\ &\preceq \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\quad + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \frac{\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{j}_{s-1}) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_s)}{\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_s)} \\ &= \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{r}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{j}_{s-1}, \mathfrak{j}_s)}{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{r}_s) + \mathfrak{o}(\mathfrak{j}_{s-1}, \mathfrak{j}_s)} \\ &= \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}). \end{aligned}$$

By proposition (1), we have

$$\begin{aligned} \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) &\preceq \kappa_1(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\quad + \kappa_2(\mathfrak{r}_s, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\quad + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) \\ &\quad + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\ &\quad + \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}). \end{aligned}$$

This implies that

$$\begin{aligned} |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq\ & \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\ & + \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|, \end{aligned}$$

which further implies that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0) + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0) + \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}{1 - \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)} |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|. \tag{16}$$

Similarly, by (14), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1},) \\
&\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1}) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{j}_{s-1}) + \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})}{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})} \\
&= \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{r}_s) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})}{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{r}_s) + \mathfrak{o}(\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})} \\
&= \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}).
\end{aligned}$$

By proposition (1), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &\preceq \kappa_1(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_{s-1})(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\preceq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}) + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}),
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| &\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)(|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| + |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|) \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\leq \kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| + \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \\
&\quad +\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|
\end{aligned}$$

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0) + \kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}{1 - \kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|. \tag{17}$$

Set $\flat = \frac{\kappa_1(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)+\kappa_3(\mathfrak{r}_0, \mathfrak{j}_0)}{1-\kappa_2(\mathfrak{r}_0, \mathfrak{j}_0)} < 1$. Then, by (16) and (17), it is easy to see that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \flat^{2s}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|. \tag{18}$$

Similarly, we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \flat^{2s-1}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| \tag{19}$$

for all $s = 1, 2, ...$ By applying the same method used in Theorem 7, we can establish that the bisequence $(\mathfrak{r}_s, \mathfrak{j}_s)$ is a Cauchy in $(\Omega, \mho, \mathfrak{o})$. $(\Omega, \mho, \mathfrak{o})$ is complete, so $(\mathfrak{r}_s, \mathfrak{j}_s) \to \mathfrak{k} \in \Omega \cap \mho$

as $s \to \infty$. Consequently, $(\mathfrak{r}_s) \to \mathfrak{k}, (\mathfrak{j}_s) \to \mathfrak{k}$ as $s \to \infty$. Hence, $\mathcal{A}\mathfrak{r}_s = \mathfrak{j}_s \to \mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$. Also, $\mathcal{A}\mathfrak{j}_s = \mathfrak{r}_{s+1} \to \mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$. Since $\mathcal{A}$ is continuous,

$$\mathcal{A}\mathfrak{k} = \lim_{s\to\infty} \mathcal{A}\mathfrak{r}_s = \lim_{s\to\infty} \mathfrak{j}_s = \mathfrak{k}.$$

Thus, $\mathcal{A}$ has an FP. Now, if $\mathfrak{v}$ is another FP of $\mathcal{A}$, then $\mathcal{A}\mathfrak{v} = \mathfrak{v}$ implies that $\mathfrak{v} \in \Omega \cap \mho$. Then,

$$\begin{aligned}
\mathfrak{o}(\mathfrak{k},\mathfrak{v}) &= \mathfrak{o}(\mathcal{A}\mathfrak{k},\mathcal{A}\mathfrak{v}) \preceq \kappa_1(\mathfrak{v},\mathfrak{k})\mathfrak{o}(\mathfrak{v},\mathfrak{k}) \\
&\quad +\kappa_2(\mathfrak{v},\mathfrak{k})(\mathfrak{o}(\mathfrak{v},\mathcal{A}\mathfrak{v}) + \mathfrak{o}(\mathcal{A}\mathfrak{k},\mathfrak{k})) \\
&\quad +\kappa_3(\mathfrak{v},\mathfrak{k})\frac{\mathfrak{o}(\mathfrak{v},\mathcal{A}\mathfrak{v})\mathfrak{o}(\mathfrak{v},\mathcal{A}\mathfrak{k}) + \mathfrak{o}(\mathcal{A}\mathfrak{k},\mathfrak{k})\mathfrak{o}(\mathfrak{k},\mathcal{A}\mathfrak{v})}{\mathfrak{o}(\mathfrak{v},\mathcal{A}\mathfrak{k}) + \mathfrak{o}(\mathfrak{k},\mathcal{A}\mathfrak{v})} \\
&= \kappa_1(\mathfrak{v},\mathfrak{k})\mathfrak{o}(\mathfrak{v},\mathfrak{k}),
\end{aligned}$$

