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On Voigt-Type Functions Extended by Neumann Function in Kernels and Their Bounding Inequalities

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Abstract: The principal aim of this paper is to introduce the extended Voigt-type function $\mathbb{V}_{\mu,\nu}(x,y)$ and its counterpart extension $\mathbb{W}_{\mu,\nu}(x,y)$, involving the Neumann function Y_ν in the kernel of the representing integral. The newly defined integral reduces to the classical Voigt functions $K(x,y)$ and $L(x,y)$, and to their generalizations by Srivastava and Miller, by the unification of Klusch. Following an approach by Srivastava and Pogány, we also present the multiparameter and multivariable versions $\mathbb{V}^{(r)}_{\mu,\nu}(x,y)$, $\mathbb{W}^{(r)}_{\mu,\nu}(x,y)$ and the r positive integer of the initial extensions $\mathbb{V}_{\mu,\nu}(x,y)$, $\mathbb{W}_{\mu,\nu}(x,y)$. Several computable series expansions are obtained for the discussed Voigt-type functions in terms of Humbert confluent hypergeometric functions $\Psi^{(r)}_2$. Furthermore, by transforming the input extended Voigt-type functions by the Grünwald–Letnikov fractional derivative, we establish representation formulae in terms of the associated Legendre functions of the second kind $Q^{-\nu}_\eta$ in the two-parameter and two-variable cases. Finally, functional bounding inequalities are given for $\mathbb{V}_{\mu,\nu}(x,y)$ and $\mathbb{W}_{\mu,\nu}(x,y)$. Particularly interesting results are presented for the Neumann function Y_ν and for the Struve $\mathbf{H}_\nu$ function in the form of several functional bounds. The article ends with a thorough discussion and closing remarks.

Keywords: Voigt function; generalized Voigt functions; extended Voigt-type functions; Neumann function; Humbert confluent hypergeometric function of several variables; Legendre function of the second kind; Grünwald–Letnikov fractional derivative; Fox–Wright generalized hypergeometric function; Struve **H** function; bounding inequalities

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1. Introduction and Preliminaries

The classical Voigt functions $K(x,y)$ and $L(x,y)$ occur frequently in several problems of physics such as astrophysical spectroscopy, emissions, the absorption and transfer of radiation in a heated atmosphere, plasma dispersion, and also in the theory of neutron reactions [1]. The functions $K(x,y)$ and $L(x,y)$, which are due essentially to Reiche [2], read as follows:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-yt-\frac{t^2}{4}} \cos(xt)\, \mathrm{d}t, \qquad x \in \mathbb{R};\ y \in \mathbb{R}^+, \tag{1}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-yt-\frac{t^2}{4}} \sin(xt)\, \mathrm{d}t, \qquad x \in \mathbb{R};\ y \in \mathbb{R}^+. \tag{2}$$

where $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ denote the sets of complex, real, positive real, non-positive, and positive integer numbers, respectively. The Bessel function of the first kind of the order ν has the following power series definition:

$$J_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \qquad -z \notin \mathbb{N};\ \nu \in \mathbb{C}. \tag{3}$$

where the principal branch of $J_\nu(z)$ should be considered (it corresponds to the principal value of z^ν) and $J_\nu(z)$ is analytic in the z-plane cut along the interval $(-\infty, 0]$. Moreover, for $\nu \in \mathbb{Z}$, the Bessel function of the first kind is entire in z in the whole complex plane [3] (p. 15). Based on the facts [4], (Eq. (10.39.2)) is as follows:

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\sin(z) \quad \text{and} \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\cos(z),$$

Srivastava and Miller [5] introduced a generalization of $K(x,y)$ and $L(x,y)$ in the following form:

$$V_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu\, \mathrm{e}^{-yt-\frac{t^2}{4}}\, J_\nu(xt)\,\mathrm{d}t, \qquad \mu,\, x,\, y \in \mathbb{R}^+;\ \Re(\mu+\nu) > -1, \tag{4}$$

so that

$$K(x,y) = V_{\frac{1}{2},-\frac{1}{2}}(x,y) \quad \text{and} \quad L(x,y) = V_{\frac{1}{2},\frac{1}{2}}(x,y).$$

Klusch [6] defined an unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the following slightly modified form:

$$\Omega_{\mu\,\nu}(x,y,z) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu\, \mathrm{e}^{-yt-zt^2}\, J_\nu(xt)\,\mathrm{d}t, \qquad \mu,\, x,\, y,\, z \in \mathbb{R}^+;\ \Re(\mu+\nu) > -1, \tag{5}$$

so that

$$V_{\mu\,\nu}(x,y) = (2\sqrt{z})^{\mu-\frac{1}{2}}\, \Omega_{\mu\,\nu}(2x\sqrt{z}, 2y\sqrt{z}, z),$$

or, equivalently,

$$\Omega_{\mu\,\nu}(x,y,z) = (2\sqrt{z})^{-\mu+\frac{1}{2}}\, V_{\mu\,\nu}\left(\frac{x}{2\sqrt{z}}, \frac{y}{2\sqrt{z}}\right).$$

It is worth mentioning that the extension of the Srivastava–Miller model (2) to a multiparameter and multivariable Voigt function was considered by Pathan et al. [7] (p. 253, Eq. (2.4)) and reconsidered by Srivastava and Pogány *viz.* [8] (p. 195, Eq. (4)), as follows:

$$V_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y) = \prod_{j=1}^r \sqrt{\frac{x_j}{2}} \int_0^\infty t^\mu \mathrm{e}^{-yt-\frac{t^2}{4}} \prod_{j=1}^r J_{\nu_j}(x_j t)\cdot \mathrm{d}t\,, \qquad r \in \mathbb{N}, \tag{6}$$

where $\mu > 0$; $x_j, y \in \mathbb{R}^+$; $\Re(\mu + |\boldsymbol{\nu}|) > -1$, $\boldsymbol{u} = (u_1, \ldots, u_r)$, $\boldsymbol{u} \in \{\boldsymbol{x}, \boldsymbol{\nu}\}$, and $|\boldsymbol{\nu}| = \sum_{j=1}^r \nu_j$. In turn, when assuming that $r = 1$, we arrive at (2). We point out that Srivastava and Pogány, in [8], concentrate on bounding inequalities upon $V_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)$.

For various other investigations involving the Voigt functions, the interested reader may be referred to several recent papers on the subject, such as [6,7,9–15] and the references cited therein.

We organize this paper as follows: (i) we firstly introduce the extended Voigt-type function $\mathbb{V}_{\mu,\nu}(x,y)$ and its counterpart extension $\mathbb{W}_{\mu,\nu}(x,y)$, involving the Neumann function Y_ν in the kernel of the representing integral; (ii) following an approach used in [8], we define the multiparameter and multivariable versions $\mathbb{V}^{(r)}_{\mu,\nu}(x,y)$ and $\mathbb{W}^{(r)}_{\mu,\nu}(x,y)$ for a positive integer $r \geq 2$; (iii) we obtain computable series expansions for discussed Voigt-type functions in terms of the Humbert confluent hypergeometric functions $\Psi_2^{(r)}$ of r variables; then, (iv) by transforming the input extended Voigt-type functions by the Grünwald–Letnikov

fractional derivative, we establish representation formulae in terms of the associated Legendre functions of the second kind $Q_\eta^{-\nu}$ in the case $r = 1$; finally, (v) functional bounding inequalities are given for $\mathbb{V}_{\mu,\nu}(x,y)$ and $\mathbb{W}_{\mu,\nu}(x,y)$ in two ways: the application of already known upper bounds for the real argument Neumann function $Y_\nu(x)$ (see [13,16,17]) and by applying newly derived upper bounds for the Struve function $\mathbf{H}_\nu(x)$. The article ends with a discussion, closing remarks, and an exhaustive references list.

