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Topological Degree via a Degree of Nondensifiability and Applications

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Abstract: The goal of this work is to introduce the notion of topological degree via the principle of the degree of nondensifiability (DND for short). We establish some new fixed point theorems, concerning, Schaefer's fixed point theorem and the nonlinear alternative of Leray–Schauder type. As applications, we study the existence of mild solution of functional semilinear integro-differential equations.

Keywords: degree of nondensifiability; α-dense curves; DND; topological degree; $k-\varphi_d-$contraction; resolvent operator; mild solution

MSC: 47H10; 47H30; 54H25

1. Introduction

Fixed point theory is a very active branch of mathematics, and plays a circular role in nonlinear analysis, since it is used for establishing the existence of solutions for many nonlinear problems arising in differential equations and inclusions in physics, economics, mechanics, and biology [1–4]. In fact, in many real problems, we seek solutions as fixed points of the original problem using hypotheses on the single and multivalued mappings involved in the problem or on the structure of the corresponding Banach space.

In 1930, Kuratowski [5] introduced the concept of measure of noncompactness (MNC), and this technique was used in functional analysis. After that, Darbo [6] developed a result on fixed point theory by using the concept of MNC and the generalized Banach principle of contraction [7]. The concept of MNC and its applications have been generalized in different directions, see [8–11] for example.

By using the notion of topological degree introduced by Brouwer [12], Leray–Schauder [13] defined this concept for compact perturbation of the identity map. Using the theory of measure of noncompactness, the different generalization of the Leray and Schauder degree was given by Nussbaum [14–16].

The theory of fixed point for multivalued applications is an important topic in set-valued analysis. For its developments and applications, one can see [17–19].

In 1997, Mora and Cherruault [20] introduced the concept of the α-dense curve and densifiable set in metric spaces. This notion is a generalization of a space-filing curve (see [21]), and the class of densifiable sets is strictly comprised between the class of Peano continua and the class of connected and precompacts sets (for more information, see [22–24]).

Very recently, several authors have proven some fixed point theorems by using the concept of degree of nondensifiability based on α-dense curves, which is an alternative method to *MNC* to obtain fixed point results (see, e.g., [25,26] and the references therein).

In [27,28], Garcia gives a version Schauder fixed point theorem via DND. This paper contains a new approach to topological degree theory by introducing the concept of "degree

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Citation: Ouahab, N.; Nieto, J.J.; Ouahab, A. Topological Degree via a Degree of Nondensifiability and Applications. *Axioms* **2024**, *13*, 482. https://doi.org/10.3390/axioms13070482

Academic Editors: Iñigo Sarría and Vladimir Rakocevic

Received: 11 June 2024
Revised: 7 July 2024
Accepted: 15 July 2024
Published: 18 July 2024

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of nondensifiability" (DND). Utilizing DND, we established some novel fixed point theorems, including a variant of the Leray–Schauder nonlinear alternative and a new version of Schaefer's fixed point theorem.

The goal of this work is to introduce the topological degree by using the concept of degree of nondensifiability. This research paper is structured as follows: Section 2 provides the definitions, notations, basic propositions, and theorems from the literature that are used throughout this paper. In Section 3, by using the degree of nondensifiability, we introduced the topological degree of Leray–Schauder type. As an application, we prove Schaefer's fixed points theorems and nonlinear alternative of Leray and Schauder. In Section 4, we apply our results to a functional semi linear integro-differential equations.

2. Preliminaries

In the first part of this work, we give several notations, definitions, and preliminary results facts that are used later.

Let X be a metric space (or normed space), and set $\mathcal{P}_*(X) = \{A \subset X\colon\ A \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}(X)\colon\ A \text{ closed}\}$, $\mathcal{P}_b(X) = \{A \in \mathcal{P}_*(X)\colon\ A \text{ bounded}\}$, $\mathcal{P}_{cv}(X) = \{A \in \mathcal{P}_*(X)\colon\ A \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{A \in \mathcal{P}_*(X)\colon\ A \text{ compact}\}$, and $\mathcal{P}_{arc}(X) = \{A \in \mathcal{P}_*(X)\colon\ A \text{ path-connected}\}$.

Let X and Y be two topological spaces, and $\mathcal{S} : X \to \mathcal{P} * (Y)$ be a multifunction. A single-valued function $h : X \to Y$ is called be a *selection* of $\mathcal{S}$, and we write $h \subset \mathcal{S}$ whenever

$$h(x) \in \mathcal{S}(x) \ \text{ for each } x \in X.$$

$\mathcal{S}$ is considered lower semi-continuous (*l.s.c.*) if, for each $x_0 \in X$, the set $\mathcal{S}(x_0)$ is a nonempty subset of Y, and if, for every open subset $\mathcal{O}$ of Y, such that $\mathcal{S}(x_0) \cap \mathcal{O} \neq \emptyset$, there exists an open U, such that $x_0 \in U$,

$$\mathcal{S}(a) \cap \mathcal{O} \neq \emptyset \ \text{ for every } a \in U.$$

Proposition 1. *Let $\mathcal{S} : X \to \mathcal{P}(Y)$ be a multivalued mapping. Then, the following statements are equivalent:*

1. *S is l.s.c.*
2. *For every open subset $\mathcal{O}$ in Y the sunset*

$$S_-^{-1}(\mathcal{O}) = \{a \in X : \mathcal{S}(a) \cap \mathcal{O} \neq \emptyset\},$$

   is an open subset of X.
3. *For all closed sunset $\mathcal{C}$ in Y the set*

$$\mathcal{S}_+^{-1}(\mathcal{C}) = \{a \in X : \mathcal{S}(a) \subset \mathcal{C}\},$$

   is any closed subset of X.

The concept of the measure of noncompactness permits us to characterize and compare the noncompactness of such sets; for more details in this direction, we refer the readers to [8–11].

Definition 1. *Consider a complete metric space $(\widetilde{X}, d)$. A function $\varphi_* : \mathcal{P}_b(X) \to [0, +\infty)$ is assigned as a measure of noncompactness (MNC) defined on $\widetilde{X}$ if:*

(a) *Regularity: for any $B_* \in \mathcal{P}_b(\widetilde{X})$, $\varphi_*(B_*) = 0 \iff B_*$ is a relatively compact set.*

(b) *Invariant under closure: $\varphi_*(B_*) = \varphi_*(\overline{B}_*)$, for any $B_* \in \mathcal{P}_b(\widetilde{X})$.*

(c) *Semi-additivity: $\varphi_*(B_1^* \cup B_2^*) = \max\{\varphi_*(B_1^*), \varphi_*(B_2^*)\}$, for any $B_1^*, B_2^* \in \mathcal{P}_b(\widetilde{X})$.*

The notion of α-dense curve was introduced by G. Mora [20] in 1997, but the notion of DND appeared in 2015 as an application of such a theory.

Definition 2. *Consider a metric space (X,d), $\alpha \in \mathbb{R}_+$ and $B_* \in \mathcal{P}_b(X)$ a function $\widehat{\gamma} \in C([0,1],\mathbb{R}_+)$ is called curve α-dense in B_* if*

- $\widehat{\gamma}([0,1]) \subset B_*$.
- *For all $x \in B_*$, there exists $\bar{t} \in [0,1]$, such that*

$$d(x,\widehat{\gamma}(\bar{t})) < \alpha.$$

The bounded subset B_* of X is said to be densifiable, if for each $\alpha > 0$ we can find an α-dense curve in B_*.

