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A Note on the Multiplicity of the Distinguished Points

Weiping Li [1,*] and Xiaoshen Wang [2,*]

[1] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
[2] Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
* Correspondence: w.li@okstate.edu (W.L.); xxwang@ualr.edu (X.W.)

Abstract: Let $P(x)$ be a system of polynomials in s variables, where $x \in C^s$. If z_0 is an isolated zero of P, then the multiplicity and its structure at z_0 can be revealed by the normal set of the quotient ring $R(< P >)$ or its dual space R^* or by certain numerical methods. In his book titled "Numerical Polynomial Algebra", Stetter described the so-called distinguished points, which are embedded in a zero manifold of P, and the author defined their multiplicities. In this note, we will generalize the definition of distinguished points and give a more appropriate definition for their multiplicity, as well as show how to calculate the multiplicity of these points.

Keywords: multiplicity; distinguished points; Puiseux series

MSC: 55-XX

Let $P(x)$ be a system of polynomials in s variables, where $x \in C^s$. If z_0 is an isolated zero of P, then the multiplicity and its structure at z_0 can be revealed by the normal set of the quotient ring $R(< P >)$ or its dual space R^* (cf. [1]), and a numerical method has been developed by Dayton and Zeng [2]. In [1], Stetter described the so-called distinguished points, which are embedded in a zero manifold of P, and defined their multiplicity. However, it turns out that in most of the cases, the points that display some kind of "distinguished" property do not satisfy the definitions for distinguished points. In addition, the multiplicity defined there is not quite appropriate. So in this note, we will generalize the definition of distinguished points and give a more appropriate definition for their multiplicity, as well as show how to calculate the multiplicity of these points.

Example 1. *For a polynomial f in one variable, if $f(x) = (x-a)^m q(x)$, where $q(a) \neq 0$, then a is a zero of f with multiplicity m. Equivalently, if there exists a maximal set of linearly independent linear functionals $\{c_1, \ldots, c_m\}$ evaluating at a such that $c_i(f) = 0$ and $c_i(gf) = 0$ for all g, then the multiplicity of f at a is m. An obvious choice of a set of such functionals is $c_i = \frac{d^{i-1}}{(i-1)!dx^{i-1}}, i = 1, \ldots, m$ evaluating at $x = a$.*

Now, we consider a perturbed equation

$$H(x,t) = f(x) - t = 0,$$

where t is a complex parameter. It is not too hard to see that the solutions to the equation can be expanded as a Puiseux series in t: $x_i(t) = a + u_i t^{(1/m)}/q(a)^{1/m} +$ higher order terms, $i = 1, \ldots, m$, where u_i is an mth primitive root of unity converging to $x = a$. Define $d_1(f) = f(x_1) = f[x_1]$, $d_2(f) = f[x_1, x_2] = (f[x_2] - f[x_1])/(x_2 - x_1), \ldots, d_{m-1} = (f[x_1, \ldots, x_{m-1}] - f[x_2, \ldots, x_m])/(x_m - x_1)$, where $f[x_1, \ldots, x_{k+1}]$ denotes the standard Newton's kth divided difference. Apparently, the d_is depend on t. When $t \to 0$, the d_is "converge" to c_is.

On the other hand, if the zero set of the perturbed polynomial $H(x,t)$ can be expanded as a Puiseux series in t—$x = x_i(t), i = 1, \ldots, m$ near $x = a$—then the multiplicity of a is m, since this means that for each $0 < |t| << 1$, $H(x,t)$ has exactly m zeros.

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For polynomials with more variables, the definition above can be generalized. The Puiseux series has long been used to derive the bound of the number of zeros for the system of polynomial equations [3,4]. We will show that the Puiseux series can be used to reveal the multiplicity of a point isolated or not.