2. Extended Voigt-Type Functions $\mathbb{V}$, $\mathbb{W}$, $\mathbb{V}^{(r)}$, and $\mathbb{W}^{(r)}$

We recall that the Bessel function of the second kind (Neumann function or Weber–Bessel function) of order ν is expressible in terms of the Bessel function of the first kind as follows [18] (p. 64, Eq. (1)):

$$Y_\nu(z) = \frac{\cos(\nu\pi)\, J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} = \cot(\nu\pi)\, J_\nu(z) - \csc(\nu\pi)\, J_{-\nu}(z), \qquad \nu \notin \mathbb{Z}. \tag{7}$$

Also, Bessel functions of half-integer order have a connection or recurrence formula, as in [19] (p. 925, Eq. (8.465)):

$$Y_{n+\frac{1}{2}}(z) = (-1)^{n-1}\, J_{-n-\frac{1}{2}}(z) \quad \text{and} \quad Y_{-n-\frac{1}{2}}(z) = (-1)^n\, J_{n+\frac{1}{2}}(z)\,.$$

On the one hand, the following is true [4] (p. 228, Eq. (10.16.1)):

$$J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\sin(z) \quad \text{and} \quad J_{-\frac{1}{2}}(z) = -Y_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\cos(z).$$

On the other hand, the asymptotic of the Neumann function when $z \to \infty$ [4] (p. 229, Entry 10.17.4) is as follows:

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\big(1 + \mathcal{O}(z^{-1})\big), \qquad |\arg(z)| \le \pi - \delta,\ |\delta| < \pi$$

This suggests extending, in the legitimate way, the Voigt-type function $\mathbb{V}_{\mu,\nu}(x,y)$ by replacing the Bessel function $J_\nu(xt)$ in (4) with the Neumann function $Y_\nu(xt)$ of the same order in the following way:

$$\mathbb{V}_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu\, \mathrm{e}^{-yt-\frac{t^2}{4}}\, Y_\nu(xt)\,\mathrm{d}t, \quad \mu,\, x,\, y \in \mathbb{R}^+;\ \Re(\nu) > 0, \Re(\mu+\nu) > -1. \tag{8}$$

The form of the multiplicative constant is chosen to agree with the pioneering introduction definitions (1) and (2).

Since the Neumann function exhibits asymptotic behavior when z approaches zero, it reads as follows [4] (p. 223, Entries 10.7.4–5):

$$\begin{aligned} Y_\nu(z) &\sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{z}\right)^\nu; \qquad \Re(\nu) > 0 \quad \text{or} \quad \nu = -k + \frac{1}{2},\, k \in \mathbb{N}, \\ Y_{-\nu}(z) &\sim -\frac{\cos(\nu\pi)\,\Gamma(\nu)}{\pi}\left(\frac{2}{z}\right)^\nu; \qquad \Re(\nu) > 0 \quad \text{and} \quad \nu \neq k - \frac{1}{2},\, k \in \mathbb{N}\,. \end{aligned} \tag{9}$$

Therefore, having in mind (9), we introduce the second variant of the extended Voigt–type function, which reads as follows:

$$\mathbb{W}_{\mu,\nu}(x,y) = -\frac{\pi}{\Gamma(\nu)}\left(\frac{x}{2}\right)^\nu \int_0^\infty t^\mu\, \mathrm{e}^{-yt-\frac{t^2}{4}}\, Y_\nu(xt)\,\mathrm{d}t, \tag{10}$$

where the parameter range consists of $\mu,\, x,\, y \in \mathbb{R}^+;\ \Re(\nu) > 0, \Re(\mu-\nu) > -1$. Furthermore, having in mind (9) and the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the constant is adapted to the multiplicative constants in (4)–(6).

Moreover, considering the generalized multiparameter Srivastava–Pogány variant (6), we also introduce the multiparameter version of the extended Voigt-type function associated with $\mathbb{V}_{\mu,\nu}(x,y)$ and $\mathbb{W}_{\mu,\nu}(x,y)$, as follows:

$$\mathbb{V}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)=\prod_{j=1}^{r}\sqrt{\frac{x_j}{2}}\cdot\int_0^\infty t^{\mu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\prod_{j=1}^{r}Y_{\nu_j}(x_jt)\,\mathrm{d}t,$$

equipped with the parametric space $\mu,\ x_j,\ y>0;\ \Re(\nu_j)>0, \Re(\mu+|\boldsymbol{\nu}|)>-1$, where $|\boldsymbol{\nu}|=\sum_{j=1}^{r}\nu_j$. Obviously, $\mathbb{V}^{(1)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)\equiv\mathbb{V}_{\mu,\nu_1}(x_1,y)$ covers (8).

Next, we define simultaneously a second multiparameter extension of the Voigt–type function, now associated with $\mathbb{W}_{\mu,\nu}(x,y)$ in the following form:

$$\mathbb{W}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)=(-1)^r\prod_{j=1}^{r}\frac{\pi}{\Gamma(\nu_j)}\left(\frac{x_j}{2}\right)^{\nu_j}\cdot\int_0^\infty t^{\mu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\prod_{j=1}^{r}Y_{\nu_j}(x_jt)\,\mathrm{d}t,\tag{11}$$

where $\mu,\ x_j,\ y>0;\ \Re(\nu_j)>0, \Re(\mu-|\boldsymbol{\nu}|)>-1$; here $\mathbb{W}^{(1)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)\equiv\mathbb{W}_{\mu,\nu_1}(x_1,y)$. Therefore,

$$K(x,y)=-\mathbb{V}_{\frac{1}{2},\frac{1}{2}}(x,y)\quad\text{and}\quad L(x,y)=\mathbb{V}_{\frac{1}{2},-\frac{1}{2}}(x,y).$$

Using the following recurrence formula [18] (p. 66, Eq. (1))

$$Y_{\nu-1}(z)+Y_{\nu+1}(z)=\frac{2\,\nu}{z}Y_{\nu}(z),$$

we obtained the following recurrence formulae for extended Voigt-type functions:

$$\frac{2\,\nu}{x}\,\mathbb{V}_{\mu,\nu}(x,y)=\mathbb{V}_{\mu+1,\nu-1}(x,y)+\mathbb{V}_{\mu+1,\nu+1}(x,y)\,.$$

For the next results, we require the definition of the Humbert confluent hypergeometric functions of n variables, the series definition of which reads as follows [20] (p. 34, Eq. (9)):

$$\Psi_2^{(n)}[\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]=\sum_{m,n\geq0}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\ldots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\qquad\max\{|x_1|,\ldots,|x_n|\}<\infty.\tag{12}$$

Clearly, the confluent Humbert function of two variables $\Psi_2^{(2)}\equiv\Psi_2$; please see [20] (p. 26, Eq. (22)).

Theorem 1. *Let $\mu,\ y\in\mathbb{R}^+,\ x\in\mathbb{R};\ |\nu|<1$ and $\Re(\mu+\nu)>-1$. Then, the following representation holds true:*

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y)=&\frac{2^{\mu-\frac{1}{2}}\,x^{\nu+\frac{1}{2}}\cot(\nu\pi)}{\Gamma(\nu+1)}\left\{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Psi_2\left[\frac{\mu+\nu+1}{2};\nu+1,\frac{1}{2};-x^2,y^2\right]\right.\\&\left.-2y\,\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Psi_2\left[\frac{\mu+\nu}{2}+1;\nu+1,\frac{3}{2};-x^2,y^2\right]\right\}\\&-\frac{2^{\mu-2\nu-\frac{1}{2}}\,x^{\frac{1}{2}-\nu}\csc(\nu\pi)}{\Gamma(1-\nu)}\left\{\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Psi_2\left[\frac{\mu-\nu}{2}+1;1-\nu,\frac{1}{2};-x^2,y^2\right]\right.\\&\left.-2y\,\Gamma\left(\frac{\mu-\nu+2}{2}\right)\Psi_2\left[\frac{\mu-\nu}{2}+1;1-\nu,\frac{3}{2};-x^2,y^2\right]\right\}.\end{aligned}$$

Moreover, for all $\mu,\ y\in\mathbb{R}^+,\ x\in\mathbb{R}$, we have the following:

$$\mathbb{W}_{\mu,\nu}(x,y) = -\frac{\pi}{\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-\frac{1}{2}} \mathbb{V}_{\mu,\nu}(x,y)\,, \tag{13}$$

provided that $\nu \in (-1,0)\cup(0,1)$ *and* $\Re(\mu-|\nu|) > -1$.

Proof. Using the representation of Neumann function (7) in the definition (8), we conclude that the following holds true:

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y) &= \sqrt{\frac{x}{2}}\int_0^\infty t^\mu\, \mathrm{e}^{-yt-\frac{t^2}{4}}\left[\cot(\nu\pi)\, J_\nu(xt) - \csc(\nu\pi)\, J_{-\nu}(xt)\right]\mathrm{d}t \\ &= \sqrt{\frac{x}{2}}\int_0^\infty t^\mu\, \mathrm{e}^{-yt-\frac{t^2}{4}}\left[\cot(\nu\pi)\sum_{m\geq 0}\frac{(-1)^m\left(\frac{xt}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)} - \csc(\nu\pi)\sum_{m\geq 0}\frac{(-1)^m\left(\frac{xt}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)}\right]\mathrm{d}t.\end{aligned}$$

The Maclaurin series expansion of e^{-yt} enables the legitimate term-wise integration of the resulting absolutely convergent double series. Hence, the following also holds true:

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y) &= 2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}}\cot(\nu\pi)\sum_{m,n\geq 0}\frac{(-x^2)^m(-2y)^n}{m!\;n!\;\Gamma(m+\nu+1)}\Gamma\Big(\frac{\mu+\nu+1}{2}+m+\frac{n}{2}\Big) \\ &\quad - 2^{\mu-\frac{1}{2}}\, x^{\frac{1}{2}-\nu}\csc(\nu\pi)\sum_{m,n\geq 0}\frac{(-x^2)^m(-2y)^n}{m!\;n!\;\Gamma(m-\nu+1)}\Gamma\Big(\frac{\mu-\nu+1}{2}+m+\frac{n}{2}\Big).\end{aligned}$$