For any $\alpha \geq 0$ and $B \in \mathcal{P}_b(X)$, we denote the sets α-dense curves by $\Gamma_{\alpha,B.}$

Definition 3. *The function $\varphi_d : \mathcal{P}_b(X) \to \mathbb{R}_+$ given by*

$$\varphi_d(B_*) = \inf\{\alpha_* \geq 0 : \Gamma_{\alpha_*,B_*} \neq \emptyset\}, \quad B_* \in \mathcal{P}_b(X),$$

which defines the degree of nondensifiability (DND).

Remark 1.

- *From Definition 3, we deduce*

$$\varphi_d(B_*) \leq \delta(B_*) \text{ for every } B_* \in \mathcal{P}_b(X),$$

  where $\delta(B_*) = \sup\{d(x,y) : x,y \in B_*\}$. This implies that φ_d is well defined.
- *From Hahn–Mazurkiewicz theorem (see, for example, [21,29]), we know that a subset $B_* \in \mathcal{P}_b(X)$ is a Peano Continuum, if and only if it is the continuous image of $[0,1]$. So, the DND φ_d quantifies the distance, in the given sense, between B and the class of Peano Continua that it contains.*

We give some neutral properties of the DND in the following result proved in [22,26,27].

Proposition 2. *Consider a complete metric space (X,d) and φ_d be DND, then*

(a) *Regularity: for any $B_* \in \mathcal{P}_{b,arc}(X)$, $\varphi_d(B_*) = 0$ if and only if $\overline{B}_*$ is a compact set.*

(b) *$\varphi_d(B) = \varphi_d(\overline{B}_*)$, for any $B_* \in \mathcal{P}_{b,arc}(X)$.*

Consider a Banach space X. Then,

(c) *$\varphi_d(x_0 + \widetilde{B}_*) = \varphi_d(\widetilde{B}_*)$, for any $\widetilde{B}_* \in \mathcal{P}_b(X)$.*

(d) *$\varphi_d(\lambda\widetilde{B}_*) = |\lambda|\varphi_d(\widetilde{B}_*)$, for all $\widetilde{B}_* \in \mathcal{P}_b(X)$ and $\lambda \in \mathbb{R}$.*

(e) *$\varphi_d(co(\widetilde{B}_*)) \leq \varphi_d(\widetilde{B}_*)$ for all $\widetilde{B}_* \in \mathcal{P}_b(X)$, co is convex hull.*

(f) *$\varphi_d(co(\widetilde{B}_* \cup C)) \leq \max(\varphi_d(co\widetilde{B}_*), \varphi_d(coC))$ for all $\widetilde{B}_*, C \in \mathcal{P}_b(X)$.*

(g) *$\varphi_d(\widetilde{B}_* + C) \leq \varphi_d(\widetilde{B}_*) + \varphi_d(C)$, for all $\widetilde{B}_*, C \in \mathcal{P}_b(X)$.*

(h) *Generalized Cantor's intersection theorem: if $(\widetilde{B}_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $\widetilde{B}_{n+1} \subseteq \widetilde{B}_n$, $(\widetilde{B}_n)_{n\in\mathbb{N}} \in \mathcal{P}_{cl,cv,b}(X)$ and $\lim_{n\to\infty} \varphi_d(\widetilde{B}_n) = 0$, then $\bigcap_{n=1}^{\infty} \widetilde{B}_n \in \mathcal{P}_{cp}(X)$.*

Example 1 ([30])**.** *Consider a Banach space X and $\bar{B}(0,1) \subset X$ to be a closed unit ball; then,*

$$\varphi_d(\bar{B}(0,1)) = \begin{cases} 1 & \text{if} \quad \dim X = +\infty; \\ 0 & \text{if} \quad \dim X < \infty. \end{cases}$$

Remark 2. *García and Mora ([26], Example 2.1), show that DND is not MNC.*

Now, we give some relationships between the Kuratowski and Hausdorff *MNC*s and the DND φ_d.

Proposition 3 ([22,31]). *Let X be a metric space and $\widetilde{B}_* \in \mathcal{P}_{b,arc}(X)$. Then,*

$$\chi(\widetilde{B}_*) \leq \varphi_d(\widetilde{B}_*) \leq 2\chi(\widetilde{B}_*)$$

where χ is a Hausdorff measure defined as follows:

$$\chi(\widetilde{B}_*) = \inf\{\epsilon > 0 : \widetilde{B}_* \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)\};$$

and

$$\frac{1}{2}\varpi(\widetilde{B}_*) \leq \varphi_d(\widetilde{B}_*) \leq \varpi(\widetilde{B}_*).$$

where ϖ is a Kuratowskii MNC defined by

$$\varpi(\widetilde{B}_*) = \inf\{\epsilon > 0 : \widetilde{B}_* \subset \bigcup_{i=1}^{n} \widetilde{B}_{*i};\ \delta(\widetilde{B}_{*i}) \leq \epsilon\}.$$

Remark 3. *Notice that for all $\widetilde{B}_* \in \mathcal{P}_b(X)$,*

$$\chi(\widetilde{B}_*) \leq \varpi(\widetilde{B}_*) \leq 2\chi(\widetilde{B}_*).$$

Definition 4. *Consider two metric spaces $\widetilde{X}, \widetilde{Y}$, $C \in \mathcal{P}_*(\widetilde{Y})$, $N : \widetilde{X} \to \widetilde{Y}$, and φ_d is a degree of nondensifiability (DND). The operator N is $k-\varphi_d-$contraction if*

$$\varphi_d(N(\widetilde{B}_*)) \leq k\varphi_d(\widetilde{B}_*), \quad \widetilde{B}_* \in \mathcal{P}_b(C),$$

where $k \in (0,1)$. The collection of these maps is shown by $\mathcal{KC}_k(\widetilde{X}, \widetilde{Y})$. For $\widetilde{X} = \widetilde{Y}$, $\mathcal{KC}_k(\widetilde{X}, \widetilde{Y}) = \mathcal{KC}_k(\widetilde{X})$.

