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One Turán Type Problem on Uniform Hypergraphs

Linlin Wang [1] and Sujuan Liu [2,*]

[1] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China; 5400@cumt.edu.cn
[2] College of Artificial Intelligence, Tianjin University of Science & Technology, Tianjin 300457, China
* Correspondence: sjliu@tust.edu.cn

Abstract: Let $n, m, p, r \in N$ with $p \geq n \geq r$. For a hypergraph, if each edge has r vertices, then the hypergraph is called an r-graph. Define $e_r(n, m; p)$ to be the maximum number of edges of an r-graph with p vertices in which every subgraph of n vertices has at most m edges. Researching this function constitutes a Turán type problem. In this paper, on the one hand, for fixed p, we present some results about the exact values of $e_r(n, m; p)$ for small m compared to n; on the other hand, for sufficient large p, we use the combinatorial technique of double counting to give an upper bound of $e(n, m; p)$ and obtain a lower bound of $e_r(n, m; p)$ by applying the lower bound of the independent set of a hypergraph.

Keywords: Turán type problem; extremal graph; uniform hypergraph; bounds

MSC: 05C15; 05C20; 05C40

1. Introduction

All graphs considered in this article are undirected and contain no loops or multiple edges. Let $r \in \mathbb{N}$ with $r \geq 2$, an r-uniform hypergraph G is a pair $(V(G), E(G))$ consisting of vertex set $V(G)$ and edge set $E(G)$ of r-element subsets of $V(G)$. When $r = 2$, G is a simple graph; when $r \geq 3$, G is also called an r-graph. We call $|V(G)|$ the order of G and $|E(G)|$ the size of G and let $e(G) = |E(G)|$.

For a vertex $v \in V(G)$, the degree of v is the number of edges in $E(G)$ containing v and is denoted by $d(v)$. We denote by $\Delta(G)$ the maximum degree of the vertices of G and by $d(G)$ their average degree, $\frac{1}{|V(G)|}\sum_{v \in V(G)} d(v)$. Let $n \in \mathbb{N}$ and $|V(G)| = n$, sometimes we use $\binom{[n]}{k}$ to denote $\{K : K \subset V(G), |K| = k\}$. If $|V(G)| = n$ and $E(G) = \binom{[n]}{r}$, then G is called a complete r-uniform hypergraph and denoted by $K_n^{(r)}$; when $r = 2$, we omit r and denote the complete graph by K_n.

For two r-graphs F and G, if any subhypergraph of G is not isomorphic to F, then we say G is F-free; otherwise, G contains at least one copy of F. Let $\mathcal{F}$ be a nonempty set of r-graphs; if G is F-free for any $F \in \mathcal{F}$, then we say G is $\mathcal{F}$-free. For a positive p and r-graph set $\mathcal{F}$, define $ex_r(p, \mathcal{F})$ as the maximum number of edges of an r-graph with p vertices which is $\mathcal{F}$-free; this is called the Turán number [1] of $\mathcal{F}$. When the set $\mathcal{F}$ only contains one element F, we write it as $ex_r(p, F)$ by simplicity. Moreover, when $r = 2$, we usually use $ex(p, F)$ rather than $ex_2(p, F)$. Meanwhile the Turán density of an r-graph F is defined as

$$\pi(F) = lim_{p \to \infty} \frac{ex_r(p, F)}{\binom{p}{r}}.$$

In the year 1941, Turán [2] firstly did research about the function $ex(p, F)$ for a general simple graph F. At present, there have been plenty of results about this function for the case $r = 2$, most of which focus on determining the bounds of the Turán number for different graphs. It seems difficult to determine the exact values of Turán numbers even for 2-graphs,

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except for several classes of graphs containing complete graphs, roads, etc. One of the classic results is the Turán number with respect to a complete graph.

Theorem 1 ([2]). *Let $p \geq n \in \mathbb{N}$, then*

$$ex(p, K_n) = \frac{(n-2)(p^2 - r^2)}{2(n-1)} + \frac{r(r-1)}{2},$$

where $p \equiv r(mod\ n-1)(r \in \{0, 1, \ldots, n-2\})$.

