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Locally Convex Spaces with Sequential Dunford–Pettis Type Properties

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Abstract: Let $p, q, q' \in [1, \infty]$, $q' \leq q$. Several new characterizations of locally convex spaces with the sequential Dunford–Pettis property of order (p, q) are given. We introduce and thoroughly study the sequential Dunford–Pettis* property of order (p, q) of locally convex spaces (in the realm of Banach spaces, the sequential $DP^*_{(p,\infty)}$ property coincides with the well-known DP^*_p property). Being motivated by the coarse p-DP^* property and the p-Dunford–Pettis relatively compact property for Banach spaces, we define and study the coarse sequential $DP^*_{(p,q)}$ property, the coarse DP^*_p property and the p-Dunford–Pettis sequentially compact property of order (q', q) in the class of all locally convex spaces.

Keywords: sequential $DP_{(p,q)}$ property; sequential $DP^*_{(p,q)}$ property; coarse sequential $DP^*_{(p,q)}$ property; coarse DP^*_p property; p-Dunford–Pettis sequentially compact property of order (q', q)

MSC: 46A03; 46E10

1. Introduction

All locally convex spaces are assumed to be Hausdorff and over the field $\mathbb{F}$ of real or complex numbers. We denote by E' the topological dual of a locally convex space (lcs for short) E.

In [1], Grothendieck defined the Dunford–Pettis property and the strict Dunford–Pettis property in the realm of locally convex spaces. For more details and historical remarks, we refer the reader to Section 9.4 of [2].

Definition 1 ([1]). *A locally convex space E is said to have*

- *The Dunford–Pettis property (the DP property) if each operator from E into a Banach space L, which transforms bounded sets into relatively weakly compact sets, transforms each absolutely convex weakly compact set into a relatively compact subset of L;*
- *The strict Dunford–Pettis property (the strict DP property) if each operator from E into a Banach space L, which transforms bounded sets into relatively weakly compact sets, transforms each weakly Cauchy sequence in E into a convergent sequence in L.*

To show that every Banach space $C(K)$ has the DP property (see Théorème 1 of [1]), Grothendieck proved in Proposition 2 of [1] that a Banach space E has the DP property if and only if for all weakly null sequences $\{x_n\}_{n\in\omega}$ and $\{\chi_n\}_{n\in\omega}$ in E and the Banach dual E'_β of E, respectively, it follows that $\lim_{n\to\infty}\langle \chi_n, x_n\rangle = 0$. It was shown in [3] that an analogous assertion holds true for a wider class of locally convex spaces, including strict (LF)-spaces. These results motivate us to introduce in [4] the following "sequential" version of the DP property in the class of all locally convex spaces.

Definition 2 ([4]). *A locally convex space E is said to have the sequential Dunford–Pettis property (the sequential DP property) if for all weakly null sequences $\{x_n\}_{n\in\omega}$ and $\{\chi_n\}_{n\in\omega}$ in E and the strong dual E'_β of E, respectively, it follows that $\lim_{n\to\infty}\langle \chi_n, x_n\rangle = 0$.*

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Let $p \in [1, \infty]$, and let E be a locally convex space. Recall (see Section 19.4 in [5]) that a sequence $\{x_n\}_{n\in\omega}$ in E is called *weakly p-summable* if for every $\chi \in E'$ it follows that $(\langle \chi, x_n\rangle) \in \ell_p$ if $p \in [1, \infty)$ and $(\langle \chi, x_n\rangle) \in c_0$ if $p = \infty$.

Unifying the notion of unconditional convergent operator and the notion of completely continuous operators (i.e., they transform weakly null sequences into norm null), Castillo and Sánchez selected in [6] the class of p-convergent operators. An operator $T : X \to Y$ between Banach spaces X and Y is called *p-convergent* if it transforms weakly p-summable sequences into norm null sequences. Using this notion they introduced and studied Banach spaces with the Dunford–Pettis property of order p for every $p \in [1, \infty]$.

Definition 3 ([6]). *Let $p \in [1, \infty]$. A Banach space X is said to have the Dunford–Pettis property of order p (the DP_p property) if every weakly compact operator from X into a Banach space Y is p-convergent.*

Therefore, a Banach space has the DP property if and only if it has the DP_∞ property. In [6], numerous distinguished examples of Banach spaces with or without the DP_p property are also constructed. The Dunford–Pettis property of order p for Banach spaces and especially Banach lattices was intensively studied by many authors; see, for example, [7–11].

Being motivated by the above-mentioned notions, we introduced in [12] the following Dunford–Pettis type properties (in which extending the aforementioned notion of p-convergent operators between Banach spaces and following [13], an operator $T : E \to L$ between locally convex spaces E and L is said to be *p-convergent* if T sends weakly p-summable sequences of E into null-sequences of L).

Definition 4 ([12]). *Let $p, q \in [1, \infty]$. A locally convex space E is said to have*

- *the quasi Dunford–Pettis property of order p (the quasi DP_p property) if for each Banach space L, every operator $T : E \to L$, which transforms bounded sets into relatively weakly compact sets, is p-convergent;*
- *the sequential Dunford–Pettis property of order (p, q) (the sequential $DP_{(p,q)}$ property) if $\lim_{n\to\infty}\langle \chi_n, x_n\rangle = 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and each weakly q-summable sequence $\{\chi_n\}_{n\in\omega}$ in the strong dual E'_β of E. If $p = q$, we shall say that E has the sequential DP_p property.*

It is clear that E has the quasi DP_∞ property if and only if it has the strict DP property, and E has the sequential DP_∞ property if and only if it has the sequential DP property. The reason for replacing "*strict*" with "*quasi*" is that there are locally convex spaces with the strict DP property but without the DP property, see Example 4.11 of [12].

The aforementioned notions also motivate us to introduce and study other sequential types of the Dunford–Pettis property in the realm of all locally convex spaces. This is the main purpose of the article.

Now we describe the content of the article and provide additional explanations for introducing sequential types of the Dunford–Pettis property, which come from the Banach space theory. In Section 2, we fix basic notions and recall some necessary results frequently used in the article.

In Section 3 (see Theorem 2), we give new characterizations of the sequential $DP_{(p,q)}$ property being motivated by some results obtained by Ghenciu in [14,15]. Taking into account Proposition 3.2 of [6], a Banach space X has the DP_p property if and only if it has the sequential $DP_{(p,\infty)}$ property. This fact motivates the study of the special case when $q = \infty$. In Corollaries 1 and 2, we obtain numerous characterizations of locally convex spaces with the sequential $DP_{(p,\infty)}$ property, which in particular generalize the corresponding results from [16].

In Section 4, we study locally convex space with the following *-version of the sequential $DP_{(p,q)}$ property.

Definition 5. *Let $p, q \in [1, \infty]$. A locally convex space E is said to have the sequential Dunford–Pettis* property of order (p, q) (the sequential $DP^*_{(p,q)}$ property) if $\lim_{n\to\infty}\langle \chi_n, x_n\rangle = 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and each weak* q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. If $p = q$ and $p = q = \infty$, we shall say that E has the sequential DP^*_p property or the sequential DP^* property, respectively.*

In Theorem 3, we give numerous characterizations of the sequential $DP^*_{(p,q)}$ property. A sufficient condition to have the sequential $DP^*_{(p,q)}$ property is given in Proposition 5. Following Carrión, Galindo, and Laurenço [17], a Banach space X has the *DP^* property* if every weakly compact set in X is limited. For $p \in [1, \infty]$, the DP^* property was generalized by Fourie and Zeekoei [18] as follows: X has the *DP^*_p property* if all weakly sequentially p-compact sets in X are limited. Taking into account a characterization of Banach spaces with the DP^*_p property given in Theorem 2.4 of [18], Corollary 6 shows that in the realm of Banach spaces the DP^*_p property in the sense of [17,18] coincides with the sequential $DP^*_{(p,\infty)}$ property. Note that Corollary 6 also generalizes Proposition 2.1 of [17] and Proposition 2.8 of [18]. In Corollary 7, we generalize Corollary 2.12 of [19] and Corollaries 20 and 21 of [14]. The class of spaces with the sequential $DP^*_{(p,q)}$ property is stable under taking direct products and direct sums; see Proposition 7. However, dense subspaces and closed subspaces of a space with the sequential $DP^*_{(p,q)}$ property may not have this property, see Examples 1 and 2.

Let $p \in [1, \infty]$. Generalizing the notion of a coarse p-limited subset of a Banach space introduced by Galindo and Miranda [20] and following [21], a non-empty subset A of a locally convex space E is called a *coarse p-limited set* if for every operator T from E to ℓ_p (or $T : E \to c_0$ if $p = \infty$), the set $T(A)$ is relatively compact. Following [20], a Banach space X is said to have the *coarse p-DP^* property* if every relatively weakly compact set in E is coarse p-limited. This notion can be naturally generalized and extended to all locally convex spaces as follows.

Definition 6. *Let $p, q \in [1, \infty]$. A locally convex space E is said to have*

(i) *The coarse sequential $DP^*_{(p,q)}$ property if every relatively weakly sequentially p-compact set in E is coarse q-limited;*

(ii) *The coarse DP^*_p property if every relatively weakly compact set in E is coarse p-limited.*

If $p = q$ and $p = q = \infty$, we shall say that E has the coarse sequential DP^*_p property and the coarse sequential DP^* property or the coarse DP^* property, respectively.

In Section 5, generalizing the corresponding results from [20], we characterize locally convex spaces with the coarse sequential $DP^*_{(p,q)}$ property, see Theorems 5 and 7. The classes of locally convex spaces with the coarse sequential $DP^*_{(p,q)}$ property and the coarse DP^*_p property are closed under taking dense subspaces, direct products and direct sums (see Propositions 11 and 14), but they are not stable under taking closed subspaces and quotients (see Remark 2).

In Section 6, we introduce a new type of the Dunford–Pettis property being motivated by the following. In [22], Emmanuele defined a Banach space E to have the *Dunford–Pettis relatively compact property* ($DPrcP$) if every weakly null sequence in E, which is an ∞-(V^*) set (=a Dunford–Pettis set) is norm null. It turns out (see [22]) that if $E = X^*$ is a dual Banach space, then E has the $DPprP$ if and only if it has the weak Radon–Nikodym property. Several characterizations of Banach spaces with the $DPprP$ were obtained by Wen and Chen [23]. For $1 \leq p < \infty$, the *p-Dunford–Pettis relatively compact property* (p-$DPprP$) was introduced by Ghenciu in [24] and studied also in [15]: the space E has the p-$DPrcP$ if every

weakly p-summable sequence that is an ∞-(V^*) set is norm null. Below, we generalize this notion to all locally convex spaces.

Definition 7. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$. A locally convex space (E, τ) is said to have the p-Dunford–Pettis sequentially compact property of order (q', q) (the p-DPscP$_{(q',q)}$ for short) if every weakly p-summable sequence, which is a (q', q)-(V^*) set is τ-null. If $q = \infty$, $q' = q = \infty$ or $p = q' = q = \infty$, we shall say simply that E has the p-DPscP$_{q'}$, the p-DPscP or the DPscP, respectively.*

Banach spaces with the *DPscP* are characterized in Proposition 16 of [22], Theorem 1.4 of [23] and in Corollary 13(i) of [15]. We essentially generalize and extend those results in Theorems 8 and 10. In Propositions 17 and 18, we show that the class of locally convex spaces with the *p-DPscP*$_{(q',q)}$ is closed under taking arbitrary subspaces, direct products and direct sums.

2. Preliminaries Results

We start with some necessary definitions and notations used in the article. Set $\omega := \{0, 1, 2, \dots\}$. All topological spaces are assumed to be Tychonoff (=completely regular and T_1). The closure of a subset A of a topological space X is denoted by $\overline{A}$. The space $C(X)$ of all continuous functions on X endowed with the pointwise topology is denoted by $C_p(X)$. A subset A of X is *functionally bounded* if $f(A)$ is a bounded subset of the field $\mathbb{F}$ for every $f \in C(X)$. A Tychonoff space X is called *Fréchet–Urysohn* if for any cluster point $a \in X$ of a subset $A \subseteq X$ there is a sequence $\{a_n\}_{n\in\omega} \subseteq A$ that converges to a. A Tychonoff space X is called an *angelic space* if (1) every relatively countably compact subset of X is relatively compact, and (2) any compact subspace of X is Fréchet–Urysohn. Note that any subspace of an angelic space is angelic, and a subset A of an angelic space X is compact if and only if it is countably compact if and only if A is sequentially compact.

Let E be a locally convex space. The span of a subset A of E and its closure are denoted by $E_A := \mathsf{span}(A)$ and $\overline{\mathsf{span}}(A)$, respectively. We denote by $\mathcal{N}_0(E)$ (resp., $\mathcal{N}_0^c(E)$) the family of all (resp., closed absolutely convex) neighborhoods of zero of E. The family of all bounded subsets of E is denoted by $\mathsf{Bo}(E)$. The value of $\chi \in E'$ on $x \in E$ is denoted by $\langle \chi, x \rangle$ or $\chi(x)$. A sequence $\{x_n\}_{n\in\omega}$ in E is said to be *Cauchy* if for every $U \in \mathcal{N}_0(E)$ there is $N \in \omega$ such that $x_n - x_m \in U$ for all $n, m \geq N$. If E is a normed space, B_E denotes the closed unit ball of E. The family of all operators from E to an lcs L is denoted by $\mathcal{L}(E, L)$.

For an lcs E, we denote by E_w and E_β the space E endowed with the weak topology $\sigma(E, E')$ and with the strong topology $\beta(E, E')$, respectively. The topological dual space E' of E endowed with weak* topology $\sigma(E', E)$ or with the strong topology $\beta(E', E)$ is denoted by E'_{w^*} or E'_β, respectively. The *polar* of a subset A of E is denoted by

$$A^\circ := \{\chi \in E' : \|\chi\|_A \leq 1\}, \quad \text{where} \quad \|\chi\|_A = \sup\{|\chi(x)| : x \in A \cup \{0\}\}.$$

A subset B of E' is *equicontinuous* if $B \subseteq U^\circ$ for some $U \in \mathcal{N}_0(E)$. If $\mathfrak{S}'$ (resp., $\mathfrak{S}$) is a directed family of subsets of E' (resp., of E) such that $\bigcup \mathfrak{S}' = E'$ ($\bigcup \mathfrak{S} = E$), then the topology on E (resp., on E') of uniform convergence on the sets of $\mathfrak{S}'$ (resp., $\mathfrak{S}$) is denoted by $\mathcal{T}_{\mathfrak{S}'}$ (resp., $\mathcal{T}_{\mathfrak{S}}$).

A subset A of a locally convex space E is called

- *Precompact* if for every $U \in \mathcal{N}_0(E)$ there is a finite set $F \subseteq E$ such that $A \subseteq F + U$;
- *Sequentially precompact* if every sequence in A has a Cauchy subsequence;
- *Weakly (sequentially) compact* if A is (sequentially) compact in E_w;
- *Relatively weakly compact* if its weak closure $\overline{A}^{\sigma(E,E')}$ is compact in E_w;
- *Relatively weakly sequentially compact* if each sequence in A has a subsequence weakly converging to a point of E;
- *Weakly sequentially precompact* if each sequence in A has a weakly Cauchy subsequence.