Theorem 1. *Consider two Banach spaces $\widetilde{X}$, $\widetilde{Y}$, $C \in \mathcal{P}_{b,cl}(\widetilde{X})$ and $f \in \mathcal{K}(C, \widetilde{Y})$; then, there is $\widetilde{f}_* \in \mathcal{K}(\widetilde{X}, \widetilde{Y})$ such that*

$$\widetilde{f}_{*|_C} = f \text{ and } \widetilde{f}_*(X) \subset cof(C).$$

Proof. Let $\mathcal{S} : \widetilde{X} \to \mathcal{P}(\widetilde{Y})$ be a multivalued map defined by

$$\mathcal{S}(a) = \begin{cases} \{f(a)\} & a \in C \\ \\ \overline{co}f(C) & a \in X \backslash C. \end{cases}$$

It is clear that, for all $a \in \widetilde{X}$, $\widetilde{S}(a) \in \mathcal{P}_{cl,cv}(\widetilde{X})$. Now, demonstrate that $\mathcal{S}$ is *l.s.c.* Indeed, let $\mathcal{O} \in \mathcal{P}_{cl}(\widetilde{Y})$, then

$$\mathcal{S}_+^{-1}(\mathcal{O}) = \{a \in \widetilde{X} : \widetilde{S}(a) \subset \mathcal{O}\}.$$

If $\overline{co}f(C) \subset \mathcal{O}$, hence

$$\mathcal{S}_+^{-1}(\mathcal{O}) = \widetilde{Y} \in \mathcal{P}_{cl}(\widetilde{X}).$$

If $\overline{co}f(C) \not\subset \widetilde{O}$, thus

$$\mathcal{S}_+^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}) \cap C \in \mathcal{P}_{cl}(\widetilde{X}).$$

Therefore, $\mathcal{S}$ is *l.s.c.* and, by the Michael Selection Theorem, there is $\widetilde{f}_* \in C(\widetilde{X}, \widetilde{Y})$ such that

$$\widetilde{f}_*(a) \in \mathcal{S}(a) \quad \text{for all } a \in \widetilde{X} \text{ and } \widetilde{f}_*(\widetilde{X}) \subset \overline{co}f(C).$$

Since f is compact, then, by the Mazur theorem,

$$\overline{co}f(C) \in \mathcal{P}_{cp}(\widetilde{Y}).$$

We conclude that

$$\widetilde{f}_* \in \mathcal{K}(\widetilde{X}, \widetilde{Y}),\ \widetilde{f}_{*|_C} = f \text{ and } \widetilde{f}_*(\widetilde{X}) \subset cof(C).$$

□

Theorem 2 ([26]). *Consider a Banach space $(X, \|\cdot\|)$ and $\widetilde{B}_* \in P_{cv,b}(\widetilde{X})$. Presume that $N{:}\widetilde{B}_* \longrightarrow \widetilde{B}_*$ is a continuous and $k-\varphi_d-$contraction of measure DND. Then, $FixN \neq \emptyset$.*

3. Topological Degree

Using the idea of the degree of nondensifiability, we define the topological degree in this section.

Lemma 1. *Consider a Banach space, where $\mathcal{O}$ an open bounded subset of $\widetilde{X}$, $f_* \in C(\overline{\mathcal{O}}, \widetilde{X}) \cap \mathcal{KC}_{\frac{k}{2}}(\overline{\mathcal{O}}, \widetilde{X})$, and $0 \notin (I-f)(\partial\mathcal{O})$. Then, there is $\widetilde{D}_* \in \mathcal{K}(\widetilde{X})$ and $\widetilde{f}_* \in C(\overline{\mathcal{O}}, \widetilde{D}_*)$ such that*

$$\widetilde{f}_{*|_{\widetilde{D}_* \cap \overline{\mathcal{O}}}} = f \text{ and } \widetilde{f}_*(\overline{\mathcal{O}}) \subset \overline{co}f(\widetilde{D}_* \cap \overline{\mathcal{O}}).$$

Proof. We set $\widetilde{D}_0 = \overline{\mathcal{O}}$, and define by induction

$$\widetilde{D}_n = \overline{co}f(\widetilde{D}_{n-1} \cap \overline{\mathcal{O}}) \text{ for all } n \in \mathbb{N}.$$

By Proposition 2, we obtain

$$\begin{aligned}
\varphi_d(\widetilde{D}_n) &= \varphi_d(\overline{co}f(\widetilde{D}_{n-1} \cap \overline{\mathcal{O}})) \\
&= \varphi_d(cof(\widetilde{D}_{n-1} \cap \overline{\mathcal{O}})) \\
&\leq \varphi_d(f(\widetilde{D}_{n-1} \cap \overline{\mathcal{O}})) \leq \chi(f(\widetilde{D}_{n-1} \cap \overline{\mathcal{O}})) \\
&\leq 2\chi(f(\widetilde{D}_{n-1})) \leq 2\varphi_d(f(\widetilde{D}_{n-1})) \\
&\leq k\varphi_d(\widetilde{D}_{n-1}).
\end{aligned}$$

Therefore,

$$\varphi_d(\widetilde{D}_n) \leq k^n \phi_d(\widetilde{D}_1) \to 0 \text{ as } n \to \infty.$$

It is clear that $(\widetilde{D}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of bounded, convex, and nonempty subsets of $\widetilde{X}$. By Proposition 2(h), $\widetilde{D}_* = \bigcap_{n\in\mathbb{N}} \widetilde{D}_n$ is nonempty, convex and compact. Also, $f \in \mathcal{K}(\widetilde{D}_* \cap \overline{\mathcal{O}}, \widetilde{X})$. Using Theorem 1, we can find $\widetilde{f}_* \in \mathcal{K}(\overline{\mathcal{O}}, \widetilde{D}_*)$, such that

$$\widetilde{f}_{*|_{\widetilde{D} \cap \overline{\mathcal{O}}}} = f \text{ and } \widetilde{f}_*(\overline{\mathcal{O}}) \subset cof(\widetilde{D} \cap \overline{\mathcal{O}}).$$

□

Now, we are in a position to give the definition topological degree based on degree of nondensifiability.

Definition 5. *Consider a Banach space $\widetilde{X}$, $\widetilde{O} \subset \widetilde{X}$, nonempty bounded, open and $f \in C(\overline{\mathcal{O}}, \widetilde{X}) \cap \mathcal{KC}_{\frac{k}{2}}(\overline{\widetilde{O}}, \widetilde{X})$ and $0 \notin (I-f)(\partial\mathcal{O})$. We define the degree of $I-f$ by*

$$deg_d(I-f, \mathcal{O}, 0) = deg_{LS}(I-\widetilde{f}, \mathcal{O}, 0),$$

where deg_{LS} is the Leray–Schauder degree and $\widetilde{f}_*$ is defined in Lemma 1.

- *If $b \notin (I-f)(\partial\mathcal{O})$ we define the topological degree by*

$$deg_d(I-f-b,\mathcal{O},b) = deg_{LS}(I-\widetilde{f}-b,\mathcal{O},b).$$

The next proposition makes Definition 5 meaningful.

Proposition 4. *The degree deg_d is well defined.*

Proof. Let $\widetilde{f}_*, \widehat{f} \in K(\overline{\mathcal{O}}, \widetilde{D})$ such that

$$\widetilde{f}_{*|_{\widetilde{D}\cap\overline{\mathcal{O}}}} = f = \widehat{f}_{|_{\widetilde{D}\cap\overline{\mathcal{O}}}}.$$

We defined the following homotopy application $\mathcal{H} : [0,1]\times\overline{\mathcal{O}} \to \widetilde{X}$ given by

$$\mathcal{H}(t,x) = t\widetilde{f}(x) + (1-t)\widehat{f}(x), \quad x \in \overline{\mathcal{O}}.$$

Let $(t,x) \in [0,1]\times\partial\mathcal{O}$ such that

$$\mathcal{H}(t,x) = x,\ x \in \partial\mathcal{O}.$$

Since $\widetilde{f}_*$ and $\widehat{f}$ both map $\overline{\mathcal{O}}$ in $\widetilde{D}_*$, then

$$x = t\widetilde{f}_*(x) + (1-t)\widehat{f}(x) \in \widetilde{D}_*.$$

For every $x \in \widetilde{D}_*$, we have

$$\widetilde{f}(x) = f(x) = \widehat{f}(x),$$

so, $0 = x - f(x)$, which is in contradiction with $0 \notin (I-f)(\partial\mathcal{O})$. Therefore,

$$x \neq \mathcal{H}(t,x) \quad \text{for all } (t,x) \in [0,1]\times\partial\mathcal{O}.$$

By using the homotopy invariance of the Leray–Schauder degree, we can conclude that

$$deg(I-\widetilde{f}_*,\mathcal{O},0) = deg(I-\widehat{f},\mathcal{O},0).$$

□

The topological degree via a degree of nondensifiability in normed space conserves the basic features of the Leray–Schauder degree.