Example 2. *Suppose that* $P(x,y) = (x^2, y^2)^t$, *where* $(x,y) \in C^2$. *It can be shown that* $z_0 = (0,0)$ *is a zero of P with a multiplicity of four. Let*

$$P_1(x,y) = -\begin{pmatrix} c_{10} + c_{11}x + c_{12}y + \cdots \\ c_{20} + c_{21}x + c_{22}y + \cdots \end{pmatrix}$$

be a generic polynomial and define a perturbed system $Q_t := P(x,y) + tP_1(x,y)$. *Then for small t, there are four zeros of* P_t *near* z_0. *To see that, we substitute* $x = x_0 t^{\alpha_1} + x_1 t^{2\alpha_1} + \cdots$, *and* $y = y_0 t^{\alpha_2} + y_1 t^{2\alpha_2} + \cdots$, *for x and y in the following equations:* $P(x,y) + tP_1(x,y) = 0$

$$\begin{array}{rcl} x_0^2 t^{2\alpha_1} + \cdots & = & t(c_{10} + c_{11}x_0 t^{\alpha_1} + c_{12}y_0 t^{\alpha_2} + \cdots), \\ y_0^2 t^{2\alpha_2} + \cdots & = & t(c_{20} + c_{21}x_0 t^{\alpha_2} + c_{22}y_0 t^{\alpha_2} + \cdots). \end{array}$$

Equating the coefficients and the exponents of the lowest order terms in t yields

$$\begin{array}{rcl} x_0^2 & = & c_{10}, \\ y_0^2 & = & c_{20}, \\ 2\alpha_1 & = & 1, \\ 2\alpha_2 & = & 1. \end{array} \tag{1}$$

It is easy to see that this system has four solutions. Thus, there are four Puiseux series in t: $x = x_i(t), i = 1, \ldots, 4$ *converging to (0,0) as* $t \to 0$.

Thus, there is a one-to-one correspondence between the multiplicity and the number of Puiseux series expansions of the zero set of the perturbed system near z_0.

Remark 1. *Write* $P_1 = C + H$, *where C is the vector containing all of the constant terms of P, and* $H = P_1 - C$. *When we derive a system of equations for solving the* $(x_{10}, \ldots, x_{n0})$ *and the* α*s as in (1), the terms in H are all higher order terms and will not contribute anything. So, in the perturbation, only the constant terms are needed.*

In general, we have the following:

Proposition 1. *Suppose that* $z_0 \in C^s$ *is an isolated zero of P. Then, there is a one-to-one correspondence between the multiplicity and the number of Puiseux series expansions of the zero set of the perturbed system* $P + tC$ *near* z_0, *where* $C \in C^s$ *is a generic vector, which is the constant term of a generic polynomial* P_1.

Proof. Assume that z_0 is a zero of P with multiplicity m. Let $x_i(t) = c_{i0}t^{\alpha} + c_{i1}t^{2\alpha} +$ *higher order terms*, $i = 1, \ldots, k$ and $y_i(t) = b_{i0}t^{\alpha} + b_{i1}t^{2\alpha} +$ *higher order terms*, and $i = 1, \ldots, K$ be the Puiseux series expansion of the solution set of $P(x) + tC = 0$ and $P(x) + tP_1 = 0$, respectively, near $t = 0$, where $c_{i0} \neq c_{j0}$ and $b_{i0} \neq b_{j0}$. According to Remark 1, $k = K$ and $b_{i0} = c_{i0}$. Since for each $t > 0$ small enough, the system $P(x) + tP_1 = 0$ has exactly m distinct solutions, and $y_i(t)$'s are distinct solutions of the system, where $m = k = K$. □

In [1], Stetter described the following:

Definition 1 (Definition 9.13, [1]). *Suppose that* P_0 *is a singular system of s polynomials in s variables with a d-dimensional zero manifold* M_0. *For a specified near singular polynomial system,*

$$P_t(z) := P_0(z) + tP_1(z),$$

let $z = z(t)$ be such that $P_t(z(t)) \equiv 0$. If $z(t) = z_0 + z_1 t + \cdots$, $rank P_0'(z_0) = s - d$, and $P_1(z_0) \in range P_0'(z_0)$, then $z_0 \in M_0$ is called a simple distinguished point of P_t.