Note that each sequentially precompact subset of E is precompact, but the converse is not true in general; see Lemma 2.2 of [13]. We shall use the next lemma repeatedly; see Lemma 4.4 in [12].

Lemma 1. *Let τ and $\mathcal{T}$ be two locally convex vector topologies on a vector space E such that $\tau \subseteq \mathcal{T}$. If $S = \{x_n\}_{n\in\omega}$ is τ-null and $\mathcal{T}$-precompact, then S is $\mathcal{T}$-null. Consequently, if S is weakly $\mathcal{T}$-null and $\mathcal{T}$-precompact, then S is $\mathcal{T}$-null.*

Recall that a locally convex space E is

- *Quasi-complete* if each closed bounded subset of E is complete;
- *Sequentially complete* if each Cauchy sequence in E converges;
- *Locally complete* if the closed absolutely convex hull of a null sequence in E is compact;
- (*Quasi*)*barrelled* if every $\sigma(E', E)$-bounded (resp., $\beta(E', E)$-bounded) subset of E' is equicontinuous;
- c_0-(*quasi*)*barrelled* if every $\sigma(E', E)$-null (resp., $\beta(E', E)$-null) sequence is equicontinuous.

It is well-known that $C_p(X)$ is quasibarrelled for every Tychonoff space X.

Denote by $\bigoplus_{i\in I} E_i$ and $\prod_{i\in I} E_i$ the locally convex direct sum and the topological product of a non-empty family $\{E_i\}_{i\in I}$ of locally convex spaces, respectively. If $0 \neq \mathbf{x} = (x_i) \in \bigoplus_{i\in I} E_i$, then the set $\operatorname{supp}(\mathbf{x}) := \{i \in I : x_i \neq 0\}$ is called the *support* of $\mathbf{x}$. The *support* of a subset A, $\{0\} \subsetneq A$, of $\bigoplus_{i\in I} E_i$ is the set $\operatorname{supp}(A) := \bigcup_{a\in A} \operatorname{supp}(a)$. We shall also consider elements $\mathbf{x} = (x_i) \in \prod_{i\in I} E_i$ as functions on I and write $\mathbf{x}(i) := x_i$.

Denote by $\operatorname{ind}_{n\in\omega} E_n$ the inductive limit of a (reduced) inductive sequence $\{(E_n, \tau_n)\}_{n\in\omega}$ of locally convex spaces. If, in addition, $\tau_m{\restriction}_{E_n} = \tau_n$ for all $n, m \in \omega$ with $n \leq m$, the inductive limit $\operatorname{ind}_{n\in\omega} E_n$ is called *strict* and is denoted by $\varinjlim\text{-}\operatorname{ind}_n E_n$. In the partial case, when all spaces E_n are Fréchet, the strict inductive limit is called a *strict* (LF)*-space*.

Let E and L be locally convex spaces. Recall that an operator $T \in \mathcal{L}(E, L)$ is called *compact* (resp., *sequentially compact*, *precompact*, *sequentially precompact*, *weakly compact*, *weakly sequentially compact*, *weakly sequentially precompact*, *bounded*) if there is $U \in \mathcal{N}_0(E)$ such that $T(U)$ a relatively compact (relatively sequentially compact, precompact, sequentially precompact, relatively weakly compact, relatively weakly sequentially compact, weakly sequentially precompact or bounded) subset of E.

Let $p \in [1, \infty]$. Then p^* is defined to be the unique element of $[1, \infty]$, which satisfies $\frac{1}{p} + \frac{1}{p^*} = 1$. For $p \in [1, \infty)$, the space ℓ_{p^*} is the dual space of ℓ_p. We denote by $\{e_n\}_{n\in\omega}$ the canonical basis of ℓ_p, if $1 \leq p < \infty$, or the canonical basis of c_0, if $p = \infty$. The canonical basis of ℓ_{p^*} is denoted by $\{e_n^*\}_{n\in\omega}$. In what follows, we usually identify ℓ_{1^*} with c_0. Denote by ℓ_p^0 and by c_0^0 the linear span of $\{e_n\}_{n\in\omega}$ in ℓ_p or in c_0 endowed with the induced norm topology, respectively. We shall use also the following well-known description of relatively compact subsets of ℓ_p and c_0, see ([25], p. 6).

Proposition 1. *(i) A bounded subset A of ℓ_p, $p \in [1, \infty)$, is relatively compact if and only if*

$$\lim_{m\to\infty} \sup\Big\{ \sum_{m\leq n} |x_n|^p : x = (x_n) \in A \Big\} = 0.$$

(ii) A bounded subset A of c_0 is relatively compact if and only if $\lim_{n\to\infty} \sup\{|x_n| : x = (x_n) \in A\} = 0$.

Let $p \in [1, \infty]$. A sequence $\{x_n\}_{n\in\omega}$ in a locally convex space E is called

- *Weakly p-convergent to $x \in E$* if $\{x_n - x\}_{n\in\omega}$ is weakly p-summable;
- *Weakly p-Cauchy* if for each pair of strictly increasing sequences $(k_n), (j_n) \subseteq \omega$, the sequence $(x_{k_n} - x_{j_n})_{n\in\omega}$ is weakly p-summable.

A sequence $\{\chi_n\}_{n\in\omega}$ in E' is called *weak* p-summable* (resp., *weak* p-convergent to* $\chi \in E'$ or *weak* p-Cauchy*) if it is weakly p-summable (resp., weakly p-convergent to $\chi \in E'$ or weakly p-Cauchy) in E'_{w^*}. Following [13], E is called *p-barrelled* (resp., *p-quasibarrelled*) if every weakly p-summable sequence in E'_{w^*} (resp., in E'_β) is equicontinuous.

Generalizing the corresponding notions in the class of Banach spaces introduced in [6,9], the following p-versions of weakly compact-type properties are defined in [13]. Let $p \in [1,\infty]$. A subset A of a locally convex space E is called

- (*relatively*) *Weakly sequentially p-compact* if every sequence in A has a weakly p-convergent subsequence with the limit in A (resp., in E);
- *Weakly sequentially p-precompact* if every sequence from A has a weakly p-Cauchy subsequence.

Following [12] an lcs E is said to have the (*weak*) *Glicksberg property* if E and E_w have the same compact (resp., absolutely convex compact) sets. Let $p \in [1,\infty]$. The p-Schur property of Banach spaces was defined in [26,27]. Generalizing this notion and following [13], an lcs E is said to have the *p-Schur property* if every weakly p-summable sequence is a null-sequence. In particular, E has the Schur property if and only if it is an ∞-Schur space.

The following classes of subsets of an lcs E were introduced and studied in [13,28] where they generalize the notions of p-(V^*), p-limited and coarse p-limited subsets of Banach spaces defined in [29], [30] and [20], respectively. Let $p, q \in [1,\infty]$. A non-empty subset A of a locally convex space E is called

- A (p,q)-(V^*) *set* if

$$\Big(\sup_{a\in A}|\langle\chi_n,a\rangle|\Big)\in\ell_q \ \text{ if } q<\infty, \quad \text{or} \quad \Big(\sup_{a\in A}|\langle\chi_n,a\rangle|\Big)\in c_0 \ \text{ if } q=\infty,$$

  for every weakly p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_β. (p,∞)-(V^*) sets and $(1,\infty)$-(V^*) sets will be called simply p-(V^*) *sets* and (V^*) *sets*, respectively.
- a (p,q)-*limited set* if

$$\Big(\|\chi_n\|_A\Big)\in\ell_q \ \text{ if } q<\infty, \quad \text{or} \quad \|\chi_n\|_A\to 0 \ \text{ if } q=\infty,$$

  for every weak* p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. (p,p)-limited sets and (∞,∞)-limited sets will be called simply p-*limited sets* and *limited sets*, respectively.

Following [13], a non-empty subset B of E' is called a (p,q)-(V) *set* if

$$\Big(\sup_{\chi\in B}|\langle\chi,x_n\rangle|\Big)\in\ell_q \ \text{ if } q<\infty, \quad \text{or} \quad \Big(\sup_{\chi\in B}|\langle\chi,x_n\rangle|\Big)\in c_0 \ \text{ if } q=\infty,$$

for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E. (p,∞)-(V) sets and $(1,\infty)$-(V) sets will be called simply p-(V) *sets* and (V) *sets*, respectively.

Generalizing the Gelfand–Phillips property of order p defined in [19] and the coarse Gelfand–Phillips property of order p introduced in [20], the following notions are defined and studied in [21]. Let $p, q \in [1,\infty]$. An lcs E is said to have

- The *precompact* (p,q)-*Gelfand–Phillips property* (the $prGP_{(p,q)}$ *property*) if every (p,q)-limited set in E is precompact;
- The *coarse p-Gelfand–Phillips property* (the *coarse* GP_p *property*) if every coarse p-limited set in E is relatively compact.

Following [4], a sequence $A=\{a_n\}_{n\in\omega}$ in an lcs E is said to be *equivalent to the standard unit basis* $\{e_n : n\in\omega\}$ *of* ℓ_1 if there exists a linear topological isomorphism R from $\overline{\mathsf{span}}(A)$ onto a subspace of ℓ_1 such that $R(a_n)=e_n$ for every $n\in\omega$ (we do not assume that the closure $\overline{\mathsf{span}}(A)$ of the $\mathsf{span}(A)$ of A is complete or that R is onto a supspace). We shall also say that A is an ℓ_1-*sequence*. A locally convex space E is said to have the *Rosenthal property* if every bounded sequence in E has a subsequence which either (1) is Cauchy in

the weak topology, or (2) is equivalent to the unit basis of ℓ_1. The following remarkable extension of the celebrated Rosenthal ℓ_1-theorem was proved by Ruess [31]: *each locally complete locally convex space E whose every separable bounded set is metrizable has the Rosenthal property.* Thus every strict (LF)-space has the Rosenthal property. Being motivated by these results, we introduce in [21] the following class of locally convex spaces. Let $p \in [1, \infty]$. A locally convex space E is said to have the *weak Cauchy subsequence property of order p* (the $wCSP_p$ for short) if every bounded sequence in E has a weakly p-Cauchy subsequence. If $p = \infty$, we shall say simply that E has the $wCSP$. It is proved in Proposition 2.9 of [21] that if $1 < p < \infty$, then ℓ_p has the $wCSP_p$ if and only if $p \geq 2$. In [21], we proved the following generalization of Odell–Stegall's theorem (which states that any ∞-(V^*) set of a Banach space is weakly sequentially precompact, see ([32], p. 377)).

Theorem 1. *Let $2 \leq p \leq q \leq \infty$, and let E be a locally convex space with the Rosenthal property. Then every (p,q)-(V^*) subset of E is weakly sequentially precompact. Consequently, each (p,q)-limited subset of E is weakly sequentially precompact.*

Let $p, q, q' \in [1, \infty]$, $q' \leq q$, and let E and L be locally convex spaces. Generalizing the corresponding notions for Banach spaces defined in [19,33] and following [34], a linear map $T : E \to L$ is called

- (q',q)-(V^*) *p-convergent* if $T(x_n) \to 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E which is a (q',q)-(V^*) subset of E;
- (p,q)-(V^*) if $T(U)$ is a (p,q)-(V^*) subset of L for some $U \in \mathcal{N}_0(E)$; if $q = \infty$ or $p = 1$ and $q = \infty$, we shall say that T is p-(V^*) or (V^*), respectively;
- *weakly sequentially p-(pre)compact* if $T(U)$ is a relatively weakly sequentially p-compact (resp., weakly sequentially p-precompact) subset of L for some $U \in \mathcal{N}_0(E)$;
- (p,q)-*limited* if $T(U)$ is a (p,q)-limited subset of L for some $U \in \mathcal{N}_0(E)$; if $p = q$ or $p = q = \infty$, we shall say that T is *p-limited* or *limited*, respectively;
- *coarse p-limited* if there is $U \in \mathcal{N}_0(E)$ such that $T(U)$ is a coarse p-limited subset of L;
- *weakly (p,q)-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weakly q-summable sequence $\{\eta_n\}_{n\in\omega}$ in L'_β and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E; if $q = \infty$, we shall say simply that T is *weakly p-convergent*.

3. The Sequential $DP_{(p,q)}$ Property

The next theorem complements numerous characterizations of the sequential $DP_{(p,q)}$ property given in Theorem 4.9 of [12]. If $1 < p < \infty$, $q = \infty$, and E is a Banach space, it is proved in Theorem 1 of [14] and Corollary 15 of [15].

Theorem 2. *Let $p, q \in [1, \infty]$, and let E be a locally convex space. Then the following assertions are equivalent:*

(i) *E has the sequential $DP_{(p,q)}$ property;*

(ii) *For each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and for each weakly q-Cauchy sequence $\{\chi_n\}_{n\in\omega}$ in E'_β, it follows $\langle \chi_n, x_n\rangle \to 0$;*

(iii) *For each weakly p-Cauchy sequence $\{x_n\}_{n\in\omega}$ in E and for each weakly q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_β, it follows $\langle \chi_n, x_n\rangle \to 0$.*

Proof. (i)⇒(ii) Suppose for a contradiction that there are $a > 0$, a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E, and a weakly q-Cauchy sequence $\{\chi_n\}_{n\in\omega}$ in E'_β such that $|\langle \chi_n, x_n\rangle| > 4a$. Taking into account that $\{x_n\}_{n\in\omega}$ is weakly null, for every $n \in \omega$ there is $i_n > n$ such that $|\langle \chi_n, x_{i_n}\rangle| < a$. We can assume that $i_0 < i_1 < \cdots$. Since $\{\chi_n\}_{n\in\omega}$ is weakly q-Cauchy, the sequence $\{\chi_{i_n} - \chi_n\}_{n\in\omega}$ is weakly q-summable. By (i), there is $N \in \omega$ such that $|\langle \chi_{i_n} - \chi_n, x_{i_n}\rangle| < a$ for every $n \geq N$. Therefore, for every $n \geq N$, we obtain

$$4a < |\langle \chi_{i_n}, x_{i_n} \rangle| \leq |\langle \chi_{i_n} - \chi_n, x_{i_n} \rangle| + |\langle \chi_n, x_{i_n} \rangle| \leq 2a.$$

This is a contradiction.

The implication (i)⇒(iii) can be proved analogously to (i)⇒(ii) replacing χ_n with x_n, and the implications (ii)⇒(i) and (iii)⇒(i) are trivial. □

Below, we give a sufficient condition to have the sequential $DP_{(p,q)}$ property.

Proposition 2. *Let $1 \leq p \leq q \leq \infty$, and let (E, τ) be a q-quasibarrelled space. If E has the p-Schur property, then E has the sequential $DP_{(p,q)}$ property.*

Proof. Since E is q-quasibarrelled, Lemma 4.8 of [12] implies that the topology $\mathcal{T}_{\mathcal{B}_{E'}} = \tau_{S_q}$ on E of uniform convergence on all weakly p-summable sequences of E'_β satisfies the inclusions $\sigma(E, E') \subseteq \tau_{S_q} \subseteq \tau$. Therefore, by the equivalence (vii)⇔(xi) of Theorem 4.9 of [12], it suffices to show that the space (E, τ_{S_q}) has the p-Schur property. However, since (E, τ) has the p-Schur property, the above inclusions trivially imply that (E, τ_{S_q}) has the p-Schur property, as desired. □

As we noticed in the introduction, in the realm of Banach spaces the DP-property of order p coincides with the sequential $DP_{(p,\infty)}$ property. This provides motivation to additionally consider the case $q = \infty$ (not only as an important partial case). If $1 < p < \infty$ and E is a Banach space, the equivalences (i)⇔(iv)⇔(vi) of the next corollary are proved in Theorem 1 of [16].