which would normally be contradictory, except $\mathfrak{k} = \mathfrak{v}$. □

Corollary 7. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and let $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be continuous. If there exist some constants $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $\kappa_1 + 2\kappa_2 + \kappa_3 < 1$ and*

$$\begin{aligned}
\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathcal{A}\mathfrak{j}) \preceq\ & \kappa_1\mathfrak{o}(\mathfrak{j},\mathfrak{r}) + \kappa_2(\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{j}) + \mathfrak{o}(\mathcal{A}\mathfrak{r},\mathfrak{r})) \\
&+\kappa_3\frac{\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{j})\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathcal{A}\mathfrak{r},\mathfrak{r})\mathfrak{o}(\mathfrak{r},\mathcal{A}\mathfrak{j})}{\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{r}) + \mathfrak{o}(\mathfrak{r},\mathcal{A}\mathfrak{j})}
\end{aligned}$$

for all $(\mathfrak{r},\mathfrak{j}) \in \Omega \times \mho$, then $\mathcal{A}$ has a UFP in $\Omega \cup \mho$.

Remark 7. *Setting the imaginary component of z to zero in Definition 5 reduces the CVbipMS to a bipMS, yielding the result of Paul et al. [16] as a corollary.*

Definition 8. *Let $(\Omega, \mho, \mathfrak{o})$ be a CVbipMS. A mapping $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ is said to be an interpolative contraction if there exist $\mathfrak{b} \in [0, 1)$ and $\sigma \in (0, 1)$ such that*

$$\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathcal{A}\mathfrak{j}) \precsim \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{j},\mathcal{A}\mathfrak{j})\mathfrak{o}(\mathcal{A}\mathfrak{r},\mathfrak{r})}{1+\mathfrak{o}(\mathfrak{j},\mathfrak{r})}\right)^{\sigma}(\mathfrak{o}(\mathfrak{j},\mathfrak{r}))^{1-\sigma} \tag{20}$$

for all $(\mathfrak{r},\mathfrak{j}) \in \Omega \times \mho \backslash Fix(\mathcal{A})$, where $Fix(\mathcal{A}) = \{\mathfrak{r} \in \Omega \cap \mho : \mathfrak{r} = \mathcal{A}\mathfrak{r}\}$.

Theorem 10. *Let $(\Omega, \mho, \mathfrak{o})$ be a complete CVbipMS and the mapping $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ be an interpolative contraction; then, $\mathcal{A}$ has an FP in $\Omega \cup \mho$.*

Proof. Let $\mathfrak{r}_0 \in \Omega$, $\mathfrak{j}_0 = \mathcal{A}\mathfrak{r}_0 \in \mho$, and $\mathfrak{r}_1 = \mathcal{A}\mathfrak{j}_0$. Define the bisequence $\{\mathfrak{r}_s, \mathfrak{j}_s\}$ on $(\Omega, \mho, \mathfrak{o})$ as

$$\mathfrak{j}_s = \mathcal{A}\mathfrak{r}_s \text{ and } \mathfrak{r}_{s+1} = \mathcal{A}\mathfrak{j}_s.$$

for each positive integer s. Now, by (20), we have

$$\begin{aligned}
\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1},\mathcal{A}\mathfrak{r}_s,) \\
&\preceq \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_s,\mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1},\mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}\right)^{\sigma}(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))^{1-\sigma} \\
&= \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_s)\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1})}\right)^{\sigma}(\mathfrak{o}(\mathfrak{r}_s,\mathfrak{j}_{s-1}))^{1-\sigma},
\end{aligned}$$

which implies that

$$\begin{aligned} |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| &\leq \mathfrak{b}\left(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|\left|\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}\right|\right)^{\sigma}(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|)^{1-\sigma} \\ &\leq \mathfrak{b}(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|)^{\sigma}(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|)^{1-\sigma}, \end{aligned}$$

which further implies that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|^{1-\sigma}. \tag{21}$$