Theorem 3. *Let $\widetilde{X}$ be a Banach space, $\mathcal{O} \subset X$ be an open bounded subset, and $f : \overline{\mathcal{O}} \longrightarrow \widetilde{X}$ be a continuous $\frac{k}{2} - \varphi_d-$contractions map. If $b \notin (I-f)(\partial\mathcal{O})$, then there exists an integer $deg_d(I-f,\mathcal{O},b)$ satisfying the following properties:*

(1) *(Solvability) $deg_d(f,\mathcal{O},b) \neq 0$ then $x - f(x) = b$ has a solution in $\mathcal{O}$;*

(2) *(Additivity). Suppose that $\mathcal{O}_1, \mathcal{O}_2$ are two disjoint open subsets of $\mathcal{O}$ and $b \notin f(\overline{\mathcal{O}}\backslash\mathcal{O}_1 \cup \mathcal{O}_2)$. Then,*

$$deg_d(I-f,\mathcal{O},b) = deg_d(I-f,\mathcal{O}_1,b) + deg_d(I-f,\mathcal{O}_2,b);$$

(3) *(Homotopy invariance) If $\mathcal{H} : [0\ 1]\times\overline{\mathcal{O}} \longrightarrow \widetilde{X}$ is continuous, $b \notin \bigcup_{t\in[0,1]} (I-\mathcal{H}(t,\partial\mathcal{O}))$ and*

$$\varphi_d(\mathcal{H}(t,B)) \leq \frac{k}{2}\varphi_d(B) \quad \text{for all } B \in \mathcal{P}_b(\overline{\mathcal{O}}),$$

then $deg_d(I-\mathcal{H}(t,\cdot),\mathcal{O},b)$ does not depend on $t \in [0,1]$;

(4) *$deg_d(I-f,\mathcal{O},b)$ is a constant on each connected component of $X \setminus f(\partial\mathcal{O})$;*

(5) *$deg_d(I-f+y_0,\mathcal{O},b+y_0) = deg_d(I-f,\mathcal{O},b)$ for all $y_0 \in X$;*

(6) *Let $g \in C(\overline{\mathcal{O}}, \widetilde{X})$ be a compact application. If $f_{|\partial\mathcal{O}} = g_{|\partial\mathcal{O}}$, then*

$$deg_d(I - f, \mathcal{O}, b) = deg_d(I - g, \mathcal{O}, b);$$

(7) *(Excision). Let $K \subset \overline{\mathcal{O}}$ is closed and $b \notin f(K)$, then*

$$deg_d(I - f, \mathcal{O}, b) = deg_d(I - f, \mathcal{O}\backslash K, b).$$

Some ramifications for this topological degree concept.

Theorem 4. *Consider a Banach space $\widetilde{X}$, where $\mathcal{O}$ is an open bounded subset of $\widetilde{X}$ with $0 \in \mathcal{O}$ and $N : \overline{\mathcal{O}} \to \widetilde{X}$ is a continuous $\frac{k}{2} - \varphi_d-$contraction map. Suppose the following Leray–Schauder condition:*

$$x \neq \lambda N(x) \text{ for every } \lambda \in [0,1),\ x \in \partial\mathcal{O}.$$

Then, N possesses at least fixed points.

Proof. Let $\mathcal{H} : [0,1] \times \overline{\mathcal{O}} \to \widetilde{X}$ a homotopy given by

$$\mathcal{H}(t,x) = x - tN(x), \quad (t,x) \in [0,1] \times \overline{\mathcal{O}}.$$

By the Leray–Schauder condition, we have

$$x \neq \lambda\mathcal{H}(t,x), \text{for all } t \in [0,1),\ x \in \partial\mathcal{O}.$$

Then,

$$deg_d(x - N(x), \mathcal{O}, 0) = deg_d d(I, \mathcal{O}, 0) = 1.$$

According to Theorem 3, there is $x \in \mathcal{O}$ such that

$$N(x) = x.$$

□

Next, as a result, we present the version Schaefer's fixed point type.

Theorem 5. *Consider a Banach space $\widetilde{X}$ and a continuous map $N : \widetilde{X} \to \widetilde{X}$ and $\frac{k}{2} - \varphi_d-$contraction map. Then, one of the following statements holds:*

1. *$\mathcal{E}(N) = \{x \in \widetilde{X} : \lambda Nx = x; \lambda \in (0,1)\}$ unbounded.*
2. *$x = Nx$ possesses at least one solution.*

Proof. Assume that $\mathcal{E}(N)$ is bounded, then there exists $r > 0$, $\mathcal{E}(N) \subset B(0,r)$. If $N(x) = x$ for some $x \in B(0,r)$, then (2) holds. Suppose that

$$N(x) \neq x \text{ for all } x \in \partial B(0,r).$$

Similar to how Theorem 4 is proven,

$$deg_d(I - N, B(0,r), 0) = 1.$$

Consequently, N possess a fixed point. □

Theorem 6. *Consider a Banach space $(\widetilde{X}, \|\cdot\|)$, $\mathcal{O} \subset \widetilde{X}$ bounded nonempty subset with $0 \in \mathcal{O}$, where $N \in C(\overline{\mathcal{O}} \to \widetilde{X})$ is a $\frac{k}{2} - \varphi-$contraction application and $L : \partial\mathcal{O} \to X$ compact continuous mapping. Suppose*

$$\lambda N(x) + (1 - \lambda_*)L(x) \neq x \text{ and } \|x\| \leq \|L(x)\| \text{ for each } x \in \partial\mathcal{O}, \lambda_* \in [0,1]. \tag{1}$$

Then,

$$deg_d(I - N, \mathcal{O}, 0) = 0.$$

Proof. By Theorem 1, there exists $\bar{S}_* : \overline{\mathcal{O}} \to C$ a compact, continuous mapping such that $\bar{S}_{*|_{\partial\mathcal{O}}} = \bar{S}_*$. Let $\mathcal{H} : [0,1] \times \overline{\mathcal{O}} \to \widetilde{X}$ be a mapping of homotopy provided by

$$\mathcal{H}(t,x) = \lambda N(x) + (1-\lambda)\bar{S}_*(x), \quad (t,x) \in [0,1] \times \overline{\mathcal{O}}.$$

