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Stability Results for Some Classes of Cubic Functional Equations

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Abstract: Applications involving functional equations (FUEQs) are commonplace. They are essential to various applications, such as fog computing. Ulam's notion of stability is highly helpful since it provides a range of estimates between exact and approximate solutions. Using Brzdęk's fixed point technique (FPT), we establish the stability of the following cubic type functional equations (CFUEQs): $\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)=2\mathcal{F}(\xi_1)+2\mathcal{F}(\xi_2)$, $2\mathcal{F}\left(\sqrt[3]{\frac{\xi_1^3+\xi_2^3}{2}}\right)=\mathcal{F}(\xi_1)+\mathcal{F}(\xi_2)$ for all $\xi_1,\xi_2\in\mathbb{R}$.

Keywords: Jensen cubic functional equation; fixed point theory; quadratic cubic functional equation; stability

MSC: 39B22; 47H10

1. Introduction

Functional equations (FUEQs) are widely used in a variety of fascinating contexts; see, e.g., [1–3]. Their applicability and inherent mathematical elegance make the theory of FUEQs attractive. FUEQs are crucial in a wide range of applications, providing an effective means of refining the models used to analyze different phenomena.

The concept of stability was introduced by Ulam [4], followed by Hyers [5,6], and it has been widely studied in mathematics and FUEQs. There is some connection between it and the topics covered in other branches of mathematics, such as shadowing (see [7]), approximation theory, and optimization. It concerns various equations (functional, differential, integral, difference, etc.), in order to determine how much an equation's approximate answer and its exact solution vary from one another. This topic has inspired many papers, and we refer the reader to [6,8–11] for further information on this subject.

In addition, many researchers have studied the stabilities of Drygas [12,13], the Pexiderized additive-quadratic equation (see [14]), the Frechet FUEQ [15], radical FUEQs [16–18], the quadratic FUEQ [19–21], and systems of FUEQs [22,23].

Many investigations on stability have been conducted in recent years, using various techniques to generate results that have been used for various functional inequalities, FUEQs, derivations, antiderivatives, and homomorphisms (see [24–28]).

In this work, applying the Brzdęk FPT, we determine the stability results of the quadratic CFUEQ

$$\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)=2\mathcal{F}(\xi_1)+2\mathcal{F}(\xi_2) \tag{1}$$

and the Jensen CFUEQ

$$2\mathcal{F}\left(\sqrt[3]{\frac{\xi_1^3+\xi_2^3}{2}}\right)=\mathcal{F}(\xi_1)+\mathcal{F}(\xi_2) \tag{2}$$

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Citation: El-hady, E.-s.; Sayyari, Y.; Dehghanian, M.; Alruwaily, Y. Stability Results for Some Classes of Cubic Functional Equations. *Axioms* **2024**, *13*, 480. https://doi.org/10.3390/axioms13070480

Academic Editors: Ravi Shankar and Qun Liu

Received: 4 June 2024
Revised: 2 July 2024
Accepted: 5 July 2024
Published: 18 July 2024

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for all $\xi_1, \xi_2 \in \mathbb{R}$.

For example, the function $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{F}(\xi_1) = c\xi_1^6$ is a solution of (1), and $\mathcal{F}(\xi_1) = c\xi_1^3$ is a solution of (2), where c is a constant.

In 2011, Brzdęk et al. [29] gave a simple FPT. Before presenting the Brzdęk FPT, let us introduce some hypotheses, which we use in this work.

H1. *A is a nonempty set, and E is a Banach space.*

H2. *$\sigma_1, \ldots, \sigma_k : A \to A$ and $\lambda_1, \ldots, \lambda_k : A \to \mathbb{R}_+$ are given maps.*

H3. *$\mathcal{T} : E^A \to E^A$ is an operator satisfying the inequality*

$$\|\mathcal{T}f(\xi_1) - \mathcal{T}g(\xi_1)\| \leq \sum_{i=1}^{k} \lambda_i(\xi_1)\|f(\sigma_i(\xi_1)) - g(\sigma_i(\xi_1))\|,$$

for all $f, g : A \to E$ and $\xi_1 \in A$.

H4. *$\Delta : \mathbb{R}_+^A \to \mathbb{R}_+^A$ is a linear operator defined by*

$$\Delta\mu(\xi_1) := \sum_{i=1}^{k} \lambda_i(\xi_1)\mu(\sigma_i(\xi_1)),$$

for $\mu : A \to \mathbb{R}_+$ and $\xi_1 \in A$.