Now, by separating the n series into even and odd terms, we obtain the following:

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y) &= \frac{2^{\mu-\frac{1}{2}}\, x^{\nu+\frac{1}{2}}\cot(\nu\pi)}{\Gamma(\nu+1)}\left\{\Gamma\Big(\frac{\mu+\nu+1}{2}\Big)\sum_{m,n\geq 0}\frac{(\frac{1}{2}(\mu+\nu+1))_{m+n}(-x^2)^m(y^2)^n}{(\nu+1)_m\,(\frac{1}{2})_n\,m!\,n!}\right. \\ &\quad \left. - 2y\,\Gamma\Big(\frac{\mu+\nu}{2}+1\Big)\sum_{m,n\geq 0}\frac{(\frac{\mu+\nu}{2}+1)_{m+n}(-x^2)^m(y^2)^n}{(\nu+1)_m\,(\frac{3}{2})_n\,m!\,n!}\right\} \\ &\quad - \frac{2^{\mu-2-\nu\frac{1}{2}}\, x^{\frac{1}{2}-\nu}\csc(\nu\pi)}{\Gamma(1-\nu)}\left\{\Gamma\Big(\frac{\mu-\nu+1}{2}\Big)\sum_{m,n\geq 0}\frac{(\frac{1}{2}(\mu-\nu+1))_{m+n}(-x^2)^m(y^2)^n}{(\nu+1)_m\,(\frac{1}{2})_n\,m!\,n!}\right. \\ &\quad \left. - 2y\,\Gamma\Big(\frac{\mu-\nu}{2}+1\Big)\sum_{m,n\geq 0}\frac{(\frac{\mu-\nu}{2}+1)_{m+n}(-x^2)^m(y^2)^n}{(\nu+1)_m\,(\frac{3}{2})_n\,m!\,n!}\right\}.\end{aligned}$$

By using the definition of Humbert confluent hypergeometric function of two variables (12), we obtain the asserted expression. □

Remark 1. *Clearly, if we put* $\mu = 1/2$, $\nu = 1/2$ *and* $\mu = 1/2$, $\nu = -1/2$ *into* Theorem 1, *we obtain the known corrected representation by Srivastava and Miller* [5] (pp. 113–114, Eq. (13–14)) *and also derived by Exton* [21] (p. L76, Eqs. (8–9)), *respectively:*

$$\begin{aligned}K(x,y) &= -\mathbb{V}_{\frac{1}{2},\frac{1}{2}}(x,y) = \Psi_2\Big[\frac{1}{2};\frac{1}{2},\frac{1}{2};-x^2,y^2\Big] - \frac{2y}{\sqrt{\pi}}\Psi_2\Big[1;\frac{1}{2},\frac{3}{2};-x^2,y^2\Big], \\ L(x,y) &= \mathbb{V}_{\frac{1}{2},-\frac{1}{2}}(x,y) = \frac{2x}{\sqrt{\pi}}\,\Psi_2\Big[1;\frac{3}{2},\frac{1}{2};-x^2,y^2\Big] - 2xy\,\Psi_2\Big[\frac{3}{2};\frac{3}{2},\frac{3}{2};-x^2,y^2\Big].\end{aligned}$$

The next result is devoted to the multiparameter extension for the multivariable Voigt-type functions.

Theorem 2. *For all* $\mu > 0, \boldsymbol{\nu} \in (-1,1)^r \setminus \{0\}$, *such that* $\mu + |\boldsymbol{\nu}| > -1$ *and* $\boldsymbol{x}, y > 0$, *we have the following:*

$$\mathbb{V}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)=2^{\mu-\frac{1}{2}}\prod_{j=1}^{r}\frac{x_j^{\nu_j+\frac{1}{2}}\cot(\nu_j\pi)}{\Gamma(\nu_j+1)}\cdot\left\{\Gamma\Big(\frac{\mu+|\boldsymbol{\nu}|+1}{2}\Big)\Psi^{(r)}_2\Big[\frac{\mu+|\boldsymbol{\nu}|+1}{2};\boldsymbol{\nu}+\mathbf{1},\frac{1}{2};-\boldsymbol{x}^2,y^2\Big]\right.$$
$$\left.-2y\Gamma\Big(\frac{\mu+|\boldsymbol{\nu}|}{2}+1\Big)\Psi^{(r)}_2\Big[\frac{\mu+|\boldsymbol{\nu}|}{2}+1;\boldsymbol{\nu}+\mathbf{1},\frac{3}{2};-\boldsymbol{x}^2,y^2\Big]\right\}$$
$$-2^{\mu-\frac{1}{2}}\prod_{j=1}^{r}\frac{x_j^{\frac{1}{2}-\nu_j}\csc(\nu_j\pi)}{\Gamma(1-\nu_j)}\cdot\left\{\Gamma\Big(\frac{\mu-|\boldsymbol{\nu}|+1}{2}\Big)\Psi^{(r)}_2\Big[\frac{\mu-|\boldsymbol{\nu}|+1}{2};\mathbf{1}-\boldsymbol{\nu},\frac{1}{2};-\boldsymbol{x}^2,y^2\Big]\right.$$
$$\left.-2y\Gamma\Big(\frac{\mu-|\boldsymbol{\nu}|}{2}+1\Big)\Psi^{(r)}_2\Big[\frac{\mu-|\boldsymbol{\nu}|}{2}+1;\mathbf{1}-\boldsymbol{\nu},\frac{3}{2};-\boldsymbol{x}^2,y^2\Big]\right\},$$

where the shorthands are as follows: $\mathbf{1}\pm\boldsymbol{\nu}=(1\pm\nu_1,\cdots,1\pm\nu_r)$ *and* $\boldsymbol{x}^2=(x_1^2,\cdots,x_r^2)$.

Moreover, for $\mu-|\boldsymbol{\nu}|>-1$, *we have the following:*

$$\mathbb{W}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)=(-1)^r\prod_{j=1}^{r}\frac{\pi}{\Gamma(\nu_j)}\Big(\frac{x_j}{2}\Big)^{\nu_j-\frac{1}{2}}\cdot\mathbb{V}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y)\,.$$

The proof is the copy of the proving procedure of Theorem 1; therefore, it is omitted.

Next, we expose another fashion computable series representation of $\mathbb{V}_{\mu,\nu}(x,y)$, for which we need the definition of the associated Legendre function of the second kind [4] (p. 354, Entry 14.3.7), which reads as follows:

$$Q^{\mu}_{\nu}(x)=\mathrm{e}^{\mu\pi\mathrm{i}}\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)(x^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})\,x^{\nu+\mu+1}}\,{}_2F_1\Big(\frac{\nu+\mu}{2}+1,\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{x^2}\Big),$$

where $-(\nu+\mu)\notin\mathbb{N}$ and $x>1$. Furthermore, we recall the definition of the Grünwald–Letnikov (GL) fractional derivative $\mathbb{D}^{\nu}_{x}[f]$ of the order ρ, with respect to x of a suitable function f [22] (p. 373, Eq. (20.7)), as follows:

$$\mathbb{D}^{\rho}_{x}[f]=\lim_{h\downarrow 0}\frac{1}{h^{\rho}}\sum_{n\geq 0}^{\infty}(-1)^n\binom{\rho}{n}f(x+(\rho-n)h),$$

where $h\downarrow 0$ means that, when approaching zero, h remains positive. The GL fractional derivative of order ρ of the exponential function is as follows [23] (p. 2510, Eq. (29)):

$$\mathbb{D}^{\rho}_{x}[\mathrm{e}^{\alpha x}]=\alpha^{\rho}\mathrm{e}^{\alpha x}. \tag{14}$$

Here and in what follows we frequently use Laplace transform notation, as follows:

$$\mathscr{L}_p[f(t)]=F(p)=\int_0^{\infty}\mathrm{e}^{-pt}f(t)\,\mathrm{d}t\,,$$

which is of a suitable input function $f(t)$.