Example 3. *Let*

$$P_0(x,y) = \begin{pmatrix} x \\ xy \end{pmatrix}, P_1(x,y) = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and $M_0 = \{(x,y)|x = 0\}$. Let $z_0 = (0, c_2/c_1)$ and $z(t) = z_0 + (c_1 t, 0)$. Then, $P_t(z(t)) \equiv 0$, and $P_1(z_0) = P_0'(z_0)(-c_1, -c_1/c_2)^t$. So, $P_1(z_0) \in range P_0'(z_0)$, and thus, z_0 is a simple distinguished point of P_t.

Note that the main characteristic of these kinds of points is that they depend on P_1. The following example shows, however, that these kinds of points do not have to be "simple".

Example 4. *Let*

$$P_0(x,y,z) = \begin{pmatrix} x^2 \\ y^2 \\ xyz \end{pmatrix}, P_1(x,y,z) = -\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then, $M_0 = \{(x,y,z)|x = y = 0\}$. It is not too hard to see that there are four curves $(x_i(t), y_j(t), z_{ij}(t))$, of $P_t(x(t), y(t), z(t)) = 0$, where $x_i(t) = (-1)^i\sqrt{c_1 t}, i = 1,2$, $y_j(t) = (-1)^j\sqrt{c_2 t}, j = 1,2$, and $z_{ij}(t) = (-1)^{i+j} c_3/\sqrt{c_1 c_2}$, thus converging to the points $z_{0k} = (0, 0, (-1)^k c_3/\sqrt{c_1 c_2}), k = 1,2$. This means for small t, the system $P_0(x) + tP_1(x) = 0$ has two solutions near $z_{0k}, k = 1,2$. Also note that z_0 depends on the perturbation P_1. In this case, $s = 3, d = 1$. While $rank(P_0'(z)) = 0 < 3 - 1$ for all $z \in M$, only two points are really "distinguished" and are dependent on the perturbation.

Motivated by the examples such as the one above, we introduce a new definition below:

Definition 2. *A generalized distinguished point $z_0 \in M_0$ with respect to P_0 under the perturbation P_1 is such that there are m (≥ 1) solution curves of $P_0(x) + tP_1(x) = 0$ converging to z_0 as $t \to 0$. Also, z_0 depends on the perturbation P_1. The multiplicity of such points z_0 with respect to this perturbation is defined as m.*

In [1], Stetter also described the following:

Definition 3 (Definition 9.14, [1])**.** *Suppose that P_0 is a singular system of s polynomials in s variables with a d-dimensional zero manifold M_0. Points of $z \in M_0$ with*

$$rank(P_0'(z)) < s - d$$

are multiple distinguished points of P_0.

Definition 4 (Definition 9.15, [1])**.** *Suppose that P_0 is a singular system of s polynomials in s variables with a d-dimensional zero manifold M_0, and $z \in M_0$ is a multiple distinguished point of P_0. For the sake of simplicity, we assume that all the zeros of P_0 are contained in M_0, and z is the only distinguished point of P_0, multiple or not. Let I_0 be the ideal of polynomials that vanish on M_0, N_0 is a normal set (see Appendix A) of I_0, and N is a normal set of $< P_0 >$ containing N_0. Then, z has multiplicity $\#(N \setminus N_0) + 1$.*

Example 5. *Let*

$$P_0(x,y) = \begin{pmatrix} x^2 \\ xy^2 \end{pmatrix}.$$

Then, $M_0 = \{(x,y)|x = 0\}$, $z_0 = (0,0)$, $I_0 =< x >$, and a normal set of I_0 and N_0 is $\{1, y, y^2, y^3, \ldots\}$. In this case, $s = 2, d = 1$, and rank $P'(z_0) = 0 < 2 - 1$. So, z_0 is the multiple

distinguished point of P_0. *However, a normal set* N *of* $< P_0 >$ *is* $\{1, x, xy, y, y^2, y^3, \ldots\}$. *Thus,* $N \setminus N_0 = \{x, xy\}$. *So, according to Definition 4, the multiplicity is suppose to be* $2 + 1 = 3$.