For convenience, we set $t_{n-1}(p) = ex(p, K_n)$ and let $Ex(p, K_n)$ be the set of graphs which is K_n-free with $ex(p, K_n)$ edges. For any natural numbers p, n with $p \geq n$, partition p vertices into $n-1$ parts $V_1, \ldots, V_{n-1}$ of nearly equal size, i.e., $||V_i| - |V_j|| \leq 1$ for different i, j. Then, this complete $(n-1)$-partite graph is called a Turán graph, denoted by $T_{n-1}(p)$. It is not hard to check that $e(T_{n-1}(p)) = t_{n-1}(p)$. So $T_{n-1}(p)$ is an extremal graph, i.e., $T_{n-1}(p) \in Ex(p, K_n)$. In addition, it was proved that the extremal graph is unique (up to isomorphism).

The exact values of Turán numbers for other types of graphs can be found in the literature [3–9]. For the graphs whose Turán number is difficult to calculate, it makes sense to know the asymptotic order of their Turán numbers. In fact, determining the Turán density of a given graph is also a classical extremal graph problem. Let $\chi(F)$ be the chromatic number of F, which means the minimum numbers of colors making any neighbors in F colored by distinct colors. Erdős, Stone [10], and Simonovits [11] proved the foundational theorem of extremal graph theory, known as the Erdős–Stone–Simonovits theorem for short.

Theorem 2 ([10,11]). *Let $p \in \mathbb{N}$ and $\mathcal{F}$ be a family of graphs, when p is large enough,*

$$ex(p, \mathcal{F}) = \frac{\chi(\mathcal{F}) - 2}{\chi(\mathcal{F}) - 1} \cdot \frac{p^2}{2} + o(p^2),$$

where $\chi(\mathcal{F}) = min\{\chi(F) : F \in \mathcal{F}\}$.

The Erdős–Stone–Simonovits theorem states that the Turán density of any χ-chromatic graph F is equal to $\frac{\chi(\mathcal{F})-2}{\chi(\mathcal{F})-1}$. For any r-graph F, a well-known fact (proved in 1964 [12]) is that the Turán density of F always exists.

In this paper, we focused on one Turán type problem for r-graphs; however, we do not consider the concrete structure of an r-graph and we are concerned about the edges of the subhypergraphs. We want to find the maximum edges or the minimum density of r-graphs only by adding a limit to the number of edges of the subhypergraphs. Now, we give the definition of a function that we mainly consider in this article.

Definition 1. *Let $p, n, m, r \in \mathbb{N}$ with $p \geq n \geq r$, define $e_r(n, m; p)$ as the maximum number of edges of an r-graph with p vertices in which n vertices induce at most m edges, and just denoted by $e(n, m; p)$ when $r = 2$.*

Obviously, $e_r(n, \binom{n}{r}; p) = \binom{p}{r}$, $e_r(n, \binom{n}{r} - 1; p) = ex_r(p, K_n^r)$.

When $0 \leq m \leq \binom{n}{r}$, we have $0 \leq e_r(n, m; p) \leq \binom{p}{2}$.

For convenience, we call G an (n, m) graph if it has the property that any n vertices induce at most m edges.

If we let $\mathcal{H} = \{H$ is an r-graph$|$ any subgraph $F \subset H$ induced by n vertices contains at least $m+1$ edges$\}$, then we have $e_r(n, m; p) = ex_r(p, \mathcal{H})$, so this is one Turán type problem. In fact, as early as 1963, G. Dirac [13] began to study this function for general simple graphs, and later there were several authors [14–16] who also did some work with this function and generalized this problem from general simple graphs to hypergraphs. It is worth mentioning that there is a similar function with respect to hypergraphs investigated

by Brown, Erdős, and T. Sós [17,18], where the function becomes much more involved and sometimes extremely deep. For more about Turán type hypergraph results, we referred to [19,20] and the surveys by Füredi [21] and Sidorenko [22].