Suppose that $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \geq |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|$. Then, from (21), we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)|^{1-\sigma}$$

which is a contradiction because $\mathfrak{b} < 1$. Thus, we we conclude that $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|$ for all $s \geq 1$. Thus, $\{|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|\}$ is a non-increasing sequence, and by inequality (21), there exists a nonnegative constant $l_1 \geq 0$ such that $\lim_{s\to\infty}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| = l_1$. Then, the inequality (21) yields

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|. \tag{22}$$

Similarly, by (1), we have

$$\begin{aligned} \mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1}) &= \mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1},) \\ &\preceq \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathcal{A}\mathfrak{r}_{s-1})\mathfrak{o}(\mathcal{A}\mathfrak{j}_{s-1}, \mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})}\right)^{\sigma}(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}))^{1-\sigma} \\ &\preceq \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})}\right)^{\sigma}(\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1}))^{1-\sigma}, \end{aligned}$$

which implies that

$$\begin{aligned} |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| &\leq \mathfrak{b}\left(\left|\frac{\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})}{1+\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})}\right||\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|\right)^{\sigma}(|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|)^{1-\sigma} \\ &\leq \mathfrak{b}(|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|)^{\sigma}(|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|)^{1-\sigma}, \end{aligned}$$

which further implies that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|^{1-\sigma}. \tag{23}$$

Suppose that $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \geq |\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|$. Then, from (23), we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|^{1-\sigma} \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})|^{1-\sigma}$$

which is a contradiction because $\mathfrak{b} < 1$. Thus, we we conclude that $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq |\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|$ for all $s \geq 1$. Thus, $\{|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|\}$ is a non-increasing sequence, and by inequality (23), we conclude that there exists a nonnegative constant $l_2 \geq 0$ such that $\lim_{s\to\infty}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})| = l_2$. Then, inequality (23) yields

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \mathfrak{b}|\mathfrak{o}(\mathfrak{r}_{s-1}, \mathfrak{j}_{s-1})|. \tag{24}$$

Then, by (22) and (24), it is easy to see that

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| \leq \mathfrak{b}^{2s}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|. \tag{25}$$

Similarly, we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_{s-1})| \leq \mathfrak{b}^{2s-1}|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|, \tag{26}$$

for all $s = 1, 2, ...$ For $t, s \in \mathbb{N}$ (if $t > s$), we have

$$\begin{aligned} |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| &\leq |\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_s)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_t)| \\ &\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| + |\mathfrak{o}(\mathfrak{r}_{s+1}, \mathfrak{j}_t)| \\ &\leq \cdots \\ &\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1} + ... + \mathfrak{b}^{2s-1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)| + |\mathfrak{o}(\mathfrak{r}_t, \mathfrak{j}_t)| \\ &\leq \left(\mathfrak{b}^{2s} + \mathfrak{b}^{2s+1} + ... + \mathfrak{b}^{2t-1} + \mathfrak{b}^{2t}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|. \end{aligned}$$

Similarly, if $s > t$, then we have

$$|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| \leq \left(\mathfrak{b}^{2t+1} + \mathfrak{b}^{2t+2} + ... + \mathfrak{b}^{2s+1}\right)|\mathfrak{o}(\mathfrak{r}_0, \mathfrak{j}_0)|.$$