Theorem 1 ([29])**.** *Suppose that* (H1)–(H4) *are satisfied. Assume that there are functions $\theta : A \to \mathbb{R}_+$ and $\psi : A \to E$: $\forall \xi_1 \in A$, such that*

$$\|\mathcal{T}\psi(\xi_1) - \psi(\xi_1)\| \leq \theta(\xi_1) \quad \text{and} \quad \theta^*(\xi_1) := \sum_{n=0}^{\infty} \Delta^n\theta(\xi_1) < \infty$$

hold. Then, for all $\xi_1 \in A$, the limit

$$S(\xi_1) := \lim_{n\to\infty} \mathcal{T}^n\psi(\xi_1)$$

exists, and the function $S : A \to E$, so defined, is a unique FP of $\mathcal{T}$ with

$$\|\psi(\xi_1) - S(\xi_1)\| \leq \theta^*(\xi_1),$$

$\forall \xi_1 \in A$.

2. Preliminaries

We use this section to recall some interesting stability results of the CFUEQs. The following FE is considered the oldest CFUEQ, introduced by Rassias (see [30]),

$$\Pi(a + 2b) + 3\Pi(a) = 3\Pi(a + b) + \Pi(a - b) + 6\Pi(b).$$

In [31], the authors investigated the general solution and the generalized Hyers–Ulam–Rassias stability for the following CFUEQ:

$$\Pi(2a + b) + \Pi(2a - b) = 2\Pi(a + b) + 2\Pi(a - b) + 12\Pi(a).$$

We highlight the papers [30,32] concerning the stability of the CFUEQ.

In this work, suppose that E is a Banach space, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3. Stability of (1)

Applying the Brzdęk FPT, we determine the stability of (1).

Theorem 2. *Suppose that a function $\mathcal{F} : \mathbb{R} \to E$ satisfies*

$$\left\|\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)-2\mathcal{F}(\xi_1)-2\mathcal{F}(\xi_2)\right\| \leq \delta(|\xi_1|^p+|\xi_2|^p) \tag{3}$$

for some $\delta, p \geq 0$, $p \neq 6$, and all $\xi_1, \xi_2 \in \mathbb{R}$. Then, there exists a unique function $\mathcal{Q} : \mathbb{R} \to E$ satisfying the quadratic CFUEQ on $\mathbb{R}$ with

$$\|\mathcal{F}(\xi_1)-\mathcal{Q}(\xi_1)\| \leq \frac{2\delta}{\left|\left(\sqrt[3]{2}\right)^p-4\right|}|\xi_1|^p$$

$\forall \xi_1 \in \mathbb{R}$.

Proof. Setting $\xi_1 = \xi_2 = 0$ in (3), we obtain $\mathcal{F}(0) = 0$. For the case $p < 6$, replacing ξ_2 by ξ_1 in (3), we obtain

$$\left\|\mathcal{F}(\xi_1)-\frac{1}{4}\mathcal{F}\left(\sqrt[3]{2}\xi_1\right)\right\| \leq \frac{\delta}{2}|\xi_1|^p, \tag{4}$$

$\forall \xi_1 \in \mathbb{R}$. Let $\mathcal{T} : E^{\mathbb{R}} \to E^{\mathbb{R}}$ and $\theta : \mathbb{R} \to \mathbb{R}_+$ be defined by

$$\mathcal{T}f(\xi_1) = \frac{1}{4}f\left(\sqrt[3]{2}\xi_1\right), \qquad f \in E^{\mathbb{R}},$$

and $\theta(\xi_1) = \frac{\delta}{2}|\xi_1|^p$ $\forall \xi_1 \in \mathbb{R}$. Thus, we can write (4) as $\|\mathcal{T}\mathcal{F}(\xi_1) - \mathcal{F}(\xi_1)\| \leq \theta(\xi_1)$ $\forall \xi_1 \in \mathbb{R}$. So,

$$\|\mathcal{T}f(\xi_1)-\mathcal{T}g(\xi_1)\| \leq \frac{1}{4}\left\|f\left(\sqrt[3]{2}\xi_1\right)-g\left(\sqrt[3]{2}\xi_1\right)\right\|,$$

$\forall f, g \in E^{\mathbb{R}}$ and $\xi_1 \in \mathbb{R}$. Hence $\mathcal{T} : E^{\mathbb{R}} \to E^{\mathbb{R}}$ satisfies the condition (H3) with $\lambda_1(\xi_1) = \frac{1}{4}$ and $\sigma_1(\xi_1) = \sqrt[3]{2}\xi_1$. By (H4), the operator $\Delta : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ is defined by

$$\Delta\mu(\xi_1) = \frac{1}{4}\mu\left(\sqrt[3]{2}\xi_1\right), \qquad \mu \in \mathbb{R}_+^{\mathbb{R}},$$