From Condition (1), we have

$$0 \notin (I - L)(\partial\mathcal{O}) \text{ and } 0 \notin (I - \mathcal{H}(t, \partial\mathcal{O})) \text{ for all } x \in \partial\mathcal{O}.$$

Thus,

$$\deg_d(I - N, \mathcal{O}, 0) = \deg_d(I - \bar{S}_*, \mathcal{O}, 0) = \deg_{LS}(I - \bar{S}_*, \mathcal{O}, 0).$$

By [32], (Lemma 2.2.11), we obtain

$$\deg_{LS}(I - \bar{S}_*, \mathcal{O}, 0) = 0.$$

□

Theorem 7. *Let X be a infinite dimensional Banach space, and $U_i \subset X, i = 1,2$ be two open bounded nonempty subset of X, $0 \in U_1 \subset U_2$, $x_0 \in X$, such that*

$$\sup_{x \in \partial U_2} \|x\| \leq \|x_0\|.$$

Let $N : \overline{U}_2 \to X$ be a continuous $\frac{k}{2} - \varphi_d$−contraction map satisfying

$(\mathcal{H}_1)$ $\|x\| \leq \|N(x)\|$ *and* $\lambda N(x) \neq x + (1-\lambda)x_0$ *for all* $x \in \partial U_2$.
$(\mathcal{H}_2)$ $\|N(x)\| \leq \|x\|$ *for all* $x \in \partial U_1$.

Then, there is $x \in \overline{U}_2 \backslash U_1$ with $x = N(x)$.

Proof. It is clear that $U_2 \backslash \overline{U}_1$ is a open set and $U_2 = U_2 \backslash \overline{U}_1 \cup U_1$. Then, by Theorem 3,

$$deg_d(I - N, U_2 \backslash \overline{U}_1, 0) = deg_d(I - N, U_2, 0) - deg_d(I - N, U_1, 0).$$

By $(\mathcal{H}_1)$, we can apply Theorem 6,

$$deg_d(I - N, U_2, 0) = deg_{LS}(I - \widetilde{N}, U_2, 0) = 0.$$

By $(\mathcal{H}_2)$, we have

$$x - \lambda N(x) \neq x, \ \lambda \in [0,1] \text{ and } x \in \partial U_1.$$

Then, by homotopy proprieties of topological degree, we obtain

$$deg_d(I - N, U_1, 0) = deg_d(I, U_2, 0) = 1,$$

which implies $deg_d(I - N, U_2 \backslash \overline{U}_1, 0) = -1$; hence, there is $x \in U_2 \backslash \overline{U}_1$ satisfying

$$x = N(x).$$

□

4. Semilinear Integro-Differential Equations with Finite Delay

In this section, we consider the following semi-linear functional differential equation problem:

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + \int_0^t S(t-s)\mathcal{U}(s)ds + f(t,\mathcal{U}_t), & \text{a.e. } t \in [0,b], \\ \mathcal{U}(t) = \phi_*(t), & t \in [-r,0], \end{cases} \tag{2}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on separable Banach space $(\mathcal{E}, \|\cdot\|)$ with domain $\mathcal{D}(\mathcal{A})$, $f : [0,b] \times \mathcal{C}([-r,0],\mathcal{E}) \times \mathcal{E} \to \mathcal{E}$, is a given function, and $\phi_* \in \mathcal{C}([-r,0],\mathcal{E})$. For any $t \in [0,b]$, $S(t)$ is a closed linear operator on $\mathcal{E}$, with domain $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(S(t))$, which is independent of t. For any function $\mathcal{U} : [-r,b] \to \mathcal{E}$ and any $t \in [0,b]$, we denote by $\mathcal{U}_t$ the element of $\mathcal{C}([-r,0],\mathcal{E})$ defined by

$$\mathcal{U}_t(\theta) = U(t+\theta),\ \theta \in [-r,0].$$

Here, $\mathcal{U}_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

The existence of mild solutions for integro-differential equations in infinite dimensional space has been investigated (see, e.g., [33–38] and the references therein).

Here, we investigate the existence of the mild solutions for the above partial integro-differential evolution equations with finite delay where the semi group is not necessarily compact.

Existence Result

We recall some knowledge on resolvent operators in Banach space.

Definition 6 ([39]). *A family of bounded linear operators $(R(t))_{t\in\mathbb{R}_+} \in \mathcal{L}(\mathcal{E})$ is called a resolvent operator associated with (2) if*

(a) *$R(0)$ is the identity map and $\|R(t)\| \leq Me^{\beta t}$ for a certain a positive real constant M and $\beta \in \mathbb{R}$.*

(b) *For each $\mathcal{U} \in \mathcal{E}$, $R(t)\mathcal{U}$ is strongly continuous.*

(c) *For all $\mathcal{U} \in \mathcal{E}$, $t \to R(t)U$ s continuously differentiable, and*

$$\begin{aligned} R'(t)\mathcal{U} &= AR(t)u + \int_0^t S(t-s)R(s)\mathcal{U}\,ds \\ &= R(t)\mathcal{A}\mathcal{U} + \int_0^t R(t-s)S(s)\mathcal{U}\,ds. \end{aligned}$$

Theorem 8 ([39]). *Let $\phi_*(0) \in D(A)$. Then, (2) possesses a resolvent operator. Also, if $\mathcal{U}$ is a solution of (2), then*

$$\mathcal{U}(t) = \begin{cases} R(t)\phi_*(0) + \int_0^t R(t-s)f(s,\mathcal{U}_s)ds, & t \in [0,b], \\ \phi_*(t), & t \in [-r,0]. \end{cases}$$

Definition 7. *A function $\mathcal{U} \in \mathcal{C}([-r,b],\mathcal{E})$ is called to be a mild solution of Problem (2) if*

$$U(t) = \begin{cases} R(t)\phi_*(0) + \int_0^t R(t-s)f(s,\mathcal{U}_s)ds, & t \in [0,b], \\ \phi_*(t), & t \in [-r,0]. \end{cases} \tag{3}$$

In the following, we will need the following lemma.

Lemma 2 ([27]). *Let $B_* \in \mathcal{P}_b(\mathcal{C}([0,c],\mathcal{E}))$, then*

$$\sup\{\varphi_d(\{x_*(t) : x_* \in B_*\}) : t \in [0,c]\} \leq \varphi_d(B_*).$$

In order to give the existence result of the problem (2), we shall need the following hypotheses:

$(\overline{\mathcal{H}}_1)$ f is a is Carathéodory from $[0,b] \times \mathcal{C}([-r,0],\mathcal{E}) \times \mathcal{E}$ to $\mathcal{E}$.

$(\overline{\mathcal{H}}_2)$ $\{T(t)\}_{t\geq 0}$ is operator-norm continuous for $t > 0$.