Theorem 3. *For all* $\mu,x-1,y>0$ *and* $\nu\in(0,1)$, *we have the following:*

$$\mathbb{V}_{\mu,\nu}(x,y)=\frac{\sqrt{2x}}{\pi}\,\mathrm{e}^{-\mu\pi\mathrm{i}}\,\Gamma(\nu+1)\sum_{n\geq 0}\frac{(-1)^{n+1}}{4^n\,n!}\,(\nu+1)_{2n}\cdot\mathbb{D}^{\mu}_{y}\left[\frac{1}{(y^2+x^2)^{n+\frac{1}{2}}}\,Q^{-\nu}_{2n}\left(\frac{y}{\sqrt{y^2+x^2}}\right)\right].$$

Moreover, for all $\mu,x-1,y>0$ *and* $\nu\in(0,1)$, *the following holds true:*

$$\mathbb{W}_{\mu,\nu}(x,y)=\sqrt{2}\,\nu\,\mathrm{e}^{-\mu\pi\mathrm{i}}\Big(\frac{x}{2}\Big)^{\nu}\sum_{n\geq 0}\frac{(-1)^n}{4^n\,n!}\,(\nu+1)_{2n}\,\mathbb{D}^{\mu}_{y}\left[\frac{1}{(y^2+x^2)^{n+\frac{1}{2}}}\,Q^{-\nu}_{2n}\left(\frac{y}{\sqrt{y^2+x^2}}\right)\right]. \tag{15}$$

Proof. We point out that the extended Voigt-type function $\mathbb{V}_{\mu,\nu}(x,y)$ is the Laplace transform, with respect to the parameter $y>0$, *viz.* as follows:

$$\mathbb{V}_{\mu,\nu}(x,y)=\sqrt{\frac{x}{2}}\,\mathscr{L}_y\left[t^{\mu}\,\mathrm{e}^{-\frac{t^2}{4}}\,Y_{\nu}(xt)\right].$$

Applying (14) to the Laplace kernel μ times, we obtain the following:

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y)&=\sqrt{\frac{x}{2}}\int_0^{\infty}(-1)^{-\mu}\mathbb{D}_y^{\mu}\left[\mathrm{e}^{-yt}\right]\,\mathrm{e}^{-\frac{t^2}{4}}Y_{\nu}(xt)\,\mathrm{d}t\\&=\sqrt{\frac{x}{2}}(-1)^{-\mu}\mathbb{D}_y^{\mu}\left[\int_0^{\infty}\mathrm{e}^{-yt-\frac{t^2}{4}}\,Y_{\nu}(xt)\,\mathrm{d}t\right]\qquad(16)\\&=\sqrt{\frac{x}{2}}(-1)^{-\mu}\sum_{n\geq 0}\frac{(-1)^n}{4^n\,n!}\,\mathbb{D}_y^{\mu}\left[\int_0^{\infty}\mathrm{e}^{-yt}\,t^{2n}\,Y_{\nu}(xt)\,\mathrm{d}t\right].\qquad(17)\end{aligned}$$

The Laplace transform, adapted to our setting above, reads as follows [24] (p. 298, Eq. (3)):

$$\mathscr{L}_p\left[t^{\mu}\,Y_{\nu}(at)\right]=-\frac{2\,\Gamma(\mu+\nu+1)}{\pi(p^2+a^2)^{\frac{\mu+1}{2}}}\,Q_{\mu}^{-\nu}\left(\frac{p}{\sqrt{p^2+a^2}}\right),\qquad \Re(\mu)>|\Re(\nu)|-1,\Re(p)>|\Im(a)|,$$

We transform (16), using $p=y$, $\mu=2n$, and $a=x$, into the following:

$$\mathbb{V}_{\mu,\nu}(x,y)=\frac{\sqrt{2x}}{\pi}(-1)^{1-\mu}\sum_{n\geq 0}\frac{(-1)^n\Gamma(2n+\nu+1)}{4^n\,n!}\,\mathbb{D}_y^{\mu}\left[\frac{1}{(y^2+x^2)^{n+\frac{1}{2}}}\,Q_{2n}^{-\nu}\left(\frac{y}{\sqrt{y^2+x^2}}\right)\right].$$

According to the definition of the associated Legendre function of the second kind, the parameter space excludes $-(2n-\nu)$ from the positive integers, and, as the Laplace transform holds for $\min_{n\geq 0}(2n-\nu+1)=1-\nu>0$, the first statement is proven. As for the claim (15), it is enough to consider (13). The rest is obvious. □

Another approach entails using the Grünwald–Letnikov derivative (14) applied to the newly introduced exponential term $\exp\{-\beta t^2\}$, which replaces $\exp\{-t^2/4\}$ in the integrand of $\mathbb{V}_{\mu,\nu}(x,y)$. The related results, which follow, contain the Kummer confluent hypergeometric function ${}_1F_1$ instead of the previously used associated Legendre function of the second kind.

Theorem 4. *For all $\mu,x,y>0$ and $\nu\in(-1,0)\cup(0,1)$, we have the following:*

$$\begin{aligned}\mathbb{V}_{\mu,\nu}(x,y)=-\frac{\cos(\nu\pi)\Gamma(-\nu)}{2\pi\,\mathrm{e}^{\mu\frac{\pi}{2}\mathrm{i}}}\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}}\sum_{n\geq 0}\frac{(-y)^n}{n!}\left\{\Gamma\left(\frac{1+n+\nu}{2}\right)\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\frac{{}_1F_1\left(\frac{n+\nu+1}{2};1+\nu;-\frac{x^2}{4\beta}\right)}{\beta^{\frac{n+\nu+1}{2}}}\right]_{\beta=\frac{1}{4}}\right.\\\left.+\frac{4^{\nu}\Gamma(\nu)\Gamma(\frac{1+n-\nu}{2})}{\Gamma(-\nu)\,x^{2\nu}}\,\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\frac{{}_1F_1\left(\frac{n-\nu+1}{2};1-\nu;-\frac{x^2}{4\beta}\right)}{\beta^{\frac{n-\nu+1}{2}}}\right]_{\beta=\frac{1}{4}}\right\}.\end{aligned}$$

Also, for all $\mu,x,y>0$, and $|\nu|<1$, the following holds true:

$$\begin{aligned}\mathbb{W}_{\mu,\nu}(x,y)=\frac{\cos(\nu\pi)}{2\,\mathrm{e}^{\mu\frac{\pi}{2}\mathrm{i}}}\frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{x}{2}\right)^{2\nu}\sum_{n\geq 0}\frac{(-y)^n}{n!}\left\{\Gamma\left(\frac{1+n+\nu}{2}\right)\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\frac{{}_1F_1\left(\frac{n+\nu+1}{2};1+\nu;-\frac{x^2}{4\beta}\right)}{\beta^{\frac{n+\nu+1}{2}}}\right]_{\beta=\frac{1}{4}}\right.\\\left.+\frac{4^{\nu}\Gamma(\nu)\Gamma(\frac{1+n-\nu}{2})}{\Gamma(-\nu)\,x^{2\nu}}\,\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\frac{{}_1F_1\left(\frac{n-\nu+1}{2};1-\nu;-\frac{x^2}{4\beta}\right)}{\beta^{\frac{n-\nu+1}{2}}}\right]_{\beta=\frac{1}{4}}\right\}.\qquad(18)\end{aligned}$$

Proof. Consider the Laplace transform, which reads as follows:

$$\begin{aligned}\mathscr{L}_y\left[t^{\mu}\,\mathrm{e}^{-\frac{t^2}{4}}\,Y_{\nu}(xt)\right] &= \mathrm{e}^{-\mu\frac{\pi}{2}\mathrm{i}}\int_0^{\infty}\mathrm{e}^{-yt}\,\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\mathrm{e}^{-\beta t^2}\right]_{\beta=\frac{1}{4}}Y_{\nu}(xt)\,\mathrm{d}t\\ &= \mathrm{e}^{-\mu\frac{\pi}{2}\mathrm{i}}\,\mathbb{D}_{\beta}^{\frac{\mu}{2}}\left[\int_0^{\infty}\mathrm{e}^{-yt}\,\mathrm{e}^{-\beta t^2}\,Y_{\nu}(xt)\,\mathrm{d}t\right]_{\beta=\frac{1}{4}}.\end{aligned}\tag{19}$$

By expanding the Laplace kernel $\exp\{-yt\}$ into a Maclaurin series, interchanging the summation and integration, and using the substitution $t=\sqrt{s}$, we obtain the following:

$$\int_0^{\infty}t^{\mu}\mathrm{e}^{-yt-\frac{t^2}{4}}\,Y_{\nu}(xt)\mathrm{d}t=\frac{1}{2}\sum_{n\geq 0}\frac{(-y)^n}{n!}\int_0^{\infty}\mathrm{e}^{-\beta s}s^{\frac{n-1}{2}}Y_{\nu}(x\sqrt{s}\,)\mathrm{d}s=\frac{1}{2}\sum_{n\geq 0}\frac{(-y)^n}{n!}\mathscr{L}_{\beta}\left[s^{\mu}\,Y_{\nu}(x\sqrt{s}\,)\right].$$