Now, let

$$P_1(x, y) = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and $Q_t = P_0 + tP_1$. *It is not too hard to see that there are four curves* $(x_i(t), y_{ij}(t))$ *such that* $Q_t(x(t), y(t)) \equiv 0$, *where* $x_i(t) = (-1)^i\sqrt{c_1 t}, i = 1, 2$, $y_{ij}(t) = (-1)^j\sqrt{(-1)^i c_2\sqrt{t/c_1}}, j = 1, 2$. *It is also easy to see that they all converge to the point* $z_0 = (0, 0)$. *Thus, the multiplicity of* z_0 *should be four instead of three.*

Based on the observations from examples such as the one above, we will modify the definition of distinguished points and give a new approach to calculate the multiplicity.

Definition 5. *Let* $P_0(x)$ *be a system of* s *polynomials with* $x \in C^s$, *and* M *is a zero manifold of* P_0. *If* $z_0 \in M$ *is such that for a generic perturbation* P_1, *the system* $P_0(x) + tP_1(x) = 0$ *has* $m \geq 1$ *solutions near* z_0 *for small* t, *then we call* z_0 *a proper distinguished point of* P_0, *and* m *is its multiplicity.*

Example 6. *Let*

$$P(x, y, z) = \begin{pmatrix} x^2 \\ y^4 \\ xyz \end{pmatrix}, P_1(x, y, z) = -\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then, $M = \{(x, y, z) | x = y = 0\}$. *It is not too hard to see that there are eight curves* $(x_i(t), y_j(t), z_{ij}(t))$, *where* $x_i(t) = (-1)^i\sqrt[2]{c_1 t}, i = 1, 2$, $y_j(t) = u^j\sqrt[4]{c_2 t}, j = 1, 2, 3, 4$, *and* $z_{ij}(t) = c_3 t/(x_i(t) * y_j^2(t))$, *where* u *is a fourth primitive root of unity converging to the proper distinguished point* $z_0 = (0, 0, 0)$. *This means for small* t, *the system* $P(x) + tP_1(x) = 0$ *has eight solutions near* z_0. *In this case,* $s = 3, d = 1$. *While* $rank(P_0'(z)) = 0 < 3 - 1$ *for all* $z \in M$, *only one point is really "distinguished" and is independent of the perturbation. So, the multiplicity of* z_0 *is eight.*

Remark 2. *Suppose that* $z_0 \in M$ *is a proper distinguished point of* P_0; *then, if we expand the zero set of* $P_0(x) + tP_1$ *near* z_0 *as Puiseux series,* $x(t) = z_0 + x_0 t^\alpha + x_1 t^{2\alpha} + \cdots$, *then each component of* α *is positive, and the number combinations of* x_0 *and* α *is the multiplicity.*

Example 7. *Let*

$$P_0(x, y, z) = \begin{pmatrix} x^2 \\ y^4 \\ xz + yz^2 \end{pmatrix}, P_1(x, y, z) = -\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then, $M = \{x = y = 0\}$. *It is not too hard to see that there are eight curves* $(x_i(t), y_j(t), z_{ij}(t))$, *with* $\alpha = (1/2, 1/4, 1/2)$ *and eight curves with* $\alpha = (1/2, 1/4, 1/4)$ *converging to the proper distinguished point* $z_0 = (0, 0, 0)$. *This means for small* t, *the system* $P_0(x) + tP_1(x) = 0$ *has 16 solutions near* z_0. *Thus, the multiplicity of* z_0 *is 16.*

On the other hand, if

$$P_0(x, y, z) = \begin{pmatrix} x^2 \\ y^4 \\ xz + y^3z^2 \end{pmatrix},$$

then, there are only eight curves $(x_i(t), y_j(t), z_{ij}(t))$, *with* $\alpha = (1/2, 1/4, 1/2)$ *converging to the proper distinguished point* $z_0 = (0, 0, 0)$. *So, the multiplicity is only eight.*

Finally, why do some of the point(s) in M become "distinguished", while the others are not? Can we find it out without making use of something "external", such as Puiseux series? We need the following:

Definition 6. *Let I and J be two ideals. Define*

$$J : I = \{f | fi \in J, \forall i \in I\}$$

Lemma 1 (Proposition 10, section 4.4 and Lemma 8, Section 4.7, [5])**.** *Let I, I_i, J, and J_i be ideals. Then,*

$$(\bigcap_{i=1}^{r} I_i) : J = \bigcap_{i=1}^{r} (I_i : J).$$

If Q is primary and $\sqrt{Q} = P$, then

$$\begin{array}{l} \text{if } f \in Q, \text{ then } Q :< f >=< 1 >, \\ \text{if } f \notin Q, \text{ then } Q :< f > \text{ is P-primary,} \\ \text{if } f \notin P, \text{ then } Q :< f >= Q. \end{array}$$

Proposition 2. *Let $< P >=< p_1, \ldots, p_s >$, and $< P_c >=< p_2 - c_2 p_1, p_3 - c_3 p_1, \ldots, p_s - c_s p_1 >$. Let $T_1 =< P_c >:< P >$, and $T_{i+1} = T_i :< P >$. Then, $T_{i+1} = T_i :< p_1 >$, and there exists an n so that $T_{n+1} = T_n$ and $V(T_n) = \overline{V(P_c) - V(P)}$.*

Proof. Since $T_{i+1} \supset T_i$ for all i, then by the ascending chain condition, there exists an n so that $T_{n+1} = T_n$. Since $< P >=< P_c > + < p_1 >$. Then, $T_1 =< P_c >:< P >= (P_c : P_c) \bigcap (< P_c >:< p_1 >) =< P_c >:< p_1 >$. Suppose $T_i = T_{i-1} :< p_1 >$. Since $T_i \supset < P_c >$, $T_{i+1} = T_i :< P_c > \bigcap T_i :< p_1 >= T_i :< p_1 >$. Let $< P_c >= \bigcap_{i=1}^{m} Q_i$ be a minimal primary decomposition for $< P_c >$. Let $S_1 = \{i | p_1 \in Q_i\}$, $S_2 = \{i | p_1 \notin Q_i, p_1 \in \sqrt{Q_i}\}$, and let $S_3 = \{i | p_1 \notin \sqrt{Q_i}\}$. By Lemma 1, $\forall i \in S_1$, $Q_i : p_1 =< 1 >$ and $\forall i \in S_3$, $Q_i : p_1 = Q_i$. Thus, $T_1 = (\bigcap_{i \in S_3} Q_i) \bigcap (\bigcap_{i \in S_2} (Q_i :< p_1 >)$. It follows from exercise 4.10 of Chapter 4 in [5] for each $i \in S_2$, $Q_i :< p_1^n >=< 1 >$. Thus, $T_n = \bigcap_{i \in S_3} Q_i$. Now, let $R_0 = \bigcap_{i=1}^{m} \sqrt{Q_i}$, and define $R_i = R_{i-1} :< p_1 >$. Since R_0 is redical, $R_1 = R_2 = \ldots = R_n$. On the other hand, $R_1 = \bigcap_{i \in S_3} \sqrt{Q_i}$. It follows from Theorem 7 of Section 4.4 in [5] that $V(R_n) = \overline{V(R_0) - V(P)} = \bigcup_{i \in S_3} V(\sqrt{Q_i}) = V(T_n)$, since $V(R_0) = V(P_c)$, $V(< P_c >:< p_1^n >) = \overline{V(P_c) - V(P)}$. □

Example 8. *In Example 6, $Q_t = (x^2 - c_1 t, y^4 - c_2 t, xyz - c_3 t)$. Now, let us look at the subideal of $< P_0 >$ generated by eliminating the variable t from the polynomials in Q_t:*

$$< Q_0(c) >=< y^4 - d_1 x^2, xyz - d_2 x^2 >, d_1 = c_2/c_1, d_2 = c_3/c_1.$$

It is not too hard to see that for generic c values, $Q_0(c)$'s zero set contains eight branches of curves not contained in M, which are the projections of the curves in Example 6, thus passing through $z_0 = (0,0,0) \in M$. To "filter" out the curves, let $Q = Q_0 :< x^2 >$. $Q =< y^4 - d_1 x^2, -yz + d_2 x, -z^2 * d_1 + y^2 d_2^2, -d_1 xz + d_2 y^3 >$. Let $R =< P_0 > +Q$. Then, the multiplicity of z_0 with respect to R is eight.