Now, taking the limit as $t, s \to \infty$, we obtain $|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_t)| \to 0$. Thus, $(\mathfrak{r}_s, \mathfrak{j}_s)$ forms a Cauchy in $(\Omega, \mho, \mathfrak{o})$. Given that $(\Omega, \mho, \mathfrak{o})$ is complete, $(\mathfrak{r}_s, \mathfrak{j}_s)$ biconverges to a point $\mathfrak{k} \in \Omega \cap \mho$. Consequently, $(\mathfrak{r}_s) \to \mathfrak{k}$ and $(\mathfrak{j}_s) \to \mathfrak{k}$. Therefore, $\mathcal{A}\mathfrak{r}_s = \mathfrak{j}_s \to \mathfrak{k} \in \Omega \cap \mho$ as $s \to \infty$, which implies $\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s) \to \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})$ as $s \to \infty$ by using Lemma (3). Now, by (1), we have

$$\begin{aligned} \mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s) &\preceq \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_s, \mathcal{A}\mathfrak{r}_s)\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})}\right)^{\sigma} (\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k}))^{1-\sigma} \\ &\preceq \mathfrak{b}\left(\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})}\right)^{\sigma} (\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k}))^{1-\sigma} \end{aligned}$$

which implies

$$|\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathcal{A}\mathfrak{r}_s)| \leq \mathfrak{b}\left(\left|\frac{\mathfrak{o}(\mathfrak{r}_s, \mathfrak{j}_s)}{1 + \mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})}\right||\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k})|\right)^{\sigma} (|\mathfrak{o}(\mathfrak{r}_s, \mathfrak{k})|)^{1-\sigma}.$$

Taking the limit as $s \to \infty$, we obtain $\mathfrak{o}(\mathcal{A}\mathfrak{k}, \mathfrak{k}) = 0$. Hence, $\mathcal{A}\mathfrak{k} = \mathfrak{k}$. Thus, $\mathcal{A}$ has an FP. □

Example 5. *Let* $\Omega = \{4, 5, 6\}$ *and* $\mho = \{5, 6, 7\}$. *Define* $\mathfrak{o} : \Omega \times \mho \to \mathbb{C}$ *by*

$$\mathfrak{o}(\mathfrak{r}, \mathfrak{j}) = |\mathfrak{r} - \mathfrak{j}| + i|\mathfrak{r} - \mathfrak{j}|,$$

for all $(\mathfrak{r}, \mathfrak{j}) \in \Omega \times \mho$; *then,* $(\Omega, \mho, \mathfrak{o})$ *is a complete CVbipMS. Define* $\mathcal{A} : (\Omega, \mho, \mathfrak{o}) \rightleftarrows (\Omega, \mho, \mathfrak{o})$ *by* $\mathcal{A}(\mathfrak{r}) = 6$ *for all* $\Omega \cup \mho$. *We can therefore verify that all the requirements of Theorem 10 hold true for* $\mathfrak{b} = \frac{1}{2}$ *and* $\sigma = \frac{1}{2}$ *and that* $\mathcal{A}$ *has an FP. Clearly, 6 is the FP of* $\mathcal{A}$.

5. Conclusions and Future Directions

In this manuscript, we have introduced some precise control functions in contractive inequalities and proved the FPs of contravariant mappings in the situation of a CVbipMS. As a consequence of our chief outcomes, we derived some well-known results from the literature, including the core achievements of Azam et al. [10], Sintunavarat et al. [12], Sitthikul et al. [13], Paul et al. [16], and Siva [21]. Some illustrative examples are also provided to verify the authenticity of the accomplished theorem. We also put forward the idea of interpolative contraction in the background of CVbipMSs and established an FP theorem for contravariant mappings.

Building on our current findings, future research will delve deeper into the intriguing behavior of FPs for set-valued functions in the intricate foundation of CVbipMSs. Furthermore, the proposed contractions can be extended to encompass a broader class of metric spaces beyond CVbipMSs. The potential applications of these results can be investigated in the context of fractional differential equations and even integral equations. Additionally, efficient computational algorithms can be designed to tackle FP problems stemming from the proposed contractions.

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