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On the Potential Vector Fields of Soliton-Type Equations

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Abstract: We highlight some properties of a class of distinguished vector fields associated to a $(1,1)$-tensor field and to an affine connection on a Riemannian manifold, with a special view towards the Ricci vector fields, and we characterize them with respect to statistical, almost Kähler, and locally product structures. In particular, we provide conditions for these vector fields to be closed, Killing, parallel, or semi-torse forming. In the gradient case, we give a characterization of the Euclidean sphere. Among these vector fields, the Ricci and torse-forming-like vector fields are particular cases.

Keywords: soliton-type equation; Ricci vector field; affine connection

MSC: 37K40; 53C21; 53C25; 53Z05

1. Introduction

The existence of certain types of vector fields on a Riemannian manifold is closely related to its topological and geometrical properties. Generalizing the concircular vector fields, Yano has introduced in [1] the notion of a torse-forming vector field, which has been further intensively used in different contexts, having many applications in geometry and physics. For example, Chen and Verstraelen [2] characterized the Euclidean hypersurfaces for which the tangential component of the position vector field is a proper torse-forming vector field. It is worth to be mentioned that, in [3], Chen and Deshmukh found characterizations of the Euclidean spheres, of the Euclidean space, and of the complex Euclidean space in terms of nontrivial concircular vector fields, in [4] the authors characterized the spheres by means of a nontrivial torse-forming vector field, and in [5] they used ρ-Ricci vector fields on a Riemannian manifold to provide characterizations of the spheres. Also, the relation between torse-forming vector fields and exterior concurrent and quasi-exterior concurrent vector fields has been established in [6]. In [7], we have introduced the notion of a torse-forming-like vector field, which is a closed vector field, and we related it to torse-forming and semi-torse-forming vector fields. The class of torse-forming-like vector fields contains the Ricci vector fields on quasi-Einstein manifolds.

In the framework of a Riemannian manifold (M,g) with a fixed affine connection, ∇, and a $(1,1)$-tensor field, φ, we study some properties of the potential vector fields, ξ, of some soliton-type equations, namely, those vector fields having the property that $\nabla\xi = f\varphi + hI$, with f and h being two smooth functions on M and I being the identity map. If ∇ is the Levi-Civita connection of g, these vector fields generalize the Ricci and the torse-forming-like vector fields. We aim to provide nontrivial conditions for these vector fields to be closed, Killing, parallel, or semi-torse forming. More precisely, we show that, under an additional assumption, if (∇, g) is a statistical structure with (or without) torsion, then these vector fields are closed, provided that the $(1,1)$-tensor field, φ, is symmetric. Any Riemannian manifold being, in particular, a statistical manifold, we particularize the results when ∇ is the Levi-Civita connection of g. We deduce that if h is a real constant and (φ, g) forms a locally product structure, then ξ is a closed vector field. On the other hand, if (φ, g) constitutes an almost Kähler structure, then ξ is a homothetic vector field. Moreover, in both of these cases, if f and h are constant, then ξ is a semi-torse-forming

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vector field. In the compact Riemannian case, if ∇ is the Levi-Civita connection of g, we find conditions for ξ to be a concircular vector field and we characterize the Euclidean spheres when ξ is of gradient-type, too. Also, we provide conditions for a Ricci vector field on a compact Riemannian manifold to be a parallel or a geodesic vector field. Recently, in [8], we have determined conditions for the potential vector field of a hyperbolic Ricci soliton or of a hyperbolic Yamabe soliton to be a parallel vector field, i.e., for the soliton to be trivial.

The motivation for studying the properties of the vector fields we will deal with in the present paper comes from the fact that they constitute a large class of vector fields that occur in concrete problems from mathematical physics, being the potential vector fields of (∇, φ)-solitons [9]. We mention that particular cases of such soliton-type equations are the Miao–Tam equation [10,11], the Fischer–Marsden equation [10,12], and the ones that describe the gradient Ricci solitons, ∇-Ricci solitons, and the shape solitons (for more details, see [9]).

2. Preliminaries

We shall recall some basic notions. Let (M, g) be an n-dimensional Riemannian manifold, let ∇ be an affine connection, let J_1 be a $(1,1)$-tensor field, and let J_2 be a symmetric $(0,2)$-tensor field on M. We consider $\{E_i\}_{1\leq i\leq n}$ to be a local orthonormal frame field on the tangent bundle TM of M, and let X, Y, Z be arbitrary vector fields tangent to M.

The torsion, T^∇, the curvature, R^∇, the Ricci curvature, Ric^∇, the Ricci operator, Q^∇, and the scalar curvature, r^∇, of ∇ are given by:

$$\begin{aligned} T^\nabla(X,Y) &:= \nabla_X Y - \nabla_Y X - [X,Y], \\ R^\nabla(X,Y)Z &:= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ \mathrm{Ric}^\nabla(X,Y) &:= \sum_{i=1}^{n} g(R^\nabla(E_i,X)Y,E_i), \\ g(Q^\nabla X,Y) &:= \mathrm{Ric}^\nabla(X,Y), \\ r^\nabla &:= \sum_{i=1}^{n} \mathrm{Ric}^\nabla(E_i,E_i), \end{aligned}$$

where $[X,Y] := X\circ Y - Y\circ X$ stands for the Lie bracket of vector fields.

