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Generalized Fuzzy-Valued Convexity with Ostrowski's, and Hermite-Hadamard Type Inequalities over Inclusion Relations and Their Applications

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Abstract: Convex inequalities and fuzzy-valued calculus converge to form a comprehensive mathematical framework that can be employed to understand and analyze a broad spectrum of issues. This paper utilizes fuzzy Aumman's integrals to establish integral inequalities of Hermite-Hahadard, Fejér, and Pachpatte types within up and down ($U{\cdot}D$) relations and over newly defined class $U{\cdot}D$-ħ-Godunova–Levin convex fuzzy-number mappings. To demonstrate the unique properties of $U{\cdot}D$-relations, recent findings have been developed using fuzzy Aumman's, as well as various other fuzzy partial order relations that have notable deficiencies outlined in the literature. Several compelling examples were constructed to validate the derived results, and multiple notes were provided to illustrate, depending on the configuration, that this type of integral operator generalizes several previously documented conclusions. This endeavor can potentially advance mathematical theory, computational techniques, and applications across various fields.

Keywords: $U{\cdot}D$-ħ–Godunova–Levin convex fuzzy-number mappings; Ostrowski's inequality; fuzzy Hermite-Hadamard type inequalities

MSC: 26A33; 26A51; 26D10

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Citation: Cortez, M.V.; Althobaiti, A.; Aljohani, A.F.; Althobaiti, S. Generalized Fuzzy-Valued Convexity with Ostrowski's, and Hermite-Hadamard Type Inequalities over Inclusion Relations and Their Applications. *Axioms* **2024**, *13*, 471. https://doi.org/10.3390/axioms13070471

Academic Editors: Loredana Ciurdariu and Simeon Reich

Received: 20 May 2024
Revised: 15 June 2024
Accepted: 21 June 2024
Published: 12 July 2024

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1. Introduction

In recent years, numerous scholars in analysis and various branches of mathematics have shown a growing interest in inequality theory [1,2]. Many real-world problems can be viewed as integral equations, emphasizing the importance of generalizing integral inequalities to address such issues [3].

In Moore's renowned book, the introductory chapter provides an interactive exploration of numerical data, serving as an initiation to interval analysis in numerical analysis (refer to [4]). Over the last five decades, numerous applications have emerged across various domains, including interval differential equations, computer graphics, aero elasticity, and optimization of neural networks. Recently, several authors have extensively investigated various integral inequalities within the context of interval-valued functions (refer to [5–7]).

It is widely acknowledged that the convexity of functions plays a pivotal role in numerous scientific disciplines, encompassing probability theory, economics, and optimal control theory. Moreover, various inequalities have been extensively documented in the literature (refer to [8,9]). The subsequent inequality is commonly known as the classical Hermite-Hadamard inequality regarding equality:

If $f : K \to \mathfrak{N}$ is a convex function defined on the interval K of real numbers, and $\theta, \lambda \in K$ with $\theta < \lambda$, then

$$f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\lambda-\theta}\int_{\theta}^{\lambda} f(б)dб \leq \frac{f(\theta)+f(\lambda)}{2}. \tag{1}$$

Both inequalities are valid in the opposite direction if f exhibits concavity.

Varosanec initially introduced the concept of h-convexity in 2007 (refer to [10]), exploring various generalizations and extensions of this inequality (refer to [11]). Several authors have subsequently developed more intricate Hermite-Hadamard inequalities involving h-convex functions (refer to [12,13]). Moreover, Costa proposed a Jensen-type inequality for fuzzy interval-valued functions (refer to [14]). In the realm of interval-valued functions, Zhao et al. presented a novel Hermite-Hadamard inequality for h-convex functions (refer to [15]).

Using the h-Godunova–Levin function (refer to [16,17]), Almutairi and Kiliman demonstrated the following inequality in 2019.

If $f : K \to \mathfrak{N}$ is a convex function defined on the interval K of real numbers, and $\theta, \lambda \in K$ with $\theta < \lambda$, then

$$f(\kappa\theta + (1-\kappa)\lambda) \leq \frac{f(\theta)}{\hbar(\kappa)} + \frac{f(\lambda)}{\hbar(1-\kappa)}, \tag{2}$$

where, $\hbar : [0,1] \subseteq K \to \mathfrak{N}^{+}$ such that $\hbar \not\equiv 0$.

Several mathematicians have expanded upon the Ostrowski inequality in various directions. Notably, several scientific articles have delved into this topic, exploring different forms of convexity. For instance, İşcan et al. [18] investigated the concept of a harmonically s-convex function. Set [19] introduced the fractional version of the Ostrowski-type inequality using Riemann–Liouville fractional operators. Liu [20] utilized the equality established by Set to devise new refinements of the Ostrowski-type inequality for an MT-convex function. Tunç [21] examined the Ostrowski-type inequality for an ħ-convex function. Ozdemir et al. [22] derived a fresh version of the Ostrowski-type inequality for an (α, m)-convex function. Agarwal et al. [23] explored a more generalized Ostrowski-type inequality using a Raina fractional integral operator. Sarikaya et al. [24,25] employed local fractional integrals to establish new generalizations of the Ostrowski-type inequality. Gürbuz et al. [26,27] utilized a Katugampola fractional operator for a generalized version of the Ostrowski inequality. Ahmad et al. [28,29] introduced some innovative generalizations of the Ostrowski inequality via an Atangana–Baleanu fractional operator for differentiable convex functions and for harmonical convexity, see [30–33]. For further details on recent advancements in the Ostrowski-type inequality, readers are referred to the following references (refer to [34–36]). Budak et al. [37] derived innovative fractional inequalities of the Ostrowski type for interval-valued functions, drawing on the definitions of gH-derivatives. Basic concepts related to fuzzy and fuzzy Aumman's integral are in the following literature (see [38,39] and the references therein). Nanda [40] introduced the concept of convexity in fuzzy environment. For interval-valued convex mapping, see [41]. Khan et al. [42] introduced log-h-convex fuzzy-interval-valued functions as a distinct class of convex fuzzy-interval-valued functions, employing a fuzzy order relation. This class facilitated the establishment of Jensen and Hermite-Hadamard inequalities (see [43–47] and the references therein).

Naturally, numerous researchers have extensively explored and examined Ostrowski's and Hermite-Hadamard inequalities in a novel context via newly defined class of $U{\cdot}D$ ħ–Godunova–Levin convex fuzzy-number mappings. Consequently, several extensions and improvements have been developed. For example, refer to [15,39,40,47] and the references therein. In this investigation, we propose some further adjustments to Fejér, Pachpatte, and Ostrowski's integral inequalities via $U{\cdot}D$-ħ–Godunova–Levin convexity and fuzzy Aumman's integrals.

2. Preliminaries

Consider $\mathbb{E}_C$ as the set comprising all closed and bounded intervals of $\mathfrak{N}$, and let V belong to $\mathbb{E}_C$, defined as:

$$\mathrm{V} = \left[\mathrm{V}_*, \mathrm{V}^*\right] = \left\{\mathfrak{o} \in \mathfrak{N} \middle| \mathrm{V}_* \leq \mathfrak{o} \leq \mathrm{V}^*\right\}, \left(\mathrm{V}_*, \mathrm{V}^* \in \mathfrak{N}\right) \tag{3}$$

It is named a positive interval $[V_*, V^*]$ if $V_* \geq 0$. The Definition of $\mathbb{E}_C^+$, which represents the set of all positive intervals, is

$$\mathbb{E}_{\mathrm{C}}^{+} = \left\{\left[\mathrm{V}_*, \mathrm{V}^*\right] : \left[\mathrm{V}_*, \mathrm{V}^*\right] \in \mathbb{E}_{\mathrm{C}} \text{ and } \mathrm{V}_* \geq 0\right\}. \tag{4}$$

Let $\imath \in \mathfrak{N}$ and $\imath \cdot V$ be defined by

$$\imath \cdot \mathrm{V} = \begin{cases} \left[\imath V_*, \imath V^*\right] \text{ if } \imath > 0, \\ \{0\} \quad \text{if } \imath = 0, \\ \left[\imath V^*, \imath V_*\right] \text{ if } \imath < 0. \end{cases} \tag{5}$$

Subsequently, the Minkowski difference $\mathfrak{K} - V$, addition $V + \mathfrak{K}$, and multiplication $V \times \mathfrak{K}$ for $V, \mathfrak{K}$ belong to $\mathbb{E}_C$ are delineated as follows:

$$\left[\mathfrak{K}_*, \mathfrak{K}^*\right] + \left[V_*, V^*\right] = \left[\mathfrak{K}_* + V_*, \mathfrak{K}^* + V^*\right], \tag{6}$$

$$\left[\mathfrak{K}_*, \mathfrak{K}^*\right] \times \left[V_*, V^*\right] = \left[min\left\{\mathfrak{K}_* V_*, \mathfrak{K}^* V_*, \mathfrak{K}_* V^*, \mathfrak{K}^* V^*\right\}, max\left\{\mathfrak{K}_* V_*, \mathfrak{K}^* V_*, \mathfrak{K}_* V^*, \mathfrak{K}^* V^*\right\}\right], \tag{7}$$

$$\left[\mathfrak{K}_*, \mathfrak{K}^*\right] - \left[V_*, V^*\right] = \left[\mathfrak{K}_* - V^*, \mathfrak{K}^* - V_*\right]. \tag{8}$$

Remark 1. *(i) For given* $\left[\mathfrak{K}_*, \mathfrak{K}^*\right], [V_*, V^*] \in \mathbb{E}_C$, *the relation "* $\supseteq_I$ *" defined on* $\mathbb{E}_C$ *by* $[V_*, V^*] \supseteq_I \left[\mathfrak{K}_*, \mathfrak{K}^*\right]$ *if and only if* $V_* \leq \mathfrak{K}_*, \mathfrak{K}^* \leq V^*$ *for all* $\left[\mathfrak{K}_*, \mathfrak{K}^*\right], [V_*, V^*] \in \mathbb{E}_C$ *is a partial interval inclusion relation. The relation* $[V_*, V^*] \supseteq_I \left[\mathfrak{K}_*, \mathfrak{K}^*\right]$ *is coincident to* $[V_*, V^*] \supseteq \left[\mathfrak{K}_*, \mathfrak{K}^*\right]$ *on* $\mathbb{E}_C$. *It can be easily seen that "*$\supseteq_I$*" looks like "up and down" on the real line* $\mathfrak{N}$, *so we call "* $\supseteq_I$ *" "up and down" (or "UD" order, in short) [44]. For* $\left[\mathfrak{K}_*, \mathfrak{K}^*\right], [V_*, V^*] \in \mathbb{E}_C$, *the Hausdorff–Pompeiu distance between intervals* $\left[\mathfrak{K}_*, \mathfrak{K}^*\right]$ *and* $[V_*, V^*]$ *is defined by*

$$d_H\left(\left[\mathfrak{K}_*, \mathfrak{K}^*\right], [V_*, V^*]\right) = max\left\{\left|\mathfrak{K}_* - V_*\right|, \left|\mathfrak{K}^* - V^*\right|\right\}. \tag{9}$$

It is a familiar fact that $(\mathbb{E}_C, d_H)$ *is a complete metric space [37–39].*

We will briefly review some essential concepts regarding fuzzy sets and fuzzy numbers since we will rely on the standard definitions of these sets.

Please note that we refer to $\mathbb{F}$ *and* $\mathbb{F}_0$ *as the set of all fuzzy subsets and fuzzy numbers of* $\mathfrak{N}$. *Given* $\widetilde{\mathfrak{K}} \in \mathbb{F}_0$, *the level sets or cut sets are given by* $\left[\widetilde{\mathfrak{K}}\right]^{\imath} = \left\{\mathfrak{o} \in \mathfrak{N} \middle| \widetilde{\mathfrak{K}}(\mathfrak{o}) > \imath\right\}$ *for all* $\imath \in [0, 1]$ *and by* $\left[\widetilde{\mathfrak{K}}\right]^{0} = \left\{ \in \mathfrak{N} \middle| \widetilde{\mathfrak{K}}(\mathfrak{o}) > 0\right\}$. *These sets are known as* $\imath$*-level sets or* $\imath$*-cut sets of* $\widetilde{\mathfrak{K}}$, *see [37].*

Proposition 1 ([44]). *Let* $\widetilde{\mathfrak{K}}, \widetilde{V} \in \mathbb{F}_0$. *Then, relation* " $\leq_{\mathbb{F}}$ " *is given on* $\mathbb{F}_0$ *by* $\widetilde{\mathfrak{K}} \leq_{\mathbb{F}} \widetilde{V}$ *when and only when* $\left[\widetilde{\mathfrak{K}}\right]' \leq_I \left[\widetilde{V}\right]'$, *for every* $\imath \in [0, 1]$, *which are left- and right-order relations.*

Proposition 2 ([44]). *Let* $\widetilde{\mathfrak{K}}, \widetilde{V} \in \mathbb{F}_0$. *Then, relation* " $\supseteq_{\mathbb{F}}$ " *is given on* $\mathbb{F}_0$ *by* $\widetilde{\mathfrak{K}} \supseteq_{\mathbb{F}} \widetilde{V}$ *when and only when* $\left[\widetilde{\mathfrak{K}}\right]' \supseteq_I \left[\widetilde{V}\right]'$ *for every* $\imath \in [0, 1]$, *which is the UD−order relation on* $\mathbb{F}_0$.