Here, let us consider the function $e_r(n, m; p)$ from two perspectives: its exact values for fixed r and p; its asymptotic values for fixed n and m and sufficiently large n.

Now, we list all the known and new results about this function from the above two perspectives.

Theorem 3 ([16]). *Let $p, n, k \in \mathbb{N}$ with $p \geq n \geq k$,*

1. $e(n, n-1; p) = ex(p, \{C_3, \ldots, C_n\})$;
2. $e(n, t_{k-1}(n); p) = t_{k-1}(p)$, *where* $k \geq 3$;
3. $e(n, \binom{n}{2} - k; p) = t_{n-k}(p)$, *where* $n \geq 2k$;
4. $t_{k-1}(p) \leq e(n, m; p) \leq t_k(p)$, *where* $t_{k-1}(n) \leq m < t_k(n)$.

For the case $m \leq n-2$, the value of $e(n, m; p)$ was obtained by different authors (refer to [14,15,23]), but the original papers are not easily found, so we give a proof of this in Section 2.

Theorem 4. *Let $m, n, p \in \mathbb{N}$ with $2 \leq m \leq n \leq p$. If $n-1 = k(m-1)+r, k \in \mathbb{N}, r \in \{0, 1, \ldots, m-2\}$, then*

$$e(n, n-m; p) = \left\lceil \frac{(k-1)p + r}{k} \right\rceil = p - m + 1 + \left\lceil \frac{n-1-p}{\lceil \frac{n-1}{m-1} \rceil} \right\rceil.$$

For a given r-graph G, if a subset S of $V(G)$ does not contain any edge in $E(G)$, i.e., $\binom{S}{r} \cap E(G) = \emptyset$, then S is called an independent set and the independence number of G is

$$\alpha(G) := max\{|S| : S \subset V(G) \text{ is an independent set}\}.$$

By the following lemma, we can give a lower bound of $e_r(n, m; p)$ in Section 3.

Lemma 1 ([24]). *For a given positive number $r \geq 2$, there exists a constant C which only depends on the value of r such that for any r-graph H*

$$\alpha(H) \geq C \frac{|V(H)|}{d(H)^{\frac{1}{r-1}}}.$$

For sufficiently large p, we use the combinatorial technique of double counting to give an upper bound of $e_r(n, m; p)$, so we have the following result.

Theorem 5. *Let $p, n, m, r \in \mathbb{N}$ with $p \geq n \geq r$. If $e_r(n, m; p) \geq m+1$, then*

$$C_1 p^{\frac{(m+1)r-n}{m}} \leq e_r(n, m; p) \leq C_2 p^{\lceil \frac{(m+1)r-n}{m} \rceil}$$

where C_1, C_2 are constants which depend only on the values of n, m, r.

2. Proof of Theorem 4

In this section, all the graphs are simple, and T_n denotes a tree with n vertices without any specific structural constraints.

Lemma 2. *Let $m, n, p \in \mathbb{N}$ with $2 \leq m \leq n \leq p$. If $n-1 = k(m-1)+r, k \in \mathbb{N}, r \in \{0, 1, \ldots, m-2\}$, then*

$$e(n, n-m; p) \geq \left\lceil \frac{(k-1)p + r}{k} \right\rceil.$$

Proof. Since

$$\begin{aligned} p - r(k+1) &\geq n - r(k+1) \\ &= k(m-1) + r + 1 - r(k+1) \\ &\geq k+1. \end{aligned}$$

we let $p - r(k+1) = ak + b, a \in \mathbb{N}, b \in \{0, 1, \ldots, k-1\}$. Set $G = rT_{k+1} \cup aT_k \cup T_b$. Then, we have

$$\begin{aligned} e(G) &= e(rT_{k+1}) + e(aT_k) + e(T_b) \\ &= rk + a(k-1) + b - 1 \\ &= rk + a(k-1) + [b - b/k] \\ &= [rk + (ak+b)(1 - 1/k)] \\ &= [rk + (p - r(k+1))\frac{k-1}{k}] \\ &= [\frac{(k-1)p + r}{k}]. \end{aligned}$$

Now, we just need to prove that this graph is an $(n, n-m)$ graph to complete the proof of the lemma.