$\forall \xi_1 \in \mathbb{R}$. Hence,

$$\Delta\theta(\xi_1) = \frac{1}{4}\theta\left(\sqrt[3]{2}\xi_1\right) = \frac{2^{\frac{p}{3}}}{4}\theta(\xi_1), \qquad \theta \in \mathbb{R}_+^{\mathbb{R}},$$

$\forall \xi_1 \in \mathbb{R}$. Using induction and linearity Δ, it follows that

$$\Delta^m\theta(\xi_1) = \left(\frac{2^{\frac{p}{3}}}{4}\right)^m\theta(\xi_1), \qquad m \in \mathbb{N}_0,$$

$\forall \xi_1 \in \mathbb{R}$. Hence, the series $\sum_{m=0}^{\infty}\Delta^m\theta(\xi_1)$ is convergent to

$$\theta^*(\xi_1) = \sum_{m=0}^{\infty}\Delta^m\theta(\xi_1) = \frac{2\delta}{4-\left(\sqrt[3]{2}\right)^p}|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$ and $p < 6$. Using Theorem 1, there is a function $\mathcal{Q} : \mathbb{R} \to E$, such that

$$\mathcal{Q}(\xi_1) = \lim_{n\to\infty}\mathcal{T}^n\mathcal{F}(\xi_1), \qquad \mathcal{Q}(\xi_1) = \frac{1}{4}\mathcal{Q}\left(\sqrt[3]{2}\xi_1\right),$$

and

$$\|\mathcal{F}(\xi_1)-\mathcal{Q}(\xi_1)\| \leq \frac{2\delta}{4-\left(\sqrt[3]{2}\right)^p}|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. From (3), we arrive at

$$\begin{aligned}
&\left\| \mathcal{TF}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right) + \mathcal{TF}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right) - 2\mathcal{TF}(\xi_1) - 2\mathcal{TF}(\xi_2) \right\| \\
&\leq \frac{1}{4}\left\| \mathcal{F}\left(\sqrt[3]{2\xi_1^3+2\xi_2^3}\right) + \mathcal{F}\left(\sqrt[3]{2\xi_1^3-2\xi_2^3}\right) - 2\mathcal{F}\left(\sqrt[3]{2}\xi_1\right) - 2\mathcal{F}\left(\sqrt[3]{2}\xi_2\right) \right\| \\
&\leq \frac{\left(\sqrt[3]{2}\right)^p \delta}{4}(|\xi_1|^p + |\xi_2|^p),
\end{aligned}$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$. By induction on $m \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
&\left\| \mathcal{T}^m\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right) + \mathcal{T}^m\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right) - 2\mathcal{T}^m\mathcal{F}(\xi_1) - 2\mathcal{T}^m\mathcal{F}(\xi_2) \right\| \\
&\leq \delta\left(\frac{\left(\sqrt[3]{2}\right)^p}{4}\right)^m (|\xi_1|^p + |\xi_2|^p),
\end{aligned}$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$. This yields

$$\mathcal{Q}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right) + \mathcal{Q}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right) = 2\mathcal{Q}(\xi_1) + 2\mathcal{Q}(\xi_2),$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$. For $p > 6$, replacing ξ_1 and ξ_2 by $\frac{x}{\sqrt[3]{2}}$ in (3), we have

$$\|\mathcal{F}(\xi_1) - 4\mathcal{F}\left(\frac{\xi_1}{\sqrt[3]{2}}\right)\| \leq \frac{2\delta}{\left(\sqrt[3]{2}\right)^p}|\xi_1|^p, \tag{5}$$

$\forall \xi_1 \in \mathbb{R}$. Define $\mathcal{T} : E^{\mathbb{R}} \to E^{\mathbb{R}}$ and $\theta : \mathbb{R} \to \mathbb{R}_+$ by

$$\mathcal{T}f(\xi_1) = 4f\left(\frac{x}{\sqrt[3]{2}}\right) \quad \text{and} \quad \theta(\xi_1) = \frac{2\delta}{\left(\sqrt[3]{2}\right)^p}|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. So, we can write (5) as $\|\mathcal{TF}(\xi_1) - \mathcal{F}(\xi_1)\| \leq \theta(\xi_1)\ \forall \xi_1 \in \mathbb{R}$. Therefore,

$$\|\mathcal{T}f(\xi_1) - \mathcal{T}g(\xi_1)\| \leq 4\left\| f\left(\frac{\xi_1}{\sqrt[3]{2}}\right) - g\left(\frac{\xi_1}{\sqrt[3]{2}}\right)\right\|,$$