$(\overline{\mathcal{H}}_3)$ There exists $h \in L^1([0,b],\mathbb{R}_+)$ such that

$$\|f(t,\xi)\| \leq h(t)(\|\xi\|_\infty + 1), \quad \text{for every } \xi \in \mathcal{C}([-r,0],\mathcal{E}), \text{ a.e. } t \in [0,b].$$

$(\overline{\mathcal{H}}_4)$ There exists $g \in L^1([0,b],\mathbb{R}_+)$ such that for all $D_* \in \mathcal{P}_b(\mathcal{C}([-r,0],\mathcal{E}))$, we have

$$\varphi_d(f(t,D_*)) \leq g(t)\left(\sup_{\theta\in[-r,0]} \varphi_d(D(\theta))\right), \text{ for a.e. } t \in [0,b]$$

where

$$D(\theta) = \{\phi(\theta) : \phi \in D\}, \quad \theta \in [-r,0].$$

Theorem 9. *If $(\overline{\mathcal{H}}_1) - (\overline{\mathcal{H}}_4)$ hold, then Problem (2) admits at least one mild solution.*

Proof. Let us consider the operator $N : \mathcal{C}([-r,b],E) \to \mathcal{C}([-r,b],E)$ as follows:

$$(N\mathcal{U})(t) = \begin{cases} R(t)\phi_*(0) + \int_0^t R(t-s)f(s,\mathcal{U}_s)ds, & t \in [0,b], \\ \phi_*(t), & t \in [-r,0]. \end{cases}$$

This is evidence that the fixed points of N are solutions of Problem (2). Utilizing Banach's fixed point theorem, we prove that N possesses a fixed point.

Step 1: Demonstrating the continuity of N.

Indeed, let $(\mathcal{U}^{(n)})_{n\in\mathbb{N}} \subset \mathcal{C}([-r,b],\mathcal{E})$ such that $\mathcal{U}^{(n)} \to \mathcal{U}$ in $\mathcal{C}([-r,b],\mathcal{E})$. Then,

$$\begin{aligned} \|(N\mathcal{U}^{(n)})(t) - (N\mathcal{U})(t)\| &= \left\| \int_0^t R(t-s)f(s,\mathcal{U}_s^{(n)})ds - \int_0^t R(t-s)f(s,\mathcal{U}_s)ds \right\| \\ &\leq \int_0^t \|R(t-s)\| \left\| f(s,\mathcal{U}_s^{(n)}) - f(s,\mathcal{U}_s) \right\| ds. \end{aligned}$$

The sequence $(f_n)_{n\in\mathbb{N}}$, defined by $f_n : t \in [0,b] \to f_n(t) = f(t,\mathcal{U}_t^{(n)})$, satisfies the conditions of the Lebesgue's theorem. Indeed,

Utilizing that f is a Carathéodory function and by the separability of $\mathcal{C}([-r,0],\mathcal{E}) \times \mathcal{E}$, we deduce that $f(\cdot,\cdot)$ is measurable. So, for any $n \in \mathbb{N}$, the function $f_n(\cdot)$.

Since the sequence $(\mathcal{U}^{(n)})_{n\in\mathbb{N}}$ converges to $\mathcal{U}$ in $\mathcal{C}([0,b],\mathcal{E})$ and f is a Carathéodory function, then there exists $M_* > 0$ such that

$$\|\mathcal{U}_n\|_\infty \leq M_*, \; f_n(t) \longrightarrow f(t,\mathcal{U}_t) \quad n \to \infty, \text{ for all } t \in [0,b],$$

and from $(\overline{\mathcal{H}}_3)$,

$$\|f_n(t)\| \leq (M_* + 1)h(t), \quad \text{for all } n \in \mathbb{N}, \; t \in [0,b].$$

By the Lebesgue theorem, we obtain

$$\|N\mathcal{U}_n - N\mathcal{U}\|_\infty \leq Me^{|\beta|b}\int_0^b \|f_n(s,\mathcal{U}_s^n) - f(s,\mathcal{U}_s)\|ds \to 0 \;\; n \to \infty.$$

Step 2: N transforms bounded sets of $\mathcal{C}([-r,b],\mathcal{E})$ into bounded subsets of $\mathcal{C}([-r,b],\mathcal{E})$.

In fact, by letting $r > 0$ and $B_r = \{u \in \mathcal{C}([-r,b],\mathcal{E}) \;/\; \|u\|_\infty \leq r\}$, we show that $N(B_r)$ is bounded. By Definition (6) and using the hypothesis $(\overline{\mathcal{H}_4})$ and $(\overline{\mathcal{H}_2})$, we obtain, for any $t \in [0,b]$,

$$\|(N\mathcal{U})(t)\| \quad \leq Me^{t\beta}\|\phi(0)\| + Me^{t\beta}\int_0^t e^{-s\beta}\|f(s,\mathcal{U}_s)\|ds.$$

Thus,

$$\|N(\mathcal{U})\|_\infty \leq Me^{|\beta|b}\left(\|\phi\|_\infty + (r+1)\int_0^b e^{-s\beta}h(s)ds\right) = l_1.$$

Therefore, $N(B_r)$ is bounded.

Step 3: Proving that N is φ_d-contractive.

Now, let $B \in \mathcal{P}_b(\mathcal{C}([-r,b],\mathcal{E}))$ and, for each $s \in [0,b]$, we obtain $\beta_s = \sup_{\theta\in[-r,0]} \varphi_d(\{\mathcal{U}_s(\theta) : \mathcal{U} \in B\})$; then, by (C_2), we have

$$\varphi_d(\{f(s,\mathcal{U}_s) : \mathcal{U} \in B\}) \leq g(s)\beta_s.$$

Thus, for $s \in [0,b]$ and $\epsilon > 0$ there exists $\widetilde{\gamma}_s \in C([0,1],\mathcal{E})$ such that

$$\widetilde{\gamma}_s([0,1]) \subset \{f(s,\mathcal{U}_s) : \mathcal{U} \in B\}$$

and for every $\mathcal{U} \in B$, there exists $\tau \in [0,1]$ with

$$\|f(s,\mathcal{U}_s) - \widetilde{\gamma}_s(\tau)\| \leq g(s)\beta_s + \epsilon.$$

Thus, for $\tau_* > 0$, we obtain

$$e^{-|\beta|s-\tau_*\int_0^s g(t)dt}\|f(s,\mathcal{U}_s) - \widetilde{\gamma}_s(\tau)\| \leq g(s)\beta_s + \epsilon.$$

We define the following mapping $\gamma_* : [0,1] \to \mathcal{C}([-r,b],\mathcal{E})$ as follows:

$$(\gamma_*(\tau))(t) = \begin{cases} R(t)\phi_*(0) + \displaystyle\int_0^t R(t-s)\widetilde{\gamma}_s(\tau)ds, & t \in [0,b], \\ \phi_*(t), & t \in [-r,0]. \end{cases}$$