The covariant derivatives ∇J_1 and ∇J_2 of J_1 and J_2 are, respectively, defined as:

$$\begin{aligned} (\nabla_X J_1)Y &:= \nabla_X J_1 Y - J_1(\nabla_X Y), \\ (\nabla_X J_2)(Y,Z) &:= X(J_2(Y,Z)) - J_2(\nabla_X Y,Z) - J_2(Y,\nabla_X Z), \end{aligned}$$

the exterior differential operators associated to ∇, by:

$$\begin{aligned} (d^\nabla J_1)(X,Y) &:= (\nabla_X J_1)Y - (\nabla_Y J_1)X + J_1(T^\nabla(X,Y)), \\ (d^\nabla J_2)(X,Y,Z) &:= (\nabla_X J_2)(Y,Z) - (\nabla_Y J_2)(X,Z) + J_2(T^\nabla(X,Y),Z), \end{aligned}$$

while the traces and the Hilbert–Schmidt norms of J_1 and J_2 are, respectively, defined as:

$$\begin{aligned} \mathrm{Trace}(J_1) &:= \sum_{i=1}^{n} g(J_1E_i,E_i), \quad \mathrm{Trace}(J_2) := \sum_{i=1}^{n} J_2(E_i,E_i), \\ \|J_1\|^2 &:= \sum_{1\leq i,j\leq n} \big(g(J_1E_i,E_j)\big)^2, \quad \|J_2\|^2 := \sum_{1\leq i,j\leq n} \big(J_2(E_i,E_j)\big)^2. \end{aligned}$$

Also, for any smooth function, u, and any vector fields X and Y tangent to M, we define the gradient, ∇u, of u, the Hessian, $\mathrm{Hess}^{\nabla}(u)$, of u, the divergence, $\mathrm{div}^{\nabla}(X)$, of X, and the Laplacian, $\Delta^{\nabla}(u)$, of u, respectively, by:

$$g(\nabla u, X) := du(X) = X(u), \ \ \mathrm{Hess}^{\nabla}(u)(X,Y) := g(\nabla_X \nabla u, Y),$$
$$\mathrm{div}^{\nabla}(X) := \sum_{i=1}^{n} g(\nabla_{E_i} X, E_i), \ \ \Delta^{\nabla}(u) := \mathrm{div}^{\nabla}(\nabla u).$$

3. On the Potential Vector Fields of Some Soliton-Type Equations

Let (M, g) be an n-dimensional Riemannian manifold, let ∇ be an affine connection, and let φ be a $(1,1)$-tensor field on M. A vector field, ξ, on M is said to be the potential vector field of a (∇, φ)-soliton [9,13] if there exist two smooth functions, f and h, on M such that

$$\nabla \xi = f\varphi + hI, \tag{1}$$

where I is the identity map on TM.

Considering the properties of ∇, φ, f, and h, we aim to characterize the vector field, ξ, that satisfies Equation (1). We will also denote by η the dual 1-form of ξ.

Let us remark the following facts: if ∇ is the Levi-Civita connection of g, and

(i) $f = 0$, then ξ is a concircular vector field [14];
(ii) $h = 0$, $\varphi = Q$, where Q is the Ricci operator, then ξ is a Ricci vector field [15];
(iii) $f \neq 0$, $h \neq 0$, $\varphi = \theta \otimes \zeta$ with ζ the dual vector field of a 1-form θ on M, then ξ is a torse-forming-like vector field [7].

Example 1. *If, in Equation (1), ∇ is the Levi-Civita connection of g, then the Reeb vector field of an α-Sasakian manifold with $f = -\alpha$ (a constant), $h = 0$, and φ the structural endomorphism, as well as the Reeb vector field of a cosymplectic manifold with $f = h = 0$ and φ the structural endomorphism, are trivial examples of potential vector fields in a soliton-type equation (see [16]).*

Example 2. *We construct now a nontrivial example. Let M be the 3-dimensional real space $\mathbb{R}^3$ with the Riemannian metric*

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + dz \otimes dz,$$

where x, y, z are the standard coordinates in $\mathbb{R}^3$. Let $\tilde{f}$ be a smooth real function on $\mathbb{R}$ and let

$$f(x,y,z) = \tilde{f}(z), \ \ h(x,y,z) = e^{z}\Big\{k(x^2+y^2) + ax + by + F(z)\Big\} + ke^{-z},$$

where k, a, b are real constants and $F = F(z)$ is an antiderivative of the function $z \mapsto \tilde{f}(z)e^{-z}$. Then,

$$\xi = (2kx + a)e^{-z}\frac{\partial}{\partial x} + (2ky + b)e^{-z}\frac{\partial}{\partial y} + (h - 2ke^{-z})\frac{\partial}{\partial z}$$

is the potential vector field of a soliton-type equation on (M, g), satisfying (1) with

$$\varphi = dz \otimes \frac{\partial}{\partial z}$$

and ∇ the Levi-Civita connection of g. Indeed, let

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \ \ E_2 = e^{-z}\frac{\partial}{\partial y}, \ \ E_3 = \frac{\partial}{\partial z}$$

be a local orthonormal frame field on TM, and let

$$\xi = \xi^1 E_1 + \xi^2 E_2 + \xi^3 E_3.$$

We have [17]:

$$\begin{gathered}\nabla_{E_1}E_1=-E_3,\ \ \nabla_{E_1}E_2=0,\ \ \nabla_{E_1}E_3=E_1,\\ \nabla_{E_2}E_1=0,\ \ \nabla_{E_2}E_2=-E_3,\ \ \nabla_{E_2}E_3=E_2,\\ \nabla_{E_3}E_1=0,\ \ \nabla_{E_3}E_2=0,\ \ \nabla_{E_3}E_3=0,\end{gathered}$$

and the condition (1) *implies that the functions* ξ^1, ξ^2, *and* ξ^3 *satisfy*

$$\begin{cases}\Big\{E_1(\xi^1)+\xi^3-h\Big\}E_1+E_1(\xi^2)E_2+\Big\{E_1(\xi^3)-\xi^1\Big\}E_3=0\\ E_2(\xi^1)E_1+\Big\{E_2(\xi^2)+\xi^3-h\Big\}E_2+\Big\{E_2(\xi^3)-\xi^2\Big\}E_3=0\\ E_3(\xi^1)E_1+E_3(\xi^2)E_2+\Big\{E_3(\xi^3)-f-h\Big\}E_3=0\end{cases},$$

which infer

$$\begin{cases}\frac{\partial\xi^1}{\partial x}=(h-\xi^3)e^z\\ \frac{\partial\xi^1}{\partial y}=0\\ \frac{\partial\xi^1}{\partial z}=0\end{cases},\quad\begin{cases}\frac{\partial\xi^2}{\partial x}=0\\ \frac{\partial\xi^2}{\partial y}=(h-\xi^3)e^z\\ \frac{\partial\xi^2}{\partial z}=0\end{cases},\quad\begin{cases}\frac{\partial\xi^3}{\partial x}=\xi^1e^z\\ \frac{\partial\xi^3}{\partial y}=\xi^2e^z\\ \frac{\partial\xi^3}{\partial z}=f+h\end{cases}.$$