Remember the approaching notions, which are offered in the literature. If $\widetilde{\mathfrak{K}}, \widetilde{V} \in \mathbb{F}_0$ *and* $\imath \in \mathfrak{N}$, *then, for every* $\imath \in [0, 1]$, *the arithmetic operations addition* "$\oplus$", *multiplication* "$\otimes$", *and scaler multiplication* "$\odot$" *are defined by*

$$\left[\widetilde{\mathfrak{K}} \oplus \widetilde{V}\right]' = \left[\widetilde{\mathfrak{K}}\right]' + \left[\widetilde{V}\right]', \tag{10}$$

$$\left[\widetilde{\mathfrak{K}} \otimes \widetilde{V}\right]' = \left[\widetilde{\mathfrak{K}}\right]' \times \left[\widetilde{V}\right]', \tag{11}$$

$$\left[t \odot \widetilde{\mathfrak{K}}\right]' = t\left[\widetilde{\mathfrak{K}}\right]', \tag{12}$$

over $[\theta, \lambda]$.

Aumann Integrals for Interval and Fuzzy Number Mappings

Now we define and discuss some properties of Aumann integrals for interval and *F·N·M*s.

Definition 1 ([37]). *If* $f : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_C$ *is an interval-valued mapping (I − V − M) satisfying that* $f(\mathfrak{6}) = [f_*(\mathfrak{6}), f^*(\mathfrak{6})]$, *then* f *is an Aumann integrable over* $[\theta, \lambda]$ *when and only when* $f_*(\mathfrak{6})$ *and* $f^*(\mathfrak{6})$ *both are lebesgue integrable over* $[\theta, \lambda]$, *such that*

$$(IA)\int_\theta^\lambda f(\mathfrak{6})d\mathfrak{6} = \left[\int_\theta^\lambda f_*(\mathfrak{6})d\mathfrak{6}, \int_\theta^\lambda f^*(\mathfrak{6})d\mathfrak{6}\right]. \tag{13}$$

The literature suggests the following conclusions, see [37,38,47]:

Definition 2 ([44]). *A fuzzy-interval-valued map* $\widetilde{f} : \Lambda \subset \mathfrak{N} \to \mathbb{F}_0$ *is named F·N·V·M. For each* $\imath \in (0, 1]$, *its* $I - V - M$*s are classified according to their* $\imath$*-levels* $f_\imath : \Lambda \subset \mathfrak{N} \to \mathbb{E}_C$ *are given by* $f_\imath(\mathfrak{6}) = [f_*(\mathfrak{6}, \imath), f^*(\mathfrak{6}, \imath)]$ *for all* $\mathfrak{6} \in \Lambda$. *Here, for each* $\imath \in (0, 1]$, *the end point real mappings* $f_*(., \imath)$, $f^*(., \imath) : \Lambda \to \mathfrak{N}$ *are called lower and upper mappings of* $\widetilde{f}(\mathfrak{6})$.

Definition 3. *Let* $\widetilde{f} : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{F}_0$ *be an F·N·V·M. Then, fuzzy integral of* $\widetilde{f}$ *over* $[\theta, \lambda]$, *denoted by* $(FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{6})d\mathfrak{6}$, *is given level-wise by*

$$\left[(FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{6})d\mathfrak{6}\right]_\imath = (IA)\int_\theta^\lambda f_\imath(\mathfrak{6})d\mathfrak{6} = \left\{\int_\theta^\lambda f(\mathfrak{6}, \imath)d\mathfrak{6} : f(\mathfrak{6}, \imath) \in \mathcal{R}_{([\theta, \lambda], \imath)}\right\}, \tag{14}$$

for all $\imath \in (0, 1]$, *where* $\mathcal{R}_{([\theta, \lambda], \imath)}$ *denotes the collection of Riemannian integrable mappings of* $I - V - M$*s. The F·N·V·M* $\widetilde{f}$ *is FA-integrable over* $[\theta, \lambda]$ *if* $(FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{6})d\mathfrak{6} \in \mathbb{F}_0$. *Note that, if* $f_*(\mathfrak{6}, \imath)$, $f^*(\mathfrak{6}, \imath)$ *are Lebesgue-integrable, then* f *is fuzzy Aumann-integrable mapping over* $[\theta, \lambda]$, *see* [44].

Theorem 1 ([39]). *Let* $\widetilde{f} : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{F}_0$ *be an F·N·V·M, it's* $I - V -$ *Ms are classified according to their* $\imath$*-levels* $f_\imath : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_C$ *are given by* $f_\imath(\text{б}) = [f_*(\text{б}, \imath), f^*(\text{б}, \imath)]$ *for all* $\text{б} \in [\theta, \lambda]$ *and for all* $\imath \in (0, 1]$. *Then,* $\widetilde{f}$ *is FA-integrable over* $[\theta, \lambda]$ *if and only if,* $f_*(\text{б}, \imath)$ *and* $f^*(\text{б}, \imath)$ *are both A-integrable over* $[\theta, \lambda]$. *Moreover, if* $\widetilde{f}$ *is FA-integrable over* $[\theta, \lambda]$, *then*

$$\left[(FA)\int_\theta^\lambda \widetilde{f}(\text{б})d\text{б}\right]' = \left[(A)\int_\theta^\lambda f_*(\text{б}, \imath)d\text{б}, (A)\int_\theta^\lambda f^*(\text{б}, \imath)d\text{б}\right] = (IA)\int_\theta^\lambda f_\imath(\text{б})d\text{б}, \tag{15}$$

for all $\imath \in (0, 1]$. *For all* $\imath \in (0, 1]$, $\mathcal{FR}_{([\theta, \lambda], \imath)}$ *denotes the collection of all FA-integrable F·N·V·Ms The family of all* (FA)*-integrable F·N·Ms over*$[\theta, \lambda]$ *are denoted by* $\mathcal{FA}_{([\theta, \lambda], \imath)}$.
Breckner discussed the coming emerging idea of interval-valued convexity in [41].
A I*·V·M* $f : \mathbb{I} = [\theta, \lambda] \to \mathbb{E}_I$ *is called convex* I*·V·M if*

$$f(\kappa\text{б} + (1 - \kappa)s) \supseteq \kappa f(\text{б}) + (1 - \kappa)f(s), \tag{16}$$

for all $\text{б}, y \in [\theta, \lambda]$, $\kappa \in [0, 1]$, *where* $\mathbb{E}_I$ *is the collection of all real valued intervals. If (16) is reversed, then* f *is called concave.*

Definition 4 ([40]). *The F·N·M* $f : [\theta, \lambda] \to \mathbb{F}_0$ *is called convex F·N·M on* $[\theta, \lambda]$ *if*

$$\widetilde{f}(\kappa\text{б} + (1 - \kappa)s) \leq_{\mathbb{F}} \kappa \odot \widetilde{f}(\text{б}) \oplus (1 - \kappa) \odot \widetilde{f}(s), \tag{17}$$

for all $\text{б}, s \in [\theta, \lambda]$, $\kappa \in [0, 1]$, *where* $\widetilde{f}(\text{б}) \geq_{\mathbb{F}} \widetilde{0}$ *for all* $\text{б} \in [\theta, \lambda]$. *If (17) is reversed then,* $\widetilde{f}$ *is called concave F·N·M on* $[\theta, \lambda]$. $\widetilde{f}$ *is affine if and only if it is both convex and concave F·N·M.*

3. Hermite-Hadamard Inequalities over U·D-ħ-Godunova–Levin Convex *F·N·M*

In this section, we start with the main Definition of U·D-ħ-Godunova–Levin convexity over fuzzy domain that will be helpful for the upcoming results. The fuzzy valued Hermite-Hadamard inequalities for U·D-ħ-Godunova–Levin convex *F·N·M*s are established in this section. Additionally, several instances are provided to support the theory produced in this study's application.

Definition 5. *Let* K *be convex set and* $\hbar : [0, 1] \subseteq K \to \mathfrak{N}^+$ *such that* $\hbar \not\equiv 0$. *Then F·N·M* $\widetilde{f} : K \to \mathbb{F}_0$ *is said to be* U·D-ħ*-Godunova–Levin convex on* K *if*

$$\widetilde{f}(\kappa\text{б} + (1 - \kappa)s) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(\text{б})}{\hbar(\kappa)} \oplus \frac{\widetilde{f}(s)}{\hbar(1 - \kappa)}, \tag{18}$$

for all $\text{б}, s \in K$, $\kappa \in [0, 1]$, *where* $\widetilde{f}(\text{б}) \geq_{\mathbb{F}} \widetilde{0}$. *The F·N·M* $\widetilde{f} : K \to \mathbb{F}_0$ *is said to be* U·D-ħ*-Godunova–Levin concave on* K *if inequality (21) is reversed. Moreover,* $\widetilde{f}$ *is known as* U·D-ħ*-Godunova–Levin affine F·N·M on* K *if*

$$\widetilde{f}(\kappa\text{б} + (1 - \kappa)s) = \frac{\widetilde{f}(\text{б})}{\hbar(\kappa)} \oplus \frac{\widetilde{f}(s)}{\hbar(1 - \kappa)}, \tag{19}$$

for all $\text{б}, s \in K$, $\kappa \in [0, 1]$, *where* $\widetilde{f}(\text{б}) \geq_{\mathbb{F}} \widetilde{0}$.

Remark 2. *The* U·D-ħ*-Godunova–Levin convex F·N·Ms have some very nice properties similar to convex F·N·M.*

(1) *if* $\widetilde{f}$ *is* U·D-ħ*-Godunova–Levin convex F·N·M, then* $\alpha\widetilde{f}$ *is also* U·D-ħ*-Godunova–Levin convex for* $\alpha \geq 0$.

(2) *if* $\widetilde{f}$ *and* $\widetilde{\mathcal{T}}$ *both are U·D-*ħ*-Godunova–Levin convex F·N·Ms, then* $max\left(\widetilde{f}(б), \widetilde{\mathcal{T}}(б)\right)$ *is also U·D-*ħ*-Godunova–Levin convex F·N·M.*

Here, we will go through a few unique exceptional cases of U·D ħ*-Godunova–Levin convex F·N·Ms:*

(i) *If* $ħ(\kappa) = \kappa^s$*, then U·D-*ħ*-Godunova–Levin convex F·N·M becomes U·D-s-Godunova–Levin convex F·N·M, that is*

$$\widetilde{f}(\kappa б + (1-\kappa)s) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(б)}{\kappa^s} \oplus \frac{\widetilde{f}(s)}{(1-\kappa)^s}, \forall б, s \in K,\ \kappa \in [0,\ 1]. \tag{20}$$

(ii) *If* $ħ(\kappa) = \kappa$*, then U·D-*ħ*-Godunova–Levin convex F·N·M becomes U·D-Godunova–Levin convex F·N·M, see* [46]*, that is*

$$\widetilde{f}(\kappa б + (1-\kappa)s) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(б)}{\kappa} \oplus \frac{\widetilde{f}(s)}{1-\kappa}, \forall б, s \in K,\ \kappa \in [0,\ 1]. \tag{21}$$

(iii) *If* $ħ(\kappa) \equiv 1$*, then U·D-*ħ*-Godunova–Levin convex F·N·M becomes U·D-Godunova–Levin P F·N·M, that is*

$$\widetilde{f}(\kappa б + (1-\kappa)s) \supseteq_{\mathbb{F}} \widetilde{f}(б) \oplus \widetilde{f}(s), \forall б, s \in K,\ \kappa \in [0,\ 1]. \tag{22}$$

Note that, there are also new special cases (i) and (iii) as well.