Corollary 1. *Let $p \in [1, \infty]$, and let E be a locally convex space. Consider the following conditions:*

(i) *E has the sequential $DP_{(p,\infty)}$ property;*
(ii) *Each (relatively) weakly sequentially p-compact subset of E is an ∞-(V^*) set;*
(iii) *Each weakly p-summable sequence in E is an ∞-(V^*) set;*
(iv) *Every weakly sequentially p-precompact subset of E is an ∞-(V^*) set;*
(v) *Every weakly sequentially precompact subset of E'_β is a p-(V) set;*
(vi) *For every lcs L and for each operator $S : X \to E$, which transforms bounded sets into weakly sequentially p-precompact sets, the adjoint operator $S^* : E'_\beta \to X'_\beta$ is completely continuous;*
(vii) *For every normed space X and each weakly sequentially p-precompact operator S from X to E, the adjoint operator $S^* : E'_\beta \to X'_\beta$ is completely continuous;*
(viii) *For each operator $R \in \mathcal{L}(\ell^0_{p^*}, E)$ (each weakly sequentially p-precompact operator $R \in \mathcal{L}(\ell^0_1, E)$ if $p = \infty$, or $R \in \mathcal{L}(c^0_0, E)$ if $p = \infty$), the adjoint operator S^* is completely continuous.*
(ix) *For each operator $R \in \mathcal{L}(\ell_{p^*}, E)$ (each weakly sequentially p-precompact operator $R \in \mathcal{L}(\ell_1, E)$ if $p = \infty$, or $R \in \mathcal{L}(c_0, E)$ if $p = \infty$), the adjoint operator S^* is completely continuous.*

Then, (i)⇔(ii)⇔(iii)⇔(iv)⇔(v)⇒(vi)⇒(vii)⇒(viii) and (v)⇒(ix). If, in addition, $1 < p < \infty$, then the conditions (i)–(viii) are equivalent. If $1 < p < \infty$ and E is sequentially complete, then the conditions (i)–(ix) are equivalent.

Proof. The equivalences (i)⇔(ii)⇔(iii) immediately follow from Theorem 4.9 of [12].

(i)⇒(iv) Suppose for a contradiction that there is a weakly sequentially p-precompact subset A of E, which is not an ∞-(V^*) set. Then, there are $a > 0$ and a weakly null sequence $\{\chi_n\}_{n\in\omega}$ in E'_β such that $\|\chi_n\|_A > 2a$ for every $n \in \omega$. For every $n \in \omega$, choose $a_n \in A$ such that $|\langle \chi_n, a_n \rangle| > a$. Since A is weakly sequentially p-precompact, passing to a subsequence if needed, we can assume that the sequence $S = \{a_n\}_{n\in\omega}$ is weakly p-Cauchy. By Theorem 2, we have $|\langle \chi_n, a_n \rangle| \to 0$, which contradicts the choice of S.

(iv)⇒(iii) is trivial.

(i)⇒(v) Suppose for a contradiction that there is a weakly sequentially precompact subset B of E'_β that is not a p-(V) set. Then, there are $a > 0$ and a weakly p-summable

sequence $\{x_n\}_{n\in\omega}$ in E such that $\sup_{\chi\in B}|\langle\chi, x_n\rangle| > 2a$ for every $n \in \omega$. For every $n \in \omega$, choose $\chi_n \in B$ such that

$$|\langle\chi_n, x_n\rangle| > a. \tag{1}$$

Since B is weakly sequentially precompact, without loss of generality, we can assume that the sequence $\{\chi_n\}_{n\in\omega}$ is weakly Cauchy in E'_β. However, (1) then contradicts the equivalence (i)⇔(ii) of Theorem 2 in which $q = \infty$.

(v)⇒(i) immediately follows from the definition of p-(V) sets and the equivalence (i)⇔(ii) of Theorem 2 (because each weakly null sequence in E'_β is weakly sequentially precompact).

(i)⇒(vi) Suppose for a contradiction that there are an lcs X and an operator $S : X \to E$, which transforms bounded sets into weakly sequentially p-precompact sets, such that $S^* : E'_\beta \to X'_\beta$ is not completely continuous. Therefore, there are a weakly null sequence $\{\chi_n\}_{n\in\omega}$ in E'_β and a bounded set $A \subseteq X$ such that $S^*(\chi_n) \notin A^\circ$ for every $n \in \omega$. For every $n \in \omega$, choose $x_n \in A$ such that

$$|\langle\chi_n, S(x_n)\rangle| = |\langle S^*(\chi_n), x_n\rangle| > 1. \tag{2}$$

Since $S(A)$ is weakly sequentially p-precompact, passing to a subsequence if needed, we can assume that the sequence $\{S(x_n)\}_{n\in\omega}$ is weakly p-Cauchy. Then, by (i) and the equivalence (i)⇔(iii) of Theorem 2, we obtain $\langle\chi_n, S(x_n)\rangle \to 0$, which contradicts (2).

(vi)⇒(vii)⇒(viii) and (v)⇒(ix) are obvious because if $1 < p < \infty$, then each operator $R \in \mathcal{L}(\ell^0_{p^*}, E)$ or $R \in \mathcal{L}(\ell_{p^*}, E)$ is weakly sequentially p-precompact, see Corollary 13.11 of [13].

(viii)⇒(i) and (ix)⇒(i): Assume that $1 < p < \infty$ (and additionally E is sequentially complete for the implication (ix)⇒(i)). Let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E, and let $\{\chi_n\}_{n\in\omega}$ be a weakly null sequence in E'_β. For every $n \in \omega$, set $S(e^*_n) := x_n$. Then, by Proposition 4.14 of [13], S defines an operator from ℓ^0_p to E (or from ℓ_p to E in the case when E is sequentially complete). Moreover, for every $\chi \in E'$, we have $S^*(\chi) = \big(\langle\chi, x_n\rangle\big)$. Since, by (viii) or (ix), the adjoint operator S^* is completely continuous, we have

$$|\langle\chi_n, x_n\rangle|^p \le \sum_{i\in\omega}|\langle\chi_n, x_i\rangle|^p = \|S^*(\chi_n)\|^p_{\ell_p} \to 0.$$

Thus, E has the sequential $DP_{(p,\infty)}$ property. □

Following [34], a locally convex space E is *weakly sequentially locally p-complete* if the closed absolutely convex hull of a weakly null sequence is weakly sequentially p-precompact. In the partial case when $p = \infty$, E is weakly sequentially locally ∞-complete if and only if the closed absolutely convex hull of a weakly null sequence is weakly sequentially precompact. For Banach spaces, the next corollary extends and generalizes Theorem 1 and Corollary 3 of [16].

Corollary 2. *Let $1 < p < \infty$, and let E be a sequentially complete locally convex space. Consider the following conditions:*

(i) *E has the sequential $DP_{(p,\infty)}$ property;*
(ii) *Each (relatively) weakly sequentially p-compact subset of E is an ∞-(V^*) set;*
(iii) *Each weakly p-summable sequence in E is an ∞-(V^*) set;*
(iv) *Every weakly sequentially p-precompact subset of E is an ∞-(V^*) set;*
(v) *Every weakly sequentially precompact subset of E'_β is a p-(V) set;*
(vi) *The identity map $\mathsf{id}_E : E \to E$ is weakly p-convergent;*
(vii) *For every normed (the same Banach) space Z, each weakly sequentially p-precompact operator R from Z to E is an ∞-(V^*) map;*
(viii) *Each operator $T \in \mathcal{L}(\ell_{p^*}, E)$ is an ∞-(V^*) operator;*
(ix) *For each operator $R \in \mathcal{L}(\ell_{p^*}, E)$, the adjoint operator S^* is completely continuous;*

(x) *For any Banach space Z and each $R \in \mathcal{L}(E,Z)$ with weakly sequentially precompact adjoint $R^* : Z'_\beta \to E'_\beta$, the operator R is p-convergent;*

(xi) *R is p-convergent for each $R \in \mathcal{L}(E,c_0)$ with weakly sequentially precompact adjoint $R^* : \ell_1 \to E'_\beta$.*

Then (i)⇔(ii)⇔(iii)⇔(iv)⇔(v)⇔(vi)⇔(vii)⇔(viii)⇔(ix)⇒(x)⇒(xi). If, in addition, E is a Mackey c_0-barrelled space and E'_β is weakly sequentially angelic, then all conditions (i)–(xi) are equivalent.

Proof. The equivalences (i)⇔(ii)⇔(iii)⇔(iv)⇔(v)⇔(ix) follow from Corollary 1.

The equivalences (iii)⇔(vi)⇔(vii)⇔(viii) follow from (E) of Theorem 3.14 of [34] applied to the identity operator $T = \mathsf{id}_E : E \to E$.

The implications (vi)⇒(x)⇒(xi) follow from (B) of Theorem 3.14 of [34] applied to the identity operator $T = \mathsf{id}_E : E \to E$.

(xi)⇒(vi) Assume in addition that E is a Mackey c_0-barrelled space such that E'_β is a weakly sequentially angelic space. Then, by Theorem 12.1.4 of [5], the space E'_β is locally complete and hence so is $(E'_\beta)_w$. Let now $S = \{\chi_n\}_{n\in\omega}$ be a weakly null sequence in E'_β. Then, the local completeness of $(E'_\beta)_w$ implies that $\overline{\mathrm{acx}}(S)$ is a weakly compact subset of E'_β. As E'_β is weakly sequentially angelic, we obtain that $\overline{\mathrm{acx}}(S)$ is a weakly sequentially compact subset of E'_β. Therefore, E'_β is weakly sequentially locally ∞-complete. Now, the implication (xi)⇒(vi) follows from (B) of Theorem 3.14 of [34]. □

The sequential $DP_{(p,q)}$ property of the range of operators implies some strong additional properties as the next assertion shows.

Proposition 3. *Let $p,q \in [1,\infty]$, and let E and L be locally convex spaces. If L has the sequential $DP_{(p,q)}$ property (for instance, L is a quasibarrelled locally complete space with the Dunford–Pettis property, e.g., $L = \ell_\infty$). Then, each operator $T \in \mathcal{L}(E,L)$ is weakly (p,q)-convergent.*

Proof. Let $\{\eta_n\}_{n\in\omega}$ be a weakly q-summable sequence in L'_β, and let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, $\{T(x_n)\}_{n\in\omega}$ is a weakly p-summable sequence in L. Now the sequential $DP_{(p,q)}$ property of L implies $\langle \eta_n, T(x_n)\rangle \to 0$. Thus, T is weakly (p,q)-convergent.

If L is a quasibarrelled locally complete space with the Dunford–Pettis property, Corollary 5.13 of [12] implies that L has the sequential $DP_{(p,q)}$ property. □

4. The Sequential $DP^*_{(p,q)}$ Property

In this section, we characterize and study locally convex spaces with the sequential $DP^*_{(p,q)}$ property. Recall that an lcs E has the *sequential $DP^*_{(p,q)}$ property* if $\lim_{n\to\infty}\langle \chi_n, x_n\rangle = 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and each weak* q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'. First, we select the next assertion.

Proposition 4. *Let $p,q \in [1,\infty]$, and let E be a locally convex space. If E has the sequential $DP^*_{(p,q)}$ property, then it has the sequential $DP_{(p,q)}$ property. The converse is true if E is semi-reflexive, but not in general.*

Proof. We need to show only that there is E, which has the sequential $DP_{(p,q)}$ property but without the sequential $DP^*_{(p,q)}$ property (other assertions follow from the corresponding definitions). Let $E = c_0$. Then, E has the DP property, and hence, by the Grothendieck theorem, it has the sequential DP property. Therefore, c_0 has the sequential $DP_{(p,q)}$ property for all $p,q \in [1,\infty]$. On the other hand, it is clear that the canonical unit basis $\{e_n\}_{n\in\omega}$ of c_0 is weakly p-summable for all $p \in [1,\infty]$, and the canonical unit basis $\{e^*_n\}_{n\in\omega}$ of ℓ_1 is weak* null. Since $\langle e^*_n, e_n\rangle = 1 \not\to 0$, the Banach space c_0 does not have the sequential $DP^*_{(p,\infty)}$ property for all $p \in [1,\infty]$. □

Remark 1. In Example 4.11, we constructed a semi-reflexive lcs H with the sequential $DP_{(p,q)}$ property for each $p, q \in [1, \infty]$ such that H does not have the DP property. Taking into account Proposition 4, it follows that there are locally convex spaces with the sequential $DP^*_{(p,q)}$ property but without the DP property.

We shall use below the following generalization of the Grothendieck property.

Definition 8. Let $p \in [1, \infty]$. A locally convex space E is said to have the *p-Grothendieck property* if the identity map $\mathrm{id}_{E'} : E'_{w^*} \to (E'_\beta)_w$ is p-convergent.

It is clear that the ∞-Grothendieck property is exactly the Grothendieck property.

Let (E, τ) be a locally convex space. Denote by $\Sigma'(E')$ the family of all absolutely convex, equicontinuous, weakly compact subsets of E'_β, and let $\tau_{\Sigma'}$ be the Grothendieck topology on E of uniform convergence on the elements of $\Sigma'(E')$. For $q \in [1, \infty]$, denote by $\tau_{S^*_q}$ the polar topology on E of uniform convergence on weak* q-summable sequences in E'. Below we list some basic properties of the "sequentially-open" topology $\tau_{S^*_q}$, which will be used repeatedly in what follows.

Lemma 2. *Let $q \in [1, \infty]$, and let (E, τ) be a locally convex space. Then:*

(i) $\sigma(E, E') \subseteq \tau_{S^*_q}$.

(ii) *E is a q-barrelled space if and only if $\tau_{S^*_q} \subseteq \tau$.*

(iii) *$\tau_{S^*_q} \subseteq \mu(E, E')$ if and only if for every weak* q-summable sequence $S = (\chi_n)$ in E', the absolutely convex hull of S is relatively weak* compact if and only if E'_{w^*} is locally complete (for example, E is barrelled).*

(iv) *$\tau_{S^*_q} \subseteq \tau_{\Sigma'}$ if and only if E is a q-barrelled q-Grothendieck space whose strong dual E'_β is locally complete.*

Proof. (i) Let $\{\chi_0, \dots, \chi_k\}$ be a finite subset of E'. For every $n > k$, set $\chi_n = 0$. It is clear that $\{\chi_n\}_{n\in\omega}$ is a weak* q-summable sequence in E'. Now it is evident that $\sigma(E, E') \subseteq \tau_{S^*_q}$.

(ii) Assume that E is a q-barrelled space. Let $S = \{\chi_n\}_{n\in\omega}$ be a weak* q-summable sequence in E'. Since E is q-barrelled, S is equicontinuous. Take a closed absolutely convex neighborhood U of zero in E such that $S \subseteq U^\circ$. Then, $U^{\circ\circ} = U \subseteq S^\circ$. Thus, $\tau_{S^*_q} \subseteq \tau$.