Example 3. *In particular, for* $k=a=b=0$ *in Example 2, the vector field*

$$\xi=f\frac{\partial}{\partial z}$$

with $f(x,y,z)=\tilde{f}(z)$, *a smooth function on* M, *is the potential vector field of a soliton-type equation on* (M,g), *which satisfies*

$$\nabla\xi=fI+(f'-f)dz\otimes\frac{\partial}{\partial z}.$$

We immediately obtain the following relations.

Lemma 1. *If* ξ *is a vector field on an* n*-dimensional Riemannian manifold* (M,g) *satisfying* (1), *then:*

$$\begin{aligned}\operatorname{div}^\nabla(\xi)&=f\operatorname{Trace}(\varphi)+nh,\\ \|\nabla\xi\|^2&=f^2\|\varphi\|^2+2fh\operatorname{Trace}(\varphi)+nh^2,\\ (d\eta)(X,Y)&=(d^\nabla g)(X,Y,\xi)+f\Big\{g(\varphi X,Y)-g(\varphi Y,X)\Big\},\\ (\pounds_\xi g)(X,Y)&=(\nabla_\xi g)(X,Y)+g(T^\nabla(\xi,X),Y)+g(T^\nabla(\xi,Y),X)\\ &\quad+f\Big\{g(\varphi X,Y)+g(\varphi Y,X)\Big\}+2hg(X,Y),\\ R^\nabla(X,Y)\xi&=(df\otimes\varphi-\varphi\otimes df+dh\otimes I-I\otimes dh)(X,Y)\\ &\quad+f(d^\nabla\varphi)(X,Y)+hT^\nabla(X,Y),\end{aligned}$$

for any vector fields X *and* Y *tangent to* M.

Proof. Let $\{E_i\}_{1\le i\le n}$ be a local orthonormal frame field on TM. We have:

$$\begin{aligned}\operatorname{div}^\nabla(\xi)&=\sum_{i=1}^n g(\nabla_{E_i}\xi,E_i)\\ &=f\sum_{i=1}^n g(\varphi E_i,E_i)+h\sum_{i=1}^n g(E_i,E_i)\end{aligned}$$

$$
\begin{aligned}
&= f\,\mathrm{Trace}(\varphi) + nh,\\
\|\nabla\xi\|^2 &= \sum_{i=1}^{n} g(\nabla_{E_i}\xi, \nabla_{E_i}\xi)\\
&= f^2 \sum_{i=1}^{n} g(\varphi E_i, \varphi E_i) + 2fh \sum_{i=1}^{n} g(\varphi E_i, E_i) + h^2 \sum_{i=1}^{n} g(E_i, E_i)\\
&= f^2\|\varphi\|^2 + 2fh\,\mathrm{Trace}(\varphi) + nh^2,\\
(d\eta)(X,Y) &= X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\\
&= X(g(Y,\xi)) - Y(g(X,\xi)) - g(\nabla_X Y - \nabla_Y X - T^{\nabla}(X,Y),\xi)\\
&= (\nabla_X g)(Y,\xi) - (\nabla_Y g)(X,\xi) + g(T^{\nabla}(X,Y),\xi)\\
&\quad + g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)\\
&= (d^{\nabla}g)(X,Y,\xi) + f\Big\{g(Y,\varphi X) - g(X,\varphi Y)\Big\},\\
(\pounds_\xi g)(X,Y) &= \xi(g(X,Y)) - g([\xi,X],Y) - g(X,[\xi,Y])\\
&= \xi(g(X,Y)) - g(\nabla_\xi X - \nabla_X\xi - T^{\nabla}(\xi,X),Y)\\
&\quad - g(X,\nabla_\xi Y - \nabla_Y\xi - T^{\nabla}(\xi,Y))\\
&= (\nabla_\xi g)(X,Y) + g(T^{\nabla}(\xi,X),Y) + g(X,T^{\nabla}(\xi,Y))\\
&\quad + g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi)\\
&= (\nabla_\xi g)(X,Y) + g(T^{\nabla}(\xi,X),Y) + g(X,T^{\nabla}(\xi,Y))\\
&\quad + f\Big\{g(\varphi X,Y) + g(X,\varphi Y)\Big\} + 2hg(X,Y),\\
R^{\nabla}(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi\\
&= X(f)\varphi Y + f\nabla_X\varphi Y + X(h)Y + h\nabla_X Y\\
&\quad - Y(f)\varphi X - f\nabla_Y\varphi X - Y(h)X - h\nabla_Y X\\
&\quad - f\varphi(\nabla_X Y) + f\varphi(\nabla_Y X) + f\varphi(T^{\nabla}(X,Y))\\
&\quad - h\nabla_X Y + h\nabla_Y X + hT^{\nabla}(X,Y)\\
&= X(f)\varphi Y - Y(f)\varphi X + X(h)Y - Y(h)X\\
&\quad + f\Big\{(\nabla_X\varphi)Y - (\nabla_Y\varphi)X + \varphi(T^{\nabla}(X,Y))\Big\} + hT^{\nabla}(X,Y)\\
&= df(X)\varphi Y - df(Y)\varphi X + dh(X)Y - dh(Y)X\\
&\quad + f(d^{\nabla}\varphi)(X,Y) + hT^{\nabla}(X,Y),
\end{aligned}
$$

for any vector fields X and Y tangent to M. □

If (∇, g) is a *statistical structure* (i.e., $d^{\nabla}g = 0$ and $T^{\nabla} = 0$, see [18]), or, if it is a *statistical structure with torsion* (i.e., $d^{\nabla}g = 0$, see [19]), we obtain the following:

Proposition 1. *Let (∇, g) be a statistical structure with (or without) torsion on a smooth manifold M. If the vector field, ξ, satisfies (1) for f nowhere zero on M, then, ξ is a closed vector field (i.e., $d\eta = 0$) if and only if φ is symmetric (i.e., $g(\varphi X, Y) = g(X, \varphi Y)$ for any vector fields X and Y tangent to M).*

Proof. In this case, $d^{\nabla}g = 0$ and we obtain:

$$(d\eta)(X,Y) = f\Big\{g(\varphi X,Y) - g(\varphi Y,X)\Big\}$$

for any vector fields X and Y tangent to M. □

Proposition 2. *Let ∇ be a torsion-free affine connection on a Riemannian manifold (M, g) and let ξ be a vector field on M satisfying (1) for φ skew-symmetric (i.e., $g(\varphi X, Y) = -g(X, \varphi Y)$ for any vector fields X and Y tangent to M). Then, ξ is a Killing vector field (i.e., $\pounds_\xi g = 0$) if and only if*

$$\nabla_\xi g = -2hg.$$

Proof. In this case, $T^\nabla = 0$ and we obtain:

$$\begin{aligned}(\pounds_\xi g)(X, Y) &= (\nabla_\xi g)(X, Y) + f\Big\{g(\varphi X, Y) + g(\varphi Y, X)\Big\} + 2hg(X, Y)\\ &= (\nabla_\xi g)(X, Y) + 2hg(X, Y)\end{aligned}$$

for any vector fields X and Y tangent to M. □

For the Levi-Civita connection, from Lemma 1, we obtain the following:

Lemma 2. *If ξ is a vector field on an n-dimensional Riemannian manifold (M, g) satisfying (1) with ∇, the Levi-Civita connection of g, then:*

$$\begin{aligned}\operatorname{div}(\xi) &= f \operatorname{Trace}(\varphi) + nh,\\ (\pounds_\xi g)(X, Y) &= f\Big\{g(\varphi X, Y) + g(\varphi Y, X)\Big\} + 2hg(X, Y),\\ R(X, Y)\xi &= (df \otimes \varphi - \varphi \otimes df + dh \otimes I - I \otimes dh)(X, Y)\\ &\quad + f\Big\{(\nabla_X \varphi)Y - (\nabla_Y \varphi)X\Big\},\end{aligned}$$

for any vector fields X and Y tangent to M.

Proof. In this case, $\nabla g = 0$, $T^\nabla = 0$ and we obtain the conclusion. □

Using the previous lemma, we can state.

Proposition 3. *Let ξ be a vector field on an n-dimensional Riemannian manifold (M, g) satisfying (1) for ∇, the Levi-Civita connection of g.*

(1) If φ is a symmetric Codazzi $(1,1)$-tensor field (i.e., φ is symmetric and $(\nabla_X \varphi)Y = (\nabla_Y \varphi)X$ for any vector fields X and Y tangent to M), then:

$$\begin{aligned}R(X, Y)\xi &= (df \otimes \varphi - \varphi \otimes df + dh \otimes I - I \otimes dh)(X, Y),\\ \operatorname{Ric}(Y, \xi) &= df(\varphi Y) - \operatorname{Trace}(\varphi) df(Y) - (n-1) dh(Y),\\ Q\xi &= \varphi(\nabla f) - \operatorname{Trace}(\varphi)\nabla f - (n-1)\nabla h,\end{aligned}$$

for any vector fields X and Y tangent to M.

(2) If φ is skew-symmetric, then ξ is a conformal vector field (i.e., $\pounds_\xi g = 2hg$ for h, a smooth function on M), and it is a Killing vector field if and only if $h = 0$.

Proof. (2) follows immediately from Lemma 2. Let $\{E_i\}_{1 \le i \le n}$ be a local orthonormal frame field on TM. For any vector field Y tangent to M, we have:

$$\begin{aligned}g(Y, Q\xi) &= \operatorname{Ric}(Y, \xi)\\ &= \sum_{i=1}^{n} g(R(E_i, Y)\xi, E_i)\\ &= \sum_{i=1}^{n} df(E_i) g(\varphi Y, E_i) - df(Y) \sum_{i=1}^{n} g(\varphi E_i, E_i)\\ &\quad + \sum_{i=1}^{n} dh(E_i) g(Y, E_i) - dh(Y) \sum_{i=1}^{n} g(E_i, E_i)\end{aligned}$$

$$\begin{aligned}
&= df(\varphi Y) - df(Y)\,\mathrm{Trace}(\varphi) + dh(Y) - ndh(Y)\\
&= df(\varphi Y) - \mathrm{Trace}(\varphi)df(Y) - (n-1)dh(Y)\\
&= g(\nabla f, \varphi Y) - \mathrm{Trace}(\varphi)g(\nabla f, Y) - (n-1)g(\nabla h, Y)\\
&= g(\varphi(\nabla f) - \mathrm{Trace}(\varphi)\nabla f - (n-1)\nabla h, Y),
\end{aligned}$$

and we obtain (1). □

Theorem 4. *Let ξ be a vector field on a Riemannian manifold (M, g) satisfying (1) for ∇, the Levi-Civita connection of g, and h constant.*

(1) If $\varphi \neq 0$ is a Codazzi $(1,1)$-tensor field, then ξ is a semi-torse-forming vector field (i.e., $R(X,\xi)\xi = 0$ for any vector field X tangent to M [20]) if and only if f is a constant.

(2) If φ is symmetric, then ξ is a closed vector field; in particular, if (φ, g) is a locally product structure (i.e., $\varphi \neq \pm I$ is symmetric and it satisfies $\varphi^2 = I$ and $\nabla\varphi = 0$), then ξ is a semi-torse-forming vector field if and only if f is a constant.