Theorem 2. *Let K be convex set, non-negative real valued function* $ħ : [0,\ 1] \subseteq K \to \mathfrak{N}$ *such that* $ħ \not\equiv 0$ *and let* $\widetilde{f} : K \to \mathbb{F}_0$ *be a F·N·M, it's I − V − Ms are classified according to their ı-levels such that,* $f_ı : K \subset \mathfrak{N} \to \mathbb{E}_I^+ \subset \mathbb{E}_I$ *are given by*

$$f_ı(б) = [f_*(б, ı),\ f^*(б, ı)], \tag{23}$$

for all $б \in K$ *and for all* $ı \in [0,\ 1]$*. Then* $\widetilde{f}$ *is U·D-*ħ*-Godunova–Levin convex on K, if and only if, for all* $ı \in [0,\ 1]$*,* $f_*(б,\ ı)$ *is* ħ*-Godunova–Levin convex and* $f^*(б,\ ı)$ *is* ħ*-Godunova–Levin concave.*

Proof. Assume that for each $ı \in [0,\ 1]$, $f_*(б,\ ı)$ and $f^*(б,\ ı)$ are ħ-Godunova–Levin convex and ħ-Godunova–Levin concave on K, respectively. Then, we have

$$f_*(\kappa б + (1-\kappa)s, ı) \leq \frac{f_*(б,\ ı)}{ħ(\kappa)} + \frac{f_*(s, ı)}{ħ(1-\kappa)},\ \forall б, s \in K,\ \kappa \in [0,\ 1],$$

and

$$f^*(\kappa б + (1-\kappa)s, ı) \geq \frac{f^*(б,\ ı)}{ħ(\kappa)} + \frac{f^*(s, ı)}{ħ(1-\kappa)}, \forall б, s \in K,\ \kappa \in [0,\ 1].$$

Then by (18), (5) and (6), we obtain

$$f_ı(\kappa б + (1-\kappa)s) = [f_*(\kappa б + (1-\kappa)s, ı),\ f^*(\kappa б + (1-\kappa)s, ı)],$$

$$\supseteq_I \left[\frac{f_*(s, ı)}{ħ(\kappa)},\ \frac{f^*(s, ı)}{ħ(\kappa)}\right] + \left[\frac{f_*(s, ı)}{ħ(1-\kappa)},\ \frac{f^*(s, ı)}{ħ(1-\kappa)}\right],$$

that is

$$\widetilde{f}(\kappa б + (1-\kappa)s) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(б)}{ħ(\kappa)} \oplus \frac{\widetilde{f}(s)}{ħ(1-\kappa)}, \forall б, s \in K,\ \kappa \in [0,\ 1].$$

Hence, $\widetilde{f}$ is *U·D-*ħ*-Godunova–Levin convex F·N·M* on K.

Conversely, let $\widetilde{f}$ is $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ on K. Then, for all $б, s \in K$ and $\kappa \in [0,\ 1]$, we have $\widetilde{f}(\kappa б + (1-\kappa)s) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(б)}{\hbar(\kappa)} \oplus \frac{\widetilde{f}(s)}{\hbar(1-\kappa)}$. Therefore, from (23), we have

$$f_ı(\kappa б + (1-\kappa)s) = [f_*(\kappa б + (1-\kappa)s, ı),\ f^*(\kappa б + (1-\kappa)s, ı)].$$

Again, from (18), (5) and (6), we obtain

$$\frac{f_ı(б)}{\hbar(\kappa)} + \frac{f_ı(б)}{\hbar(1-\kappa)} = \left[\frac{f_*(s, ı)}{\hbar(\kappa)},\ \frac{f^*(s, ı)}{\hbar(\kappa)}\right] + \left[\frac{f_*(s, ı)}{\hbar(1-\kappa)},\ \frac{f^*(s, ı)}{\hbar(1-\kappa)}\right],$$

for all $б, s \in K$ and $\kappa \in [0,\ 1]$. Then by $U{\cdot}D$-ħ-Godunova–Levin convexity of $\widetilde{f}$, we have for all $б, s \in K$ and $\kappa \in [0,\ 1]$ such that

$$f_*(\kappa б + (1-\kappa)s, ı) \leq \frac{f_*(б,\ ı)}{\hbar(\kappa)} + \frac{f_*(s, ı)}{\hbar(1-\kappa)},$$

and

$$f^*(\kappa б + (1-\kappa)s, ı) \geq \frac{f^*(б,\ ı)}{\hbar(\kappa)} + \frac{f^*(s, ı)}{\hbar(1-\kappa)},$$

for each $ı \in [0,\ 1]$. Hence, the result follows. □

Remark 3. *If $f_*(б,\ ı) = f^*(б,\ ı)$ with $ı = 1$, then $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ reduces to the $U{\cdot}D$-ħ-Godunova–Levin convex function.*

If $f_*(б,\ ı) = f^*(б,\ ı)$ with $ı = 1$ and $\hbar(\kappa) = \kappa^s$ with $s \in (0,\ 1)$, then $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ reduces to the s-Godunova–Levin convex function.

If $f_*(б,\ ı) = f^*(б,\ ı)$ with $ı = 1$ and $\hbar(\kappa) = \kappa$, then $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ reduces to the ħ-Godunova–Levin convex function.

If $f_*(б,\ ı) = f^*(б,\ ı)$ with $ı = 1$ and $\hbar(\kappa) = 1$, then $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ reduces to the P-convex function.

Example 1. *We consider $\hbar(\kappa) = \kappa$, for $\kappa \in [0,\ 1]$ and the $F{\cdot}N{\cdot}M$ $\widetilde{f} : [0,\ 1] \to \mathbb{F}_0$ defined by*

$$\widetilde{f}(б)(\sigma) = \begin{cases} \frac{\sigma}{2б^2} & \sigma \in \left[0,\ 2б^2\right] \\ \frac{4б^2-\sigma}{2б^2} & \sigma \in \left(2б^2,\ 4б^2\right] \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

Then, for each $ı \in [0,\ 1]$, we have $f_ı(б) = \left[2ıб^2, (4-2ı)б^2\right]$. Since end point functions $f_*(б, ı)$, $f^*(б, ı)$ are ħ-Godunova–Levin convex and ħ-Godunova–Levin concave functions for each $ı \in [0,\ 1]$, respectively. Hence $\widetilde{f}(б)$ is $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$.

Definition 6. *Let $\widetilde{f} : [\theta,\ \lambda] \to \mathbb{F}_0$ be a $F{\cdot}N{\cdot}M$, it's $I - V -$ Ms are classified according to their $ı$-levels such that, $f_ı : [\theta,\ \lambda] \to \mathbb{E}_C^+ \subset \mathbb{E}_C$ are given by*

$$f_ı(б) = [f_*(б, ı),\ f^*(б, ı)],$$

for all $ϰ \in [\theta,\ \lambda]$ and for all $ı \in [0,\ 1]$. Then, $\widetilde{f}$ is lower ħ-Godunova–Levin convex (upper ħ-Godunova–Levin concave) $F{\cdot}N{\cdot}M$ o $n[\theta,\ \lambda]$, if and only if, for all $ı \in [0,\ 1]$, $f_*(ϰ,\ ı)$ is a ħ-Godunova–Levin convex (ħ-Godunova–Levin concave) mapping and $f^*(ϰ,\ ı)$ is a ħ-Godunova–Levin affine mapping.

Definition 7. *Let $\widetilde{f} : [\theta, \lambda] \to \mathbb{F}_0$ be a F·N·M, it's $I-V-$ Ms are classified according to their $\imath$-levels such that, $f_\imath : [\theta, \lambda] \to \mathbb{E}_C^+ \subset \mathbb{E}_C$ are given by*

$$f_\imath(\mathfrak{б}) = [f_*(\mathfrak{б}, \imath), f^*(\mathfrak{б}, \imath)],$$

for all $\varkappa \in [\theta, \lambda]$ and for all $\imath \in [0, 1]$. Then, $\widetilde{f}$ is an upper ħ*-Godunova–Levin convex (*ħ*-Godunova–Levin concave) F·N·M on $[\theta, \lambda]$, if and only if, for all $\imath \in [0, 1]$, $f_*(\varkappa, \imath)$ is an* ħ*-Godunova–Levin affine mapping and $f^*(\varkappa, \imath)$ is a* ħ*-Godunova–Levin convex (*ħ*-Godunova–Levin concave) mapping.*

Remark 4. *If $\hbar(\kappa) = \kappa$, then both concepts "U·D-*ħ*-Godunova–Levin convex F·N·M" and "*ħ*-Godunova–Levin convex F·N·M", are behave alike when f is lower U·D-*ħ*-Godunova–Levin convex F·N·M.*

*Both concepts "U·D-*ħ*-Godunova–Levin convex fuzzy number mapping", and "*ħ*-Godunova–Levinconvex interval-valued mapping" are coincident when f is lower* ħ*-Godunova–Levin convex F·N·M with $\imath = 1$.*

*The following result discuss the Hermite-Hadamard inequality over U·D-*ħ*-Godunova–Levin convex F·N·M.*

Theorem 3. *Let $\widetilde{f} : [\theta, \lambda] \to \mathbb{F}_0$ be a U·D-*ħ*-Godunova–Levin convex F·N·M with non-negative real valued function $\hbar : [0, 1] \to \mathfrak{N}^+$ and $\hbar\left(\frac{1}{2}\right) \neq 0$, it's $I-V-$ Ms are classified according to their $\imath$-levels such that, $f_\imath : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_I^+$ are given by $f_\imath(\mathfrak{б}) = [f_*(\mathfrak{б}, \imath), f^*(\mathfrak{б}, \imath)]$ for all $\mathfrak{б} \in [\theta, \lambda]$ and for all $\imath \in [0, 1]$. If $\widetilde{f} \in \mathcal{FA}_{([\theta, \lambda], \imath)}$, then*

$$\frac{\hbar\left(\frac{1}{2}\right)}{2} \odot \widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{б})d\mathfrak{б} \supseteq_{\mathbb{F}} \left[\widetilde{f}(\theta) \oplus \widetilde{f}(\lambda)\right] \odot \int_0^1 \frac{1}{\hbar(\kappa)} d\kappa. \quad (25)$$

If $\widetilde{f}$ is ħ*-Godunova–Levin concave F·N·M, then (25) is reversed.*

$$\frac{\hbar\left(\frac{1}{2}\right)}{2} \odot \widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \subseteq_{\mathbb{F}} \frac{1}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{б})d\mathfrak{б} \subseteq_{\mathbb{F}} \left[\widetilde{f}(\theta) \oplus \widetilde{f}(\lambda)\right] \odot \int_0^1 \frac{1}{\hbar(\kappa)} d\kappa. \quad (26)$$

Proof. Let $\widetilde{f} : [\theta, \lambda] \to \mathbb{F}_0$ be a U·D-ħ-Godunova–Levin convex F·N·M. Then, for $a, b \in [\theta, \lambda]$, we have

$$\widetilde{f}(\kappa a + (1-\kappa)b) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(a)}{\hbar(\kappa)} \oplus \frac{\widetilde{f}(b)}{\hbar(1-\kappa)}$$

If $\kappa = \frac{1}{2}$, then we have

$$\widetilde{f}\left(\frac{a+b}{2}\right) \supseteq_{\mathbb{F}} \frac{\widetilde{f}(a)}{\hbar\left(\frac{1}{2}\right)} \oplus \frac{\widetilde{f}(b)}{\hbar\left(\frac{1}{2}\right)}.$$

Let $a = \kappa\theta + (1-\kappa)\lambda$ and $b = (1-\kappa)\theta + \kappa\lambda$. Then, above inequality we have

$$\hbar\left(\frac{1}{2}\right) \odot \widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \widetilde{f}(\kappa\theta + (1-\kappa)\lambda) \oplus \widetilde{f}((1-\kappa)\theta + \kappa\lambda).$$

Therefore, for every $\imath \in [0,1]$, we have

$$\hbar\left(\tfrac{1}{2}\right) f_*\left(\tfrac{\theta+\lambda}{2}, \imath\right) \leq f_*(\kappa\theta + (1-\kappa)\lambda, \imath) + f_*((1-\kappa)\theta + \kappa\lambda, \imath),$$
$$\hbar\left(\tfrac{1}{2}\right) f^*\left(\tfrac{\theta+\lambda}{2}, \imath\right) \geq f^*(\kappa\theta + (1-\kappa)\lambda, \imath) + f^*((1-\kappa)\theta + \kappa\lambda, \imath).$$

Then

$$\hbar\left(\frac{1}{2}\right)\int_0^1 f_*\left(\frac{\theta+\lambda}{2}, \imath\right)d\kappa \leq \int_0^1 f_*(\kappa\theta+(1-\kappa)\lambda, \imath)d\kappa + \int_0^1 f_*((1-\kappa)\theta+\kappa\lambda, \imath)d\kappa,$$
$$\hbar\left(\frac{1}{2}\right)\int_0^1 f^*\left(\frac{\theta+\lambda}{2}, \imath\right)d\kappa \geq \int_0^1 f^*(\kappa\theta+(1-\kappa)\lambda, \imath)d\kappa + \int_0^1 f^*((1-\kappa)\theta+\kappa\lambda, \imath)d\kappa.$$

It follows that

$$\hbar\left(\frac{1}{2}\right)f_*\left(\frac{\theta+\lambda}{2}, \imath\right) \leq \frac{2}{\lambda-\theta}\int_\theta^\lambda f_*(б, \imath)dб,$$
$$\hbar\left(\frac{1}{2}\right)f^*\left(\frac{\theta+\lambda}{2}, \imath\right) \geq \frac{2}{\lambda-\theta}\int_\theta^\lambda f^*(б, \imath)dб.$$

That is

$$\hbar\left(\frac{1}{2}\right)\left[f_*\left(\frac{\theta+\lambda}{2}, \imath\right), f^*\left(\frac{\theta+\lambda}{2}, \imath\right)\right] \supseteq_I \frac{2}{\lambda-\theta}\left[\int_\theta^\lambda f_*(б, \imath)dб, \int_\theta^\lambda f^*(б, \imath)dб\right].$$

Thus,

$$\frac{\hbar\left(\frac{1}{2}\right)}{2}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб. \tag{27}$$

In a similar way as above, we have

$$\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб \supseteq_{\mathbb{F}} \left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa. \tag{28}$$

Combining (27) and (28), we have

$$\frac{\hbar\left(\frac{1}{2}\right)}{2}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб \supseteq_{\mathbb{F}} \left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa.$$

Hence, the required result.