Example 9. *In Example 4, $Q_t = (x^2 - c_1 t, y^2 - c_2 t, xyz - c_3 t)$. Let*

$$< Q_0(c) >=< y^4 - d_1 x^2, xyz - d_2 x^2 >, d_1 = c_2/c_1, d_2 = c_3/c_1.$$

Let $Q = Q_0 :< x^2 >$. Then, $Q =< -y^2 + d_1 x^2, -y d_2 + xzd1, yz - d_2 x >$. The intersection of the zero set of Q and M contains two points $z_{0k} = (0, 0, (-1)^k c_3/\sqrt{c_1 c_2}), k = 1, 2$. Let $R =< P_0 > +Q$. Then, the multiplicity of z_{0k} with respect to R is four for $k = 1, 2$.

In general, we have the following.

Theorem 1. *Let P be a system of s polynomials with s variables. Suppose that M is a zero manifold of P. Let c be a generic vector, and let $Q_0(c) = (q_2, \ldots, q_s)^t$, where $q_i = p_i - d_i p_1, i = 2, \ldots, s$, $d_i = c_i/c_1$, $Q_1 = Q_0 :< p_1 >, \ldots, Q_i = Q_{i-1} :< p_1 >$. Then, there exists an n so that $Q_{n+1} = Q_n$. Denote $Q_n = Q$. Suppose that $z_0 \in M$ is a zero of $R =< P > +Q$. Then, z_0 is an isolated zero of $R =< P > +Q$, and the multiplicity of z_0 as a zero of R is the same as the multiplicity of z_0 as a distinguished point of P. In addition, if z_0 is independent of c, then z_0 is a proper distinguished point of P; otherwise, it is a generalized distinguished point of P.*

Proof. According to Proposition 2, $z_0 \in \overline{V(P_c) - V(P)} \bigcap V(P)$. However, the only points in $\overline{V(P_c) - V(P)} \bigcap V(P)$ are the end points $z(0)$ of the curves $z(t)$, where $P(z(t)) = ct$. There are only finitely many of them, so they are isolated. Suppose that the multiplicity of z_0 as an isolated zero of R is m. Let $P_t = (p_1 - c_1 t, p_2 - c_2 t, \ldots, p_s - c_s t)$ and $R_t =< P_t > +Q$. For each t, $V(P_t) = V(Q_t) \subset V(Q)$ consists of isolated points. Since the multiplicity of z_0 as a zero of R_0 is m, for small t, there are m points of the zero set of P_t near z_0. Thus, by definition, the multiplicity of z_0 as a distinguished point of P is m. □

Example 10. *Let $Q_t(x, y, z) = (x^2 - c_1 t, y^4 - c_2 t, xz + yz^2 - c_3 t)$, where $(x, y, z) \in C^3$. Let $< Q_0(c) >=< y^4 - c_2 x^2/c_1, xyz - c_3 x^2/c_1 >$ and $Q = Q_0 :< x^2 >$. Then,*

$$Q =< y^4 - d_2 x^2, -xz - yz^2 + d_3 x^2, -xz^2 d_2 - y^3 z + xy^3 d_3, zy^2 - y^2 x d_3 + zy^3 d_3 - z^3 d_2 >,$$

where $d_i = c_i/c_1$. The multiplicity of $z_0 = (0, 0, 0)$ as an isolated zero of $R =< P_0 > +Q_1$ is 16, and so is the multiplicity of z_0 as a distinguished point of P_0.

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Appendix A

Definition A1. *A set of monomials $N = \{x^j | j \in J\}$, where $x \in C^s$ and $j = (j_1, \ldots, j_s)$, is called closed if*

$$x^j \in N \Rightarrow x^{j'} \in N \ \forall j' : x^{j'} | x^j.$$

Definition A2. *Let I be a polynomial ideal and $R(I)$ be the quotient ring. A closed set of monomials $N(I)$ is a normal set of I if the monomials in $N(I)$ form a basis of $R(I)$.*

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