(3) If φ is skew-symmetric, then ξ is a homothetic vector field (i.e., $\pounds_\xi g = cg$, with c a real constant); in particular, if (φ, g) is an almost Kähler structure (i.e., $\varphi \neq \pm I$ is skew-symmetric and it satisfies $\varphi^2 = -I$ and $\nabla\varphi = 0$), then ξ is a semi-torse-forming vector field if and only if f is a constant.

Proof. (1) follows from Proposition 3, and (2) and (3) follow from Lemma 1. □

A sufficient condition for ξ to reduce to a concircular vector field is further provided.

Theorem 5. *Let ξ be a vector field on an n-dimensional compact Riemannian manifold (M, g) satisfying (1) for ∇, the Levi-Civita connection of g.*

(1) If φ is symmetric and

$$\int_M \mathrm{Ric}(\xi,\xi) \geq \int_M \Big\{ (f(\mathrm{Trace}(\varphi) + (n-1)h)^2 + (n-1)h^2 \Big\},$$

or

(2) if φ is skew-symmetric and

$$\int_M \mathrm{Ric}(\xi,\xi) \geq n(n-1)\int_M h^2,$$

then ξ is a concircular vector field.

Proof. On a compact Riemannian manifold, we have [14]:

$$\int_M \Big\{ \mathrm{Ric}(\xi,\xi) + \frac{1}{2}\|\pounds_\xi g\|^2 - \|\nabla\xi\|^2 - (\mathrm{div}(\xi))^2 \Big\} = 0.$$

Let $\{E_i\}_{1\leq i\leq n}$ be a local orthonormal frame field on TM. Then:

$$\begin{aligned}
\|\pounds_\xi g\|^2 &= \sum_{1\leq i,j\leq n} ((\pounds_\xi g)(E_i, E_j))^2\\
&= f^2\Big\{ \sum_{1\leq i,j\leq n} (g(\varphi E_i, E_j))^2 + 2\sum_{1\leq i,j\leq n} g(\varphi E_i, E_j)g(\varphi E_j, E_i)\\
&\quad + \sum_{1\leq i,j\leq n} (g(\varphi E_j, E_i))^2 \Big\} + 4h^2 \sum_{1\leq i,j\leq n} (g(E_i, E_j))^2\\
&\quad + 4fh \sum_{1\leq i,j\leq n} \Big\{ g(\varphi E_i, E_j) + g(\varphi E_j, E_i) \Big\} g(E_i, E_j)\\
&= 2f^2\Big\{ \sum_{1\leq i,j\leq n} (g(\varphi E_i, E_j))^2 + \sum_{1\leq i,j\leq n} g(\varphi E_i, E_j)g(\varphi E_j, E_i) \Big\}
\end{aligned}$$

$$
\begin{aligned}
&+ 8fh \sum_{1\le i,j\le n} g(\varphi E_i, E_j) g(E_i, E_j) + 4h^2 \sum_{1\le i,j\le n} (g(E_i, E_j))^2 \\
&= 2f^2 \Big\{ \|\varphi\|^2 + \sum_{1\le i,j\le n} g(\varphi E_i, E_j) g(\varphi E_j, E_i) \Big\} \\
&+ 8fh \,\mathrm{Trace}(\varphi) + 4nh^2.
\end{aligned}
$$

If φ is symmetric, then:

$$
\begin{aligned}
&\mathrm{div}(\xi) = f \,\mathrm{Trace}(\varphi) + nh, \\
&\|\nabla \xi\|^2 = f^2 \|\varphi\|^2 + 2fh \,\mathrm{Trace}(\varphi) + nh^2, \\
&\|\pounds_\xi g\|^2 = 4 \Big\{ f^2 \|\varphi\|^2 + 2fh \,\mathrm{Trace}(\varphi) + nh^2 \Big\},
\end{aligned}
$$

and we infer:

$$
\int_M f^2 \|\varphi\|^2 = \int_M \Big\{ (f(\mathrm{Trace}(\varphi) + (n-1)h)^2 + (n-1)h^2 - \mathrm{Ric}(\xi, \xi) \Big\} \le 0.
$$

If φ is skew-symmetric, then:

$$
\begin{aligned}
&\mathrm{div}(\xi) = nh, \\
&\|\nabla \xi\|^2 = f^2 \|\varphi\|^2 + nh^2, \\
&\|\pounds_\xi g\|^2 = 4nh^2,
\end{aligned}
$$

and we infer:

$$
\int_M f^2 \|\varphi\|^2 = \int_M \Big\{ n(n-1)h^2 - \mathrm{Ric}(\xi, \xi) \Big\} \le 0;
$$

hence the proof is complete. □

In the gradient case, i.e., if $\xi = \nabla u$ for u, a smooth function on M, and assuming that ∇ is the Levi-Civita connection of g, we can characterize the Euclidean spheres in the following way.

Let us recall the well-known theorem of Obata, which provides a characterization for a Euclidean sphere.

Theorem 6 ([21]). *If there exists a nonconstant smooth function, u, on a complete Riemannian manifold (M, g) satisfying* $\mathrm{Hess}(u) = -cug$, *with c a positive constant, then the manifold is isometric to the Euclidean sphere of radius $\frac{1}{\sqrt{c}}$.*

We remark that, if $\xi = \nabla u$, then Equation (1) becomes

$$
\mathrm{Hess}(u)(X, Y) = fg(\varphi X, Y) + hg(X, Y) \tag{2}
$$

for any vector fields X and Y tangent to M. By taking the trace in (2), we obtain:

$$
\Delta(u) = f \,\mathrm{Trace}(\varphi) + nh. \tag{3}
$$

Theorem 7. *Let $\xi = \nabla u$ for u, a smooth function on M, be a gradient vector field on an n-dimensional compact Riemannian manifold (M, g) satisfying (1) for ∇, the Levi-Civita connection of g, and $h = -cu$, $c > 0$ (a constant). If* $\mathrm{Trace}(\varphi) = 0$ *(in particular, if φ is skew-symmetric) and*

$$
\int_M \mathrm{Ric}(\nabla u, \nabla u) \ge n(n-1)c^2 \int_M u^2,
$$

then (M, g) is isometric to the Euclidean sphere of radius $\frac{1}{\sqrt{c}}$.