Let G_0 be an arbitrary subgraph of G induced by n vertices. Obviously, G_0 is a forest, and then we suppose that G_0 has t components, among which there are c trees with $k+1$ vertices; thereby $c \leq r$. If $t \leq m-1$, then $n = v(G_0) \leq c(k+1) + (t-c)k = kt + c \leq k(m-1) + r = n - 1$, which is impossible! Hence, $t \geq m$ and $e(G_0) = n - t \leq n - m$, which means that G is an $(n, n-m)$ graph. □

Lemma 3. *Let $m, n, p \in \mathbb{N}$ with $2 \leq m \leq n \leq p$. If $n - 1 = k(m-1) + r, k \in \mathbb{N}, r \in \{0, 1, \ldots, m-2\}$, G is an $(n, n-m)$ graph with p vertices, and G does not contain subtree with at least $k+1$ vertices, then*

$$e(n, n-m; p) \leq \left[\frac{(k-1)p + r}{k}\right].$$

Proof. Case 1: G is a forest.

Assume that G has t subtrees, then by the condition, we know each component contains at most k vertices; thereby, $p \leq kt$. Hence,

$$e(G) = p - t \leq p - p/k = \frac{(k-1)p}{k} \leq \left[\frac{(k-1)p + r}{k}\right].$$

Case 2: G is not a forest.

Suppose that G has t components, and let $G = \cup_{i=1}^{t} G_i$, where $G_1, \ldots, G_s$ are trees and $G_{s+1}, \ldots, G_t$ are not; obviously, $0 \leq s < t$. Let $v(G_i) = p_i, e(G_i) = q_i$, for $i = 1, \ldots, t$.

When $p_i \leq q_i$ for $s+1 \leq i \leq t$, without loss of generality, we may assume that $p_1 \leq p_2 \leq \cdots p_s \leq p_{s+1} \leq \cdots \leq p_t$.

If $\sum_{i=s+1}^{t} p_i \geq n$, then denote the minimal integer by u such that $u \in \{s+1, \cdots, t\}$ and $\sum_{i=s+1}^{u} p_i \geq n$. Let T be a subtree of G_u, and $v(T) = n - \sum_{i=s+1}^{u-1} p_i$. Then, $G_{s+1} \cup \cdots \cup G_{u-1} \cup T$ induces a subgraph with vertices $\sum_{i=s+1}^{u-1} p_i + n - \sum_{i=s+1}^{u-1} p_i = n$, and edges

$$e(G_{s+1} \cup \cdots G_{u-1}) + e(T) \geq \sum_{i=s+1}^{u-1} p_i + n - \sum_{i=s+1}^{u-1} p_i - 1 = n - 1 > n - m.$$

This contradicts that G is an $(n, n-m)$ graph. So, we have $\sum_{i=s+1}^{t} p_i \leq n - 1$. Hence, we have $s \geq 1$.

Let $l = \sum_{i=s+1}^{t}(q_i - p_i)$. For $s+1 \le i \le t$, we have $q_i \ge p_i$, so $l \ge 0$. Because $\sum_{i=1}^{s} p_i = p - \sum_{i=s+1}^{t} p_i \ge n - \sum_{i=s+1}^{t} p_i \ge 1$, we can choose one subgraph denoted by G_0 among the components $G_1 \cup \cdots G_s$ such that $v(G_0) = n - \sum_{i=s+1}^{t} p_i$ and $e(G_0) = n - \sum_{i=s+1}^{t} p_i - s$. Now, $G_0 \cup G_{s+1} \cup \cdots G_t$ is a subgraph of G with vertices n and edges at least $n - \sum_{i=s+1}^{t} p_i - s + \sum_{i=s+1}^{t} q_i = n - s + l$. Because G is an $(n, n-m)$ graph, we have $n - s + l \le n - m$ that is $l \le s - m$.