Note that, by using same steps, the Formula (26) can be proved with the help of ħ-Godunova–Levin concave $F{\cdot}N{\cdot}M$. □

Remark 5. *If* $\hbar(\kappa)=\frac{1}{\kappa^s}$, *then Theorem 3 simplifies to the outcome for* $U{\cdot}D$*-s-convex* $F{\cdot}N{\cdot}M$ *which is also new one:*

$$2^{s-1}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб \supseteq_{\mathbb{F}} \frac{1}{s+1}\odot\left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]. \tag{29}$$

If $\hbar(\kappa)=\frac{1}{\kappa}$, *then Theorem 3 simplifies to the outcome for* $U{\cdot}D$*-convex* $F{\cdot}N{\cdot}M$ *which is also new one:*

$$\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб \supseteq_{\mathbb{F}} \frac{\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)}{2}. \tag{30}$$

If $\hbar(\kappa)\equiv 1$ *then Theorem 3 simplifies to the outcome for* $U{\cdot}D$*-P-*$F{\cdot}N{\cdot}M$ *which is also new one:*

$$\frac{1}{2}\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(б)dб \supseteq_{\mathbb{F}} \widetilde{f}(\theta)\oplus\widetilde{f}(\lambda). \tag{31}$$

If $f_*(б, \imath)\neq f^*(б, \imath)$ *with* $\imath=1$ *and* $\hbar(\kappa)=\frac{1}{\kappa}$, *then Theorem 3 simplifies to the outcome for classical convex function, see* [15]:

$$f\left(\frac{\theta+\lambda}{2}\right) \supseteq \frac{1}{\lambda-\theta}(IA)\int_\theta^\lambda f(б)dб \supseteq \frac{f(\theta)+f(\lambda)}{2}. \tag{32}$$

If $f_*(б,\ ı) = f^*(б,\ ı)$ *with* $ı = 1$, *then Theorem 3 simplifies to the outcome for classical* ħ*-convex function, see* [46]:

$$\frac{ħ\left(\frac{1}{2}\right)}{2} f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\lambda-\theta}\int_\theta^\lambda f(б)dб \leq [f(\theta)+f(\lambda)]\int_0^1 \frac{1}{ħ(\kappa)}d\kappa. \tag{33}$$

If $f_*(б,\ ı) = f^*(б,\ ı)$ *with* $ı = 1$ *and* $ħ(\kappa) = \frac{1}{\kappa^s}$, *then Theorem 3 simplifies to the outcome for classical s-convex function, see* [46]:

$$2^{s-1} f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\lambda-\theta}\int_\theta^\lambda f(б)dб \leq \frac{1}{s+1}[f(\theta)+f(\lambda)]. \tag{34}$$

If $f_*(б,\ ı) = f^*(б,\ ı)$ *with* $ı = 1$ *and* $ħ(\kappa) = \frac{1}{\kappa}$, *then Theorem 3 simplifies to the outcome for classical convex function:*

$$f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\lambda-\theta}\int_\theta^\lambda f(б)dб \leq \frac{f(\theta)+f(\lambda)}{2}. \tag{35}$$

If $f_*(б,\ ı) = f^*(б,\ ı)$ *with* $ı = 1$ *and* $ħ(\kappa) \equiv 1$, *then Theorem 3 simplifies to the outcome for classical P-convex function:*

$$\frac{1}{2} f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\lambda-\theta}\int_\theta^\lambda f(б)dб \leq f(\theta)+f(\lambda). \tag{36}$$

Example 2. *We consider* $ħ(\kappa) = \frac{1}{\kappa}$, *for* $\kappa \in [0,\ 1]$, *and the F·N·M* $\widetilde{f} : [\theta,\ \lambda] = [2,\ 3] \to \mathbb{F}_0$ *defined by,*

$$\widetilde{f}(б)(\theta) = \begin{cases} \frac{\theta-2+б^{\frac{1}{2}}}{1-б^{\frac{1}{2}}} & \theta \in \left[2-б^{\frac{1}{2}},\ 3\right] \\ \frac{2+б^{\frac{1}{2}}-\theta}{б^{\frac{1}{2}}-1} & \theta \in \left(3,\ 2+б^{\frac{1}{2}}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Then, for each $ı \in [0,\ 1]$, *we have* $f_ı(б) = \left[(1-ı)\left(2-б^{\frac{1}{2}}\right)+3ı, (1-ı)\left(2+б^{\frac{1}{2}}\right)+3ı\right]$. *Since left and right end point mappings* $f_*(б, ı) = (1-ı)\left(2-б^{\frac{1}{2}}\right)+3ı$, $f^*(б,\ ı) = (1-ı)\left(2+б^{\frac{1}{2}}\right)+3ı$, *are U·D-*ħ*-Godunova–Levin convex mappings for each* $ı \in [0,\ 1]$, *then* $\widetilde{f}(б)$ *is* ħ*-Godunova–Levin convex F·N·M. We clearly see that* $\widetilde{f} \in L([\theta,\ \lambda], \mathbb{F}_0)$. *Now computing the following*

$$\frac{ħ\left(\frac{1}{2}\right)}{2} f_*\left(\frac{\theta+\lambda}{2},\ ı\right) \leq \frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(б,\ ı)dб \leq [f_*(\theta,\ ı)+f_*(\lambda,\ ı)]\int_0^1 \frac{1}{ħ(\kappa)}d\kappa.$$

$$\frac{ħ\left(\frac{1}{2}\right)}{2} f_*\left(\frac{\theta+\lambda}{2},\ ı\right) = f_*\left(\frac{5}{2},\ ı\right) = \frac{4-\sqrt{10}}{2}(1-ı)+3ı,$$

$$\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(б,\ ı)dб = \int_2^3 \left((1-ı)\left(2-б^{\frac{1}{2}}\right)+3ı\right)dб = \frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-ı)+3ı,$$

$$[f_*(\theta,\ ı)+f_*(\lambda,\ ı)]\int_0^1 \frac{1}{ħ(\kappa)}d\kappa = \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-ı)+3ı,$$

for all $\imath \in [0,\ 1]$.*That means*

$$\frac{4-\sqrt{10}}{2}(1-\imath)+3\imath \leq \frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\imath)+3\imath \leq \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\imath)+3\imath.$$

Similarly, it can be easily show that

$$\frac{\hbar\left(\frac{1}{2}\right)}{2}f^*\left(\frac{\theta+\lambda}{2},\ \imath\right) \geq \frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\ \imath)d\text{б} \geq [f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa.$$

for all $\imath \in [0,\ 1]$, *such that*

$$\frac{\hbar\left(\frac{1}{2}\right)}{2}f^*\left(\frac{\theta+\lambda}{2},\ \imath\right) = f^*\left(\frac{5}{2},\ \imath\right) = \frac{4+\sqrt{10}}{2}(1-\imath)+3\imath,$$

$$\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\ \imath)d\text{б} = \int_2^3 \left((1-\imath)\left(2+\text{б}^{\frac{1}{2}}\right)+3\imath\right)d\text{б} = \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\imath)+3\imath,$$

$$[f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa = \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\imath)+3\imath.$$

From which, we have

$$\frac{4+\sqrt{10}}{2}(1-\imath)+3\imath \geq \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\imath)+3\imath \geq \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\imath)+3\imath,$$

that is

$$\left[\frac{4-\sqrt{10}}{2}(1-\imath)+3\imath, \frac{4+\sqrt{10}}{2}(1-\imath)+3\imath\right] \supseteq_I \left[\frac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\imath)+3\imath, \frac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\imath)+3\imath\right]$$

$$\supseteq_I \left[\frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\imath), \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\imath)+3\imath\right]$$

for all $\imath \in [0,\ 1]$.

Hence,

$$\frac{\hbar\left(\frac{1}{2}\right)}{2}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(\text{б})d\text{б} \supseteq_{\mathbb{F}} \left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa.$$

Theorem 4. *Let* $\widetilde{f}:[\theta,\ \lambda]\to\mathbb{F}_0$ *be a U·D-ħ-Godunova–Levin convex F·N·M with non-negative real valued function* $\hbar:[0,\ 1]\to\mathfrak{N}^+$ *and* $\hbar\left(\frac{1}{2}\right)\neq 0$, *it's* $I-V-M$*s are classified according to their* $\imath$*-levels such that,* $f_\imath:[\theta,\ \lambda]\subset\mathfrak{N}\to\mathbb{E}_I^+$ *are given by* $f_\imath(\text{б}) = [f_*(\text{б},\imath),\ f^*(\text{б},\imath)]$ *for all* $\text{б}\in[\theta,\ \lambda]$ *and for all* $\imath\in[0,\ 1]$. *If* $\widetilde{f}\in\mathcal{FA}_{([\theta,\ \lambda],\ \imath)}$, *then*

$$\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(\text{б})d\text{б}\supseteq_{\mathbb{F}} \rhd_1 \supseteq_{\mathbb{F}} \left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\left[\frac{1}{2}+\frac{1}{\hbar\left(\frac{1}{2}\right)}\right]\int_0^1 \hbar(\kappa)d\kappa, \quad (38)$$

If $\widetilde{f}$ *is a* ħ*-Godunova–Levin concave F·N·M, then (38) is reversed.*

$$\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)\subseteq_{\mathbb{F}}\ \rhd_2\subseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda\widetilde{f}(б)dб\subseteq_{\mathbb{F}}\rhd_1\subseteq_{\mathbb{F}}\left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\left[\frac{1}{2}+\frac{1}{\hbar\left(\frac{1}{2}\right)}\right]\int_0^1\hbar(\kappa)d\kappa,\quad(39)$$

where

$$\rhd_1=\left[\frac{\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)}{2}\oplus\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)\right]\odot\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,$$

$$\rhd_2=\frac{\left[\hbar\left(\frac{1}{2}\right)\right]}{4}\odot\left[\widetilde{f}\left(\frac{3\theta+\lambda}{4}\right)\oplus\widetilde{f}\left(\frac{\theta+3\lambda}{4}\right)\right],$$

and $\rhd_1=\left[\rhd_{1*},\ \rhd_1{}^*\right]$,$\rhd_2=\left[\rhd_{2*},\ \rhd_2{}^*\right]$.