Proof. On a compact Riemannian manifold, we have [14]:

$$\int_M \left\{ \operatorname{Ric}(\nabla u, \nabla u) + \|\operatorname{Hess}(u)\|^2 - (\Delta(u))^2 \right\} = 0.$$

From (2), with $h = -cu$, we obtain:

$$\begin{aligned} \|\operatorname{Hess}(u)\|^2 &= f^2\|\varphi\|^2 - 2cfu\operatorname{Trace}(\varphi) + nc^2u^2 \\ &= f^2\|\varphi\|^2 + nc^2u^2, \end{aligned}$$

which, together with $\Delta(u)$ from (3) replaced in the previous relation, give:

$$\int_M f^2\|\varphi\|^2 = \int_M \left\{ n(n-1)c^2u^2 - \operatorname{Ric}(\nabla u, \nabla u) \right\} \leq 0.$$

Therefore, $f = 0$ or $\varphi = 0$; hence $\operatorname{Hess}(u) = -cug$ and we obtain the conclusion by means of Obata's theorem. □

4. Ricci Vector Fields

We shall now consider the special case when the potential vector field is a Ricci vector field. We recall that ξ is called a *Ricci vector field* [15] on a Riemannian manifold (M, g) if there exists a smooth function, f, on M such that $\nabla\xi = fQ$, where ∇ is the Levi-Civita connection of g and Q is the Ricci operator.

We immediately obtain the following relations.

Lemma 3. *If ξ is a Ricci vector field on an n-dimensional Riemannian manifold (M, g), $\nabla\xi = fQ$, with f a smooth function on M, then:*

$$\operatorname{div}(\xi) = fr, \quad \pounds_\xi g = 2f\operatorname{Ric},$$

$$\|\nabla\xi\|^2 = f^2\|Q\|^2, \quad \|\pounds_\xi g\|^2 = 4f^2\|Q\|^2,$$

where Ric *is the Ricci curvature tensor field and r is the scalar curvature of (M, g).*

Proof. Indeed, for any vector fields X and Y tangent to M, we have:

$$\begin{aligned} \operatorname{div}(\xi) &= \sum_{i=1}^n g(\nabla_{E_i}\xi, E_i) = f\sum_{i=1}^n \operatorname{Ric}(E_i, E_i) = fr, \\ (\pounds_\xi g)(X, Y) &= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 2f\operatorname{Ric}(X, Y), \\ \|\nabla\xi\|^2 &= \sum_{i=1}^n g(\nabla_{E_i}\xi, \nabla_{E_i}\xi) = f^2\sum_{i=1}^n g(QE_i, QE_i) = f^2\|Q\|^2, \\ \|\pounds_\xi g\|^2 &= \sum_{1\leq i,j\leq n} ((\pounds_\xi g)(E_i, E_j))^2 = 4f^2\sum_{1\leq i,j\leq n} (\operatorname{Ric}(E_i, E_j))^2 \\ &= 4f^2\|Q\|^2, \end{aligned}$$

where $\{E_i\}_{1\leq i\leq n}$ is a local orthonormal frame field on TM. □

Let us notice that a compact Riemannian manifold possessing a Ricci vector field, ξ, such that $\nabla\xi = aQ$ with $a \in \mathbb{R}\setminus\{0\}$, is either a manifold of nonconstant scalar curvature, r, satisfying $\int_M r = 0$ or of zero scalar curvature, by means of the divergence theorem.

We shall further determine some nontrivial conditions under which a Ricci vector field reduces to a parallel vector field (in this case, the manifold will be Ricci-flat, i.e., $\operatorname{Ric} = 0$).

Theorem 8. *Let ξ be a Ricci vector field on an n-dimensional compact Riemannian manifold (M, g) such that $\nabla\xi = aQ$, with $a \in \mathbb{R} \setminus \{0\}$ and $n > 2$, satisfying*

$$\int_M \mathrm{Ric}(\xi, \xi) \geq \frac{(n-1)a^2}{n} \int_M r^2.$$

Then, M is a Ricci-flat manifold and ξ is a parallel vector field.

Proof. Since on a compact Riemannian manifold we have [14]:

$$\int_M \Big\{ \mathrm{Ric}(\xi, \xi) + \frac{1}{2}\|\pounds_\xi g\|^2 - \|\nabla\xi\|^2 - (\mathrm{div}(\xi))^2 \Big\} = 0,$$

we obtain:

$$\int_M \Big\{ \mathrm{Ric}(\xi, \xi) + a^2(\|Q\|^2 - r^2) \Big\} = 0,$$

and we infer:

$$a^2 \int_M \Big(\|Q\|^2 - \frac{r^2}{n} \Big) = -\int_M \Big\{ \mathrm{Ric}(\xi, \xi) - \frac{n-1}{n}(ar)^2 \Big\} \leq 0,$$

which, by means of the Schwartz's inequality, $\|Q\|^2 \geq \frac{r^2}{n}$, gives $Q = \frac{r}{n} I$. Since $n > 2$, the manifold, M, is a compact Einstein manifold, which implies that the constant scalar curvature must be 0, due to the divergence theorem. Therefore, $Q = 0$ and $\nabla\xi = 0$, and we obtain the conclusion. □

Theorem 9. *Let ξ be a Ricci vector field on an n-dimensional compact Riemannian manifold (M, g) such that $\nabla\xi = aQ$, with $a \in \mathbb{R} \setminus \{0\}$ and $n > 2$, satisfying*

$$\int_M \mathrm{Ric}(\xi, \xi) \leq \frac{1}{n} \int_M \Big\{ (ar)^2 - na\,\mathrm{Trace}(\pounds_\xi \mathrm{Ric}) \Big\}.$$

Then, M is a Ricci-flat manifold and ξ is a parallel vector field.