$\forall f, g \in E^{\mathbb{R}}$ and $\xi_1 \in \mathbb{R}$. We apply Theorem 1 with $\lambda_1(\xi_1) = 4$, $\sigma_1(\xi_1) = \frac{\xi_1}{\sqrt[3]{2}}$ and $\Delta : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ by

$$\Delta\mu(\xi_1) = 4\mu\left(\frac{\xi_1}{\sqrt[3]{2}}\right), \qquad \mu \in \mathbb{R}_+^{\mathbb{R}},$$

$\forall \xi_1 \in \mathbb{R}$. Accordingly,

$$\Delta\theta(\xi_1) = 4\theta\left(\frac{\xi_1}{\sqrt[3]{2}}\right) = \frac{4}{\left(\sqrt[3]{2}\right)^p}\theta(\xi_1), \qquad \theta \in \mathbb{R}_+^{\mathbb{R}},$$

$\forall \xi_1 \in \mathbb{R}$; thus,

$$\Delta^m\theta(\xi_1) = \left(\frac{4}{\left(\sqrt[3]{2}\right)^p}\right)^m \theta(\xi_1),$$

$\forall \xi_1 \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Since $p > 6$,

$$\theta^*(\xi_1) := \sum_{m=0}^{\infty} \Delta^m \theta(\xi_1) = \frac{2\delta}{\left(\sqrt[3]{2}\right)^p - 4} |\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. Using Theorem 1, there exists a function $\mathcal{Q} : \mathbb{R} \to E$, such that

$$\mathcal{Q}(\xi_1) = \lim_{n\to\infty} \mathcal{T}^n \mathcal{F}(\xi_1), \qquad \mathcal{Q}(\xi_1) = 4\mathcal{Q}\left(\frac{\xi_1}{\sqrt[3]{2}}\right),$$

and

$$\|\mathcal{Q}(\xi_1) - \mathcal{F}(\xi_1)\| \leq \frac{2\delta}{\left(\sqrt[3]{2}\right)^p - 4} |\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$.

Similar to the previous case, it can be easily shown that $\mathcal{Q}$ satisfies FE (1).

Finally, we prove, only for case $p < 6$, that $\mathcal{Q}$ is unique. The proof of the case $p > 6$ is similar to the proof of the case $p < 6$.

Suppose that $\mathcal{Q}_1, \mathcal{Q}_2 : \mathbb{R} \to E$ satisfy the quadratic CFUEQs on $\mathbb{R}$ and

$$\|\mathcal{Q}_1(\xi_1) - \mathcal{F}(\xi_1)\| \leq \delta_1 |\xi_1|^p, \qquad \|\mathcal{Q}_2(\xi_1) - \mathcal{F}(\xi_1)\| \leq \delta_2 |\xi_1|^p,$$

for some $\delta_1, \delta_2 \geq 0$ and for all $\xi_1 \in \mathbb{R}$. Thus,

$$\|\mathcal{Q}_1(\xi_1) - \mathcal{Q}_2(\xi_1)\| \leq (\delta_1 + \delta_2)|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. Hence,

$$\mathcal{Q}_1(\xi_1) = \frac{1}{4}\mathcal{Q}_1\left(\sqrt[3]{2}\xi_1\right), \qquad \mathcal{Q}_2(\xi_1) = \frac{1}{4}\mathcal{Q}_2\left(\sqrt[3]{2}\xi_1\right),$$

$\forall \xi_1 \in \mathbb{R}$. Therefore,

$$\|\mathcal{Q}_1(\xi_1) - \mathcal{Q}_2(\xi_1)\| \leq \frac{1}{4}\left\|\mathcal{Q}_1\left(\sqrt[3]{2}\xi_1\right) - \mathcal{Q}_2\left(\sqrt[3]{2}\xi_1\right)\right\| \leq \frac{\left(\sqrt[3]{2}\right)^p}{4}(\delta_1 + \delta_2)|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. By induction on $m \in \mathbb{N}_0$, we see that

$$\|\mathcal{Q}_1(\xi_1) - \mathcal{Q}_2(\xi_1)\| \leq \left(\frac{\left(\sqrt[3]{2}\right)^p}{4}\right)^m (\delta_1 + \delta_2)|\xi_1|^p,$$

which tends to 0 as $m \to \infty$ for all $\xi_1 \in \mathbb{R}$. This implies $\mathcal{Q}_1(\xi_1) = \mathcal{Q}_2(\xi_1)$ $\forall \xi_1 \in \mathbb{R}$. □

In the following example, we prove that, for $p = 6$, the quadratic CFUEQ is not stable.