Proof. Take $\left[\theta,\ \frac{\theta+\lambda}{2}\right]$, we have

$$\hbar\left(\frac{1}{2}\right)\odot\widetilde{f}\left(\frac{\kappa\theta+(1-\kappa)\frac{\theta+\lambda}{2}}{2}+\frac{(1-\kappa)\theta+\kappa\frac{\theta+\lambda}{2}}{2}\right)\supseteq_{\mathbb{F}}\widetilde{f}\left(\kappa\theta+(1-\kappa)\frac{\theta+\lambda}{2}\right)\oplus\widetilde{f}\left((1-\kappa)\theta+\kappa\frac{\theta+\lambda}{2}\right).$$

Therefore, for every $\imath\in[0,1]$, we have

$$\begin{aligned}&\hbar\left(\tfrac{1}{2}\right)f_*\left(\tfrac{\kappa\theta+(1-\kappa)\frac{\theta+\lambda}{2}}{2}+\tfrac{(1-\kappa)\theta+\kappa\frac{\theta+\lambda}{2}}{2},\ \imath\right)\\&\quad\le f_*\left(\kappa\theta+(1-\kappa)\tfrac{\theta+\lambda}{2},\ \imath\right)+f_*\left((1-\kappa)\theta+\kappa\tfrac{\theta+\lambda}{2},\ \imath\right),\\&\hbar\left(\tfrac{1}{2}\right)f^*\left(\tfrac{\kappa\theta+(1-\kappa)\frac{\theta+\lambda}{2}}{2}+\tfrac{(1-\kappa)\theta+\kappa\frac{\theta+\lambda}{2}}{2},\ \imath\right)\\&\quad\ge f^*\left(\kappa\theta+(1-\kappa)\tfrac{\theta+\lambda}{2},\ \imath\right)+f^*\left((1-\kappa)\theta+\kappa\tfrac{\theta+\lambda}{2},\imath\right).\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}&\tfrac{\hbar\left(\frac{1}{2}\right)}{4}f_*\left(\tfrac{3\theta+\lambda}{4},\ \imath\right)\le\tfrac{1}{\lambda-\theta}\int_\theta^{\frac{\theta+\lambda}{2}}f_*(б,\ \imath)dб,\\&\tfrac{\hbar\left(\frac{1}{2}\right)}{4}f^*\left(\tfrac{3\theta+\lambda}{4},\ \imath\right)\ge\tfrac{1}{\lambda-\theta}\int_\theta^{\frac{\theta+\lambda}{2}}f^*(б,\ \imath)dб.\end{aligned}$$

That is

$$\frac{\hbar\left(\frac{1}{2}\right)}{4}\left[f_*\left(\frac{3\theta+\lambda}{4},\ \imath\right),\ f^*\left(\frac{3\theta+\lambda}{4},\ \imath\right)\right]\supseteq_I\frac{1}{\lambda-\theta}\left[\int_\theta^{\frac{\theta+\lambda}{2}}f_*(б,\ \imath)dб,\ \int_\theta^{\frac{\theta+\lambda}{2}}f^*(б,\ \imath)dб\right].$$

It follows that

$$\frac{\hbar\left(\frac{1}{2}\right)}{4}\odot\widetilde{f}\left(\frac{3\theta+\lambda}{4}\right)\supseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot\int_\theta^{\frac{\theta+\lambda}{2}}\widetilde{f}(б)dб.\quad(40)$$

In a similar way as above, we have

$$\frac{\hbar\left(\frac{1}{2}\right)}{4}\odot\widetilde{f}\left(\frac{\theta+3\lambda}{4}\right)\supseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot\int_{\frac{\theta+\lambda}{2}}^\lambda\widetilde{f}(б)dб.\quad(41)$$

Combining (40) and (41), we have

$$\frac{\hbar\left(\frac{1}{2}\right)}{4}\odot\left[\widetilde{f}\left(\frac{3\theta+\lambda}{4}\right)\oplus\widetilde{f}\left(\frac{\theta+3\lambda}{4}\right)\right]\supseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot\int_\theta^\lambda\widetilde{f}(б)dб.$$

By using Theorem 3, we have

$$\frac{\hbar\left(\frac{1}{2}\right)}{4}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)=\frac{\hbar\left(\frac{1}{2}\right)}{4}\odot\widetilde{f}\left(\frac{1}{2}.\frac{3\theta+\lambda}{4}+\frac{1}{2}.\frac{\theta+3\lambda}{4}\right).$$

Therefore, for every $\imath\in[0,1]$, we have

$$\begin{aligned}
\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}f_*\left(\tfrac{\theta+\lambda}{2},\,\imath\right)&=\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}f_*\left(\tfrac{1}{2}.\tfrac{3\theta+\lambda}{4}+\tfrac{1}{2}.\tfrac{\theta+3\lambda}{4},\,\imath\right),\\
\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}f^*\left(\tfrac{\theta+\lambda}{2},\,\imath\right)&=\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}f^*\left(\tfrac{1}{2}.\tfrac{3\theta+\lambda}{4}+\tfrac{1}{2}.\tfrac{\theta+3\lambda}{4},\,\imath\right),\\
&\leq\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}\left[\frac{f_*\left(\frac{3\theta+\lambda}{4},\,\imath\right)}{\hbar\left(\frac{1}{2}\right)}+\frac{f_*\left(\frac{\theta+3\lambda}{4},\,\imath\right)}{\hbar\left(\frac{1}{2}\right)}\right],\\
&\geq\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4}\left[\frac{f^*\left(\frac{3\theta+\lambda}{4},\,\imath\right)}{\hbar\left(\frac{1}{2}\right)}+\frac{f^*\left(\frac{\theta+3\lambda}{4},\,\imath\right)}{\hbar\left(\frac{1}{2}\right)}\right],\\
&=\rhd_{2*},\\
&=\rhd_2{}^*,\\
&\leq\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda f_*(\text{б},\,\imath)d\text{б},\\
&\geq\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda f^*(\text{б},\,\imath)d\text{б},\\
&\leq\left[\tfrac{f_*(\theta,\,\imath)+f_*(\lambda,\,\imath)}{2}+f_*\left(\tfrac{\theta+\lambda}{2},\,\imath\right)\right]\textstyle\int_0^1\hbar(\kappa)d\kappa,\\
&\geq\left[\tfrac{f^*(\theta,\,\imath)+f^*(\lambda,\,\imath)}{2}+f^*\left(\tfrac{\theta+\lambda}{2},\,\imath\right)\right]\textstyle\int_0^1\hbar(\kappa)d\kappa,\\
&=\rhd_{1*},\\
&=\rhd_1{}^*,\\
&\leq\left[\tfrac{f_*(\theta,\,\imath)+f_*(\lambda,\,\imath)}{2}+\tfrac{1}{\hbar\left(\frac{1}{2}\right)}(f_*(\theta,\,\imath)+f_*(\lambda,\,\imath))\right]\textstyle\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,\\
&\geq\left[\tfrac{f^*(\theta,\,\imath)+f^*(\lambda,\,\imath)}{2}+\tfrac{1}{\hbar\left(\frac{1}{2}\right)}(f^*(\theta,\,\imath)+f^*(\lambda,\,\imath))\right]\textstyle\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,\\
&=[f_*(\theta,\,\imath)+f_*(\lambda,\,\imath)]\left[\tfrac{1}{2}+\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\right]\textstyle\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,\\
&=[f^*(\theta,\,\imath)+f^*(\lambda,\,\imath)]\left[\tfrac{1}{2}+\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\right]\textstyle\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,
\end{aligned}$$

that is

$$\frac{\left[\hbar\left(\frac{1}{2}\right)\right]}{4}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)\supseteq_{\mathbb{F}}\rhd_2\supseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda\widetilde{f}(\text{б})d\text{б}\supseteq_{\mathbb{F}}\rhd_1\supseteq_{\mathbb{F}}\left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\left[\frac{1}{2}+\frac{1}{\hbar\left(\frac{1}{2}\right)}\right]\int_0^1\frac{1}{\hbar(\kappa)}d\kappa,$$

hence, the result follows. □

Example 3. *We consider* $\hbar(\kappa)=\kappa$, *for* $\kappa\in[0,\,1]$, *and the F·N·M* $\widetilde{f}:[\theta,\,\lambda]=[2,\,3]\to\mathbb{F}_0$ *defined by,* $f_\imath(\text{б})=\left[(1-\imath)\left(2-\text{б}^{\frac{1}{2}}\right)+3\imath,(1-\imath)\left(2+\text{б}^{\frac{1}{2}}\right)+3\imath\right]$, *as in Example 2, then* $\widetilde{f}(\text{б})$ *is*

U·D-ħ-Godunova–Levin convex F·N·M. We have $f_*(б, \imath) = (1-\imath)\left(2-б^{\frac{1}{2}}\right)+3\imath$ *and* $f^*(б, \imath) = (1-\imath)\left(2+б^{\frac{1}{2}}\right)+3\imath$. *We now compute the following:*

$$\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4} f_*\left(\frac{\theta+\lambda}{2}, \imath\right) = f_*\left(\frac{5}{2}, \imath\right) = \frac{4-\sqrt{10}}{2}(1-\imath)+3\imath,$$
$$\frac{\left[\hbar\left(\frac{1}{2}\right)\right]^2}{4} f^*\left(\frac{\theta+\lambda}{2}, \imath\right) = f^*\left(\frac{5}{2}, \imath\right) = \frac{4+\sqrt{10}}{2}(1-\imath)+3\imath,$$

$$\triangleright_{2*} = \frac{\hbar\left(\frac{1}{2}\right)}{4}\left[f_*\left(\frac{3\theta+\lambda}{4}, \imath\right)+f_*\left(\frac{\theta+3\lambda}{4}, \imath\right)\right] = \frac{5-\sqrt{11}}{4}(1-\imath)+3\imath,$$
$$\triangleright_2{}^* = \frac{\hbar\left(\frac{1}{2}\right)}{4}\left[f^*\left(\frac{3\theta+\lambda}{4}, \imath\right)+f^*\left(\frac{\theta+3\lambda}{4}, \imath\right)\right] = \frac{7+\sqrt{11}}{4}(1-\imath)+3\imath.$$

$$\triangleright_{1*} = \left[\frac{f_*(\theta, \imath)+f_*(\lambda, \imath)}{2}+f_*\left(\frac{\theta+\lambda}{2}, \imath\right)\right]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa = \frac{\left(8-\sqrt{2}-\sqrt{3}-\sqrt{10}\right)}{4}(1-\imath)+3\imath,$$
$$\triangleright_1{}^* = \left[\frac{f^*(\theta, \imath)+f^*(\lambda, \imath)}{2}+f^*\left(\frac{\theta+\lambda}{2}, \imath\right)\right]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa = \frac{\left(8+\sqrt{2}+\sqrt{3}+\sqrt{10}\right)}{4}(1-\imath)+3\imath,$$

$$[f_*(\theta, \imath)+f_*(\lambda, \imath)]\left[\frac{1}{2}+\frac{1}{\hbar\left(\frac{1}{2}\right)}\right]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa = \frac{\left(4-\sqrt{2}-\sqrt{3}\right)}{2}(1-\imath)+3\imath,$$
$$[f^*(\theta, \imath)+f^*(\lambda, \imath)]\left[\frac{1}{2}+\frac{1}{\hbar\left(\frac{1}{2}\right)}\right]\int_0^1 \frac{1}{\hbar(\kappa)}d\kappa = \frac{\left(4+\sqrt{2}+\sqrt{3}\right)}{2}(1-\imath)+3\imath.$$

Then we obtain that

$$(1-\imath)\tfrac{4-\sqrt{10}}{2}+3\imath \le \tfrac{5-\sqrt{11}}{4}(1-\imath)+3\imath \le \tfrac{\left(6+4\sqrt{2}-6\sqrt{3}\right)}{3}(1-\imath)+3\imath$$
$$\le \tfrac{\left(8-\sqrt{2}-\sqrt{3}-\sqrt{10}\right)}{4}(1-\imath)+3\imath \le \tfrac{(1-\imath)\left(4-\sqrt{2}-\sqrt{3}\right)}{2}+3\imath$$
$$(1-\imath)\tfrac{4+\sqrt{10}}{2}+3\imath \ge \tfrac{7+\sqrt{11}}{4}(1-\imath)+3\imath \ge \tfrac{\left(6-4\sqrt{2}+6\sqrt{3}\right)}{3}(1-\imath)+3\imath$$
$$\ge \tfrac{\left(8+\sqrt{2}+\sqrt{3}+\sqrt{10}\right)}{4}(1-\imath)+3\imath \ge \tfrac{(1-\imath)\left(4+\sqrt{2}+\sqrt{3}\right)}{2}+3\imath.$$

Hence, Theorem 4 is verified.

The novel fuzzy Hermite-Hadamard inequalities for the product of two U·D-ħ-Godunova–Levin convex F·N·Ms are found in the results.