If $\sum_{i=s+1}^{t} p_i + \sum_{i=1}^{l+m-1} p_i \ge n$, then, similar to above, we can find a subgraph also denoted by G_0 among the components $\cup_{i=1}^{l+m-1} G_i$ with vertices $|V(G_0)| = n - \sum_{i=s+1}^{t} p_i$ and edges $e(G_0) \ge n - \sum_{i=s+1}^{t} p_i - (l+m-1)$. Now, we have that $G_0 \cup G_{s+1} \cup \cdots \cup G_t$ is a subgraph of G with vertices n and edges at least

$$n - \sum_{i=s+1}^{t} p_i - (l+m-1) + \sum_{i=s+1}^{t} q_i = n - m + 1 > n - m,$$

which contradicts the property of G. Hence,

$$\sum_{i=s+1}^{t} p_i + \sum_{i=1}^{l+m-1} p_i \le n - 1.$$

So

$$\sum_{i=l+m}^{s} p_i = p - \sum_{i=1}^{t} p_i - \sum_{i=1}^{l+m-1} p_i \ge p - (n-1).$$

On the other hand,

$$\sum_{i=l+m}^{s} p_i \le \sum_{i=l+m}^{s} k = k(s - l - m + 1).$$

Combining the above two inequalities, we have that

$$p - (n-1) \le k(s - l - m + 1).$$

Therefore, $l \le s - m + 1 - \frac{p-n+1}{k}$. Now, we can estimate the edges of G like this,

$$\begin{aligned} e(G) &= \sum_{i=s+1}^{t} q_i + \sum_{i=1}^{s}(p_i - 1) \\ &= \sum_{i=s+1}^{t} p_i + l + \sum_{i=1}^{s} p_i - s = p + l - s \\ &\le p + s - m + 1 - \frac{p-n+1}{k} - s \\ &= p - m + 1 - \frac{p-n+1}{k} = \frac{(k-1)p + r}{k}. \end{aligned}$$

Therefore, we obtain that $e(G) \le [\frac{(k-1)p+r}{k}]$. □

Now, we prove Theorem 4

Proof. Since $k = [\frac{n-1}{m-1}]$, by simple derivation, we have that

$$\begin{aligned} \left[\frac{(k-1)p + r}{k}\right] &= p + \left[\frac{r-p}{k}\right] = p + \left[\frac{n - 1 - k(m-1) - p}{k}\right] \\ &= p - m + 1 + \left[\frac{n-1-p}{[\frac{n-1}{m-1}]}\right]. \end{aligned}$$

Now, we prove the theorem by induction on m. When $m = 2$, $n-1 = (n-1)\cdot(2-1)+0$ and so $k = n-1, r = 0$. Then, by Lemma 2, we know that $e(n, n-2; p) \geq [\frac{(n-2)p}{n-1}]$. Furthermore, G is an $(n, n-2)$ graph, so G cannot contain one subtree with vertices $n = k+1$; this, together with Lemma 3, implies that

$$e(n, n-2; p) \leq \left[\frac{(n-2)p}{n-1}\right].$$

So, the theorem is true for $m = 2$.

Now, we assume that $m \geq 3$ and the theorem is true for $m-1$.

Suppose that G is an $(n, n-m)$ graph and $v(G) = p$. Let $G_1, \ldots, G_t$ be all components of G and $v(G_i) = p_i$ for $i \in \{1, \ldots, t\}$. By Lemma 2, it suffices to prove that $e(G) \leq [\frac{(k-1)p+r}{k}]$. If G does not contain a subtree with at least $k+1$ vertices, then, by Lemma 3, we have completed it. So, from now, we assume G contains some component denoted by G_i such that $v(G_i) = p_i \geq k+1$ and $e(G_i) \geq p_i - 1$; then, we claim that $p_i \leq n-m+1$; otherwise, $p_i \geq n-m+2$ and $e(G_i) \geq n-m+1$; then we can find a subgraph in G_i denoted by H satisfying that $v(H) = n-m+2$ and $e(H) = n-m+1$. Again, this contradicts the property of G.