In turn, the Laplace transform reads as follows [24] (p. 300, Eq. 3.13.2.4.):

$$\begin{aligned}\mathscr{L}_p\left[s^{\mu}\,Y_{\nu}(a\sqrt{s}\,)\right] &= -\frac{a^{\nu}\cos(\nu\pi)\Gamma(-\nu)\Gamma(\mu+\frac{\nu}{2}+1)}{\pi 2^{\nu}\,p^{\,\mu+\frac{\nu}{2}+1}}\,{}_1F_1\left(\mu+\frac{\nu}{2}+1;1+\nu;-\frac{a^2}{4p}\right)\\ &\quad-\frac{2^{\nu}\cos(\nu\pi)\Gamma(\nu)\Gamma(\mu-\frac{\nu}{2}+1)}{\pi a^{\nu}\,p^{\,\mu-\frac{\nu}{2}+1}}\,{}_1F_1\left(\mu-\frac{\nu}{2}+1;1-\nu;-\frac{a^2}{4p}\right),\end{aligned}$$

provided $2|\Re(\mu)|>|\Re(\nu)|-2$ and $\Re(\beta)>0$. For $2\mu=n-1$, $p=\beta$, $a=x$, we conclude that the following holds true:

$$\begin{aligned}\int_0^{\infty}t^{\mu}\mathrm{e}^{-yt-\frac{t^2}{4}}\,Y_{\nu}(xt)\mathrm{d}t &= -\frac{1}{2\pi}\sum_{n\geq 0}\frac{(-y)^n}{n!}\left\{\frac{x^{\nu}\cos(\nu\pi)\Gamma(-\nu)\Gamma(\frac{n+\nu+1}{2})}{2^{\nu}\,\beta^{\,\frac{n+\nu+1}{2}}}\,{}_1F_1\left(\frac{n+\nu+1}{2};1+\nu;-\frac{x^2}{4\beta}\right)\right.\\ &\quad\left.+\frac{2^{\nu}\cos(\nu\pi)\Gamma(\nu)\Gamma(\frac{n-\nu+1}{2})}{x^{\nu}\,\beta^{\,\frac{n-\nu+1}{2}}}\,{}_1F_1\left(\frac{n-\nu+1}{2};1-\nu;-\frac{x^2}{4\beta}\right)\right\}.\end{aligned}\tag{20}$$

By now inserting (20) into (19), we complete the proving procedure of the statement.

The second statement (18) is the immediate consequence of relation (13). □

3. Functional Bounding Inequalities

First, we recall the integral definition of the Struve function, which reads as follows [18] (p. 328, Eq. (1)):

$$\mathbf{H}_{\nu}(z)=\frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\,\sin(zt)\,\mathrm{d}t\,,\qquad 2\Re(\nu)>-1.\tag{21}$$

It is worth mentioning that Struve considered only $\mathbf{H}_0$ and $\mathbf{H}_1$. Now, the Gubler–Weber formula reads as follows: [18] (p. 165, Eq. (5))

$$Y_{\nu}(z)=\frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\left(\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\,\sin(zt)\,\mathrm{d}t\,-\int_0^{\infty}(1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-zt}\,\mathrm{d}t\right)\tag{22}$$

which implies the following:

$$Y_{\nu}(z)=\mathbf{H}_{\nu}(z)-\frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty}(1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-zt}\,\mathrm{d}t\,,\tag{23}$$

where both representation formulae hold for $\Re(z)>0$ and $\nu>-1/2$, due to the convergence control of the second integral.

By splitting the ν domain into the disjoint intervals, we obtain the following:

$$\left(-\tfrac{1}{2},\,\infty\right)=\left(-\tfrac{1}{2},\,\tfrac{1}{2}\right]\cup\left(\tfrac{1}{2},\,\tfrac{3}{2}\right]\cup\left(\tfrac{3}{2},\,\infty\right)=:U_1\cup U_2\cup U_3,$$

Baricz et al. [16] (pp. 957–958) derive the functional bounds for $Y_\nu(x)$ as follows (see also [25] (pp. 7–8) and the monograph [17]):

$$Y_\nu(x)+\frac{x^\nu}{2^\nu\Gamma(\nu+1)}\leq\begin{cases}\dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}, & -\frac{1}{2}<\nu\leq\frac{1}{2},\\ \dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}+\dfrac{2^\nu\,\Gamma(\nu)}{\pi x^\nu}, & \frac{1}{2}<\nu\leq\frac{3}{2},\\ \dfrac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+\frac{1}{2})}+\dfrac{2^{2\nu-\frac{3}{2}}\,\Gamma(\nu)}{\pi x^\nu}, & \nu>\frac{3}{2}\end{cases}\tag{24}$$

where $x>0$. To present our recent findings, we need the definition of the parabolic (or Weber) cylinder function $D_a(x)$ (Whittaker's notation), which is a principal solution of the Weber differential equation [4] (p. 303, Entry 12.2.4), which reads as follows:

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\nu+\frac{1}{2}-\frac{1}{4}\,z^2\right)w=0;\qquad w=D_\nu(z).$$

The solution is entire function of z and ν. Furthermore, in the derivation procedure, we apply the integral representation for parabolic cylinder function, as follows [19] (p. 503, Eq. 3.953.2):

$$D_{-\mu}\Big(\frac{\gamma}{\sqrt{2\beta}}\Big)=\frac{(2\beta)^{\frac{\mu}{2}}}{\Gamma(\mu)}\,\mathrm{e}^{-\frac{\gamma^2}{8\beta}}\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-\gamma t-\beta t^2}\,\mathrm{d}t,\qquad \mu,\gamma,\beta\in\mathbb{C};\ \min\{\Re(\beta),\Re(\mu)\}>0\,.\tag{25}$$

Theorem 5. *For all μ, $\nu+\frac{1}{2}>0$, with $\Re(\mu)>|\Re(\nu)|>0$ and for all positive x,y, we have the following:*

$$\mathbb{V}_{\mu,\nu}(x,y)\leq\begin{cases}\kappa_1(\mu,\nu)\,D_{-(\mu+\nu)}(\sqrt{2}y)+\kappa_2(\mu,\nu)\,D_{-(\mu+\nu-1)}(\sqrt{2}y), & -\frac{1}{2}<\nu\leq\frac{1}{2},\\ \kappa_1(\mu,\nu)\,D_{-(\mu+\nu)}(\sqrt{2}y)+\kappa_2(\mu,\nu)\,D_{-(\mu+\nu-1)}(\sqrt{2}y) & \\ \quad+\kappa_3(\mu,\nu)\,D_{-(\mu-\nu)}(\sqrt{2}y), & \frac{1}{2}<\nu\leq\frac{3}{2},\\ \kappa_1(\mu,\nu)\,D_{-(\mu+\nu)}(\sqrt{2}y)+\kappa_4(\mu,\nu)\,D_{-(\mu+\nu-1)}(\sqrt{2}y) & \\ \quad+\kappa_5(\mu,\nu)\,D_{-(\mu-\nu)}(\sqrt{2}y), & \nu>\frac{3}{2}\end{cases}$$

where

$$\kappa_1(\mu,\nu)=\frac{x^{\nu+\frac{1}{2}}\,\mathrm{e}^{\frac{y^2}{2}}\Gamma(\mu+\nu)}{2^{\frac{\nu-\mu+1}{2}}\,\Gamma(\nu+1)},\ \kappa_2(\mu,\nu)=\frac{x^{\nu-\frac{1}{2}}\,\mathrm{e}^{\frac{y^2}{2}}\Gamma(\mu+\nu-1)}{2^{\frac{\nu-\mu}{2}}\,\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})},\ \kappa_3(\mu,\nu)=\frac{x^{\frac{1}{2}-\nu}\,\mathrm{e}^{\frac{y^2}{2}}\Gamma(\nu)\Gamma(\mu-\nu)}{2^{\frac{1-\nu-\mu}{2}}\,\pi\,\Gamma(\nu+1)},$$

$$\kappa_4(\mu,\nu)=\frac{x^{\nu-\frac{1}{2}}\,\mathrm{e}^{\frac{y^2}{2}}\Gamma(\mu+\nu-1)}{\sqrt{\pi}\,2^{\frac{3-\nu-\mu}{2}}\,\Gamma(\nu+\frac{1}{2})},\ \kappa_5(\mu,\nu)=\frac{x^{\frac{1}{2}-\nu}\,\mathrm{e}^{\frac{y^2}{2}}\Gamma(\nu)\Gamma(\mu-\nu)}{2^{\frac{4-3\nu-\mu}{2}}\,\pi\,\Gamma(\nu+1)}.$$