Theorem 5. *Let* $\widetilde{f}, \widetilde{\mathcal{J}} : [\theta, \lambda] \to \mathbb{F}_0$ *be two U·D-ħ-Godunova–Levin convex F·N·Ms with non-negative real valued functions* $\hbar_1, \hbar_2 : [0, 1] \to \mathfrak{N}^+$ *and* $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \neq 0$, *it's* $I-V-$*Ms are classified according to their ı-levels such that,* $f_\imath, \mathcal{J}_\imath : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_I^+$ *are given by* $f_\imath(б) = [f_*(б, \imath), f^*(б, \imath)]$ *and* $\mathcal{J}_\imath(б) = [\mathcal{J}_*(б, \imath), \mathcal{J}^*(б, \imath)]$ *for all* $б \in [\theta, \lambda]$ *and for all* $\imath \in [0, 1]$. *If* $\widetilde{f}$, $\widetilde{\mathcal{J}}$ *and* $\widetilde{f} \otimes \widetilde{\mathcal{J}} \in \mathcal{FA}_{([\theta, \lambda], \imath)}$, *then*

$$\frac{1}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \widetilde{f}(б) \otimes \widetilde{\mathcal{J}}(б)dб \supseteq_{\mathbb{F}} \widetilde{\mathcal{M}}(\theta, \lambda) \odot \int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa \oplus \widetilde{\mathcal{N}}(\theta, \lambda) \odot \int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa, \tag{42}$$

If $\widetilde{f}$ *is ħ-Godunova–Levin concave F·N·M, then (42) is reversed.*

$$\frac{1}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \widetilde{f}(б) \otimes \widetilde{\mathcal{J}}(б)dб \subseteq_{\mathbb{F}} \widetilde{\mathcal{M}}(\theta, \lambda) \odot \int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa \oplus \widetilde{\mathcal{N}}(\theta, \lambda) \odot \int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa, \tag{43}$$

where $\widetilde{\mathcal{M}}(\theta, \lambda) = \widetilde{f}(\theta) \otimes \widetilde{\mathcal{J}}(\theta) \oplus \widetilde{f}(\lambda) \otimes \widetilde{\mathcal{J}}(\lambda), \widetilde{\mathcal{N}}(\theta, \lambda) = \widetilde{f}(\theta) \otimes \widetilde{\mathcal{J}}(\lambda) \oplus \widetilde{f}(\lambda) \otimes \widetilde{\mathcal{J}}(\theta)$, *and* $\mathcal{M}_\imath(\theta, \lambda) = [\mathcal{M}_*((\theta, \lambda), \imath), \mathcal{M}^*((\theta, \lambda), \imath)]$ *and* $\mathcal{N}_\imath(\theta, \lambda) = [\mathcal{N}_*((\theta, \lambda), \imath), \mathcal{N}^*((\theta, \lambda), \imath)]$.

Proof. Let $\widetilde{f}, \widetilde{\mathcal{J}} : [\theta, \lambda] \to \mathbb{F}_0$ be two $U{\cdot}D$-$\hbar_1$-Godunova–Levin convex and $U{\cdot}D$-$\hbar_2$-Godunova–Levin convex $F{\cdot}N{\cdot}M$s. Then, we have

$$f_*(\kappa\text{б} + (1-\kappa)s, \imath) \leq \frac{f_*(\text{б}, \imath)}{\hbar(\kappa)} + \frac{f_*(s, \imath)}{\hbar(1-\kappa)},$$

and

$$f^*(\kappa\text{б} + (1-\kappa)s, \imath) \geq \frac{f^*(\text{б}, \imath)}{\hbar(\kappa)} + \frac{f^*(s, \imath)}{\hbar(1-\kappa)},$$

$$\begin{aligned} f_*(\kappa\theta + (1-\kappa)\lambda, \imath) &\leq \tfrac{f_*(\text{б}, \imath)}{\hbar_1(\kappa)} + \tfrac{f_*(s,\imath)}{\hbar_1(1-\kappa)} \\ f^*(\kappa\theta + (1-\kappa)\lambda, \imath) &\geq \tfrac{f^*(\text{б}, \imath)}{\hbar_1(\kappa)} + \tfrac{f^*(s,\imath)}{\hbar_1(1-\kappa)}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) &\leq \tfrac{\mathcal{J}_*(\text{б}, \imath)}{\hbar_2(\kappa)} + \tfrac{\mathcal{J}_*(s,\imath)}{\hbar_2(1-\kappa)}, \\ \mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) &\geq \tfrac{\mathcal{J}^*(\text{б}, \imath)}{\hbar_2(\kappa)} + \tfrac{\mathcal{J}^*(s,\imath)}{\hbar_2(1-\kappa)}. \end{aligned}$$

From the Definition of $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ it follows that $\widetilde{f}(x) \geq_{\mathbb{F}} 0$ and $\widetilde{\mathcal{J}}(x) \geq_{\mathbb{F}} 0$, so

$$\begin{aligned} &f_*(\kappa\theta + (1-\kappa)\lambda, \imath)\mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ &\quad \leq \left(\tfrac{f_*(\text{б}, \imath)}{\hbar_1(\kappa)} + \tfrac{f_*(s,\imath)}{\hbar_1(1-\kappa)}\right)\left(\tfrac{\mathcal{J}_*(\text{б}, \imath)}{\hbar_2(\kappa)} + \tfrac{\mathcal{J}_*(s,\imath)}{\hbar_2(1-\kappa)}\right) \\ &= f_*(\theta, \imath)\mathcal{J}_*(\theta, \imath)\left[\tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}\right] + f_*(\lambda, \imath)\mathcal{J}_*(\lambda, \imath)\left[\tfrac{1}{\hbar_1(1-\kappa)\hbar_2(1-\kappa)}\right] \\ &\quad + f_*(\theta, \imath)\mathcal{J}_*(\lambda, \imath)\left[\tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}\right] + f_*(\lambda, \imath)\mathcal{J}_*(\theta, \imath)\left[\tfrac{1}{\hbar_1(1-\kappa)\hbar_2(\kappa)}\right], \\ &f^*(\kappa\theta + (1-\kappa)\lambda, \imath)\mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ &\quad \geq \left(\tfrac{f^*(\text{б}, \imath)}{\hbar_1(\kappa)} + \tfrac{f^*(s,\imath)}{\hbar_1(1-\kappa)}\right)\left(\tfrac{\mathcal{J}^*(\text{б}, \imath)}{\hbar_2(\kappa)} + \tfrac{\mathcal{J}^*(s,\imath)}{\hbar_2(1-\kappa)}\right) \\ &= f^*(\theta, \imath)\mathcal{J}^*(\theta, \imath)\left[\tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}\right] + f^*(\lambda, \imath)\mathcal{J}^*(\lambda, \imath)\left[\tfrac{1}{\hbar_1(1-\kappa)\hbar_2(1-\kappa)}\right] \\ &\quad + f^*(\theta, \imath)\mathcal{J}^*(\lambda, \imath)\left[\tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}\right] + f^*(\lambda, \imath)\mathcal{J}^*(\theta, \imath)\left[\tfrac{1}{\hbar_1(1-\kappa)\hbar_2(\kappa)}\right]. \end{aligned}$$

Integrating both sides of above inequality over $[0, 1]$ we get

$$\begin{aligned} \int_0^1 f_*(\kappa\theta + (1-\kappa)\lambda, \imath)\mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) &= \tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(x, \imath)\mathcal{J}_*(x, \imath)dx \\ &\leq (f_*(\theta, \imath)\mathcal{J}_*(\theta, \imath) + f_*(\lambda, \imath)\mathcal{J}_*(\lambda, \imath))\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa \\ &\quad + (f_*(\theta, \imath)\mathcal{J}_*(\lambda, \imath) + f_*(\lambda, \imath)\mathcal{J}_*(\theta, \imath))\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa, \\ \int_0^1 f^*(\kappa\theta + (1-\kappa)\lambda, \imath)\mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) &= \tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(x, \imath)\mathcal{J}^*(x, \imath)dx \\ &\geq (f^*(\theta, \imath)\mathcal{J}^*(\theta, \imath) + f^*(\lambda, \imath)\mathcal{J}^*(\lambda, \imath))\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa \\ &\quad + (f^*(\theta, \imath)\mathcal{J}^*(\lambda, \imath) + f^*(\lambda, \imath)\mathcal{J}^*(\theta, \imath))\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa. \end{aligned}$$

It follows that,

$$\begin{aligned}\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(x,\imath)\mathcal{J}_*(x,\imath)dx &\le \mathcal{M}_*((\theta,\lambda),\imath)\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa\\ &+\mathcal{N}_*((\theta,\lambda)),\imath\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa,\\ \tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(x,\imath)\mathcal{J}^*(x,\imath)dx &\ge \mathcal{M}^*((\theta,\lambda),\imath)\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa\\ &+\mathcal{N}^*((\theta,\lambda),\imath)\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa,\end{aligned}$$

that is

$$\begin{aligned}\tfrac{1}{\lambda-\theta}\left[\int_\theta^\lambda f_*(x,\imath)\mathcal{J}_*(x,\imath)dx,\ \int_\theta^\lambda f^*(x,\imath)\mathcal{J}^*(x,\imath)dx\right]\supseteq_I [\mathcal{M}_*((\theta,\lambda),\imath),\ \mathcal{M}^*((\theta,\lambda),\imath)]\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa\\ +[\mathcal{N}_*((\theta,\lambda),\imath),\ \mathcal{N}^*((\theta,\lambda),\imath)]\int_0^1 \tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa.\end{aligned}$$

Thus,

$$\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \widetilde{f}(x)\otimes\widetilde{\mathcal{J}}(x)dx\supseteq_{\mathbb{F}} \widetilde{\mathcal{M}}(\theta,\lambda)\odot\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa\oplus\widetilde{\mathcal{N}}(\theta,\lambda)\odot\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa,$$

and Theorem 5 has been established. □

Example 4. *We consider* $\hbar_1(\kappa)=\frac{1}{\kappa}$, $\hbar_2(\kappa)=\frac{1}{\kappa}$, *for* $\kappa\in[0,\,1]$, *and the F·N·Ms* $\widetilde{f},\ \widetilde{\mathcal{J}}:[\theta,\,\lambda]=[0,\,1]\to\mathbb{F}_0$ *defined by,*

$$\widetilde{f}(б)(\sigma)=\begin{cases}\frac{\theta}{б} & \theta\in[0,\,б]\\ \frac{2б-\theta}{б} & \theta\in(б,\,2б]\\ 0 & \text{otherwise,}\end{cases}$$

$$\widetilde{\mathcal{J}}(б)(\sigma)=\begin{cases}\frac{\theta-б}{2-б} & \theta\in[б,\,2]\\ \frac{8-e^б-\theta}{8-e^б-2} & \theta\in\left(2,\,8-e^б\right]\\ 0 & \text{otherwise.}\end{cases}$$

Then, for each $\imath\in[0,\,1]$, *we have* $f_\imath(б)=[\imath б,(2-\imath)б]$ *and* $\mathcal{J}_\imath(б)=\left[(1-\imath)б+2\imath,(1-\imath)\left(8-e^б\right)+2\imath\right]$. *Since end point functions* $f_*(б,\imath)=\imath б$, $f^*(б,\,\imath)=(2-\imath)б$ *and* $\mathcal{J}_*(б,\imath)=(1-\imath)б+2\imath$,$\mathcal{J}^*(б,\,\imath)=(1-\imath)\left(8-e^б\right)+2\imath$ $\hbar_1,\hbar_2$*-Godunova–Levin convex functions for each* $\imath\in[0,\,1]$. *Hence* $\widetilde{f},\ \widetilde{\mathcal{J}}$ *both are U·D-*$\hbar_1$,*U·D-*$\hbar_2$*-Godunova–Levin convex F·N·Ms, respectively. We now computing the following*

$$\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(б,\imath)\times\mathcal{J}_*(б,\imath)dб=\tfrac{1}{2}\int_0^2\left(\imath(1-\imath)б^2+2\imath^2б\right)dб=\tfrac{2}{3}\imath(2+\imath)$$

$$\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(б,\imath)\times\mathcal{J}^*(б,\imath)dб=\tfrac{1}{2}\int_0^2\left((1-\imath)(2-\imath)б\left(8-e^б\right)+2\imath(2-\imath)б\right)dб\approx\tfrac{(2-\imath)}{2}\left(\tfrac{1903}{250}-\tfrac{903}{250}\imath\right),$$

$$\begin{aligned}\mathcal{M}_*((\theta,\lambda),\,\imath)\int_0^1\tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa&=\tfrac{4\imath}{3}\\ \mathcal{M}^*((\theta,\lambda),\,\imath)\int_0^1\tfrac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa&=\tfrac{2(2-\imath)\left[(1-\imath)\left(8-e^2\right)+2\imath\right]}{3},\\ \mathcal{N}_*((\theta,\lambda),\,\imath)\int_0^1\tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa&=\tfrac{2\imath^2}{3}\\ \mathcal{N}^*((\theta,\lambda),\,\imath)\int_0^1\tfrac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa&=\tfrac{(2-\imath)(7-5\imath)}{3},\end{aligned}$$

for each $\imath\in[0,\,1]$,*that means*

$$\left[\frac{2}{3}\imath(1+2\imath),\frac{(2-\imath)}{2}\left(\frac{1903}{250}-\frac{903}{250}\imath\right)\right]\supseteq_I\frac{1}{3}\left[2\imath(2+\imath),(2-\imath)\left[2(1-\imath)\left(8-e^2\right)-\imath+7\right]\right]$$

Hence, Theorem 5 is demonstrated.