It clear that γ_* is a continuous mapping and $\gamma_*([0,1]) \subset N(B)$. Furthermore, for ally $\mathcal{U} \in B$, there exists $\tau \in [0,1]$ such that

$$\begin{aligned} \|(N\mathcal{U})(t) - (\gamma_*(\tau))(t)\| &\leq M\int_0^t e^{\beta(-s+t)}e^{\tau_*\int_0^s g(\xi)d\xi}e^{-\tau_*\int_0^s g(\xi)d\xi}\|f(s,\mathcal{U}_s) - \widetilde{\gamma}_s(\tau)\|ds \\ &\leq M\int_0^t e^{|\beta|(-s+t)}e^{\tau_*\int_0^s g(\xi)d\xi}e^{-\tau_*\int_0^s g(\xi)d\xi}\|f(s,\mathcal{U}_s) - \widetilde{\gamma}_s(\tau)\|ds \\ &\leq M\int_0^t e^{|\beta|(-s+t)}e^{\tau_*\int_0^s g(\xi)d\xi}e^{-\tau_*\int_0^s g(\xi)d\xi}\|f(s,\mathcal{U}_s) - \widetilde{\gamma}_s(\tau)\|ds \\ &\leq e^{|\beta|t}M\int_0^t g(s)e^{\tau_*\int_0^s g(\xi)d\xi}ds\varphi_d(B) + \epsilon e^{|\beta|t}M\int_0^t e^{\tau_*\int_0^s g(\xi)d\xi}ds. \end{aligned}$$

Thus,

$$e^{-|\beta|t-\tau_*\int_0^s g(\xi)d\xi}\|(N\mathcal{U})(t) - (\gamma_*(\tau))(t)\| \quad \leq \quad \frac{M}{\tau_*}\varphi_d(B) + \epsilon Mb.$$

Therefore,

$$\|(N\mathcal{U}) - (\gamma_*(\tau))\|_* \quad \leq \quad \frac{M}{\tau_*}\varphi_d(B) + \epsilon M b,$$

where

$$\|\mathcal{U}\|_* = \sup_{t\in[-r,b]} e^{-|\beta|t-\tau_*\int_0^s g(\xi)d\xi}\|\mathcal{U}(t)\|.$$

Letting $\epsilon \to 0$, we obtain

$$\|N(\mathcal{U}) - \gamma_*(\tau)\|_* \quad \leq \quad \frac{M}{\tau_*}\alpha.$$

Since τ_* is arbitrary, we choose τ_* such that $0 < \frac{2M}{\tau_*} < 1$, then

$$\varphi_d(B) \leq \frac{k}{2}\varphi_d(B),\ k = \frac{2M}{\tau_*} \quad B \in \mathcal{P}_b(\mathcal{C}([-r,b],\mathcal{E})).$$

Consequently, N is a $\frac{k}{2} - \varphi_d-$contraction. It is clear that $(\mathcal{C}([-r,b],\mathcal{E}), \|\cdot\|_*)$ is a Banach space, and $N : (\mathcal{C}([-r,b],\mathcal{E}), \|\cdot\|_*) \to (\mathcal{C}([-r,b],\mathcal{E}), \|\cdot\|_*)$ are continuous operators.

Step 4: A priori estimation.

Let $\mathcal{U} \in \mathcal{C}([-r,b],\mathcal{E})$ such that $\mathcal{U} = \lambda N(\mathcal{U})$, and $0 < \lambda < 1$. Then, for all $t \in [0,b]$, we obtain

$$\mathcal{U}(t) \quad = \quad \lambda\left(R(t)\phi_*(0)\int_0^t R(t-s)f(s,\mathcal{U}_s)ds\right), \quad u(t) = \phi(t), \quad t \in [-r,0].$$

Thus,

$$\begin{aligned}
\|\mathcal{U}(t)\| &\leq Me^{\beta t}\|\phi\|_\infty + \int_0^t Me^{\beta(-s+t)}\|f(s,\mathcal{U}_s)\|ds \\
&\leq Me^{\beta t}\|\phi\|_\infty + \int_0^t Me^{\beta(-s+t)}h(s)(\|\mathcal{U}_s\|_\infty + 1)ds \\
&\leq Me^{|\beta|t}\|\phi\|_\infty + Me^{|\beta|b}\|h\|_{L^1} + \int_0^t Me^{\beta(-s+t)}h(s)\|\mathcal{U}_s\|_\infty ds \\
&\leq Me^{|\beta|b}\|\phi\|_\infty + Me^{|\beta|b}\|h\|_{L^1} + \int_0^t Me^{\beta(-s+t)}h(s)\sup_{\theta\in[-r,0]}\|\mathcal{U}(\theta+s)\|ds.
\end{aligned}$$

Therefore,

$$\|\mathcal{U}(t)\| \leq Me^{|\beta|b}\|\phi\|_\infty + Me^{|\beta|b}\|h\|_{L^1} + \int_0^t Me^{\beta(-s+t)}h(s)\sup_{\theta\in[-r,0]}\|\mathcal{U}(\theta+s)\|ds. \tag{4}$$

Set $\mathcal{V}(t) = \sup_{s\in[0,t]}\|\mathcal{U}(s)\|$. Hence, the inequality (4) implies

$$\mathcal{V}(t) \leq B_* + Me^{|\beta|b}\int_0^t h(s)\mathcal{V}(s)ds,$$

where

$$B_* = Me^{|\beta|b}\|\phi\|_\infty + Me^{|\beta|b}\|\phi\|_\infty\|h\|_{L^1}.$$

Gronwall's inequality,

$$\mathcal{V}(t) \leq B_*\exp\left(Me^{|\beta|b}\int_0^t h(s)ds\right), \quad t \in [-r,b].$$

Therefore,

$$\|\mathcal{U}\|_\infty \leq B_* \exp\left(Me^{|\beta|b}\int_0^b h(s)ds\right) := \overline{M}_*.$$

Set

$$\mathfrak{F} = \{\mathcal{U} \in \mathcal{C}([-r,b],\mathcal{E}) : \|\mathcal{U}\|_* < M_* + 1\}.$$

So, $0 \in \mathfrak{F}$ is a bounded open. Then, from above steps, we can conclude that N satisfies all the conditions of Theorem 4. As a result, N has at least a fixed point $\mathcal{U} \in \mathfrak{F}$, representing the solution to Problem (2). □

5. Conclusions

This paper contains a new approach to topological degree theory by introducing the concept of "degree of nondensifiability" (DND). Utilizing DND, we established some novel fixed point theorems, including a variant of the Leray–Schauder nonlinear alternative and a new version of Schauder's fixed point theorem. In the end, this work shows that, without the compactness of the Nemytskii operator, some class of semi linear integro differential with delay hast at least one solution under some sufficient conditions. I hope that these results extend some previous ones in the literature.

Author Contributions: Conceptualization, N.O. and A.O.; Methodology, N.O., J.J.N. and A.O.; Validation, J.J.N. and A.O.; Formal analysis, J.J.N.; Investigation, N.O. and J.J.N.; Resources, J.J.N.; Writing—original draft, N.O. and A.O.; Writing—review & editing, N.O., J.J.N. and A.O.; Supervision, J.J.N. and A.O.; Funding acquisition, J.J.N. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received partial funding from the Mathematics Laboratory and faculty of exact sciences at university Djillali Liabés of Sid-Bel-Abbés (Algeria), funding primarily includes office space, two computers, and necessary materials for our research work.

Data Availability Statement: Data sharing does not apply to this article, as the research predominantly involves mathematical analysis and does not involve the generation, collection, or analysis of specific data sets. The results presented in this paper are derived from mathematical proofs.

Acknowledgments: The research of J.J. Nieto was supported by the Agencia Estatal de Investigaciòn (AEI) of Spain Grant PID2020-113275GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe, by the European Union and Xunta de Galicia, grant ED431C 2023/12 for Competitive Reference Research Groups (2023–2026).