Now, we have that $G - G_i$ is an $(n-p_i, n-p_i-m+1)$ graph. Since $n-p_i-1 \geq m-2$, we may assume that $n-p_i-1 = k'(m-2)+r', r' \in \mathbb{N}, r' \in \{0, 1, \ldots, m-3\}$. Note that $2 \leq m-1 \leq n-p_i \leq p-p_i$; by the induction, we have

$$e(n-p_i, n-p_i-m+1; p-p_i) = \left[\frac{(k'-1)(p-p_i)+r'}{k'}\right].$$

As $p_i \geq k+1 = [\frac{n-1}{m-1}]+1 > \frac{n-1}{m-1}$, we have that $\frac{n-p_i-1}{m-2} - \frac{n-1}{m-1} = \frac{n-1-(m-1)p_i}{(m-2)(m-1)} < 0$, and so

$$k' = \left[\frac{n-p_i-1}{m-2}\right] \leq \frac{n-p_i-1}{m-2} < \frac{n-1}{m-1}.$$

Therefore, $k' \leq [\frac{n-1}{m-1}] = k$. Since

$$\begin{aligned}
&p_i - 1 + \left[\frac{(k'-1)(p-p_i)+r'}{k'}\right] - \frac{(k-1)p+r}{k} \\
&\leq p_i - 1 + \frac{(k'-1)(p-p_i)+r'}{k'} - \frac{(k-1)p+r}{k} \\
&= \frac{(k'-k)p + kp_i + kr' - k'r - kk'}{kk'} \\
&= \frac{(k'+k)p + kp_i + k(n-p_i-1-k'(m-2)) - k'(n-1-k(m-1)) - kk'}{kk'} \\
&= \frac{(k-k')(p-n+1)}{kk'} \leq 0,
\end{aligned}$$

we have that

$$\begin{aligned}
e(G) &= e(G_i) + e(G - G_i) \\
&\leq p_i - 1 + e(n-p_i, n-p_i-m+1; p-p_i) \\
&= p_i - 1 + \left[\frac{(k'-1)(p-p_i)+r'}{k'}\right] \\
&\leq \frac{(k-1)p+r}{k}.
\end{aligned}$$

Therefore, we can obtain that $e(G) \leq [\frac{(k-1)p+r}{k}]$. □

Next, we give several corollaries of Theorem 4, which mainly give a simpler expression of $e(n, m; p)$ for small m.

Corollary 1. *Let $k, n, p \in \mathbb{N}$ and $k+1 \leq n \leq p$. If a graph G satisfies $v(G) = p, e(G) \geq [\frac{(k-1)p}{n-1}]$, then G must contain a subgraph induced by n vertices with at least k edges.*

Proof. Let $m = n + 1 - k$, then $n \geq m \geq 2$. By Lemma 2, we can deduce that

$$\begin{aligned} e(n, k-1; p) &= e(n, n-m; p) \\ &= p - (m-1) + \left[-\frac{p-(n-1)}{[\frac{n-1}{m-1}]} \right] \\ &\leq p - (m-1) + [-\frac{p-(n-1)}{\frac{n-1}{m-1}}] \\ &= [\frac{(n-m)p}{n-1}], \end{aligned}$$

this, together with the given condition, implies the result. □

Corollary 2. *Let $n, p \in \mathbb{N}$ with $p \geq n$ and $v(G) = p$. If $e(G) = q \leq p$, then G must contain a subgraph induced by n vertices with edges large than $\lceil \frac{(n-1)q}{p} \rceil$.*

Proof. As $\lceil \frac{(n-1)q}{p} \rceil < 1 + \frac{(n-1)q}{p}$, we deduce that $(\lceil \frac{(n-1)q}{p} \rceil - 1)p < (n-1)q$; thereby, $q > \frac{(\lceil \frac{(n-1)q}{p} \rceil - 1)p}{n-1}$. In Corollary 1, let $k = [\frac{(n-1)q}{p}]$, then we can obtain the desired result. □

Corollary 3. *Let $p, m, n, r \in \mathbb{N}$ with $r(m+1) \leq n \leq p$. Then $e_r(n, m; p) = m$.*

Proof. Suppose that G is an (n, m) graph and $e(G) = e_r(n, m; p)$. Assume to the contrary that $e(G) \geq m + 1$. Since $m + 1$ edges were at most induced by $r(m+1)(\leq n)$ vertices, then we can find a subgraph induced by n vertices with at least $m + 1$ edges, which is a contradiction.