Conversely, assume that $\tau_{S^*_q} \subseteq \tau$. To show that E is q-barrelled, let $S = \{\chi_n\}_{n\in\omega}$ be a weak* q-summable sequence in E'_β. Then, the inclusion $\tau_{S^*_q} \subseteq \tau$ implies that there is $U \in \mathcal{N}_0(E)$ such that $U \subseteq S^\circ$. Then, $S \subseteq U^\circ$, and hence, the sequence S is equicontinuous. Thus, E is a q-barrelled space.

(iii) By the Mackey–Arens theorem, the inclusion $\tau_{S^*_q} \subseteq \mu(E, E')$ holds if and only if for every weak* q-summable sequence $S = (\chi_n)$ in E', there is a weak* compact, absolutely convex subset K of E' such that $K^\circ \subseteq S^\circ$, and hence, if and only if $S^{\circ\circ} \subseteq K^{\circ\circ} = K$, as desired. The last assertion is equivalent to the second one by the fact that E'_{w^*} carries its weak topology and Theorem 10.2.4 of [5]. If E is barrelled, then by Proposition 11.1.4 of [5], E'_{w^*} is quasi-complete and hence locally complete.

(iv) Assume that $\tau_{S^*_q} \subseteq \tau_{\Sigma'}$. Let $S = (\chi_n)$ be a weak* q-summable sequence in E'. Then, there is $K \in \Sigma'(E')$ such that $K^\circ \subseteq S^\circ$, and hence, $S \subseteq S^{\circ\circ} \subseteq K$. Since K is equicontinuous, it follows that also S is equicontinuous and hence E is q-barrelled. Since K is weakly compact and absolutely convex in E'_β, we obtain that the closed, absolutely convex hull of S is a weakly compact subset of E'_β. Since every weakly q-summable sequence in E'_β is weak* q-summable, Theorem 10.2.4 of [5] implies that E'_β is locally complete. As $S \subseteq K$, it follows that S is weakly precompact in E'_β. Since the weak topology of E'_β is finer than the weak* topology, Lemma 1 implies that $\chi_n \to 0$ in the weak topology of E'_β. Thus, E has the q-Grothendieck property.

Conversely, assume that E is a q-barrelled q-Grothendieck space whose strong dual E'_β is locally complete. If $S = (\chi_n)$ is a weak* q-summable sequence in E', then the q-Grothendieck property implies that S is weakly null in E'_β. Whence, by the local completeness of E'_β, the closed absolutely convex hull $K := \overline{\mathrm{acx}}(S)$ of S is a weakly compact subset of E'_β. Since E is q-barrelled, S and hence also K are equicontinuous. Therefore, $K \in \Sigma'(E')$. It follows that $K^\circ \subseteq S^\circ$ and hence $\tau_{S^*_q} \subseteq \tau_{\Sigma'}$, as desired. □

Below we characterize locally convex spaces with the sequential $DP^*_{(p,q)}$ property.

Theorem 3. *Let $p, q \in [1, \infty]$, (E, τ) be a locally convex space, $\mathcal{B}_E = \ell^w_p(E)$ (or $c^w_0(E)$ if $p = \infty$), $\mathcal{B}_{E'} = \ell^{w^*}_q(E')$ (or $c^{w^*}_0(E')$ if $q = \infty$), and let $\mathsf{id} : E \to E$ be the identity map. Then, the following assertions are equivalent:*

(i) *For each $A = (x_n) \in \mathcal{B}_E$, the sequence A is precompact (in fact, null) For $\mathcal{T}_{\mathcal{B}_{E'}}$;*
(ii) *For each $B = (\chi_n) \in \mathcal{B}_{E'}$, the sequence B is precompact (in fact, null) for $\mathcal{T}_{\mathcal{B}_E}$;*
(iii) *For each $A \in \mathcal{B}_E$, $\mathsf{id}\restriction_A : \big(A, \sigma(E, E')\restriction_A\big) \to \big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ is uniformly continuous;*
(iv) *For each $B \in \mathcal{B}_{E'}$, $\mathsf{id}^*\restriction_B : \big(B, \sigma(E', E)\restriction_B\big) \to \big(E', \mathcal{T}_{\mathcal{B}_E}\big)$ is uniformly continuous;*
(v) *For each $A = (x_n) \in \mathcal{B}_E$ and every $B = (\chi_n) \in \mathcal{B}_{E'}$, the restriction to $B \times A$ of $\langle \eta, x\rangle$ is uniformly continuous for the product topology $\beta(E', E)\restriction_B \times \sigma(E, E')\restriction_A$ (or vice versa for $\sigma(E', E)\restriction_B \times \beta(E, E')\restriction_A$);*
(vi) *For each $A = (x_n) \in \mathcal{B}_E$ and every $B = (\chi_n) \in \mathcal{B}_{E'}$, the restriction to $B \times A$ of $\langle \eta, x\rangle$ is uniformly continuous for the product topology $\sigma(E', E)\restriction_B \times \sigma(E, E')\restriction_A$;*
(vii) *E has the sequential $DP^*_{(p,q)}$ property;*
(viii) *Each relatively weakly sequentially p-compact set in E is a (q, ∞)-limited set;*
(ix) *Each weakly sequentially p-compact set in E is a (q, ∞)-limited set;*
(x) *Each weakly p-summable sequence in E is a (q, ∞)-limited set;*
(xi) *For each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and for each weak* q-Cauchy sequence $\{\chi_n\}_{n\in\omega}$ in E', it follows $\langle \chi_n, x_n\rangle \to 0$;*
(xii) *For each weakly p-Cauchy sequence $\{x_n\}_{n\in\omega}$ in E and for each weak* q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E', it follows $\langle \chi_n, x_n\rangle \to 0$.*
(xiii) *Each weakly sequentially p-precompact set in E is a (q, ∞)-limited set;*

Moreover, if $\mathcal{T}_{\mathcal{B}_{E'}}$ is compatible with τ (i.e., if E'_{w^*} is locally complete; for example, E is barrelled), then (i)-(xii) *are equivalent to*

(xiv) *$\big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ has the p-Schur property.*

Proof. Set $L := E$ and define $T : E \to L$ by $T := \mathsf{id}$.

By (i) of Lemma 2, we have $\sigma(E, E') \subseteq \mathcal{T}_{\mathcal{B}_{E'}}$. Hence, by Lemma 1, if $A \in \mathcal{B}_E$ is $\mathcal{T}_{\mathcal{B}_{E'}}$-precompact, then $A \cup \{0\}$ is $\mathcal{T}_{\mathcal{B}_{E'}}$-null. Analogously, it is clear that $\sigma(E', E) \subseteq \mathcal{T}_{\mathcal{B}_E}$. Since each $B = (\chi_n) \in \mathcal{B}_{E'}$ is $\sigma(E', E)$-null, Lemma 1 implies that B is $\mathcal{T}_{\mathcal{B}_E}$-precompact if and only if $B \cup \{0\}$ is $\mathcal{T}_{\mathcal{B}_E}$-null. Then, the equivalences (i)⇔(ii)⇔(iii)⇔(iv)⇔(v)⇔(vi) follow from Theorem 9.2.1 of [2] in which $\mathfrak{S} = \mathcal{B}_E$ and $\mathfrak{S}' = \mathcal{B}_{E'}$.

(vi)⇒(vii) immediately follows from the continuity of $\langle \eta, x\rangle$ at zero $(0, 0)$ with respect to $\sigma(E', E)\restriction_B \times \sigma(E, E')\restriction_A$.

(vii)⇒(vi) Assume that E has the sequential $DP^*_{(p,q)}$ property. We claim that the bilinear map $\langle \eta, x\rangle$ is weak*–weak continuous at $(0, 0)$ for every $(x_n) \in \mathcal{B}_E$ and each $(\chi_n) \in \mathcal{B}_{E'}$. Indeed, suppose for a contradiction that there are $(x_n) \in \mathcal{B}_E$ and $(\chi_n) \in \mathcal{B}_{E'}$ such that $\langle \eta, x\rangle$ is weak*–weak discontinuous at $(0, 0)$. Then, there exists $\varepsilon > 0$ such that for every $i \in \omega$ there are $n_i > i$ and $m_i > i$ such that $|\langle \eta_{n_i}, x_{m_i}\rangle| \geq \varepsilon$. Without loss of generality, we can assume that $\{n_i\}_{i\in\omega}$ and $\{m_i\}_{i\in\omega}$ are strictly increasing. Then, $(x_{n_i}) \in \mathcal{B}_E$ and $(\chi_{n_i}) \in \mathcal{B}_{E'}$ but $\langle \eta_{n_i}, x_{m_i}\rangle \not\to 0$, which contradicts the sequential $DP^*_{(p,q)}$ property of E.

Since all sequences $A = (x_n) \in \mathcal{B}_E$ are weakly null and all sequences $B = (\chi_n) \in \mathcal{B}_{E'}$ are weak* null, the claim implies that the bilinear map $\langle \eta, x\rangle$ is continuous on the (weak*-

weak) *compact* space $B \times A$. Therefore, $\langle \eta, x \rangle$ is uniformly continuous on $B \times A$ for the product topology $\sigma(E', E)\restriction_B \times \sigma(E, E')\restriction_A$.

(vii)⇒(viii) Assume that E has the sequential $DP^*_{(p,q)}$ property. Suppose for a contradiction that there is a relatively weakly sequentially p-compact subset A of E, which is not a (q, ∞)-limited set. Then, there exists a weak* q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E' and $\varepsilon > 0$ such that $\sup_{a\in A} |\langle \chi_n, a\rangle| \geq 2\varepsilon$ for every $n \in \omega$. For each $n \in \omega$, choose $a_n \in A$ such that $|\langle \chi_n, a_n\rangle| \geq \varepsilon$. Since A is relatively weakly sequentially p-compact, the sequence $\{a_n\}_{n\in\omega}$ has a subsequence $\{a_{n_k}\}_{k\in\omega}$, which weakly p-converges to a point $z \in E$. Then

$$\varepsilon \leq |\langle \chi_{n_k}, a_{n_k}\rangle| \leq |\langle \chi_{n_k}, a_{n_k} - z\rangle| + |\langle \chi_{n_k}, z\rangle| \to 0 \quad \text{as } k \to \infty,$$

a contradiction.

The implications (viii)⇒(ix)⇒(x) are clear because every weakly p-summable sequence in E is a weakly sequentially p-compact set.

(x)⇒(vii) Assume that each weakly p-summable sequence in E is a (q, ∞)-limited set. Let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E, and let $\{\chi_n\}_{n\in\omega}$ be a weak* q-summable sequence in E'_β. By assumption, the set $S := \{x_n\}_{n\in\omega}$ is a (q, ∞)-limited set. Therefore, by the definition of (q, ∞)-limited sets, we have

$$0 \leq \lim_{n\to\infty} |\langle \chi_n, x_n\rangle| \leq \lim_{n\to\infty} \sup_{i\in\omega} |\langle \chi_n, x_i\rangle| = 0.$$

Thus, E has the sequential $DP^*_{(p,q)}$ property.

(vii)⇒(xi) Suppose for a contradiction that there are $a > 0$, a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E, and a weak* q-Cauchy sequence $\{\chi_n\}_{n\in\omega}$ in E' such that $|\langle \chi_n, x_n\rangle| > 4a$. Taking into account that $\{x_n\}_{n\in\omega}$ is weakly null, for every $n \in \omega$ there is $i_n > n$ such that $|\langle \chi_n, x_{i_n}\rangle| < a$. We can assume that $i_0 < i_1 < \cdots$. Since $\{\chi_n\}_{n\in\omega}$ is weak* q-Cauchy, the sequence $\{\chi_{i_n} - \chi_n\}_{n\in\omega}$ is weak* q-summable. By (vii), there is $N \in \omega$ such that $|\langle \chi_{i_n} - \chi_n, x_{i_n}\rangle| < a$ for every $n \geq N$. Therefore, for every $n \geq N$, we obtain

$$4a < |\langle \chi_{i_n}, x_{i_n}\rangle| \leq |\langle \chi_{i_n} - \chi_n, x_{i_n}\rangle| + |\langle \chi_n, x_{i_n}\rangle| \leq 2a.$$

This is a contradiction.

(xi)⇒(vii) follows from the fact that every weak* q-summable sequence is weak* q-Cauchy.

(vii)⇒(xii) and (xii)⇒(vii) can be proved analogously to (vii)⇒(xi) and (xi)⇒(vii).

(xii)⇒(xiii) Suppose for a contradiction that there is a weakly sequentially p-precompact subset A of E that is not a (q, ∞)-limited set. Then, there exists a weak* q-summable sequence $\{\chi_n\}_{n\in\omega}$ in E' and $\varepsilon > 0$ such that $\sup_{a\in A} |\langle \chi_n, a\rangle| \geq 2\varepsilon$ for every $n \in \omega$. For each $n \in \omega$, choose $a_n \in A$ such that $|\langle \chi_n, a_n\rangle| \geq \varepsilon$. Since A is weakly sequentially p-precompact, the sequence $\{a_n\}_{n\in\omega}$ has a subsequence $\{a_{n_k}\}_{k\in\omega}$ that is weakly p-Cauchy. Then, (xii) implies

$$\varepsilon \leq |\langle \chi_{n_k}, a_{n_k}\rangle| \to 0 \quad \text{as } k \to \infty,$$

a contradiction.

(xiii)⇒(xii) is evident.

Below we assume that $\mathcal{T}_{\mathcal{B}_{E'}}$ is compatible with τ (which is equivalent to the condition that E'_{w^*} is locally complete, see (iii) of Lemma 2). Therefore, $\sigma(E, E')$ is also the weak topology of $\left(E, \mathcal{T}_{\mathcal{B}_{E'}}\right)$.

(iii)⇒(xiv) Let $(x_n) \in \ell_p^w\left(E, \mathcal{T}_{\mathcal{B}_{E'}}\right)$ (or $\in c_0^w\left(E, \mathcal{T}_{\mathcal{B}_{E'}}\right)$ if $p = \infty$). Since $\mathcal{T}_{\mathcal{B}_{E'}}$ is compatible with τ, it follows that $(x_n) \in \mathcal{B}_E$. Then (iii) implies $x_n \to 0$ in $\mathcal{T}_{\mathcal{B}_{E'}}$, that is, $\left(E, \mathcal{T}_{\mathcal{B}_{E'}}\right)$ has the p-Schur property.

(xiv)⇒(iii) Let $A = (x_n) \in \mathcal{B}_E$. Since $\mathcal{T}_{\mathcal{B}_{E'}}$ and τ are compatible, it follows that $A \in \ell_p^w\big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ (or $\in c_0^w\big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ if $p = \infty$). Then, the p-Schur property of $\big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ implies that $\sigma(E, E')\restriction_{A\cup\{0\}} = \mathcal{T}_{\mathcal{B}_{E'}}\restriction_{A\cup\{0\}}$. Taking into account that $A \cup \{0\}$ is weakly compact, it follows that the identity map $\big(A, \sigma(E, E')\restriction_{A\cup\{0\}}\big) \to \big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$ is uniformly continuous, and hence, so is the identity map $\big(A, \sigma(E, E')\restriction_A\big) \to \big(E, \mathcal{T}_{\mathcal{B}_{E'}}\big)$. □

Setting $p = q = \infty$ in Theorem 3, we obtain the following corollary, which generalizes a characterization of Banach spaces with the DP^* property obtained in [17].