Example 1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\psi(\xi_1) = \begin{cases} \xi_1^6 & |\xi_1| < 1 \\ 1 & |\xi_1| \geq 1 \end{cases},$$

and suppose that $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{F}(\xi_1) = \sum_{n=0}^{\infty} \frac{\psi\left(2^{\frac{n}{3}}\xi_1\right)}{4^n},$$

for all $\xi_1 \in \mathbb{R}$. *We prove that*

$$\left|\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)-2\mathcal{F}(\xi_1)-2\mathcal{F}(\xi_2)\right| \leq 32\left(\xi_1^6+\xi_2^6\right),$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$; *but, there is no constant* $k \geq 0$ *and no function* $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ *satisfying* (1), *and*

$$|\mathcal{F}(\xi_1)-\mathcal{Q}(\xi_1)| \leq k\xi_1^6,$$

$\forall \xi_1 \in \mathbb{R}$.

Proof. Clearly, we see that $|\mathcal{F}(\xi_1)| \leq \frac{4}{3}$, for all $\xi_1 \in \mathbb{R}$. Now, we consider the two following cases.

Case 1: If $\xi_1, \xi_2 \in \mathbb{R}$ and $\xi_1^6+\xi_2^6 \geq \frac{1}{4}$, then

$$\left|\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)-2\mathcal{F}(\xi_1)-2\mathcal{F}(\xi_2)\right| \leq 8 \leq 32\left(\xi_1^6+\xi_2^6\right).$$

Case 2: Suppose that $\xi_1, \xi_2 \in \mathbb{R}$ and $\xi_1^6+\xi_2^6 < \frac{1}{4}$; hence, $\xi_1^6-\xi_2^6 < \frac{1}{4}$, $\xi_1^6 < \frac{1}{4}$, and $\xi_2^6 < \frac{1}{4}$. Then, there exists $N \in \mathbb{N}$, such that

$$\frac{1}{4^{N+1}} \leq \xi_1^6+\xi_2^6 < \frac{1}{4^N}.$$

Therefore, $2^n\left(\xi_1^6+\xi_2^6\right) < 1$, $\left|2^n\left(\xi_1^6-\xi_2^6\right)\right| < 1$, $2^n\xi_1^6 < 1$, and $2^n\xi_2^6 < 1$, for all $n = 0, 1, \ldots, N-1$. Hence,

$$\begin{aligned}
&\left|\mathcal{F}\left(\sqrt[3]{\xi_1^3+\xi_2^3}\right)+\mathcal{F}\left(\sqrt[3]{\xi_1^3-\xi_2^3}\right)-2\mathcal{F}(\xi_1)-2\mathcal{F}(\xi_2)\right| \\
&= \left|\sum_{n=0}^{\infty}\frac{\psi\left(2^{\frac{n}{3}}\sqrt[3]{\xi_1^3+\xi_2^3}\right)}{4^n}+\sum_{n=0}^{\infty}\frac{\psi\left(2^{\frac{n}{3}}\sqrt[3]{\xi_1^3-\xi_2^3}\right)}{4^n}-2\sum_{n=0}^{\infty}\frac{\psi\left(2^{\frac{n}{3}}\xi_1\right)}{4^n}-2\sum_{n=0}^{\infty}\frac{\psi\left(2^{\frac{n}{3}}\xi_2\right)}{4^n}\right| \\
&\leq 6\sum_{n=N}^{\infty}\frac{1}{4^n} = \frac{2}{4^{N-1}} \leq 32\left(\xi_1^6+\xi_2^6\right),
\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$.