Proof. Starting with (8), using the decomposition $(-\frac{1}{2},\infty)=U_1\cup U_2\cup U_3$ and the estimates (24), we conclude as follows:

$$\mathbb{V}_{\mu,\nu}(x,y)\leq\sqrt{\frac{x}{2}}\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,Y_\nu(xt)\,\mathrm{d}t\leq\begin{cases}\mathbb{V}_{\mu,U_1}(x,y), & -\frac{1}{2}<\nu\leq\frac{1}{2},\\ \mathbb{V}_{\mu,U_2}(x,y), & \frac{1}{2}<\nu\leq\frac{3}{2},\\ \mathbb{V}_{\mu,U_3}(x,y), & \nu>\frac{3}{2}\end{cases}\tag{26}$$

where

$$\mathbb{V}_{\mu,U_1}(x,y) \le \sqrt{\frac{x}{2}}\left\{\frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma(\nu+1)}\int_0^{\infty} t^{\mu+\nu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t + \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty} t^{\mu+\nu-1}\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t\right\},$$

$$\mathbb{V}_{\mu,U_2}(x,y) \le \sqrt{\frac{x}{2}}\left\{\frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma(\nu+1)}\int_0^{\infty} t^{\mu+\nu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t + \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty} t^{\mu+\nu-1}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t + \frac{2^{\nu}\,\Gamma(\nu)}{\pi\,x^{\nu}\,\Gamma(\nu+1)}\int_0^{\infty} t^{\mu-\nu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t\right\},$$

and

$$\mathbb{V}_{\mu,U_3}(x,y) \le \sqrt{\frac{x}{2}}\left\{\frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma(\nu+1)}\int_0^{\infty} t^{\mu+\nu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t + \frac{x^{\nu-1}}{\sqrt{2\,\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty} t^{\mu+\nu-1}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t + \frac{2^{2\nu-\frac{3}{2}}\,\Gamma(\nu)}{\pi\,x^{\nu}\,\Gamma(\nu+1)}\int_0^{\infty} t^{\mu-\nu}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t\right\},$$

which is equivalent to the statement of the theorem. In the derivation procedure, we apply (25) with $\gamma = y$ and $\beta = 1/4$. □

The Fox–Wright generalized hypergeometric function with p numerator parameter couples $(a_1, A_1), \cdots, (a_p, A_p)$ and q denominator parameter pairs $(b_1, B_1), \cdots, (b_q, B_q)$ has the following power series definition [26] (pp. 286–287):

$${}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{array}\middle|\,z\right] = {}_p\Psi_q\left[\begin{array}{c}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{array}\middle|\,z\right] = \sum_{n\ge 0}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\,\frac{z^n}{n!}, \tag{27}$$

where $A_j,\ B_k \ge 0$, $j = 1,\ldots,p$, and $k = 1,\ldots,q$. The series (27) converges for all $z \in \mathbb{C}$ when the following is true:

$$\Delta := 1 + \sum_{j=1}^{q} B_j - \sum_{k=1}^{p} A_k > 0.$$

When $\Delta = 0$, the series in (27) converges for $|z| < \nabla$ and $|z| = \nabla$ under the condition $\Re(\Xi) > 1/2$, where the following holds true:

$$\nabla := \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right), \qquad \Xi = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

The function ${}_1\Psi_0[\cdot]$ is a confluent Fox–Wright function.

Theorem 6. *For all μ, $\nu + \frac{1}{2} > 0$ and for all positive $x, y > 0$, we have the following:*

$$\mathbb{V}_{\mu,\nu}(x,y) \leq \begin{cases} \lambda_1(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu+1}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big]+\lambda_2(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big], & -\frac{1}{2}<\nu\leq\frac{1}{2},\\ \lambda_1(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu+1}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big]+\lambda_2(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big] & \\ +\lambda_3(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu-\nu+1}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big], & \frac{1}{2}<\nu\leq\frac{3}{2},\\ \lambda_1(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu+1}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big]+\lambda_4(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu+\nu}{2},\frac{1}{2})\\ \text{---}\end{matrix}\Big|-2y\Big] & \\ +\lambda_5(\mu,\nu)\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu-\nu+1}{2},\frac{2}{2})\\ \text{---}\end{matrix}\Big|-2y\Big], & \nu>\frac{3}{2} \end{cases}$$

where

$$\lambda_1(\mu,\nu)=\frac{x^{\nu+\frac{1}{2}}\,2^{\mu-\frac{1}{2}}}{\Gamma(\nu+1)},\ \lambda_2(\mu,\nu)=\frac{x^{\nu-\frac{1}{2}}\,2^{\mu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})},\ \lambda_3(\mu,\nu)=\frac{x^{\frac{1}{2}-\nu}\,2^{\mu+\frac{1}{2}}}{\pi\,\Gamma(\nu+\frac{1}{2})},$$

$$\lambda_4(\mu,\nu)=\frac{x^{\nu-\frac{1}{2}}\,2^{\mu+\nu-2}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})},\ \lambda_5(\mu,\nu)=\frac{x^{\frac{1}{2}-\nu}\,2^{\mu+\nu-2}\Gamma(\nu)}{\pi\,\Gamma(\nu+1)}.$$

Proof. In the derivation procedure of Theorem 5, we apply the following integral representation for the Fox–Wright function [8] (p. 198, Equation (15)):

$$\mathcal{I}_\mu(y)=\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-yt-\frac{t^2}{4}}\,\mathrm{d}t=2^{\mu-1}\,{}_1\Psi_0\Big[\begin{matrix}(\frac{\mu}{2},\frac{1}{2});\\ \text{---};\end{matrix}\,-2y\Big],\qquad \Re(\mu),\Re(y)>0.$$

Using (26), we firstly evaluate the input extended Voigt function $\mathbb{V}_{\mu,\nu}(x,y)$, then obtain the claim of the theorem. □

In the following, we refine the bounds (24) focusing on the representation of Formula (23), which simultaneously gives novel bounds for real parameters and positive variables for either the Neumann function $Y_\nu(x)$ or the Struve function $\mathbf{H}_\nu(x)$.

Theorem 7. *For all $\nu\geq\frac{1}{2}$ and $x>0$, we have the following:*

$$Y_\nu(x)\leq\begin{cases}\mathbf{H}_\nu(x)-\sqrt{\dfrac{2}{\pi\,x}}, & \frac{1}{2}\leq\nu\leq\frac{3}{2}\\ \mathbf{H}_\nu(x)-\dfrac{x^{\nu-3}(x^2+2\nu-1)}{\sqrt{\pi}\,2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})} & \nu>\frac{3}{2}\end{cases}.\tag{28}$$

Proof. For all $\nu\geq\frac{1}{2}$ and $x>0$, the function $\mathbf{H}_\nu(x)\geq 0$ (see [4] (p. 291, Entry 11.4.13)). Indeed, the case $\nu=\frac{1}{2}$ is obvious, since the following holds true:

$$\mathbf{H}_{\frac{1}{2}}(x)=\sqrt{\frac{2x}{\pi}}\int_0^1\sin(xt)\,\mathrm{d}t=\frac{2\sqrt{2}}{\sqrt{\pi x}}\,\sin^2\Big(\frac{x}{2}\Big)\geq 0\,.$$

In the case $\nu>\frac{1}{2}$, consider three consecutive zeros $t_k=(\pi/x)k$ of $\sin(xt)$, t_{2k},t_{2k+1},t_{2k+2}. Then, having in mind that $(1-t^2)^{\nu-\frac{1}{2}}\searrow$ for $t\in[0,1]$, we have the following:

$$\int_{t_{2k}}^{t_{2k+1}}\sin(xt)(1-t^2)^{\nu-\frac{1}{2}}\mathrm{d}t-\int_{t_{2k+1}}^{t_{2k+2}}\sin(xt)(1-t^2)^{\nu-\frac{1}{2}}\mathrm{d}t\geq 0,\qquad k\in\mathbb{N}_0\,.$$

Therefore, a fortiori, the following holds true:

$$\mathbf{H}_\nu(x) = \sum_{k\geq 0}\left(\int_{t_{2k}}^{t_{2k+1}} - \int_{t_{2k+1}}^{t_{2k+2}}\right)\sin(xt)(1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t \geq 0\,.$$

Now, recalling (23), we see that, for $2\nu > 3$, we have, by virtue of the Bernoulli inequality [27] (p. 317), the following:

$$(1+h)^\alpha \geq 1+\alpha h, \qquad h > -1,\ \alpha \notin (0,1)\,,$$

and the following estimate:

$$\begin{aligned} Y_\nu(x) &\leq \mathbf{H}_\nu(x) - \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty\left[1+\left(\nu-\frac{1}{2}\right)t^2\right]\mathrm{e}^{-xt}\,\mathrm{d}t \\ &= \mathbf{H}_\nu(x) - \frac{x^{\nu-3}(x^2+2\nu-1)}{\sqrt{\pi}\,2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})}\,. \end{aligned} \tag{29}$$

The Bernoulli inequality is reversed for $\alpha \in (0,1)$; therefore, assuming $\nu \in [\frac{1}{2}, \frac{3}{2}]$, we conclude by the A–G inequality $1+t^2 \geq 2t$, that the following holds true:

$$Y_\nu(x) \leq \mathbf{H}_\nu(x) - \frac{2\,x^\nu}{\sqrt{2\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty t^{\nu-\frac{1}{2}}\mathrm{e}^{-xt}\,\mathrm{d}t = \mathbf{H}_\nu(x) - \sqrt{\frac{2}{\pi\,x}}\,. \tag{30}$$