Theorem 6. *Let* $\widetilde{f}, \widetilde{\mathcal{J}} : [\theta, \lambda] \to \mathbb{F}_0$ *be two U·D-*$\hbar_1$*-Godunova–Levin convex and U·D-*$\hbar_2$*-Godunova–Levin convex F·N·Ms with non-negative real valued functions* $\hbar_1, \hbar_2 : [0, 1] \to \mathfrak{N}^+$*, respectively and* $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \neq 0$*, respectively, it's* $I-V-$ *Ms are classified according to their* $\imath$*-levels such that,* $f_\imath, \mathcal{J}_\imath : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_I^+$ *are given, respectively, by* $f_\imath(\mathfrak{б}) = [f_*(\mathfrak{б}, \imath), f^*(\mathfrak{б}, \imath)]$ *and* $\mathcal{J}_\imath(\mathfrak{б}) = [\mathcal{J}_*(\mathfrak{б}, \imath), \mathcal{J}^*(\mathfrak{б}, \imath)]$ *for all* $\mathfrak{б} \in [\theta, \lambda]$ *and for all* $\imath \in [0, 1]$*. If* $\widetilde{f} \otimes \widetilde{\mathcal{J}} \in \mathcal{FA}_{([\theta, \lambda], \imath)}$*, then*

$$\begin{aligned}\frac{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}{2} \odot \widetilde{f}\left(\tfrac{\theta+\lambda}{2}\right) \otimes \widetilde{\mathcal{J}}\left(\tfrac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \tfrac{1}{\lambda-\theta} \odot (FA) \textstyle\int_\theta^\lambda \widetilde{f}(\mathfrak{б}) \otimes \widetilde{\mathcal{J}}(\mathfrak{б}) d\mathfrak{б} \\ \oplus \widetilde{\mathcal{M}}(\theta, \lambda) \odot \textstyle\int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)} d\kappa \oplus \widetilde{\mathcal{N}}(\theta, \lambda) \odot \int_0^1 \frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)} d\kappa,\end{aligned} \tag{44}$$

where $\widetilde{\mathcal{M}}(\theta, \lambda) = \widetilde{f}(\theta) \otimes \widetilde{\mathcal{J}}(\theta) \oplus \widetilde{f}(\lambda) \otimes \widetilde{\mathcal{J}}(\lambda), \widetilde{\mathcal{N}}(\theta, \lambda) = \widetilde{f}(\theta) \otimes \widetilde{\mathcal{J}}(\lambda) \oplus \widetilde{f}(\lambda) \otimes \widetilde{\mathcal{J}}(\theta)$*, and* $\mathcal{M}_\imath(\theta, \lambda) = [\mathcal{M}_*((\theta, \lambda), \imath), \mathcal{M}^*((\theta, \lambda), \imath)]$ *and* $\mathcal{N}_\imath(\theta, \lambda) = [\mathcal{N}_*((\theta, \lambda), \imath), \mathcal{N}^*((\theta, \lambda), \imath)]$*.*

Proof. By hypothesis, for each $\imath \in [0, 1]$, we have

$$\begin{aligned}&f_*\left(\tfrac{\theta+\lambda}{2}, \imath\right) \times \mathcal{J}_*\left(\tfrac{\theta+\lambda}{2}, \imath\right) \\ &f^*\left(\tfrac{\theta+\lambda}{2}, \imath\right) \times \mathcal{J}^*\left(\tfrac{\theta+\lambda}{2}, \imath\right)\end{aligned}$$

$$\begin{aligned}&\leq \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f_*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f_*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}_*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &+\tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f_*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f_*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}_*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix}, \\ &\geq \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f^*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f^*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}^*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &+\tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f^*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f^*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}^*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix}, \\ &\leq \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f_*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f_*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}_*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &\quad +\tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} \big(\hbar_1(\kappa) f_*(\theta, \imath) + \hbar_1(1-\kappa) f_*(\lambda, \imath)\big) \\ \times \big(\hbar_2(1-\kappa)\mathcal{J}_*(\theta, \imath) + \hbar_2(\kappa)\mathcal{J}_*(\lambda, \imath)\big) \\ + \big(\hbar_1(1-\kappa) f_*(\theta, \imath) + \hbar_1(\kappa) f_*(\lambda, \imath)\big) \\ \times \big(\hbar_2(\kappa)\mathcal{J}_*(\theta, \imath) + \hbar_2(1-\kappa)\mathcal{J}_*(\lambda, \imath)\big)\end{bmatrix}, \\ &\geq \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f^*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f^*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}^*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &\quad +\tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} \big(\hbar_1(\kappa) f^*(\theta, \imath) + \hbar_1(1-\kappa) f^*(\lambda, \imath)\big) \\ \times \big(\hbar_2(1-\kappa)\mathcal{J}^*(\theta, \imath) + \hbar_2(\kappa)\mathcal{J}^*(\lambda, \imath)\big) \\ + \big(\hbar_1(1-\kappa) f^*(\theta, \imath) + \hbar_1(\kappa) f^*(\lambda, \imath)\big) \\ \times \big(\hbar_2(\kappa)\mathcal{J}^*(\theta, \imath) + \hbar_2(1-\kappa)\mathcal{J}^*(\lambda, \imath)\big)\end{bmatrix}, \\ &= \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f_*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}_*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f_*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}_*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &+\tfrac{2}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} \big\{\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(1-\kappa)\hbar_2(1-\kappa)\big\}\mathcal{N}_*((\theta, \lambda), \imath) \\ + \big\{\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(1-\kappa)\hbar_2(\kappa)\big\}\mathcal{M}_*((\theta, \lambda), \imath)\end{bmatrix}, \\ &= \tfrac{1}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} f^*(\kappa\theta + (1-\kappa)\lambda, \imath) \times \mathcal{J}^*(\kappa\theta + (1-\kappa)\lambda, \imath) \\ + f^*((1-\kappa)\theta + \kappa\lambda, \imath) \times \mathcal{J}^*((1-\kappa)\theta + \kappa\lambda, \imath)\end{bmatrix} \\ &+\tfrac{2}{\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\begin{bmatrix} \big\{\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(1-\kappa)\hbar_2(1-\kappa)\big\}\mathcal{N}^*((\theta, \lambda), \imath) \\ + \big\{\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(1-\kappa)\hbar_2(\kappa)\big\}\mathcal{M}^*((\theta, \lambda), \imath)\end{bmatrix},\end{aligned}$$

Integrating over $[0, 1]$, we have

$$\begin{aligned}\frac{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{2}f_*\left(\tfrac{\theta+\lambda}{2},\imath\right)\times\mathcal{J}_*\left(\tfrac{\theta+\lambda}{2},\imath\right)&\leq\tfrac{1}{\lambda-\theta}(R)\textstyle\int_\theta^\lambda f_*(\text{б},\imath)\times\mathcal{J}_*(\text{б},\imath)d\text{б}\\&+\mathcal{M}_*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa\\&+\mathcal{N}_*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa,\\\frac{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{2}f^*\left(\tfrac{\theta+\lambda}{2},\imath\right)\times\mathcal{J}^*\left(\tfrac{\theta+\lambda}{2},\imath\right)&\geq\tfrac{1}{\lambda-\theta}(R)\textstyle\int_\theta^\lambda f^*(\text{б},\imath)\times\mathcal{J}^*(\text{б},\imath)d\text{б}\\&+\mathcal{M}^*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa\\&+\mathcal{N}^*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa,\end{aligned}$$

that is

$$\frac{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{2}\odot\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)\otimes\widetilde{\mathcal{J}}\left(\frac{\theta+\lambda}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda\widetilde{f}(\text{б})\otimes\widetilde{\mathcal{J}}(\text{б})d\text{б}$$

$$\oplus\widetilde{\mathcal{M}}(\theta,\lambda)\odot\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa\oplus\widetilde{\mathcal{N}}(\theta,\lambda)\odot\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa,$$

Hence, the required result. □

Example 5. *We consider* $\hbar_1(\kappa)=\frac{1}{\kappa}$, $\hbar_2(\kappa)=\frac{1}{\kappa}$, *for* $\kappa\in[0,1]$, *and the F·N·Ms* $\widetilde{f},\widetilde{\mathcal{J}}:[\theta,\lambda]=[0,1]\to\mathbb{F}_0$, *as in Example 4. Then, for each* $\imath\in[0,1]$, *we have* $f_\imath(\text{б})=[\text{б},(2-\imath)\text{б}]$ *and* $\mathcal{J}_\imath(\text{б})=\left[(1-\imath)\text{б}+2\imath,(1-\imath)\left(8-e^{\text{б}}\right)+2\imath\right]$ *and,* $\widetilde{f}(\text{б})$, $\widetilde{\mathcal{J}}(\text{б})$ *are U·D-*$\hbar_1$*-Godunova–Levin convex and U·D-*$\hbar_2$*-Godunova–Levin convex F·N·Ms, respectively. We have* $f_*(\text{б},\imath)=\imath\text{б}$, $f^*(\text{б},\imath)=(2-\imath)\text{б}$ *and* $\mathcal{J}_*(\text{б},\imath)=(1-\imath)\text{б}+2\imath$,$\mathcal{J}^*(\text{б},\imath)=(1-\imath)\left(8-e^{\text{б}}\right)+2\imath$. *We now computing the following*

$$\begin{aligned}\tfrac{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{2}f_*\left(\tfrac{\theta+\lambda}{2},\imath\right)\times\mathcal{J}_*\left(\tfrac{\theta+\lambda}{2},\imath\right)&=2\imath(1+\imath),\\\tfrac{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{2}f^*\left(\tfrac{\theta+\lambda}{2},\imath\right)\times\mathcal{J}^*\left(\tfrac{\theta+\lambda}{2},\imath\right)&=2\left[16-20\imath+6\imath^2+\left(2-3\imath+\imath^2\right)e\right].\end{aligned}$$

$$\begin{aligned}\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda f_*(\text{б},\imath)\times\mathcal{J}_*(\text{б},\imath)d\text{б}&=\tfrac{1}{2}\int_0^2\left(\imath(1-\imath)\text{б}^2+2\imath^2\text{б}\right)d\text{б}=\tfrac{4}{3}\imath(3-\imath),\\\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda f^*(\text{б},\imath)\times\mathcal{J}^*(\text{б},\imath)d\text{б}&=\tfrac{1}{2}\int_0^2\left((1-\imath)(2-\imath)\text{б}\left(8-e^{\text{б}}\right)+2\imath(2-\imath)\text{б}\right)d\text{б}\\&\approx\tfrac{(2-\imath)}{2}\left(\tfrac{1903}{250}-\tfrac{903}{250}\imath\right).\end{aligned}$$

$$\begin{aligned}\mathcal{M}_*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa&=\tfrac{2\imath}{3},\\\mathcal{M}^*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(1-\kappa)}d\kappa&=\tfrac{(2-\imath)\left[(1-\imath)\left(8-e^2\right)+2\imath\right]}{3},\\\mathcal{N}_*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa&=\tfrac{4\imath^2}{3},\\\mathcal{N}^*((\theta,\lambda),\imath)\textstyle\int_0^1\frac{1}{\hbar_1(\kappa)\hbar_2(\kappa)}d\kappa&=\tfrac{2(2-\imath)(7-5\imath)}{3},\end{aligned}$$

for each $\imath\in[0,1]$, *that means*

$$\begin{aligned}2\left[\imath(1+\imath),\left[16-20\imath+6\imath^2+\left(2-3\imath+\imath^2\right)e\right]\right]\supseteq_I&\left[\tfrac{2}{3}\imath(2+\imath),\tfrac{(2-\imath)}{2}\left(\tfrac{1903}{250}-\tfrac{903}{250}\imath\right)\right]\\&+\tfrac{1}{3}\left[2\imath(1+2\imath),(2-\imath)\left[(1-\imath)\left(8-e^2\right)-8\imath+14\right]\right],\end{aligned}$$

hence, Theorem 6 is demonstrated.