For the case $r = 2$, the result also follows from Theorem 4. □

At the end of this section, we give a formula that is easier to use for the case $m = n - 3$, and the proof is simple and so omitted.

Corollary 4. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 3$. Then*

$$e(n, n-3; p) = \begin{cases} [\frac{(n-3)p}{n-1}], & 2 \nmid n \\ [\frac{(n-4)p+2}{n-2}], & 2 \mid n. \end{cases}$$

3. Proof of Theorem 5

Proof. Suppose that H is an (n, m) graph with p vertices. To prove the lower bound, firstly, construct an $m + 1$-graph denoted by F as follows:

$V(F) = \binom{[p]}{r}$, and $m + 1$ different vertices $A_1, A_2, \ldots, A_{m+1} \in \binom{[p]}{r}$ form an edge of F if and only if $|\cup_{i=1}^{m+1} A_i| \leq n$.

Because H is an (n, m) graph, n vertices induce at most m edges, then we claim that

$$\text{any } m+1 \text{ edges contain at least } n+1 \text{ vertices,}$$

otherwise, there exist $m + 1$ edges denoted by $A_1, A_2, \ldots, A_{m+1}$ such that $|\cup_{i=1}^{m+1} A_i| \leq n$,

which contradicts the definition of H. The claim together with the construction of F shows that $A_1, \ldots, A_{m+1}$ do not form an edge of F. Therefore, $\alpha(F) = e(H)$.

On the other hand, for an edge $\{A_1, \ldots, A_{m+1}\}$ of F, it follows that $\cup_{i=1}^{m+1} A_i$ contains at most n vertices of H. Then, for a vertex A_i of F, since $|A_i| = r$, there are at most $\binom{p-r}{n-r}\binom{n}{r}^m$ edges of F containing A_i. Therefore, we have

$$d(F) \leq \Delta(F) \leq \binom{p-r}{n-r}\binom{n}{r}^m.$$

By Lemma 1, we have

$$\begin{aligned} e(H) = \alpha(F) &\geq C_1 \frac{|V(F)|}{d^{\frac{1}{m}}(F)} \\ &= C_1 \binom{p}{r}\binom{p-r}{n-r}^{-1/m} \cdot \binom{n}{r}^{-1} \\ &= \Omega(p^{\frac{(m+1)r-n}{m}}), \end{aligned}$$

where C_1 is a constant which depends only on the values of n, m, r.

Now, we show the upper bound of $e_r(n, m; p)$. Suppose that H is an (n, m) graph with p vertices.

Setting $t = \lceil \frac{(m+1)r-n}{m} \rceil$, for any subset $T \in \binom{[p]}{t}$, there at most exist m edges of H which contain T simultaneously; otherwise, suppose $T \in \binom{[p]}{t}$ and $A_1, A_2, \ldots, A_{m+1} \in E(H)$ such that $T \subset \cap_{i=1}^{m+1} A_i$, then

$$|\cup_{i=1}^{m+1} A_i| \leq |T| + \sum_{i=1}^{m+1} |A_i \setminus T| = t + (m+1)(r-t) \leq n,$$

which contradicts the above claim. Now, we consider the set $\{(T, A) : T \in \binom{[p]}{t}, A \in E(H), T \subset A\}$. By counting its size in different ways, we obtain that

$$e(H) \cdot \binom{r}{t} \leq \binom{p}{t} \cdot m.$$

Then

$$e_r(n, m; p) \leq C_2 p^{\lceil \frac{(m+1)r-n}{m} \rceil},$$

where C_2 is a constant which depends only on the values of n, m, r. □

Remark 1. *From the bounds of $e_r(n, m; p)$, we can evaluate the Turán density of an (n, m) graph, so maybe it is interesting to search some concrete graphs which are (n, m) graphs.*

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