Therefore, the proof is complete. □

Now it remains to apply (28) in bounding the extended Voigt and allied functions. Before this, we recall the following power series definition:

$$\mathbf{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1}\sum_{n\geq 0}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})}\,, \qquad 2\Re(\nu) > -3\,. \tag{31}$$

Theorem 8. *For all $\mu,\ x,\ y > 0$ and $\nu \in \left[\frac{1}{2}, \frac{3}{2}\right]$, the following holds true:*

$$\mathbb{V}_{\mu,\nu}(x,y) \leq \mathcal{A}_{\mu,\nu}(x,y) - \frac{2^{\frac{\mu}{2}+\frac{1}{4}}}{\sqrt{\pi}}\,\Gamma\left(\mu+\tfrac{1}{2}\right)\mathrm{e}^{\frac{y^2}{2}}\,D_{-(\mu+\frac{1}{2})}\left(\sqrt{2}\,y\right), \tag{32}$$

where

$$\mathcal{A}_{\mu,\nu}(x,y) = \frac{x^{\nu+\frac{3}{2}}\,\mathrm{e}^{\frac{y^2}{2}}}{2^{\frac{\nu-\mu+1}{2}}}\sum_{n\geq 0}\frac{(-1)^n x^{2n}\Gamma(\mu+\nu+2+2n)}{2^n\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})}\,D_{-(\mu+\nu+2+2n)}\left(\sqrt{2}\,y\right).$$

For $\nu > \frac{3}{2}$ and $\mu+\nu > 2$, we have the following:

$$\begin{aligned} \mathbb{V}_{\mu,\nu}(x,y) \leq \mathcal{A}_{\mu,\nu}(x,y) - \frac{2^{\frac{\mu-\nu+1}{2}}\,x^{\nu-\frac{5}{2}}\,\mathrm{e}^{\frac{y^2}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\Big\{&\Gamma(\mu+\nu)\,x^2\,D_{-(\mu+\nu)}(\sqrt{2}y) \\ &+ (\nu-\tfrac{1}{2})\,\Gamma(\mu+\nu-2)\,D_{-(\mu+\nu-2)}(\sqrt{2}y)\Big\}\,. \end{aligned} \tag{33}$$

Proof. When $\frac{1}{2} \leq \nu \leq \frac{3}{2}$, the integral definition (8), under upper bound (28) applied to the Neumann function in the kernel, becomes the following:

$$\begin{aligned}
\mathbb{V}_{\mu,\nu}(x,y) &= \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, Y_\nu(xt) \, \mathrm{d}t \\
&\leq \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} \left(\mathbf{H}_\nu(xt) - \sqrt{\frac{2}{\pi\, x}} \, t^{-\frac{1}{2}} \right) \mathrm{d}t \\
&= \left(\frac{x}{2}\right)^{\nu+\frac{3}{2}} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})} \int_0^\infty t^{\mu+\nu+1+2n} \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, \mathrm{d}t \\
&\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty t^{\mu-\frac{1}{2}} \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, \mathrm{d}t \\
&= \frac{x^{\nu+\frac{3}{2}} \, \mathrm{e}^{\frac{y^2}{2}}}{2^{\frac{\nu-\mu+1}{2}}} \sum_{n\geq 0} \frac{(-1)^n x^{2n} \Gamma(\mu+\nu+2+2n)}{2^n \Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})} \, D_{-(\mu+\nu+2+2n)}\big(\sqrt{2}\, y\big) \\
&\quad - \frac{2^{\frac{\mu}{2}+\frac{1}{4}}}{\sqrt{\pi}} \, \Gamma\big(\mu+\tfrac{1}{2}\big) \, \mathrm{e}^{\frac{y^2}{2}} \, D_{-(\mu+\frac{1}{2})}\big(\sqrt{2}\, y\big) \,,
\end{aligned} \tag{34}$$

where the parabolic cylinder function formula (25) is employed twice.

Next, for $\nu > \frac{3}{2}$, the calculation reduces to the following:

$$\begin{aligned}
I_{\mu,\nu}(x,y) &= \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, \frac{(xt)^{\nu-3}[(xt)^2+2\nu-1]}{\sqrt{\pi}\, 2^{\nu-1}\, \Gamma(\nu+\frac{1}{2})} \, \mathrm{d}t \\
&= \frac{x^{\nu-\frac{5}{2}}}{2^{\nu-\frac{1}{2}} \, \sqrt{\pi} \, \Gamma(\nu+\frac{1}{2})} \left\{ x^2 \int_0^\infty t^{\mu+\nu-1} \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, \mathrm{d}t + (2\nu-1) \int_0^\infty t^{\mu+\nu-3} \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, \mathrm{d}t \right\} \\
&= \frac{2^{\frac{\mu-\nu+1}{2}} \, x^{\nu-\frac{5}{2}} \, \mathrm{e}^{\frac{y^2}{2}}}{\sqrt{\pi} \, \Gamma(\nu+\frac{1}{2})} \Big\{ \Gamma(\mu+\nu) \, x^2 \, D_{-(\mu+\nu)}(\sqrt{2}\, y) \\
&\quad + (\nu-\tfrac{1}{2}) \, \Gamma(\mu+\nu-2) \, D_{-(\mu+\nu-2)}(\sqrt{2}\, y) \Big\},
\end{aligned}$$

where we also use the aid of (25), provided $\mu+\nu > 2$. □

The defining relation reads as follows (13):

$$\mathbb{W}_{\mu,\nu}(x,y) = -\frac{\pi}{\Gamma(\nu)} \left(\frac{x}{2}\right)^{\nu-\frac{1}{2}} \mathbb{V}_{\mu,\nu}(x,y) \,,$$

which, when combined with Theorem 8, readily provides the refined lower bound for the counterpart extended Voigt function.

Corollary 1. *For all μ, x, $y > 0$, and $\nu \in \left[\frac{1}{2}, \frac{3}{2}\right]$, the following holds true:*

$$\mathbb{W}_{\mu,\nu}(x,y) \geq \frac{\sqrt{\pi}\, \Gamma\big(\mu+\frac{1}{2}\big)}{\Gamma(\nu)\, 2^{\nu-\frac{\mu}{2}-\frac{3}{4}}} \, x^{\nu-\frac{1}{2}} \, \mathrm{e}^{\frac{y^2}{2}} \, D_{-(\mu+\frac{1}{2})}\big(\sqrt{2}\, y\big) - \mathcal{B}_{\mu,\nu}(x,y) \,, \tag{35}$$

where

$$\mathcal{B}_{\mu,\nu}(x,y) = \frac{\pi}{\Gamma(\nu)} \, \frac{x^{2\nu+1} \, \mathrm{e}^{\frac{y^2}{2}}}{2^{\frac{3\nu-\mu}{2}}} \sum_{n\geq 0} \frac{(-1)^n x^{2n} \Gamma(\mu+\nu+2+2n)}{2^n \Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})} \, D_{-(\mu+\nu+2+2n)}\big(\sqrt{2}\, y\big) \,.$$

In turn, for $\nu > \frac{3}{2}$ *and* $\mu + \nu > 2$, *we have the following:*

$$\mathbb{W}_{\mu,\nu}(x,y) \geq \sqrt{\pi}\,\frac{2^{\frac{\mu-3\nu}{2}+1}\,x^{2\nu-3}\,\mathrm{e}^{\frac{y^2}{2}}}{\Gamma(\nu)\,\Gamma(\nu+\frac{1}{2})}\Big\{\Gamma(\mu+\nu)\,x^2\,D_{-(\mu+\nu)}(\sqrt{2}\,y) + (\nu-\tfrac{1}{2})\,\Gamma(\mu+\nu-2)\,D_{-(\mu+\nu-2)}(\sqrt{2}\,y)\Big\} - \mathcal{B}_{\mu,\nu}(x,y)\,. \tag{36}$$

Now, we significantly refine the non-negativity result $\mathbf{H}_\nu(x) \geq 0$ listed in [4] (p. 291, Entry 11.4.13), proving a more efficient functional lower bound for the Struve $\mathbf{H}_\nu(x)$. With the aid of the resulting bound, we establish a lower bound for the Neumann function $Y_\nu(x)$.