Suppose that there exists a constant $k \geq 0$ and a function $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ satisfying (1) with

$$|\mathcal{F}(\xi_1)-\mathcal{Q}(\xi_1)| \leq k\xi_1^6, \tag{6}$$

for all $\xi_1 \in \mathbb{R}$. Hence, $|\mathcal{Q}(\xi_1)| \leq k\xi_1^6+\frac{4}{3}$, for all $\xi_1 \in \mathbb{R}$. Substituting $2^{\frac{n}{3}}\xi_1$ in the place of ξ_1 in the above inequality yields

$$\left|\mathcal{Q}\left(2^{\frac{n}{3}}\xi_1\right)\right| \leq 4^n k\xi_1^6+\frac{4}{3},$$

for all $\xi_1 \in \mathbb{R}$. Dividing the expression obtained by 4^n, we obtain

$$\left|\frac{1}{4^n}\mathcal{Q}\left(2^{\frac{n}{3}}\xi_1\right)\right| \leq k\xi_1^6+\frac{1}{3\times 4^{n-1}}, \tag{7}$$

for all $\xi_1 \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $\mathcal{Q}(\xi_1) = \frac{1}{4^n}\mathcal{Q}\left(2^{\frac{n}{3}}\xi_1\right)$, for all $\xi_1 \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$|\mathcal{Q}(\xi_1)| \leq k\xi_1^6, \tag{8}$$

for all $\xi_1 \in \mathbb{R}$. From (6) and (8), we have

$$\left|\frac{\mathcal{F}(\xi_1)}{\xi_1^6}\right| \leq 2k,$$

for all $\xi_1 \in \mathbb{R}$.

Choose $N \in \mathbb{N}$, such that $N > 2k$, and take $\xi_1 > 0$ with $\xi_1 < \frac{1}{2^{\frac{N-1}{3}}}$. Thus, $2^{\frac{n}{3}}\xi_1 < 1$ for $n = 0, 1, \ldots, N-1$, and

$$\mathcal{F}(\xi_1) = \sum_{n=0}^{N-1} \frac{\psi\left(2^{\frac{n}{3}}\xi_1\right)}{4^n} + \sum_{n=N}^{\infty} \frac{\psi\left(2^{\frac{n}{3}}\xi_1\right)}{4^n} \geq N\xi_1^6,$$

which implies

$$\frac{\mathcal{F}(\xi_1)}{\xi_1^6} \geq N > 2k,$$

which is a contradiction. □

4. Stability of the Jensen Cubic Functional Equation

Using the Brzdęk FPT we determine the stability of the Jensen CFUEQs (2) for $\bigcap p > 3$.

Theorem 3. *Assume that E is a Banach space. Suppose that there is a function $h : \mathbb{R} \to E$ satisfying*

$$\left\|2\mathcal{F}\left(\sqrt[3]{\frac{\xi_1^3 + \xi_2^3}{2}}\right) - \mathcal{F}(\xi_1) - \mathcal{F}(\xi_2)\right\| \leq \delta(|\xi_1|^p + |\xi_2|^p), \tag{9}$$

for some $\delta \geq 0$, $p > 3$ and all $\xi_1, \xi_2 \in \mathbb{R}$. Then, there exists a unique function $\mathcal{J} : \mathbb{R} \to E$ satisfying the Jensen CFUEQs on $\mathbb{R}$:

$$\|\mathcal{F}(\xi_1) - \mathcal{J}(\xi_1)\| \leq \frac{2 + 2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}}{1 - 2^{\frac{3-p}{3}} - 2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}}\delta|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$.

Proof. Fix $p > 3$. The replacement of ξ_2 by $\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1$ in (9) yields

$$\begin{aligned}&\left\|\mathcal{F}(\xi_1) + \mathcal{F}\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1\right) - 2\mathcal{F}\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right)\right\| \\ &\leq \left[1 + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}\right]\delta|\xi_1|^p\end{aligned}, \tag{10}$$

$\forall \xi_1 \in \mathbb{R}$. Let $\mathcal{T} : E^{\mathbb{R}} \to E^{\mathbb{R}}$ and $\theta : \mathbb{R} \to \mathbb{R}_+$ be defined by

$$\mathcal{T}f(\xi_1) = 2f\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right) - f\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1\right), \quad f \in E^{\mathbb{R}}$$

and

$$\theta(\xi_1) = \left[1 + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}\right]\delta|\xi_1|^p,$$

$\forall \xi_1 \in \mathbb{R}$. Then, we can write (10) as $\|\mathcal{F}(\xi_1) - \mathcal{T}\mathcal{F}(\xi_1)\| \leq \theta(\xi_1)\ \forall \xi_1 \in \mathbb{R}$. Thus,

$$\begin{aligned}\|\mathcal{T}f(\xi_1) - \mathcal{T}g(\xi_1)\| \leq & 2\left\|f\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right) - g\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right)\right\| \\ & + \left\|f\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1\right) - g\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1\right)\right\|,\end{aligned}$$