Proof. We know that [22]:

$$\mathrm{Trace}(\pounds_\xi \pounds_\xi g) = 2\Big\{ a^2\|Q\|^2 + a\,\mathrm{div}(Q\xi) - \mathrm{Ric}(\xi, \xi) \Big\},$$

which, in our case, implies

$$\|Q\|^2 - \frac{r^2}{n} = \frac{1}{a^2}\Big\{ a\,\mathrm{Trace}(\pounds_\xi \mathrm{Ric}) - a\,\mathrm{div}(Q\xi) + \mathrm{Ric}(\xi, \xi) - \frac{(ar)^2}{n} \Big\}.$$

By integrating the previous relation, we obtain:

$$\int_M \Big(\|Q\|^2 - \frac{r^2}{n} \Big) = \frac{1}{a^2} \int_M \Big\{ a\,\mathrm{Trace}(\pounds_\xi \mathrm{Ric}) + \mathrm{Ric}(\xi, \xi) - \frac{(ar)^2}{n} \Big\} \leq 0,$$

which, by means of the Schwartz's inequality, $\|Q\|^2 \geq \frac{r^2}{n}$, gives $Q = \frac{r}{n} I$, and, with the same arguments that we have used in Theorem 8, we obtain the conclusion. □

A Ricci vector field belonging to the kernel of the Ricci operator is a geodesic vector field. In particular, if the manifold is Ricci-flat, then any Ricci vector field is a parallel vector field. We shall provide a nontrivial necessary and sufficient condition for a Ricci vector field, ξ, to be a geodesic vector field (i.e., $\nabla_\xi \xi = 0$).

Let us first remark that, for any vector fields X, Y, Z tangent to M, we have:

$$(\nabla_X \mathrm{Ric})(Y, Z) = X(\mathrm{Ric}(Y, Z)) - \mathrm{Ric}(\nabla_X Y, Z) - \mathrm{Ric}(Y, \nabla_X Z)$$

$$
\begin{aligned}
&= X(g(QY,Z)) - g(Q(\nabla_X Y),Z) - g(QY,\nabla_X Z)\\
&= g(\nabla_X QY,Z) - g(Q(\nabla_X Y),Z)\\
&= g((\nabla_X Q)Y,Z).
\end{aligned}
$$

Theorem 10. *Let (M,g) be an n-dimensional Riemannian manifold with θ-recurrent Ricci tensor field (i.e., $\nabla \mathrm{Ric} = \theta \otimes \mathrm{Ric}$ for θ, a 1-form on M) and nonzero scalar curvature, r. Then, a Ricci vector field, ξ, such that $\nabla\xi = fQ$, with f a nowhere zero smooth function, is a geodesic vector field if and only if $f\zeta + \nabla f$ is an eigenvector of the Ricci operator corresponding to the eigenvalue function r, where ζ is the dual vector field of θ.*

Proof. The θ-recurrence condition is equivalent to $\nabla Q = \theta \otimes Q$. Then, for any vector fields X and Y tangent to M, we have:

$$
\begin{aligned}
\theta(X)QY &= (\nabla_X Q)Y\\
&= \nabla_X QY - Q(\nabla_X Y)\\
&= -\frac{X(f)}{f^2}\nabla_Y\xi + \frac{1}{f}\nabla_X\nabla_Y\xi - \frac{1}{f}\nabla_{\nabla_X Y}\xi\\
&= -\frac{X(f)}{f}QY + \frac{1}{f}\nabla^2_{X,Y}\xi,
\end{aligned}
$$

and we obtain:

$$
\begin{aligned}
R(X,Y)\xi &= \nabla^2_{X,Y}\xi - \nabla^2_{Y,X}\xi\\
&= f\Big\{\theta(X)QY - \theta(Y)QX\Big\} + X(f)QY - Y(f)QX\\
&= \Big\{f\theta(X) + X(f)\Big\}QY - \Big\{f\theta(Y) + Y(f)\Big\}QX,
\end{aligned}
$$

and

$$
\begin{aligned}
g(Y,Q\xi) &= \mathrm{Ric}(Y,\xi)\\
&= f\sum_{i=1}^{n}\theta(E_i)g(QY,E_i) + \sum_{i=1}^{n}E_i(f)g(QY,E_i)\\
&\quad - \Big\{f\theta(Y) + Y(f)\Big\}\sum_{i=1}^{n}g(QE_i,E_i)\\
&= fg(QY,\zeta) + g(QY,\nabla f) - r\Big\{fg(Y,\zeta) + g(Y,\nabla f)\Big\}\\
&= g(Y,Q(f\zeta)) + g(Y,Q(\nabla f)) - r\Big\{g(Y,f\zeta) + g(Y,\nabla f)\Big\}\\
&= g(Y,Q(f\zeta + \nabla f)) - rg(Y,f\zeta + \nabla f),
\end{aligned}
$$

where $\{E_i\}_{1\le i\le n}$ is an orthonormal frame field on TM. Then,

$$
Q\xi = Q(f\zeta + \nabla f) - r(f\zeta + \nabla f);
$$

hence $\nabla_\xi\xi = fQ\xi = f\Big\{Q(f\zeta + \nabla f) - r(f\zeta + \nabla f)\Big\}$, and we obtain the conclusion. □

In particular, we deduce the following:

Corollary 11. *Let (M,g) be a Riemannian manifold with θ-recurrent Ricci tensor field and nonzero scalar curvature, r, for θ, a 1-form on M. Then, a Ricci vector field, ξ, such that $\nabla\xi = aQ$, with $a \in \mathbb{R}\setminus\{0\}$, is a geodesic vector field if and only if the dual vector field, ζ, of θ is an eigenvector of the Ricci operator corresponding to the eigenvalue function r.*

Proof. In this case, $\nabla_\xi \xi = a^2(Q\zeta - r\zeta)$, and we obtain the conclusion. □