Theorem 9. *For all* $2\Re(\nu) \geq -1$ *and* $x \in (0,\pi]$, *there holds the following lower bound:*

$$\mathbf{H}_\nu(x) \geq \frac{x^\nu\,\sin(x)}{2^\nu\,\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}\,. \tag{37}$$

Proof. Let $t_1 = \pi/x$ be the first positive zero of $\sin(xt)$ in the integrand of (21). In the case $t_1 \geq 1$, the secant line containing the origin and the point $T_1(1,\sin(x))$ is completely below the sine arc; that is, $\sin(xt) \geq t\cdot\sin(x)$, $t \in [0,1]$. Accordingly, taking into account the integral representation which holds true for $\Re(\nu) > -\frac{1}{2}$, we conclude the following:

$$\mathbf{H}_\nu(x) = \frac{2\,(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\sin(xt)\,\mathrm{d}t \geq \frac{2\,(\frac{x}{2})^\nu\,\sin(x)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 t(1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t\,,$$

which evidently gives (37). □

Theorem 10. *For all* $-\frac{1}{2} < \nu < 1$, *there holds the following lower bound:*

$$Y_\nu(x) \geq \frac{1}{2^\nu\,\sqrt{\pi x}\,\Gamma(\nu+\frac{3}{2})}\big[x^{\nu+\frac{1}{2}}\,\sin(x) - 2^{\nu+\frac{1}{2}}\,\Gamma(\nu+\tfrac{3}{2})\big], \qquad x \in [x_1,x_2] \subset (0,\pi]\,. \tag{38}$$

Here, x_0, x_1 *are the first two positive solutions of the following equation:*

$$x^{\nu+\frac{1}{2}}\,\sin(x) = 2^{\nu+\frac{1}{2}}\,\Gamma(\nu+\tfrac{3}{2})\,.$$

Proof. Letting $\nu \in (-\frac{1}{2},1)$, we apply the lower bound (37) to the relation (23) as follows:

$$\begin{aligned} Y_\nu(x) &= \mathbf{H}_\nu(x) - \frac{2\,(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt}\,\mathrm{d}t \\ &\geq \frac{x^\nu\,\sin(x)}{2^\nu\,\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})} - \frac{2\,(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-xt}\,\mathrm{d}t\,. \end{aligned}$$

By virtue of the A–G inequality $(1+t^2)^{\nu-\frac{1}{2}} \leq (2t)^{\nu-\frac{1}{2}}$, $|\nu| < \frac{1}{2}$, we clearly conclude that the estimate reads as follows:

$$Y_\nu(x) \geq \frac{1}{2^\nu\,\sqrt{\pi x}\,\Gamma(\nu+\frac{3}{2})}\big[x^{\nu+\frac{1}{2}}\,\sin(x) - 2^{\nu+\frac{1}{2}}\,\Gamma(\nu+\tfrac{3}{2})\big]\,,$$

which is definitely non-negative on the interval $[x_0,x_1]$. □

4. Discussion and Further Comments

A. The formula applied for the half-integer $\nu = n+\frac{1}{2}$ is [3] (p. 3, Eq. (2)), as follows:

$$\Gamma\Big(n+\frac{1}{2}\Big) = \frac{(2n-1)!!}{2^n}\,\sqrt{\pi}, \qquad n \in \mathbb{N}_0\,.$$

Specifying all parameters $\nu, \boldsymbol{\nu}$ as half-integers in (10) and (11), we infer the following expressions:

$$\mathbb{W}_{\mu,\nu}(x,y) = -\frac{1}{\sqrt{2}} \frac{x^{n+\frac{1}{2}}}{(2n-1)!!} \int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} \, Y_{n+\frac{1}{2}}(xt) \, \mathrm{d}t,$$

$$\mathbb{W}^{(r)}_{\mu,\boldsymbol{\nu}}(\boldsymbol{x},y) = \frac{(-1)^r}{2^{\frac{r}{2}}} \prod_{j=1}^{r} \frac{x_j^{n_j+\frac{1}{2}}}{(2n_j-1)!!} \cdot \int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} \prod_{j=1}^{r} Y_{n_j+\frac{1}{2}}(x_j t) \, \mathrm{d}t.$$

B. We point out the integral expression [19] (p. 1018, Eq. 9.240), which reads as follows:

$$\int_0^\infty t^{p-1} \mathrm{e}^{-yt-\frac{t^2}{4}} \, \mathrm{d}t = 2^{p-1} \left[\Gamma\left(\frac{p}{2}\right) {}_1F_1\left(\frac{p}{2}; \frac{1}{2}; y^2\right) - 2(p-1) \, y \, \Gamma\left(\frac{p+1}{2}\right) {}_1F_1\left(\frac{p+1}{2}; \frac{3}{2}; y^2\right) \right]. \tag{39}$$

which holds for all p, $\Re(p) > 0$. This formula connects the parabolic cylinder function D_η with the Kummer confluent hypergeometric ${}_1F_1$. Applying this expression for both integrals in (34), we can infer other computational series expansion formulae for the functional upper bound in Theorem 8.

C. Theorem 9 implies the questions of sharpness and possible argument and parameter extensions in (37); therefore, so does (38). Moreover, we point out that the domain $[x_1, x_2]$ in Theorem 10 is not empty; see, for instance, the case $\nu = 0$, to which the numerical values $x_1 \approx 1.57079633$ and $x_2 \approx 2.09439510$ are associated, while the third root of the related equation $\sqrt{x} \, \sin(x) = \sqrt{2} \, \Gamma\left(\frac{3}{2}\right)$ becomes $x_3 > 6.785$.

D. Future research can be focused on exhausting the Bessel–Struve functions class, which also contains the modified Bessel functions of the first and second kind, modified Struve functions of the first and second kind, the Bessel–Struve kernel function, and their various generalizations (see, e.g., the article [28] and the relevant references therein) in unifying the multiparameter and/or multivariable Voigt functions. However, we leave these study topics to another addressees.

5. Conclusions

The classical Voigt functions $K(x,y)$ and $L(x,y)$ are involved in several physics problems, such as astrophysical spectroscopy, emissions, the absorption and transfer of radiation in an atmosphere, plasma dispersion, and in the theory of neutron reactions, for instance. These functions were introduced by Voigt himself in 1889 and Reiche in 1913, respectively, in [1,2]. Their structure, integral forms, and the fact that both are Laplace transformations of $\exp\left\{-\frac{t^2}{4}\cos(xt)\right\}$—$\exp\left\{-\frac{t^2}{4}\sin(xt)\right\}$ or Fourier cosine and/or sine transforms (up to a multiplicative constant) of input functions $\exp\left\{-yt-\frac{t^2}{4}\right\}$—motivated the authors either to study more general oscillatory special functions, the special cases of which are the cosine and sine, or to also consider their finite products. These studies also result in multiparameter extensions; please consult the thorough introduction and section A. in the discussion section.

Our results are grouped in the following way: in Section 2. we introduce the extended Voigt-type function $\mathbb{V}_{\mu,\nu}(x,y)$ and its counterpart function $\mathbb{W}_{\mu,\nu}(x,y)$ using the Neumann function Y_ν multiplied with the power function, obtaining $t^\mu Y_\nu(xt)$ in the kernel of the observed integral. Additionally, products from kernels of $r \geq 2$ such functions are also used to define the multiparameter variants $\mathbb{V}^{(r)}_{\mu,\nu}(x,y)$, and $\mathbb{W}^{(r)}_{\mu,\nu}(x,y)$ are introduced. In Theorem 1, we establish computable series representations of $\mathbb{V}_{\mu,\nu}(x,y)$ and $\mathbb{W}_{\mu,\nu}(x,y)$ in terms of Humbert's confluent hypergeometric function of Ψ_2 of two variables, while Theorem 2 is devoted to the same fashion representations in terms of Humbert's confluent hypergeometric $\Psi^{(r)}_2$ of $r+1$ variables. Theorem 3 expresses $\mathbb{V}_{\mu,\nu}(x,y)$ and $\mathbb{W}_{\mu,\nu}(x,y)$ by virtue of the Grünwald–Letnikov (GL) fractional derivative of the Legendre function of the second kind, and Theorem 4 describes the GL-derivative representation of Voigt-type functions applied to the Kummer confluent hypergeometric function ${}_1F_1$.

Section 3 contains functional bounding inequalities for the Voigt-type functions $\mathbb{V}, \mathbb{W}$ using the Gubler–Weber integral Formula (22). Theorem 5 contains the upper bound, consisting of a linear combination of parabolic cylinder function $D_{-\mu}$, while the linear combination of Fox–Wright generalized confluent hypergeometric ${}_1Psi_0$ functions build another form functional upper bound; see Theorem 6. Theorems 7, 8, 10, and the Corollary 1 are devoted to obtain functional upper or lower bounds for the Neumann Y_ν; finally, Theorem 9 is nothing other than a generalization of the famous Jordan inequality for the real variable Struve function $\mathbf{H}_\nu$. The application of already known upper bounds for the real argument Neumann function $Y_\nu(x)$ (see [13,16,17]) and applying the here-derived upper bound for the Struve function $\mathbf{H}_\nu(x)$ in a new way, together with introducing the GL fractional derivative in the investigation methodology, provide constructive contributions for the study of Voigt-type and allied special functions.

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