Corollary 3. *A locally convex space E has the sequential DP^* property if and only if every weakly sequentially (pre)compact subset of E is limited if and only if each weakly null sequence in E is limited.*

Corollary 4. *Let $p \in [1, \infty]$, and let E be a p-barrelled space. If E has the p-Schur property, then E has the sequential DP_p^* property.*

Proof. Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. By the p-Schur property, S is a compact subset of E. Therefore, by (ii) of Proposition 3.6 of [28], S is a (p, ∞)-limited set. Thus, by the equivalence (vii)⇔(x) of Theorem 3, E has the sequential DP_p^* property. □

Corollary 5. *Let $p, q \in [1, \infty]$, and let E be a locally convex space such that the identity map $\mathsf{id}_{E'} : E'_{w^*} \to E'_\beta$ is q-convergent. Then, E has the sequential $DP^*_{(p,q)}$ property.*

Proof. Since $\mathsf{id}_{E'} : E'_{w^*} \to E'_\beta$ is q-convergent, Corollary 5.7 of [28] implies that every bounded subset of E is a (q, ∞)-limited set. Now the equivalence (vii)⇔(viii) of Theorem 3 applies. □

Below, we give another sufficient condition to have the sequential $DP^*_{(p,q)}$ property.

Proposition 5. *Let $p, q \in [1, \infty]$, and let (E, τ) be a q-barrelled locally complete space such that E'_{w^*} is locally complete. If E has the DP property, then it has the sequential $DP^*_{(p,q)}$ property.*

Proof. Since E'_{w^*} is locally complete, by the equivalence (vii)⇔(xiv) of Theorem 3, to show that the space E has the sequential $DP^*_{(p,q)}$ property, it suffices to prove that the space $(E, \tau_{S_q^*})$ has the p-Schur property. Since E is q-barrelled, (i) and (ii) of Lemma 2 imply $\sigma(E, E') \subseteq \tau_{S_q^*} \subseteq \tau_{\Sigma'}$. By (i) of Lemma 4.1 of [12], we also have $\tau_{\Sigma'} \subseteq \tau$.

Since E is locally complete and has the DP property, (i) of Proposition 5.12 of [12] implies that E has the quasi DP_p property. Therefore, by Theorem 4.5 of [12], the space $\big(E, \tau_{\Sigma'}\big)$ has the p-Schur property. Then, the proved inclusions $\sigma(E, E') \subseteq \tau_{S_q^*} \subseteq \tau_{\Sigma'}$ immediately imply that the space $(E, \tau_{S_q^*})$ has the p-Schur property. □

Let $p, q \in [1, \infty]$, and let E and L be locally convex spaces. Following [34], a linear map $T : E \to L$ is called *weak* (p, q)-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weak* q-summable sequence $\{\eta_n\}_{n\in\omega}$ in L' and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E. The next assertion is similar to Proposition 3.

Proposition 6. *Let $p, q \in [1, \infty]$, and let E and L be locally convex spaces. If L has the sequential $DP^*_{(p,q)}$ property, then each operator $T \in \mathcal{L}(E, L)$ is weak* (p, q)-convergent.*

Proof. Let $\{\eta_n\}_{n\in\omega}$ be a weak* q-summable sequence in L'_β, and let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, $\{T(x_n)\}_{n\in\omega}$ is a weakly p-summable sequence in L. Now, the sequential $DP^*_{(p,q)}$ property of L implies $\langle \eta_n, T(x_n)\rangle \to 0$. Thus, T is weak* (p, q)-convergent. □

As we noticed in the introduction, in the realm of Banach spaces the DP^*-property of order p coincides with the sequential $DP^*_{(p,\infty)}$ property. This motivates additionally to consider the case $q = \infty$. We start from the following characterization of locally convex spaces with the sequential $DP^*_{(p,\infty)}$ property.

Corollary 6. *Let $p \in [1, \infty]$. For a locally convex space E, consider the following assertions:*

(i) *E has the sequential $DP^*_{(p,\infty)}$ property;*
(ii) *Each (relatively) weakly sequentially p-compact subset of E is limited;*
(iii) *Every weakly sequentially p-precompact subset of E is limited;*
(vi) *Each weakly p-summable sequence in E is limited;*
(v) *Every operator $T \in \mathcal{L}(E, c_0)$ is p-convergent.*

Then, (i)⇔(ii)⇔(iii)⇔(iv)⇒(v). If, additionally, E is barrelled, then all assertions (i)–(v) are equivalent and they are equivalent to the following condition:

(vi) *the identity operator $\mathsf{id}_E : E \to E$ is weak* p-convergent.*

Proof. The equivalences (i)⇔(ii)⇔(iii)⇔(iv) immediately follow from Theorem 3.

(iv)⇒(v) Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, S is a limited subset of E. Therefore, $T(S)$ is a limited subset of c_0. Since c_0 has the GP property, $T(S)$ is relatively compact in c_0. As $T(S)$ is also a weakly null sequence, Lemma 1 implies $T(x_n) \to 0$ in c_0.

(v)⇒(iv) Assume that E is barrelled. Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. We have to show that S is a limited subset of E. Suppose for a contradiction that S is not limited. Then, there is a weak* null sequence $\{\chi_n\}_{n\in\omega}$ in E' such that $\sup_{k\in\omega} |\langle \chi_n, x_k\rangle| \not\to 0$. Passing to a subsequence of $\{\chi_n\}_{n\in\omega}$ if needed, we assume that $\sup_{k\in\omega} |\langle \chi_n, x_k\rangle| > \varepsilon$ for some $\varepsilon > 0$. For every $n \in \omega$, choose $k_n \in \omega$ such that $|\langle \chi_n, x_{k_n}\rangle| \geq \varepsilon$.

Define an operator $T : E \to (c_0)_p$ by $T(x) := \big(\langle \chi_n, x\rangle\big)_n$ $(x \in E)$, where $(c_0)_p$ denotes the Banach space c_0 endowed with the topology induced from $\mathbb{F}^\omega$. By the choice of the sequence $\{x_{k_n}\}_{n\in\omega}$, we have $\|T(x_{k_n})\|_{c_0} \geq \varepsilon$ for every $n \in \omega$. Since E is barrelled, we apply Lemma 2.8 of [35] to get that T is also continuous as a linear map from E to the Banach space c_0. By (v), T is p-convergent. In particular, we have $\|T(x_{k_n})\|_{c_0} \to 0$, a contradiction.

If E is barrelled, the equivalence (v)⇔(vi) follows from (B) of Theorem 3.12 of [34]. □

If $1 < p < \infty$, we can extend Corollary 6.

Corollary 7. *Let $1 < p < \infty$. For a sequentially complete locally convex space E, the following assertions are equivalent:*

(i) *E has the sequential $DP^*_{(p,\infty)}$ property;*
(ii) *Each (relatively) weakly sequentially p-compact subset of E is limited;*
(iii) *Every weakly sequentially p-compact operator from a Banach space Z to E is limited;*
(iv) *Each $T \in \mathcal{L}(\ell_{p^*}, E)$ is a limited operator;*
(v) *Every weakly sequentially p-precompact subset of E is limited;*
(vi) *For every normed (the same Banach) space Z, each weakly sequentially p-compact operator S from Z to E is limited;*
(vii) *The identity operator $\mathsf{id}_E : E \to E$ is weak* p-convergent.*

Proof. The equivalence (i)⇔(ii) is proved in Corollary 6, and the implication (ii)⇒(iii) is trivial.

(iii)⇒(iv) follows from the fact that each operator $T = T \circ \mathsf{id}_{\ell_{p^*}} \in \mathcal{L}(\ell_{p^*}, E)$ is weakly sequentially p-compact by Proposition 1.4 of [6] (or Corollary 13.11 of [13]).

(iv)⇒(i) Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E, and let $\{\chi_n\}_{n\in\omega}$ be a weak* null sequence in E'. Then, by Proposition 4.14 of [13], there is a bounded operator $T : \ell_{p^*} \to E$ such that

$$T(a_0e_0^* + \cdots + a_ne_n^*) := a_0x_0 + \cdots + a_nx_n, \quad \text{for } (a_n) \in \ell^0_{p^*},$$

Therefore, by (iv), T is a limited operator and hence the sequence $S = \{T(e_n^*)\}_{n\in\omega}$ is a limited subset of E. Whence

$$0 \le \lim_{n\to\infty} |\langle \chi_n, x_n\rangle| \le \lim_{n\to\infty} \sup_{x\in S} |\langle \chi_n, x\rangle| = 0$$

which means that E has the sequential $DP^*_{(p,\infty)}$ property.

The equivalences (iv)⇔(v)⇔(vi)⇔(vii) follow from (A) and (D) of Theorem 3.12 of [34] applied to the identity operator $\mathsf{id}_E : E \to E$. □

The next corollary generalizes Corollary 23 of [15].

Corollary 8. *Let $1 < p < \infty$, and let E and L be locally convex spaces.*

(i) If either E or L is sequentially complete and has the sequential $DP_{(p,\infty)}$ property, then each operator $T : E \to L$ is weakly p-convergent.

(ii) If either E or L is sequentially complete and has the sequential $DP^*_{(p,\infty)}$ property, then each operator $T : E \to L$ is weak p-convergent.*

Proof. (i) By Corollary 2, either the identity operator id_E or id_L is weakly p-convergent. Therefore, $T = T \circ \mathsf{id}_E = \mathsf{id}_L \circ T$ is weakly p-convergent.

(ii) The proof is similar to that of (i) using Corollary 7. □

The next proposition shows that the class of locally convex spaces with the sequential $DP^*_{(p,q)}$ property is stable under taking direct products and direct sums.

Proposition 7. *Let $p, q \in [1,\infty]$, and let $\{E_i\}_{i\in I}$ be a nonempty family of locally convex spaces.*

(i) $E = \prod_{i\in I} E_i$ has the sequential $DP^*_{(p,q)}$ property if and only if for every $i \in I$, the factor E_i has the sequential $DP^*_{(p,q)}$ property.

(ii) $E = \bigoplus_{i\in I} E_i$ has the sequential $DP^*_{(p,q)}$ property if and only if for every $i \in I$, the summand E_i has the sequential $DP^*_{(p,q)}$ property.

Proof. The proposition immediately follows from (i)-(ii) and (iii)-(iv) of Lemma 4.25 of [13], respectively, and the definition of the sequential $DP^*_{(p,q)}$ property. □

We need some definitions. The *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map f from X to a locally convex space E gives rise to a unique continuous linear operator $\Psi_E(f) : L(X) \to E$ with $f = \Psi_E(f) \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique. For $\chi = a_1x_1 + \cdots + a_nx_n \in L(X)$ with distinct $x_1, \ldots, x_n \in X$ and nonzero $a_1, \ldots, a_n \in \mathbb{F}$, we set $\chi[x_i] := a_i$ and

$$\|\chi\| := |a_1| + \cdots + |a_n|, \text{ and } \operatorname{supp}(\chi) := \{x_1, \ldots, x_n\}.$$

From the definition of $L(X)$, it easily follows the well-known fact that the dual space $L(X)'$ of $L(X)$ is linearly isomorphic to the space $C(X)$ with the pairing

$$\langle f, \chi\rangle = a_1f(x_1) + \cdots + a_nf(x_n) \quad \text{for } f \in C(X).$$

Below we show that the sequential $DP^*_{(p,q)}$ property is not preserved by taking dense subspaces and closed subspaces. This example is interesting also because it gives a C_p-

example of a space without the sequential $DP^*_{(p,\infty)}$ property, although any space $C_p(X)$ has the sequential $DP_{(p,q)}$ property for all $p, q \in [1, \infty]$, see Corollary 5.3 of [12].

Example 1. *Let X be a Tychonoff space containing a non-trivial convergent sequence $\{x_n\}_{n\leq\omega}$. Then, the dense subspace $C_p(X)$ of $\mathbb{F}^X$ does not have the sequential $DP^*_{(p,\infty)}$ property for every $p \in [1, \infty]$.*

Proof. Passing to a subsequence in needed, for every $n \in \omega$, choose a neighborhood U_n of x_n such that $U_n \cap U_m = \emptyset$ and $x_\omega \notin U_n$ for all distinct $n, m \in \omega$ (recall that $x_n \to x_\omega$). For every $n \in \omega$, choose a continuous function $f_n : X \to [0, 1]$ such that $f_n(X \setminus U_{2n}) = \{0\}$ and $f_n(x_{2n}) = 1$. Since $C_p(X)' = L(X)$, the construction of f_n implies that the sequence $\{f_n\}_{n\in\omega}$ is weakly p-summable for every $p \in [1, \infty]$. For every $n \in \omega$, let $\chi_n := \delta_{x_{2n}} - \delta_{x_{2n+1}} \in L(X)$. Since $x_n \to x_\omega$, it follows that the sequence $\{\chi_n\}_{n\leq\omega}$ is weak* null in $L(X)$. By construction, $\langle \chi_n, f_n \rangle = 1$ for all $n \in \omega$. Therefore, $C_p(X)$ does not have the sequential $DP^*_{(p,\infty)}$ property. It remains to note that $\mathbb{F}^X$ has the sequential $DP^*_{(p,\infty)}$ property by Proposition 7. □

We do not know a characterization of Tychonoff spaces X for which $C_p(X)$ has the sequential $DP^*_{(p,q)}$ property. However, we note the following.

Proposition 8. *A barrelled space $C_p(X)$ has the sequential $DP^*_{(p,q)}$ property for all $p, q \in [1, \infty]$.*

Proof. Since $C_p(X)$ is barrelled, the Buchwalter–Schmets theorem implies that X has no infinite functionally bounded subsets. Let $\{f_n\}_{n\in\omega} \subseteq C_p(X)$ be a weakly p-summable sequence and let $\{\chi_n\}_{n\in\omega} \subseteq L(X)$ be a weak* q-summable sequence. Since $\{\chi_n\}_{n\in\omega}$ is a bounded subset of $\big(L(X), \sigma(L(X), C(X))\big)$, Proposition 2.7 of [36] implies that there are a finite subset $F = \{x_1, \dots, x_m\}$ of X and $C > 0$ such that for every $n \in \omega$, the functional χ_n has a decomposition

$$\chi_n = a_{1,n}x_1 + \cdots + a_{m,n}x_m \quad \text{with} \quad |a_{1,n}| + \cdots + |a_{m,n}| \leq C.$$

Moreover, since $\{\chi_n\}_{n\in\omega}$ is weak* q-summable, it is easy to see that $\sum_{i=1}^m \sum_n |a_{i,n}|^q < \infty$ (or $\sum_{i=1}^m |a_{i,n}| \to 0$ if $q = \infty$). Since $\{f_n\}_{n\in\omega}$ is bounded, there is $K > 0$ such that $|f_n(x_i)| \leq K$ for all $n \in \omega$ and $1 \leq i \leq m$. Therefore

$$|\langle \chi_n, f_n \rangle| \leq \sum_{i=1}^m |a_{i,n}| \cdot |f_n(x_i)| \leq K \cdot \sum_{i=1}^m |a_{i,n}| \to 0 \quad \text{as } n \to \infty,$$

which means that $C_p(X)$ has the sequential $DP^*_{(p,q)}$ property. □

Example 2. *Let $p \in [1, \infty]$. There is a barrelled space $E = C_p(Z)$ and a closed subspace H of E such that E has the sequential $DP^*_{(p,\infty)}$ property, but H does not have the sequential $DP^*_{(p,\infty)}$ property.*

Proof. Let X be a Tychonoff space containing a non-trivial convergent sequence $\{x_n\}_{n\leq\omega}$, and let $Z = \mathcal{R}(X)$ be the compact-finite resolution of X defined in ([37], p. 27). Then, by Theorem 4.5 of [37], $C_p(X)$ is a closed subspace of $C_p(Z)$ and the space $C_p(Z)$ is barrelled. Therefore, by Proposition 8, $C_p(Z)$ has the sequential $DP^*_{(p,\infty)}$ property. However, by Example 1, its closed subspace $C_p(X)$ does not have the sequential $DP^*_{(p,\infty)}$ property. □

Below, we characterize free locally convex spaces with the sequential $DP^*_{(p,q)}$ property.