Also, we have the following:

Proposition 12. *A Ricci vector field, ξ, such that $\nabla\xi = aQ$, with $a \in \mathbb{R} \setminus \{0\}$, on a Riemannian manifold (M, g) with Codazzi Ricci operator is a geodesic vector field.*

Proof. Indeed, for any vector fields X and Y tangent to M, we have:

$$0 = (\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{a}(\nabla^2_{X,Y}\xi - \nabla^2_{X,Y}\xi) = R(X, Y)\xi;$$

hence

$$\mathrm{Ric}(Y, \xi) = 0, \;\; Q\xi = 0;$$

therefore, $\nabla_\xi \xi = 0$. □

In particular, we deduce the following:

Corollary 13. *A Ricci vector field, ξ, such that $\nabla\xi = aQ$, with $a \in \mathbb{R} \setminus \{0\}$, on a Ricci symmetric Riemannian manifold (M, g) (i.e., satisfying $\nabla \mathrm{Ric} = 0$) is a geodesic vector field.*

Proof. The Ricci symmetry condition is equivalent to $\nabla Q = 0$. In this case, for any vector fields X and Y tangent to M, we have:

$$0 = (\nabla_X Q)Y = \frac{1}{a}\nabla^2_{X,Y}\xi;$$

hence

$$\nabla^2_{X,Y}\xi = 0, \;\; R(X, Y)\xi = 0, \;\; \mathrm{Ric}(Y, \xi) = 0, \;\; Q\xi = 0;$$

therefore, $\nabla_\xi \xi = 0$. □

5. Conclusions

We highlighted some properties of a special class of vector fields, ξ, attached to an affine connection, ∇, and to a $(1, 1)$-tensor field, φ, on a Riemannian manifold, namely, those vector fields, ξ, such that $\nabla\xi = f\varphi + hI$, with f and h being two smooth real functions on the manifold and I the identity map. The vector fields we have described here encompass the Reeb vector fields of cosymplectic and α-Sasakian manifolds [16], and also the torse-forming-like vector fields that we have recently considered and whose properties we have studied in [7]. They constitute a large class of vector fields, being the potential vector fields of (∇, φ)-solitons; therefore, they occur in concrete problems from mathematical physics.

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References

1. Yano, K. On torse-forming direction in a Riemannian space. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 340–346. [CrossRef]
2. Chen, B.-Y.; Verstraelen, L. A link between torse-forming vector fields and rotational hypersurfaces. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750177. [CrossRef]
3. Chen, B.-Y.; Deshmukh, S. Some results about concircular vector fields on Riemannian manifolds. *Filomat* **2020**, *34*, 835–842. [CrossRef]
4. Ishan, A.; Deshmukh, S. Torse-forming vector fields on m-spheres. *AIMS Math.* **2022**, *7*, 3056–3066. [CrossRef]
5. Alohali, H.; Deshmukh, S. Ricci vector fields. *Mathematics* **2023**, *11*, 4622. [CrossRef]
6. Mihai, A.; Mihai, I. Torse-forming vector fields and exterior concurrent vector fields on Riemannian manifolds and applications. *J. Geom. Phys.* **2013**, *73*, 200–208. [CrossRef]
7. Blaga, A.M.; Özgür, C. On torse-forming-like vector fields. *submitted*.

8. Blaga, A.M. On trivial gradient hyperbolic Ricci and gradient hyperbolic Yamabe solitons. *J. Geom.* **2024**, *115*, 26. [CrossRef]
9. Crasmareanu, M. A new approach to gradient Ricci solitons and generalizations. *Filomat* **2018**, *32*, 3337–3346. [CrossRef]
10. Turki, N.B.; Blaga, A.M.; Deshmukh, S. Soliton-type equations on a Riemannian manifold. *Mathematics* **2022**, *10*, 633. [CrossRef]
11. Miao, P.; Tam, L.-F. On the volume functional of compact manifolds with boundary with constant scalar curvature. *Calc. Var. Partial Differ. Equ.* **2009**, *36*, 14–171. [CrossRef]
12. Fischer, A.E.; Marsden, J.E. Manifolds of Riemannian metrics with prescribed scalar curvature. *Bull. Am. Math. Soc.* **1974**, *80*, 47–484. [CrossRef]
13. Blaga, A.M. Remarks on almost η-solitons. *Matematički Vesnik* **2019**, *71*, 244–249.
14. Yano, K. *Integral Formulas in Riemannian Geometry*; Marcel Dekker Inc.: New York, NY, USA, 1970.
15. Hinterleitner, I.; Kiosak, V.A. ϕ(Ric)-vector fields in Riemannian spaces. *Arch. Math.* **2008**, *44*, 385–390. Available online: http://dml.cz/dmlcz/127124 (accessed on 1 May 2024).
16. Oubina, J.A. New classes of almost contact metric structures. *Publ. Math. Debrecen*. **1985**, *32*, 187–193. [CrossRef]
17. Blaga, A.M.; Laţcu, D.R. Remarks on Einstein solitons with certain types of potential vector field. *Int. Electron. J. Geom.* **2021**, *14*, 100–105. [CrossRef]
18. Amari, S.I. *Differential-Geometrical Methods in Statistics*; Lecture Notes in Statistics 28; Springer: New York, NY, USA, 1985.
19. Kurose, T. *Statistical Manifolds Admitting Torsion. Geometry and Something*; Fukuoka Univ.: Fukuoka-shi, Japan, 2007. (In Japanese)
20. Rachunek, L.; Mikes, J. On tensor fields semiconjugated with torse-forming vector fields. *Acta Univ. Palacki Olomuc Fac. Rerum Nat. Math.* **2005**, *44*, 151–160.
21. Obata, M. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan* **1962**, *14*, 333–340. [CrossRef]
22. Blaga, A.M.; Özgür, C. Killing and 2-Killing vector fields on doubly warped products. *Mathematics* **2023**, *11*, 4983. [CrossRef]

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