$\forall f, g \in E^{\mathbb{R}}$ and $\xi_1 \in \mathbb{R}$. Hence, $\mathcal{T} : E^{\mathbb{R}} \to E^{\mathbb{R}}$ satisfies the condition (H3) with $\lambda_1(\xi_1) = 2$, $\lambda_2(\xi_1) = 1$,

$$\sigma_1(\xi_1) = \left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1 \quad \text{, and} \quad \sigma_2(\xi_1) = \sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1.$$

By (H4), the operator $\Delta : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ is defined by

$$\Delta\mu(\xi_1) = 2\mu\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right) + \mu\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1}\xi_1\right),$$

$\forall \xi_1 \in \mathbb{R}$. Hence,

$$\Delta\theta(\xi_1) = \left(\frac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}\right)\theta(\xi_1), \quad \theta \in \mathbb{R}^{\mathbb{R}},$$

$\forall \xi_1 \in \mathbb{R}$. Since Δ is linear, by induction on $m \in \mathbb{N}_0$,

$$\Delta^m\theta(\xi_1) = \left(\frac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}}\right)^m \theta(\xi_1),$$

$\forall \xi_1 \in \mathbb{R}$. Since

$$\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{1}{3}} < \left(\frac{1-2^{\frac{3-p}{3}}}{2}\right)^{\frac{1}{p}},$$

$\forall p > 3$,

$$\frac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}} - 1\right)^{\frac{p}{3}} < 1.$$

Consequently, the series $\sum_{m=0}^{\infty}\Delta^m\theta(\xi_1)$ is convergent for all $\xi_1\in\mathbb{R}$, and

$$\theta^*(\xi_1)=\sum_{m=0}^{\infty}\Delta^m\theta(\xi_1)=\sum_{m=0}^{\infty}\left(\frac{1+2^{\frac{3-p}{3}}}{2}+\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}\right)^m\theta(\xi_1)$$
$$=\frac{2+2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}}{1-2^{\frac{3-p}{3}}-2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}}\delta|\xi_1|^p,$$

$\forall\xi_1\in\mathbb{R}$. By the use of Theorem 1, there is a function $\mathcal{J}:\mathbb{R}\to E$: $\mathcal{J}(\xi_1)=\lim_{n\to\infty}\mathcal{T}^n\mathcal{F}(\xi_1)$,

$$\mathcal{J}(\xi_1)=2\mathcal{J}\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right)-\mathcal{J}\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1}\xi_1\right),$$

and

$$\|\mathcal{J}(\xi_1)-\mathcal{F}(\xi_1)\|\le\frac{2+2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}}{1-2^{\frac{3-p}{3}}-2\left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}}\delta|\xi_1|^p,$$

$\forall\xi_1\in\mathbb{R}$. Now, we show that $\mathcal{J}$ satisfies the Jensen CFUEQs. From (9), we have

$$\begin{aligned}
&\left\|2\mathcal{T}\mathcal{F}\left(\sqrt[3]{\tfrac{\xi_1^3+\xi_2^3}{2}}\right)-\mathcal{T}\mathcal{F}(\xi_1)-\mathcal{T}\mathcal{F}(\xi_2)\right\|\\
&=\left\|4\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\sqrt[3]{\tfrac{\xi_1^3+\xi_2^3}{2}}\right)-2\mathcal{F}\left(\sqrt[3]{\left(2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)\left(\tfrac{\xi_1^3+\xi_2^3}{2}\right)}\xi_1\right)\right.\\
&\quad-2\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right)+\mathcal{F}\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1}\xi_1\right)\\
&\quad\left.-2\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_2\right)+\mathcal{F}\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1}\xi_2\right)\right\|\\
&\le 2\left\|2\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\sqrt[3]{\tfrac{\xi_1^3+\xi_2^3}{2}}\right)-\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_1\right)-\mathcal{F}\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{1}{p}}\xi_2\right)\right\|\\
&\le\left\|2\mathcal{F}\left(\sqrt[3]{\left(2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)\left(\tfrac{\xi_1^3+\xi_2^3}{2}\right)}\xi_1\right)-\mathcal{F}\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1}\xi_1\right)\right.\\
&\quad\left.-\mathcal{F}\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1}\xi_2\right)\right\|\\
&\le\delta\left(\tfrac{1+2^{\frac{3-p}{3}}}{2}+\left(2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4}\right)^{\frac{3}{p}}-1\right)^{\frac{p}{3}}\right)(|\xi_1|^p+|\xi_2|^p),
\end{aligned}$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$. By induction on $m \in \mathbb{N}_0$, we obtain

$$\begin{aligned}&\left\| 2\mathcal{T}^m \mathcal{F}\left(\sqrt[3]{\frac{\xi_1^3+\xi_2^3}{2}}\right) - \mathcal{T}^m \mathcal{F}(\xi_1) - \mathcal{T}^m \mathcal{F}(\xi_2) \right\| \\ &\leq \delta \left(\frac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\frac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1 \right)^{\frac{p}{3}} \right)^m (|\xi_1|^p + |\xi_2|^p),\end{aligned}$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$. Letting $m \to \infty$, we conclude that

$$2\mathcal{J}\left(\sqrt[3]{\frac{\xi_1^3+\xi_2^3}{2}}\right) = \mathcal{J}(\xi_1) + \mathcal{J}(\xi_2),$$