Theorem 4. *Let $p, q \in [1, \infty]$. Then, the free locally convex space $L(X)$ over a Tychonoff space X has the sequential $DP^*_{(p,q)}$ property if and only if X has no infinite functionally bounded subsets.*

Proof. Assume that $L(X)$ has the sequential $DP^*_{(p,q)}$ property, and suppose for a contradiction that X has an infinite functionally bounded subset A. Since A is infinite, one can find a sequence $\{a_n\}_{n\in\omega}$ in A and a sequence $\{U_n\}_{n\in\omega}$ of open subsets of X such that $a_n \in U_n$ and $U_n \cap U_m = \emptyset$ for all distinct $n, m \in \omega$.

For every $n \in \omega$, set $\chi_n := \frac{1}{n^2} \cdot a_n$. To show that the sequence $\{\chi_n\}_{n\in\omega}$ is weakly p-summable, fix an arbitrary $f \in C(X)$. Since A is functionally bounded, there is $C > 0$ such that $|f(a)| \leq C$ for every $a \in A$. Taking into account that $|\langle f, \chi_n\rangle| = \frac{1}{n^2}|f(a_n)| \leq \frac{C}{n^2}$, it follows that the sequence $\{\chi_n\}_{n\in\omega}$ is weakly p-summable. Now, for every $n \in \omega$, choose a continuous function $f_n : X \to [0, n^2]$ such that $f_n(a_n) = n^2$ and $f(X\backslash U_n) = \{0\}$. Since U_n are pairwise disjoint and the support of any $\chi \in L(X)$ is finite, the sequence $\{f_n\}_{n\in\omega}$ is weak* q-summable. As $\langle f_n, \chi_n\rangle = 1 \not\to 0$ it follows that $L(X)$ has no the sequential $DP^*_{(p,q)}$ property, a contradiction.

Conversely, assume that X has no infinite functionally bounded subsets. Let $S = \{\chi_n\}_{n\in\omega} \subseteq L(X)$ be weakly p-summable and $\{f_n\}_{n\in\omega}$ be a weak* q-summable sequence in $L(X)' = C(X)$. Since S is bounded, Proposition 2.7 of [36] implies that the support $\mathrm{supp}(S)$ of S is finite, i.e., there are distinct $x_1, \ldots, x_m \in X$ such that for every $n \in \omega$, χ_n has a representation

$$\chi_n = a_{n,1}x_1 + \cdots + a_{n,m}x_m \quad \text{where } a_{n,1}, \ldots, a_{n,m} \in \mathbb{F}.$$

Since S is weakly null it follows that $a_{n,i} \to 0$ for each $1 \leq i \leq m$. Choose $C > 0$ such that $|f_n(x_i)| \leq C$ for all $n \in \omega$ and $1 \leq i \leq m$. Then

$$|\langle f_n, \chi_n\rangle| \leq \sum_{i=1}^{m} |a_{n,i}| \cdot |f_n(x_i)| \leq C \cdot \sum_{i=1}^{m} |a_{n,i}| \to 0.$$

Thus, $L(X)$ has the sequential $DP^*_{(p,q)}$ property. □

Theorem 1.7 of [36] implies that if X is a metrizable space, then $L(X)$ has the sequential DP property if and only if X is discrete. This result and Theorem 4 motivate the following problem.

Problem 1. *Let $p, q \in [1, \infty]$. Characterize Tychonoff spaces X for which the free locally convex space $L(X)$ has the sequential $DP_{(p,q)}$ property.*

It is known (see Proposition 5.2 of [12]) that the sequential $DP_{(p,q)}$ property is the property of the duality (E, E'). Since weakly p-summable sequences and weak* q-summable sequences depend only on the duality (E, E'), an analogous result holds true also for the sequential $DP^*_{(p,q)}$ property.

Proposition 9. *Let $p, q \in [1, \infty]$, and let (E, τ) be a locally convex space. If $\mathcal{T}$ is a locally convex topology on E compatible with τ, then the spaces (E, τ) and $(E, \mathcal{T})$ have the sequential $DP^*_{(p,q)}$ property simultaneously.*

5. The Coarse Sequential $DP^*_{(p,q)}$ Property and the Coarse DP^*_p Property

We start this section with the next characterization of spaces with the coarse sequential $DP^*_{(p,q)}$ property; it extends and generalizes Theorems 1 and 2 of [20] and has a similar proof.

Theorem 5. *Let $p, q \in [1, \infty]$. Then, for a locally convex space E, the following assertions are equivalent:*

(i) E has the coarse sequential $DP^*_{(p,q)}$ property;

(ii) Each weakly sequentially p-precompact subset of E is coarse q-limited;

(iii) Each weakly p-summable sequence in E is coarse q-limited;

(iv) Each operator $T : E \to \ell_q$ (or $T : E \to c_0$ if $q = \infty$) is p-convergent.

Proof. For simplicity, we consider only the case $q < \infty$ because the case $q = \infty$ can be considered analogously.

(i)⇒(ii) Assume that E has the coarse sequential $DP^*_{(p,q)}$ property, and suppose for a contradiction that there is a weakly sequentially p-precompact subset A of E that is not coarse q-limited. Then, there is an operator $T : E \to \ell_q$ such that $T(A)$ is not relatively compact in ℓ_q. For every $n \in \omega$, set $\chi_n := T^*(e^*_n) \in E'$. Then, by (i) of Proposition 4.17 of [13], the sequence $\{\chi_n\}_{n\in\omega}$ is weak* q-summable and $T(x) = (\langle \chi_n, x\rangle)$. Therefore, by Proposition 1, there is $\varepsilon > 0$ such that

$$\sup\Big\{ \sum_{m\le n} |\langle \chi_n, x\rangle|^q : x \in A\Big\} > (2\varepsilon)^q \quad \text{for all } \ m \in \omega.$$

For every $m \in \omega$, choose $x_m \in A$ such that $\sum_{m\le n} |\langle \chi_n, x_m\rangle|^q \ge (2\varepsilon)^q$. Since A is weakly sequentially p-precompact, we can find a weakly p-Cauchy subsequence $\{x_{m_k}\}_{k\in\omega}$ of $\{x_m\}_{m\in\omega}$. For every $k \in \omega$, choose $i_k > k$ such that $\sum_{m_{i_k}\le n} |\langle \chi_n, x_{m_k}\rangle|^q < \varepsilon^q$. Then, by the triangle inequality, we obtain

$$\Big(\sum_{m_{i_k}\le n} |\langle \chi_n, x_{m_{i_k}} - x_{m_k}\rangle|^q\Big)^{1/q} \ge \Big(\sum_{m_{i_k}\le n} |\langle \chi_n, x_{m_{i_k}}\rangle|^q\Big)^{1/q} - \Big(\sum_{m_{i_k}\le n} |\langle \chi_n, x_{m_k}\rangle|^q\Big)^{1/q} \ge \varepsilon. \tag{3}$$

On the other hand, since the sequence $S = \{x_{m_{i_k}} - x_{m_k}\}_{m\in\omega}$ is weakly p-summable, it is weakly sequentially p-compact. Therefore, S must be coarse q-limited and hence

$$\lim_{j\to\infty} \sup\Big\{ \sum_{j\le n} |\langle \chi_n, x_{m_{i_k}} - x_{m_k}\rangle|^q : \ k \in \omega\Big\} = 0$$

which contradicts (3).

(ii)⇒(iii) is trivial.

(iii)⇒(iv) Let $T : E \to \ell_q$ be an operator, and let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. By (iii), S is a coarse q-limited subset of E. Therefore, the sequence $T(S)$ is a relatively compact subset of ℓ_q. On the other hand, $T(S)$ is also a weakly null sequence. Whence, by Lemma 1, $T(x_n) \to 0$ in ℓ_q. Thus, T is a p-convergent operator.

(iv)⇒(i) Assume that each operator $T : E \to \ell_q$ is p-convergent. Let A be a relatively weakly sequentially p-compact set in E. To show that A is coarse q-limited, fix an arbitrary operator $T : E \to \ell_q$. If $S = \{x_n\}_{n\in\omega}$ is a sequence in A, take an $x \in E$ and a subsequence $\{x_{n_k}\}_{k\in\omega}$ of S which weakly p-converges to x. Since T is p-convergent, we obtain that $T(x_{n_k}) \to T(x)$ in ℓ_q. Therefore, $T(A)$ is relatively compact in ℓ_q. Thus, A is coarse q-limited, and hence E has the coarse sequential $DP^*_{(p,q)}$ property. □

Corollary 9. *Let $q \in [1,\infty]$, and let E be a locally convex space with the wCSP. If E has the coarse sequential $DP^*_{(\infty,q)}$ property, then every bounded subset of E is a coarse q-limited set.*

Proof. Let A be a bounded subset of E. Since E has the *wCSP*, each sequence in A has a weakly Cauchy subsequence, i.e., A is a weakly sequentially (∞-)precompact subset of E. Thus, by Theorem 5, A is a coarse q-limited set. □

Corollary 10. *For every $p \in [1,\infty]$, each locally convex space E has the coarse sequential $DP^*_{(p,1)}$ property.*

Proof. Let $S = \{x_n\}_{n\in\omega}$ be weakly p-summable sequence in E, and let $T : E \to \ell_1$ be an operator. Since T is weakly continuous and ℓ_1 is a Schur space, we obtain $T(x_n) \to 0$ in

ℓ_1. Therefore, S is a coarse 1-limited set. Thus, by Theorem 5, E has the coarse sequential $DP^*_{(p,1)}$ property. □

Since in angelic spaces, the class of relatively compact sets coincides with the class of relatively sequentially compact sets, we note the following assertion.

Proposition 10. *Let $p \in [1,\infty]$, and let E be a weakly angelic locally convex space. Then, E has the coarse DP^*_p property if and only if it has the coarse sequential $DP^*_{(\infty,p)}$ property.*

The next proposition shows that dense subspaces inherit the coarse DP^* type properties.

Proposition 11. *Let $p, q \in [1,\infty]$, and let H be a dense subspace of a locally convex space E. If E has the coarse DP^*_p property (resp., the coarse sequential $DP^*_{(p,q)}$ property), then also H has the same property.*

Proof. Let A be a relatively weakly compact (resp., relatively weakly sequentially p-compact) subset of H and hence of E. Therefore, by assumption, A is a coarse p-limited (resp., coarse q-limited) set in E. Let T be an operator from H to ℓ_p or to c_0 if $p = \infty$ (resp., to ℓ_q or to c_0 if $q = \infty$). Since H is dense in E, T can be extended to an operator $\bar{T}$ from E. Then, $T(A) = \bar{T}(A)$ is relatively compact in ℓ_p (resp., in c_0 or ℓ_q). Hence, A is a coarse p-limited (resp., coarse q-limited) set in H. Thus, H has the coarse DP^*_p property (resp., the coarse sequential $DP^*_{(p,q)}$ property). □

It is known (see Proposition 3.4 of [4]) that a barrelled space E has the Schur property if and only if every weak* bounded subset of E' is an ∞-(V) set. We know (see [21]) that there is a natural relationship between the $spGP_{(1,\infty)}$ property and the Schur property. For Banach spaces, an analogous relationship exists also between the coarse GP_p property, the coarse DP^*_p property and the Schur property, see Corollary 2 of [20]. Below, we generalize this result.

Proposition 12. *Let $p, q \in [1,\infty]$, and let (E,τ) be a locally convex space.*

(i) If E has both the coarse DP^*_p property and the coarse $prGP_p$ property, then E has the p-Schur property.

(ii) If E has both the coarse sequential $DP^*_{(p,q)}$ property and the coarse $prGP_q$ property, then E has the p-Schur property.

(iii) If E has the p-Schur property, then E has the coarse sequential $DP^*_{(p,q)}$ property.

(iv) If $2 \le p \le \infty$ and E is a weakly angelic locally convex space with the Rosenthal property (for example, E is a strict (LF)-space), then E has the Schur property if and only if it has the coarse DP^*_p property and the coarse $prGP_p$ property.

(v) If E has the Glicksberg property, then E has the coarse DP^*_p property.

Proof. (i) Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, by the coarse DP^*_p property, S is a coarse p-limited set. Therefore, by the coarse $prGP_p$ property, S is precompact in E. Then, by Lemma 1, S is τ-null. Thus E has the p-Schur property.

(ii) Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, by the coarse sequential $DP^*_{(p,q)}$ property, S is a coarse q-limited set. Therefore, by the coarse $prGP_q$ property, S is precompact in E. Then, by Lemma 1, S is τ-null. Thus E has the p-Schur property.

(iii) Let A be a relatively weakly sequentially p-compact set in E. To show that A is coarse q-limited, let $T \in \mathcal{L}(E,\ell_q)$ (or $T \in \mathcal{L}(E,c_0)$ if $q = \infty$). We have to prove that the image $T(A)$ is precompact. To this end, let $\{a_n\}_{n\in\omega}$ be a sequence in A. Since A is relatively weakly sequentially p-compact, there is a subsequence $\{a_{n_k}\}_{k\in\omega}$ of $\{a_n\}_{n\in\omega}$, which weakly p-converges to some $x \in E$. By the p-Schur property, we have $a_{n_k} \to x$ in E. Therefore, $T(a_{n_k}) \to T(x)$, which means that $T(A)$ is relatively sequentially compact and hence precompact in ℓ_q (or in c_0), as desired.

(iv) To prove the necessity, we prove first that E has the coarse DP_p^* property. Let A be a relatively weakly compact subset of E. Since E is weakly angelic, A is a relatively weakly sequentially compact subset of E. Then, every sequence $\{a_n\}_{n\in\omega}$ in A has a subsequence $\{a_{n_k}\}_{k\in\omega}$ that weakly converges to some $x \in E$. To show that A is coarse p-limited, let $T \in \mathcal{L}(E, \ell_p)$ (or $T \in \mathcal{L}(E, c_0)$ if $p = \infty$). Then, the Schur property of E implies that $a_{n_k} \to x$ in E, and hence $T(a_{n_k}) \to T(x)$ in ℓ_p (or in c_0). Thus, $T(A)$ is a relatively sequentially compact set, and hence, E has the coarse DP_p^* property. To show that E has the coarse $prGP_p$ property, let $\{x_n\}_{n\in\omega}$ be a weakly null coarse p-limited sequence in E. By the Schur property, we have $x_n \to 0$ in E. Therefore, by (iv) of Theorem 4.5 of [21], the space E has the coarse $prGP_p$ property.