Conflicts of Interest: The authors confirm that there are no known conflicts of interest or personal relationships that could have influenced the work reported in this paper.

References

1. Amar, M.; Andreucci, D.; Bisegna, P.; Gianni, R. Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics. *Nonlinear Anal. Real World Appl.* **2005**, *6*, 367–380. [CrossRef]
2. Bainov, D.D.; Lakshmikantham, V.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
3. Kohlberg, E.; Mertens, J.F. On the strategic stability of equilibrium points. *Econometrica* **1986**, *54*, 1003–1037. [CrossRef]
4. Joshi, M.C.; Tomar, A. *Fixed Point Theory and Its Applications to Real World Problems*; Nova Science Pub Inc.: Hauppauge, NY, USA, 2021.
5. Kuratowski, K. Sur les espaces. *Fund. Math.* **1930**, *15*, 301–309. [CrossRef]
6. Darbo, G. Punti uniti in trasformazioni a codominio non compatto. *Rend Sem. Mat. Univ. Padova* **1955**, *24*, 84–92.
7. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
8. Akhmerov, R.R.; Kamenskii, M.I.; Potapov, A.S.; Rodkina, A.E.; Sadovskij, B.N. *Measures of Noncompactness and Condensing Operators*; Nauka: Novosibirsk, Russia, 1986. (In Russian)
9. Banas, J.; Goebel, K. *Measure of Noncompactness in Banach Spaces*; Lecture Notes in Pure and Applied Mathematics; Dekker: New York, NY, USA, 1980.
10. Gabeleh, M.; Malkowsky, E.; Mursaleen, M.; Rakočević, V. A New Survey of Measures of Noncompactness and Their Applications. *Axioms* **2022**, *11*, 299. [CrossRef]

11. Toledano, J.M.A.; Benavides, T.D.; Acedo, G.L. *Measures of Noncompactness in Metric Fixed Point Theory*; Springer: Birkhäuser, Basel, 1997.
12. Brouwer, L.E.J. Über Abbildung von Mannigfaltigkeiten. *Math. Ann.* **1911**, *71*, 97–115. (In German) [CrossRef]
13. Leary, J.; Scauder, J. Topologie et equations fonctionelles. *Ann. Sci. l'École Norm. Supér.* **1934**, *51*, 45–78. [CrossRef]
14. Nussbaum, R.D. The fixed point index for local condensing maps. *Ann. Math.* **1971**, *89*, 217–258. [CrossRef]
15. Nussbaum, R.D. The fitted point index and asymptotic fixed point theorems for *k*-set contractions. *Bull. Amer. Math. Soc.* **1969**, *75*, 490–495. [CrossRef]
16. Nussbaum, R.D. Degree theory for local condensing maps. *J. Math. Anal. Appl.* **1972**, *37*, 741–766. [CrossRef]
17. Graef, J.R.; Henderson, J.; Ouahab, A. *Topological Methods for Differential Equations and Inclusions*, 1st ed.; CRC Press: Boca Raton, FL, USA, 2018.
18. Górniewicz, L. *Topological Fixed Point Theory of Multi-Valued Mappings, Mathematics and Its Applications*; Kluwer: Dordrecht, The Netherlands, 1999; Volume 495.
19. Jeribi, J.; Krichen, B. *Nonlinear Functional Analysis in Banach Spaces and Banach Algebras. Fixed Point Theory under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications*; Monographs and Research Notes in Mathematics; CRC Press: Boca Raton, FL, USA, 2016.
20. Mora, G.; Cherruault, Y. Characterization and generation of α-dense curves. *Comput. Math. Appl.* **1997**, *33*, 83–91. [CrossRef]
21. Sagan, H. *Space-Filling Curves*; Springer: New York, NY, USA, 1994.
22. García, G.; Mora, G. The degree of convex nondensifiability in Banach spaces. *J. Convex Anal.* **2015**, *22*, 871–888.
23. Mora, G.; Cherruault, Y. The theoretic calculation time associated to α-dense curves. *Kybernetes* **1998**, *27*, 919–939. [CrossRef]
24. Mora, G.; Redtwitz, D.A. Densifiable metric spaces. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM* **2011**, *105*, 71–83. [CrossRef]
25. Ali, A.; Eskandar, N. On the degree of nondensifiability and fixed point theorems in Banach spaces. *J. Nonlinear Convex Anal.* **2024**, *24*, 2143–2153.
26. García, G.; Mora, G. A fixed point result in Banach algebras based on the degree of nondensifiability and applications to quadratic integral equations. *J. Math. Anal. Appl.* **2019**, *472*, 1220–1235. [CrossRef]
27. García, G. Solvability of initial value problems with fractional order differential equations in Banach spaces by α-dense curves. *Fract. Calc. Appl. Anal.* **2017**, *20*, 646–661. [CrossRef]
28. García, G.; Mora, G. The degree of nondensifiability of linear bounded operators and its applications. *Appl. Gen. Topol.* **2024**, *25*, 213–228. [CrossRef]
29. Cherruault, Y.; Mora, G. *Optimisation Globale. Théorie des Courbes α-Denses*; Económica: Paris, France, 2005.
30. Mora, G.; Mira, J.A. Alpha-dense curves in infinite dimensional spaces. *Int. J. Pure Appl. Math.* **2003**, *5*, 257–266.
31. García, G. The minimal displacement problem of DND-Lipschitzian mappings. *Bol. Soc. Mat. Mex.* **2023**, *29*, 38. [CrossRef]
32. O'Regan, D.; Cho, Y.J.; Chen, Y.Q. *Topological Degree Theory and Applications*; Chapman & Hall_CRC: Boca Raton, FL, USA, 2006.
33. Balachandran, K.; Anandhi, E.R. Boundary controllability of integrodifferential systems in Banach spaces. *Proc. Indian Acad. Sci. (Math. Sci.)* **2001**, *111*, 127–135. [CrossRef]
34. Balachandran, K.; Anandhi, E.R. Controllability of neutral integrodifferential infinite delay systems in Banach spaces. *Taiwan J. Math.* **2004**, *8*, 689–702. [CrossRef]
35. Balachandran, K.; Leelamani, A. Null controllability of neutral integrodifferential systems with infinite delay. *Math. Probl. Eng.* **2006**, *2006*, 1–18. [CrossRef]
36. Bete, K.H.; Ogouyandjoua, C.; Diop, A.; Diop, M.A. On the attractivity of an integrodifferential system with state-dependent delay. *J. Nonlinear Sci. Appl.* **2019**, *12*, 611–620.
37. Bahuguna, D.; Srivastava, S.K. Semilinear integro-differential equations with compact semigroups. *Appl. Math. Stoch. Anal.* **1998**, *11*, 507–517. [CrossRef]
38. Sahraoui, F.; Ouahab, A. Existence and compactness of solutions of semilinear integrodifferential equations with finite delay. *Ukr. Math. J.* **2023**, *74*, 1404–1432. [CrossRef]
39. Grimmer, R.C. Resolvent opeators for integral equations in a Banach space. *Trans. Am. Math. Soc.* **1982**, *273*, 333–349. [CrossRef]

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