$\forall \xi_1, \xi_2 \in \mathbb{R}$.

Uniqueness: Consider two functions $\mathcal{J}_1, \mathcal{J}_2 : \mathbb{R} \to E$ satisfying (2) with

$$\|\mathcal{F}(\xi_1) - \mathcal{J}_i(\xi_1)\| \leq L|\xi_1|^p, \quad i = 1, 2$$

for some $L > 0$ and all $\xi_1 \in \mathbb{R}$. Therefore,

$$\|\mathcal{J}_1(\xi_1) - \mathcal{J}_2(\xi_1)\| \leq 2L|\xi_1|^p, \tag{11}$$

$\forall \xi_1 \in \mathbb{R}$.

Also, $\mathcal{J}_1$ and $\mathcal{J}_2$ satisfy (2); thus,

$$\mathcal{J}_i(\xi_1) = 2\mathcal{J}_i\left(\left(\frac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{1}{p}} \xi_1 \right) - \mathcal{J}_i\left(\sqrt[3]{2\left(\frac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1}\,\xi_1 \right), \qquad i = 1, 2,$$

$\forall \xi_1 \in \mathbb{R}$. Therefore,

$$\begin{aligned}\|\mathcal{J}_1(\xi_1) - \mathcal{J}_2(\xi_1)\| &\leq 2\left\| \mathcal{J}_1\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{1}{p}} \xi_1 \right) - \mathcal{J}_2\left(\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{1}{p}} \xi_1 \right) \right\| \\ &\quad + \left\| \mathcal{J}_1\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1}\,\xi_1 \right) - \mathcal{J}_2\left(\sqrt[3]{2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1}\,\xi_1 \right) \right\| \\ &\leq \left(\tfrac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1 \right)^{\frac{p}{3}} \right) 2L|\xi_1|^p,\end{aligned}$$

$\forall \xi_1 \in \mathbb{R}$. By induction on $m \in \mathbb{N}_0$, we obtain

$$\|\mathcal{J}_1(\xi_1) - \mathcal{J}_2(\xi_1)\| \leq \left(\tfrac{1+2^{\frac{3-p}{3}}}{2} + \left(2\left(\tfrac{1+2^{\frac{3-p}{3}}}{4} \right)^{\frac{3}{p}} - 1 \right)^{\frac{p}{3}} \right)^m 2L|\xi_1|^p \ ,$$

$\forall \xi_1 \in \mathbb{R}$ and all $m \in \mathbb{N}_0$. This means that $\mathcal{J}_1 = \mathcal{J}_2$. □

5. Conclusions

We employed a well-known FPT to study the Ulam stability of some classes of symmetric CFUEQ. Stability helped us to obtain estimates between the exact and approximate

solutions in many cases. One can find such results for a more generalized version of the FUEQ. Future directions will involve studying the Hyers–Ulam–Rassias stability of much more complicated FUEQs or in some new function spaces.

Author Contributions: Conceptualization, Y.S., M.D. and E.-s.E.-h.; methodology, Y.S., M.D. and E.-s.E.-h.; software, Y.S., M.D., Y.A. and E.-s.E.-h.; validation, Y.S., M.D. and E.-s.E.-h.; formal analysis, Y.S., M.D. and Y.A.; investigation, Y.S. and M.D.; data curation, Y.S., M.D., Y.A. and E.-s.E.-h.; writing—original draft preparation, Y.S. and M.D.; writing—review and editing, Y.S., M.D. and E.-s.E.-h.; visualization, Y.S. and M.D.; supervision, Y.S. and M.D.; project administration, Y.S., M.D. and E.-s.E.-h.; funding acquisition, E.-s.E.-h. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the Deanship of Graduate Studies and Scientific Research at Jouf University under grant No. (DGSSR-2024-02-01049).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding authors.

Conflicts of Interest: The authors declare no conflicts of interest.

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