To prove the sufficiency, let $S = \{x_n\}_{n\in\omega}$ be a weakly null sequence in E. Since S is relatively weakly compact, the coarse DP_p^* property implies that S is a coarse p-limited set. Then, the coarse $prGP_p$ property implies that S is precompact in E. Applying Lemma 1, we obtain that S is τ-null. Thus, E has the Schur property.

(v) Let $T \in \mathcal{L}(E, \ell_p)$ (or $T \in \mathcal{L}(E, c_0)$ if $p = \infty$), and let K be a weakly compact subset of E. By the Glicksberg property, K is a compact subset of E, and hence, $T(K)$ is a compact subset of ℓ_p (or c_0). Thus, K is a coarse p-limited set. □

Proposition 13. *Let $p, q \in [1, \infty]$, and let E be a weakly angelic space. If E has the coarse DP_q^* property, then E has the coarse sequential $DP_{(p,q)}^*$ property.*

Proof. Let A be a relatively weakly sequentially p-compact set in E. Then A is relatively weakly sequentially compact, and hence, by the weak angelicity of E, A is a relatively weakly compact set in E. By the coarse DP_q^* property of E, A is coarse q-limited. Thus, E has the coarse sequential $DP_{(p,q)}^*$ property. □

Theorem 6. *Let $s, q \in [1, \infty]$ and $1 < p < \infty$.*

(i) ℓ_p has the coarse DP_s^* property if and only if $s < p$.
(ii) ℓ_p has the coarse sequential $DP_{(s,q)}^*$ property if and only if either $q < p$ or $q \geq p$ and $s < p^*$.

Proof. (i) Observe that B_{ℓ_p} is a weakly compact subset of the reflexive space ℓ_p. If $s \geq p$ and $I_s : \ell_p \to \ell_s$ is the identity inclusion, then $I_s(B_{\ell_p})$ is not relatively compact in ℓ_s. Therefore, ℓ_p does not have the coarse DP_s^* property. If $s < p$, then, by the Pitt theorem, each operator $T : \ell_p \to \ell_s$ is compact and hence B_{ℓ_p} is coarse s-limited. Thus, ℓ_p has the coarse DP_s^* property.

(ii) Assume that $q < p$. Then, by (i), ℓ_p has the coarse DP_q^* property. Thus, by Proposition 13, ℓ_p has the coarse sequential $DP_{(s,q)}^*$ property.

Assume that $q \geq p$ and $s \geq p^*$. By Proposition 1.4 of [6] (or by Corollary 13.11 of [13]), B_{ℓ_p} is weakly sequentially p^*-compact. Since $s \geq p^*$, it follows that B_{ℓ_p} is weakly sequentially s-compact. As $q \geq p$, it follows from the proof of (i) that B_{ℓ_p} is not coarse q-limited. Thus, ℓ_p does not have the coarse sequential $DP_{(s,q)}^*$ property.

Assume that $q \geq p$ and $s < p^*$. To show that ℓ_p has the coarse sequential $DP_{(s,q)}^*$ property, it suffices to prove that each relatively weakly sequentially s-compact subset A of ℓ_p is (norm) precompact. Suppose for a contradiction that there is a relatively weakly sequentially s-compact subset A of ℓ_p that is not precompact. Then, there is a sequence $S = \{x_n\}_{n\in\omega}$ in A and $\varepsilon > 0$ such that $\|x_n - x_m\|_{\ell_p} \geq \varepsilon$ for all distinct $n, m \in \omega$. Without loss of generality, we assume that $\{x_n\}_{n\in\omega}$ weakly s-converges to some $x \in \ell_p$. Hence,

$$\|(x_n - x) - (x_m - x)\|_{\ell_p} \geq \varepsilon$$

for all distinct $n, m \in \omega$. For every $n \in \omega$, set $b_n := \|x_n - x\|_{\ell_p}$. Observe that if $\|x_{n_k} - x\|_{\ell_p} \to 0$ for some sequence $(n_k) \subseteq \omega$, then

$$\|x_{n_k} - x_{n_j}\|_{\ell_p} \leq \|x_{n_k} - x\|_{\ell_p} + \|x_{n_j} - x\|_{\ell_p} \to 0$$

which contradicts the choice of S. Therefore, there is $\lambda > 1$ such that

$$\tfrac{1}{\lambda} \le b_n = \|x_n - x\|_{\ell_p} \le \lambda \quad \text{for every } n \in \omega. \tag{4}$$

Then (4) implies that the normalized sequence $\{\frac{1}{b_n}(x_n - x)\}_{n\in\omega}$ is weakly s-summable.

Proposition 2.1.3 of [38] implies that there are a basic subsequence $\{x_{n_k} - x\}_{k\in\omega} \subseteq \ell_p$ of $\{x_n - x\}_{n\in\omega}$ and a linear topological isomorphism $R : \overline{\{x_{n_k} - x\}_{k\in\omega}}^{\ell_p} \to \ell_p$ such that

$$R(x_{n_k} - x) = a_k e_k \quad \text{for every } k \in \omega, \text{ where } a_k := b_{n_k} = \|x_{n_k} - x\|_{\ell_p},$$

and such that the subspace $\overline{\{x_{n_k} - x\}_{k\in\omega}}^{\ell_p}$ is complemented in ℓ_p. In particular, the canonical basis $\{\frac{1}{a_k}R(x_{n_k} - x) = e_k\}_{k\in\omega}$ of ℓ_p is weakly s-summable. However, since $s < p^*$, $\{e_k\}_{k\in\omega}$ is not weakly s-summable (see Example 4.4 of [13]). This contradiction finishes the proof. □

Remark 2. *(i) Quotients of spaces with the coarse (sequential) DP^*_p property may not have this property. Indeed, let $1 < p \le q < \infty$, and consider the Banach spaces $E = \ell_1$ and $H = \ell_p$. Then H is a quotient of E. By (iii) and (v) of Proposition 12, the space E has the coarse sequential $DP^*_{(p^*,q)}$ property and the coarse DP^*_p property. However, by Theorem 6, the space H has neither the coarse DP^*_p property nor the coarse sequential $DP^*_{(p^*,q)}$ property.*

(ii) Closed subspaces of spaces with the coarse DP^*_p property or the coarse sequential $DP^*_{(s,p)}$ property may not have these properties. Indeed, let $p, s \in [2, \infty]$ and let $E = L_1[0, 1]$. It was noticed after Proposition 7 of [20] that E has the coarse DP^*_p property (however, E does not have the DP^* property). Therefore, by Proposition 13, E has the coarse sequential $DP^*_{(s,p)}$ property. On the other hand, by Proposition 6.4.2 of [38], E contains a closed subspace H isomorphic to ℓ_2. If, in addition, $s \ge 2$, Theorem 6 implies that H has neither the coarse DP^*_p property nor the coarse sequential $DP^*_{(s,p)}$ property.

(iii) The coarse DP^*_p property and the coarse sequential $DP^*_{(p,q)}$ property depend on the duality (E, E') (cf. Proposition 9). Indeed, let $p, r \in (1, \infty)$. Consider the spaces $H = \ell_p$ and H_w. Then, by (i), H has neither the coarse DP^*_p property nor the coarse sequential $DP^*_{(p^*,q)}$ property. On the other hand, since any operator $T : H_w \to \ell_r$ is finite-dimensional it follows that any bounded subset of H_w is coarse r-limited. Thus, H_w has the coarse DP^*_p property and the coarse sequential $DP^*_{(p^*,q)}$ property.

(iv) Let $1 < p < 2$ (so $p < p^*$) and $q \in [1, \infty]$. Then, by Proposition 5.10 of [12], ℓ_p has the quasi DP_p property and the sequential $DP_{(p,q)}$ property. Hence, by the reflexivity, ℓ_p has the sequential $DP^*_{(p,q)}$ property. By Theorem 6, ℓ_p has the coarse sequential $DP^*_{(p,q)}$ property. However, if additionally $q \ge p$, Theorem 6 implies that ℓ_p does not have the coarse DP^*_q property.

(v) Let $2 < r < \infty$, $r^* \le p < r$ and $q < r$. Then, by Proposition 5.10 of [12], ℓ_r has neither the quasi DP_p property nor the sequential $DP_{(p,q)}$ property. On the other hand, by Theorem 6, the Banach space ℓ_r has the coarse DP^*_p property and the coarse sequential $DP^*_{(p,q)}$ property.

Below we show that the classes of locally convex spaces with the coarse sequential $DP^*_{(p,q)}$ property and the coarse DP^*_p property are stable with respect to direct products and direct sums.

Proposition 14. *Let $p, q \in [1, \infty]$, and let $\{E_i\}_{i\in I}$ be a nonempty family of locally convex spaces.*

(i) $E = \prod_{i\in I} E_i$ has the coarse DP^*_p property (resp., the coarse sequential $DP^*_{(p,q)}$ property) if and only if for every $i \in I$, the factor E_i has the same property.

(ii) $E = \bigoplus_{i\in I} E_i$ has the coarse DP^*_p property (resp., the coarse sequential $DP^*_{(p,q)}$ property) if and only if for every $i \in I$, the summand E_i has the same property.

Proof. Let E has the coarse DP_p^* property (resp., the coarse sequential $DP_{(p,q)}^*$ property). Fix an arbitrary $j \in I$, and let A_j be a relatively weakly compact (resp., relatively weakly sequentially p-compact) subset of E_j. Considering E_j as a direct summand of E, we see that A_j is relatively weakly compact (resp., relatively weakly sequentially p-compact) in E. Therefore, by the coarse DP_p^* property (resp., the coarse sequential $DP_{(p,q)}^*$ property), the set A_j is coarse p-limited (resp., coarse q-limited) in E and hence also in E_j (see (iii) of Lemma 4.1 of [28]). Thus, E_j has the coarse DP_p^* property (resp., the coarse sequential $DP_{(p,q)}^*$ property).

Conversely, assume that all spaces E_i have the coarse DP_p^* property (resp., the coarse sequential $DP_{(p,q)}^*$ property). Let A be a relatively weakly compact (resp., relatively weakly sequentially p-compact) subset of E. Then, each projection A_i of A onto E_i is relatively weakly compact (resp., relatively weakly sequentially p-compact) in E_i, and for the case (ii), all but finitely many of A_i are equal to zero. By assumption, all A_i are coarse p-limited (resp., coarse q-limited). Therefore, by Proposition 4.3 of [28], the set A is coarse p-limited (resp., coarse q-limited). Thus, E has the coarse DP_p^* property (resp., the coarse sequential $DP_{(p,q)}^*$ property). □

Now, we consider the coarse DP_p^* property separately. We start from the next proposition, which shows that the case $p = 1$ is not of interest.

Proposition 15. *Each locally convex space E has the coarse DP_1^* property.*

Proof. Let A be a relatively weakly compact subset of E. To show that A is coarse 1-limited, let $T \in \mathcal{L}(E, \ell_1)$. Since T is weak-weak continuous, $T(A)$ is relatively weakly compact in the Schur space ℓ_1, and hence, $T(A)$ is a relatively compact subset of ℓ_1. Thus, A is coarse 1-limited. □

Following [4], an lcs E has the *Krein property* if $\overline{\text{acx}}(K)$ is weakly compact for every weakly compact subset K of E. By Proposition 2.12 of [4], E is a Krein space if and only if $(E', \mu(E', E))$ is a subspace of $C_k(E_w)$. Note also that, by the Krein theorem ([39], § 24.5(4)), if $(E, \mu(E, E'))$ is quasi-complete, then E is a Krein space. In particular, every quasibarrelled quasi-complete space (for example, each strict (LF)-space) is a Krein space. Below we characterize locally convex spaces with the coarse DP_p^* property; this result extends and generalizes Theorem 4 of [20].

Theorem 7. *Let $1 < p \leq \infty$. For a locally convex space E with the Krein property, the following assertions are equivalent:*

(i) *E has the coarse DP_p^* property;*

(ii) *For every locally convex space L, each operator $T : L \to E$, which transforms bounded subsets of L into relatively weakly compact sets in E and transforms bounded sets in L into coarse p-limited subsets of E;*

(iii) *For every normed space L, each weakly compact operator $T : L \to E$ is coarse p-limited;*

(iv) *each weakly compact operator $T : \ell_1^0 \to E$ is coarse p-limited.*

If, in addition, E is locally complete, then (i)-(iv) *are equivalent to the following*

(v) *Each weakly compact operator $T : \ell_1 \to E$ is coarse p-limited.*

Proof. (i)⇒(ii) Assume that E has the coarse DP_p^* property, and let $T : L \to E$ be an operator that transforms bounded subsets of L into relatively weakly compact sets in E. Fix an arbitrary bounded subset A of L. Then, $T(A)$ is a relatively weakly compact subset of E, and hence, by the coarse DP_p^* property, $T(A)$ is a coarse p-limited set.

The implications (ii)⇒(iii)⇒(iv) and (iii)⇒(v) are obvious.

(iv)⇒(i) and (v)⇒(i): Let A be a relatively weakly compact subset of E. We have to prove that A is coarse p-limited. By Lemma 4.1(iv) of [28], we can assume that $A = \{x_n\}_{n\in\omega}$

is countable. Since A is a bounded sequence, Proposition 14.9 of [13] implies that the linear map $T : \ell_1^0 \to E$ (or $T : \ell_1 \to E$ if E is locally complete) defined by

$$T(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \ldots, a_n \in \mathbb{F})$$

which is continuous. It is clear that $T(B_{\ell_1^0}) \subseteq \overline{\operatorname{acx}}(A)$ (or $T(B_{\ell_1}) \subseteq \overline{\operatorname{acx}}(A)$ if E is locally complete). Since $\overline{A}^w$ is weakly compact, the Krein property of E implies that the set $\overline{\operatorname{acx}}(A)$ is weakly compact as well. Therefore, T is a weakly compact operator, and hence, by (iv) or (v), T is a coarse p-limited operator. Thus, the set $T(\{e_n\}_{n\in\omega}) = A$ is coarse p-limited, as desired. □

6. p-Dunford–Pettis Sequentially Compact Property of Order (q', q)

We start from the following characterization of spaces with the p-$DPscP_{(q',q)}$.

Theorem 8. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$. Then for a locally convex space (E, τ), the following assertions are equivalent:*

(i) *E has the p-$DPscP_{(q',q)}$;*
(ii) *Each operator $T \in \mathcal{L}(E, \ell_\infty)$ is (q', q)-(V^*) p-convergent;*
(iii) *Each weakly (p, q)-convergent operator is (q', q)-(V^*) p-convergent.*

Proof. (i)⇒(ii) Assume that E has the p-$DPscP_{(q',q)}$, and let $\{x_n\}_{n\in\omega}$ be weakly p-summable sequence in E which is a (q', q)-(V^*) set. By the p-$DPscP_{(q',q)}$, we have $x_n \to 0$ in E. Therefore, also $T(x_n) \to 0$ in ℓ_∞. Thus, T is a (q', q)-(V^*) p-convergent operator.

(ii)⇒(i) Assume that each operator $T \in \mathcal{L}(E, \ell_\infty)$ is (q', q)-(V^*) p-convergent. To show that E has the p-$DPscP_{(q',q)}$, let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E which is a (q', q)-(V^*) set. Assuming that $x_n \not\to 0$ in E and passing to a subsequence if needed, we can assume that there is $U \in \mathcal{N}_0^c(E)$ such that $x_n \notin U$ for all $n \in \omega$. For every $n \in \omega$, choose $\chi_n \in U^\circ$ such that $\langle \chi_n, x_n \rangle > 1$. Define a linear map $T : E \to \ell_\infty$ by

$$T(x) := \big(\langle \chi_n, x \rangle\big)_n \quad (x \in E).$$

Since $T(U) \subseteq B_{\ell_\infty}$, the map T is bounded and hence continuous. As $\|T(x_n)\|_{\ell_\infty} \geq \langle \chi_n, x_n \rangle > 1 \not\to 0$, it follows that T is not (q', q)-(V^*) p-convergent. This is a contradiction.

(ii)⇔(iii) immediately follows from Proposition 3. □

Theorem 8 motivates solving the problem of characterizing (q', q)-(V^*) p-convergent operators. If E and L are Banach spaces and $q' = q = \infty$, the next theorem is proved in Theorem 1.1 of [23].

Theorem 9. *Let $2 \leq q' \leq q \leq \infty$, and let E be a locally convex space with the Rosenthal property. Then, for an operator T from E to a locally convex space L, the following assertions are equivalent:*

(i) *T is (q', q)-(V^*) ∞-convergent;*
(ii) *For each (q', q)-(V^*) set $A \subseteq E$, the image $T(A)$ is sequentially precompact in L;*
(iii) *For every locally convex (the same, normed) space H and each (q', q)-(V^*) operator $R : H \to E$, the operator $T \circ R$ is sequentially precompact;*
(iv) *For each (q', q)-(V^*) operator $R : \ell_1^0 \to E$, the operator $T \circ R$ is sequentially precompact.*

If, in addition, E is locally complete, then (i)–(iv) *are equivalent to the following:*

(v) *For each (q', q)-(V^*) operator $R : \ell_1 \to E$, the operator $T \circ R$ is sequentially precompact.*

Proof. (i)⇒(ii) Assume that $T \in \mathcal{L}(E, L)$ is (q', q)-(V^*) ∞-convergent, and let A be a (q', q)-(V^*) subset of E. To show that $T(A)$ is sequentially precompact in L, let $\{a_n\}_{n\in\omega}$ be a sequence in A. Since, by Theorem 1, A is weakly sequentially precompact, we can assume that $\{a_n\}_{n\in\omega}$ is weakly Cauchy. Therefore, for every strictly increasing sequence $\{n_k\}_{k\in\omega}$ in

ω, the sequence $\{a_{n_{k+1}} - a_{n_k}\}_{k\in\omega}$ is weakly null and also a (q', q)-(V^*) set. Since T is (q', q)-(V^*) ∞-convergent it follows that $T(a_{n_{k+1}}) - T(a_{n_k}) \to 0$ in L, and hence the sequence $\{T(a_{n_k})\}_{k\in\omega}$ is Cauchy in L. Thus, $T(A)$ is sequentially precompact in E.

(ii)⇒(iii) Take $U \in \mathcal{N}_0(H)$ such that $R(U)$ is a (q', q)-(V^*) subset of E. By (ii), we have $T(R(U))$ is sequentially precompact in L. Thus, $T \circ R$ is sequentially precompact.

(iii)⇒(iv) and (iii)⇒(v) are obvious.

(iv)⇒(i) and (v)⇒(i): Let $S = \{x_n\}_{n\in\omega}$ be a weakly null sequence in E which is a (q', q)-(V^*) set. Since S is bounded, by Proposition 14.9 of [13], the linear map $R : \ell_1^0 \to E$ (or $R : \ell_1 \to E$ if E is locally complete) defined by

$$R(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \ldots, a_n \in \mathbb{F}),$$

which is continuous. It is clear that $R(B_{\ell_1^0}) \subseteq \overline{\text{acx}}(S)$ (or $R(B_{\ell_1}) \subseteq \overline{\text{acx}}(S)$ if E is locally complete). Observe that $\overline{\text{acx}}(S)$ is also a (q', q)-(V^*) set. Therefore, the operator R is (q', q)-(V^*), and hence, by (iv) or (v), $T \circ R$ is sequentially precompact. In particular, the weakly null sequence $T(S)$ is (sequentially) precompact in L. Therefore, by Lemma 1, $T(x_n) \to 0$ in L. Thus T is (q', q)-(V^*) ∞-convergent. □

Below, we obtain a sufficient condition on locally convex spaces to have the p-$DPscP_q$.

Proposition 16. *Let $p, q \in [1, \infty]$, and let E be a locally convex space. Then, E has the p-$DPscP_q$ if one of the following conditions holds:*

(i) For every locally convex space X and for each $T \in \mathcal{L}(X, E)$ whose adjoint $T^* : E'_\beta \to X'_\beta$ is q-convergent, the operator T transforms bounded subsets of X into sequentially precompact sets in E;

(ii) For every normed space X and for each $T \in \mathcal{L}(X, E)$ whose adjoint $T^* : E'_\beta \to X'_\beta$ is q-convergent, the operator T is sequentially precompact;

(iii) The same as (ii) *with $X = \ell_1^0$;*

(iv) If E is locally complete, the same as (ii) *with X a Banach space;*

(v) If E is locally complete, the same as (ii) *with $X = \ell_1$;*

Proof. Since the implications (i)⇒(ii)⇒(iii) and (i)⇒(ii)⇒(iv)⇒(v) are obvious, it suffices to prove that (iii) and (v) imply the p-$DPscP_q$. Suppose for a contradiction that E has no the p-$DPscP_q$. Then there is a weakly p-summable sequence $S = \{x_n\}_{n\in\omega}$ in E which is a q-(V^*) set such that $x_n \not\to 0$ in E. Without loss of generality, we assume that $x_n \notin U$ for some $U \in \mathcal{N}_0(E)$ and all $n \in \omega$.

Since S is a q-(V^*) set, Proposition 14.9 of [13] implies that the linear map $T : \ell_1^0 \to E$ (or $T : \ell_1 \to E$ if E is locally complete) defined by

$$T(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \ldots, a_n \in \mathbb{F}),$$

is continuous and its adjoint $T^* : E'_\beta \to \ell_\infty$ is q-convergent. Therefore, by (iii) or (v), the operator T is sequentially precompact. In particular, the set $T(\{e_n\}_{n\in\omega}) = S$ is (sequentially) precompact in E. Since S is also weakly null, Lemma 1 implies that $x_n \to 0$ in E. However, this contradicts the choice of S. □

Theorem 10. *Let $2 \le q \le \infty$. Then for a locally convex space E with the Rosenthal property, the following assertions are equivalent:*

(i) E has the $DPscP_q$;

(ii) For every normed space X and each $T \in \mathcal{L}(X, E)$ whose adjoint $T^* : E'_\beta \to X'_\beta$ is q-convergent, the operator T is sequentially precompact;

(iii) The same as (ii) with $X = \ell_1^0$;

If, in addition, E is locally complete, then (i)–(iii) are equivalent to

(iv) The same as (ii) with X a Banach space;

(v) The same as (ii) with $X = \ell_1$;

Proof. By (the proof of) Proposition 16, we only have to prove the implication (i)⇒(ii). Therefore, let X be a normed space, and let $T \in \mathcal{L}(X, E)$ be such that the adjoint $T^* : E'_\beta \to X'_\beta$ is q-convergent. We have to show that the set $T(B_X)$ is sequentially precompact in E. To this end, let $\{\chi_n\}_{n\in\omega}$ be a weakly q-summable sequence in E'_β. Since T^* is q-convergent, we have

$$\sup_{y\in B_X} |\langle \chi_n, T(y)\rangle| = \sup_{y\in B_X} |\langle T^*(\chi_n), y\rangle| = \|T^*(\chi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, $T(B_X)$ is a q-(V^*) set in E. Since $q \geq 2$ and E has the Rosenthal property, Theorem 1 implies that $T(B_X)$ is weakly sequentially precompact. To show that $T(B_X)$ is sequentially precompact in E, it suffices to prove that for any sequence $S = \{x_n\}_{n\in\omega}$ in B_X, the image $T(S)$ has a Cauchy subsequence. To this end, taking into account that $T(B_X)$ is weakly sequentially precompact, we can assume that $T(S)$ is weakly Cauchy. For any strictly increasing sequence $\{n_k\}_{k\in\omega}$ in ω, the sequence $\{T(x_{n_k}) - T(x_{n_{k+1}})\}_{k\in\omega}$ is weakly null and a q-(V^*) set. Therefore, by the $DPscP_q$ of E, we obtain $T(x_{n_k}) - T(x_{n_{k+1}}) \to 0$ in E. Therefore, $\{T(x_{n_k})\}_{k\in\omega}$ is Cauchy in E. □

Although, by Example 2 and (i) of Remark 2, the classes of spaces with the sequential $DP^*_{(p,q)}$ property and with the coarse (sequential) DP^*_p property are not closed under taking even closed subspaces, and the class of spaces with the p-$DPscP_{(q',q)}$ is stable under taking arbitrary subspaces.

Proposition 17. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$, and let E be a locally convex space with the p-$DPscP_{(q',q)}$. Then any subspace H of E has the p-$DPscP_{(q',q)}$.*

Proof. Let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in H which is a (q', q)-(V^*) set. Then, it is also a weakly p-summable sequence in E, which is a (q', q)-(V^*) set (see (iii) of Lemma 4.6 and (iv) of Lemma 7.2 of [13]). Therefore, $x_n \to 0$ in E, and hence $x_n \to 0$ also in H. Thus, H has the p-$DPscP_{(q',q)}$. □

The class of locally convex spaces with the p-$DPscP_{(q',q)}$ is stable also under taking direct products and direct sums.

Proposition 18. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$, and let $\{E_i\}_{i\in I}$ be a non-empty family of locally convex spaces. Then, the following assertions are equivalent:*

(i) *$\prod_{i\in I} E_i$ has the p-$DPscP_{(q',q)}$;*
(ii) *$\bigoplus_{i\in I} E_i$ has the p-$DPscP_{(q',q)}$;*
(iii) *For every $i \in I$, the space E_i has the p-$DPscP_{(q',q)}$.*

Proof. (i)⇒(ii) Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in $\bigoplus_{i\in I} E_i$, which is a (q', q)-(V^*) set. Then, S has finite support F. By (i) and Proposition 17, the subspace $\prod_{j\in F} E_j$ of $\prod_{i\in I} E_i$ has the p-$DPscP_{(q',q)}$. Therefore, $x_n \to 0$ in $\prod_{j\in F} E_j$ and hence also in $\bigoplus_{i\in I} E_i$. Thus, $\bigoplus_{i\in I} E_i$ has the p-$DPscP_{(q',q)}$.

(ii)⇒(iii) Since E_i is isomorphic to a subspace of $\bigoplus_{i\in I} E_i$, it has the p-$DPscP_{(q',q)}$ by Proposition 17.

(iii)⇒(i) Let $\{x_n = (x_{i,n})_{i\in I}\}_{n\in\omega}$ be a weakly p-summable sequence in $\prod_{i\in I} E_i$, which is a (q', q)-(V^*) set. Then, by (iii) of Lemma 4.6 and (iv) of Lemma 7.2 of [13], for every $i \in I$, the sequence $\{x_{i,n}\}_{n\in\omega}$ is a weakly p-summable sequence in E_i, which is a (q', q)-(V^*) set. Therefore, by (iii), $x_{i,n} \to 0_i$ for every $i \in I$. Hence, $x_n \to 0$ in E. Thus $\prod_{i\in I} E_i$ has the p-$DPscP_{(q',q)}$. □

Propositions 17 and 18 immediately imply the next corollary.

Corollary 11. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$, and let E be a dense subspace of $\mathbb{F}^\kappa$ for some cardinal κ. Then E has the p-$DPscP_{(q',q)}$. In particular, for every locally convex space L and for each Tychonoff space X, the spaces L_w and $C_p(X)$ have the p-$DPscP_{(q',q)}$.*

Let $p, q, q' \in [1, \infty]$, $q' \leq q$. Following [21], a locally convex space E has the p-$GPscP_{(q',q)}$, if any weakly p-summable sequence in E which is a (q', q)-limited set is a null sequence in E.

Proposition 19. *Let $p, q, q' \in [1, \infty]$, $q' \leq q$, and let E be a locally convex space.*

(i) If E has the p-Schur property (for example, $E = E_w$), then E has the p-$DPscP_{(q',q)}$.

(ii) If E has the p-$DPscP_{(q',q)}$, then it has the p-$GPscP_{(q',q)}$. The converse is true if E is semi-reflexive, but not in general.

Proof. (i) is trivial.

(ii) follows from the two easy facts that any (q', q)-limited set is a (q', q)-(V^*) set, and if E is semi-reflexive, then the classes of (q', q)-limited sets and (q', q)-(V^*) sets are coincide.

To show that in general the converse is not true, let $q = q' = \infty$ and $E = c_0$. We claim that E does not have the p-$DPscP_{(q,\infty)}$. Indeed, first we observe that B_E is a q-(V^*) set because any weakly null sequence in $E'_\beta = \ell_1$ is norm null. Therefore, the unit canonical basis $S = \{e_n\}_{n\in\omega}$ is also a q-(V^*) set. Observe also that S is weakly p-summable (if $\chi = (a_n) \in \ell_1$, then $\sum_{n\in\omega} |\langle \chi, e_n\rangle|^p = \sum_{n\in\omega} |a_n|^p < \infty$ or $|\langle \chi, e_n\rangle| = |a_n| \to 0$ if $p = \infty$). Since $e_n \not\to 0$ in E, we obtain that E does not have the p-$DPscP_{(q,\infty)}$. On the other hand, since E has the Gelfand–Phillips property, any weakly p-summable sequence Q that is limited is precompact in E, and hence, by Lemma 1, Q is a null sequence in E. Therefore, E has the p-$GPscP$. □

Remark 3. *Note that Lemma 1 implies that an lcs E has the p-$DPscP_{(q',q)}$ or the p-$GPscP_{(q',q)}$ if and only if every weakly p-summable sequence, which is a (q', q)-(V^*) set or a (q', q)-limited set, respectively, is precompact in E.*

7. Conclusions

In this work, we obtained several new characterizations of locally convex spaces with the sequential Dunford–Pettis property of order (p, q), where $p, q, q' \in [1, \infty]$, $q' \leq q$. Being motivated by known Dunford–Pettis type properties widely studied in the Banach Space Theory, we introduced and studied new types of the sequential Dunford–Pettis property in the realm of all locally convex spaces. Numerous characterizations of the introduced notions have been given. For future research, we suggest studying the introduced notions in some important subclasses of Banach, Fréchet or strict (LF)-spaces as well as in the classes of function spaces or free locally convex spaces.

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