The H-H Fejér inequalities for $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$s are now presented. The second H-H Fejér inequality for $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ is first obtained.

Theorem 7. *Let $\widetilde{f} : [\theta, \lambda] \to \mathbb{F}_0$ be an $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$ with $\hbar : [0, 1] \to \mathfrak{N}^+$, it's $I - V - M$s are classified according to their $\imath$-levels such that, $f_\imath : [\theta, \lambda] \subset \mathfrak{N} \to \mathbb{E}_I^+$ are given by $f_\imath(\mathfrak{6}) = [f_*(\mathfrak{6}, \imath), f^*(\mathfrak{6}, \imath)]$ for all $\mathfrak{6} \in [\theta, \lambda]$ and for all $\imath \in [0, 1]$. If $\widetilde{f} \in \mathcal{FR}_{([\theta, \lambda], \imath)}$ and $\mathfrak{B} : [\theta, \lambda] \to \mathfrak{N}, \mathfrak{B}(\mathfrak{6}) \geq 0$, symmetric with respect to $\frac{\theta+\lambda}{2}$,then*

$$\frac{1}{\lambda - \theta} \odot (FA) \int_\theta^\lambda f(\mathfrak{6}) \odot \mathfrak{B}() d\mathfrak{6} \supseteq_{\mathbb{F}} [f(\theta) \oplus f(\lambda)] \odot \int_0^1 \frac{\mathfrak{B}((1-\kappa)\theta + \kappa\lambda)}{\hbar(\kappa)} d\kappa. \tag{45}$$

Proof. Let $\widetilde{f}$ be an $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$. Then, for each $\imath \in [0, 1]$, we have

$$\begin{aligned} &f_*(\kappa\theta + (1-\kappa)\lambda, \imath) B(\kappa\theta + (1-\kappa)\lambda) \\ &\quad \leq \left(\frac{f_*(\theta, \imath)}{\hbar(\kappa)} + \frac{f_*(\lambda, \imath)}{\hbar(1-\kappa)} \right) B(\kappa\theta + (1-\kappa)\lambda), \\ &f^*(\kappa\theta + (1-\kappa)\lambda, \imath) B(\kappa\theta + (1-\kappa)\lambda) \\ &\quad \geq \left(\frac{f^*(\theta, \imath)}{\hbar(\kappa)} + \frac{f^*(\lambda, \imath)}{\hbar(1-\kappa)} \right) B(\kappa\theta + (1-\kappa)\lambda). \end{aligned} \tag{46}$$

And

$$\begin{aligned} &f_*((1-\kappa)\theta + \kappa\lambda, \imath) B((1-\kappa)\theta + \kappa\lambda) \\ &\quad \leq \left(\frac{f_*(\theta, \imath)}{\hbar(1-\kappa)} + \frac{f_*(\lambda, \imath)}{\hbar(\kappa)} \right) B((1-\kappa)\theta + \kappa\lambda), \\ &f^*((1-\kappa)\theta + \kappa\lambda, \imath) B((1-\kappa)\theta + \kappa\lambda) \\ &\quad \geq \left(\frac{f^*(\theta, \imath)}{\hbar(1-\kappa)} + \frac{f^*(\lambda, \imath)}{\hbar(\kappa)} \right) B((1-\kappa)\theta + \kappa\lambda). \end{aligned} \tag{47}$$

After adding (46) and (47), and integrating over $[0, 1]$, we get

$$\begin{aligned} &\int_0^1 f_*(\kappa\theta + (1-\kappa)\lambda, \imath) \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) d\kappa \\ &\quad + \int_0^1 f_*((1-\kappa)\theta + \kappa\lambda, \imath) \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) d\kappa \\ &\leq \int_0^1 \left[\begin{array}{l} f_*(\theta, \imath) \left\{ \frac{1}{\hbar(\kappa)} \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) + \frac{1}{\hbar(1-\kappa)} \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) \right\} \\ + f_*(\lambda, \imath) \left\{ \frac{1}{\hbar(1-\kappa)} \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) + \frac{1}{\hbar(\kappa)} \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) \right\} \end{array} \right] d\kappa, \\ &\int_0^1 f^*((1-\kappa)\theta + \kappa\lambda, \imath) \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) d\kappa \\ &\quad + \int_0^1 f^*(\kappa\theta + (1-\kappa)\lambda, \imath) \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) d\kappa \\ &\geq \int_0^1 \left[\begin{array}{l} f^*(\theta, \imath) \left\{ \frac{1}{\hbar(\kappa)} \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) + \frac{1}{\hbar(1-\kappa)} \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) \right\} \\ + f^*(\lambda, \imath) \left\{ \frac{1}{\hbar(1-\kappa)} \mathfrak{B}(\kappa\theta + (1-\kappa)\lambda) + \frac{1}{\hbar(\kappa)} \mathfrak{B}((1-\kappa)\theta + \kappa\lambda) \right\} \end{array} \right] d\kappa, \end{aligned}$$

$$\begin{aligned} &= 2 f_*(\theta, \imath) \int_0^1 \frac{1}{\hbar(\kappa)} B(\kappa\theta + (1-\kappa)\lambda) d\kappa + 2 f_*(\lambda, \imath) \int_0^1 \frac{1}{\hbar(\kappa)} B((1-\kappa)\theta + \kappa\lambda) d\kappa, \\ &= 2 f^*(\theta, \imath) \int_0^1 \frac{1}{\hbar(\kappa)} B(\kappa\theta + (1-\kappa)\lambda) d\kappa + 2 f^*(\lambda, \imath) \int_0^1 \frac{1}{\hbar(\kappa)} B((1-\kappa)\theta + \kappa\lambda) d\kappa. \end{aligned}$$

Since $\mathfrak{B}$ is symmetric, then

$$\begin{aligned}
&\int_0^1 f_*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa\\
&\quad+\int_0^1 f_*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa\\
&\quad\le 2[f_*(\theta,\ \imath)+f_*(\lambda,\ \imath)]\int_0^1 \frac{1}{\hbar(\kappa)}B((1-\kappa)\theta+\kappa\lambda)d\kappa,\\
&\int_0^1 f^*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa\\
&\quad+\int_0^1 f^*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa\\
&\quad\ge 2[f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1 \frac{1}{\hbar(\kappa)}B((1-\kappa)\theta+\kappa\lambda)d\kappa.
\end{aligned}\tag{48}$$

Since

$$\begin{aligned}
&\int_0^1 f_*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa\\
&=\int_0^1 f_*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa=\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}()d\text{б}\\
&\int_0^1 f^*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa\\
&=\int_0^1 f^*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa=\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\ \imath)\mathfrak{B}()d\text{б}
\end{aligned}\tag{49}$$

Then from (49), (48) we have

$$\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}()d\text{б}\le[f_*(\theta,\ \imath)+f_*(\lambda,\ \imath)]\int_0^1\frac{1}{\hbar(\kappa)}B((1-\kappa)\theta+\kappa\lambda)d\kappa,$$

$$\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}()d\text{б}\ge[f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1\frac{1}{\hbar(\kappa)}B((1-\kappa)\theta+\kappa\lambda)d\kappa,$$

that is

$$\left[\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б},\ \frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}\right]$$

$$\supseteq_I[f_*(\theta,\ \imath)+f_*(\lambda,\ \imath),\ f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1\frac{1}{\hbar(\kappa)}B((1-\kappa)\theta+\kappa\lambda)d\kappa$$

hence

$$\frac{1}{\lambda-\theta}\odot(FA)\int_\theta^\lambda\widetilde{f}(\text{б})\odot\mathfrak{B}()d\text{б}\supseteq_{\mathbb{F}}\left[\widetilde{f}(\theta)\oplus\widetilde{f}(\lambda)\right]\odot\int_0^1\frac{\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)}{\hbar(\kappa)}d\kappa.$$

Now, generalizing the first H-H Fejér inequalities for classical Godunova–Levin convex functions and we build the first H-H Fejér inequality for *U·D*-ħ-Godunova–Levin convex *F·N·M*. □

Theorem 8. *Let* $\widetilde{f}:[\theta,\ \lambda]\to\mathbb{F}_0$ *be an U·D-ħ-Godunova–Levin convex F·N·M with* $\hbar:[0,\ 1]\to\mathfrak{N}^+$, *it's* $I-V-M$*s are classified according to their ı-levels such that,* $f_\imath:[\theta,\ \lambda]\subset\mathfrak{N}\to\mathbb{E}_I^+$ *are given by* $f_\imath(\text{б})=[f_*(\text{б},\imath),\ f^*(\text{б},\imath)]$ *for all* $\text{б}\in[\theta,\ \lambda]$ *and for all* $\imath\in[0,\ 1]$. *If* $\widetilde{f}\in\mathcal{FR}_{([\theta,\ \lambda],\ \imath)}$ *and* $\mathfrak{B}:[\theta,\ \lambda]\to\mathfrak{N},\mathfrak{B}(\text{б})\ge 0$, *symmetric with respect to* $\frac{\theta+\lambda}{2}$, *and* $\int_\theta^\lambda\mathfrak{B}()d\text{б}>0$, *then*

$$\widetilde{f}\left(\frac{\theta+\lambda}{2}\right)\supseteq_{\mathbb{F}}\frac{2}{\hbar\left(\frac{1}{2}\right)\int_\theta^\lambda\mathfrak{B}()d\text{б}}\odot(FA)\int_\theta^\lambda\widetilde{f}(\text{б})\odot\mathfrak{B}()d\text{б}.\tag{50}$$

Proof. Since $\widetilde{f}$ is an *U·D*-ħ-Godunova–Levin convex, then for $\imath\in[0,\ 1]$, we have

$$\begin{aligned}
f_*\left(\tfrac{\theta+\lambda}{2},\ \imath\right)&\le\frac{1}{\hbar\left(\frac{1}{2}\right)}(f_*(\kappa\theta+(1-\kappa)\lambda,\ \imath)+f_*((1-\kappa)\theta+\kappa\lambda,\ \imath)),\\
f^*\left(\tfrac{\theta+\lambda}{2},\ \imath\right)&\ge\frac{1}{\hbar\left(\frac{1}{2}\right)}(f^*(\kappa\theta+(1-\kappa)\lambda,\ \imath)+f^*((1-\kappa)\theta+\kappa\lambda,\ \imath)),
\end{aligned}\tag{51}$$

Since $\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda) = \mathfrak{B}((1-\kappa)\theta+\kappa\lambda)$, then by multiplying (51) by $\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)$ and integrate it with respect to κ over $[0,\ 1]$, we obtain

$$\begin{aligned} & f_*\left(\tfrac{\theta+\lambda}{2},\ \imath\right)\int_0^1 \mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa \\ \leq & \frac{1}{\hbar\left(\frac{1}{2}\right)}\begin{pmatrix}\int_0^1 f_*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa \\ +\int_0^1 f_*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa\end{pmatrix}, \\ & f^*\left(\tfrac{\theta+\lambda}{2},\ \imath\right)\int_0^1 \mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa \\ \geq & \frac{1}{\hbar\left(\frac{1}{2}\right)}\begin{pmatrix}\int_0^1 f^*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa \\ +\int_0^1 f^*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa\end{pmatrix}. \end{aligned} \tag{52}$$

Since

$$\begin{aligned} & \int_0^1 f_*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa \\ & \quad = \int_0^1 f_*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa \\ & \quad = \tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}()d\text{б} \\ & \int_0^1 f^*((1-\kappa)\theta+\kappa\lambda,\ \imath)\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa \\ & \quad = \int_0^1 f^*(\kappa\theta+(1-\kappa)\lambda,\ \imath)\mathfrak{B}(\kappa\theta+(1-\kappa)\lambda)d\kappa \\ & \quad = \tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}()d\text{б}. \end{aligned} \tag{53}$$

And

$$\int_0^1 \mathfrak{B}((1-\kappa)\theta+\kappa\lambda)d\kappa = \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathfrak{B}()d\text{б} \tag{54}$$

Then from (53) and (54), (52) we have

$$\begin{aligned} f_*\left(\tfrac{\theta+\lambda}{2},\ \imath\right) & \leq \tfrac{2}{\hbar\left(\frac{1}{2}\right)\int_\theta^\lambda \mathfrak{B}(\text{б})d\text{б}}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}, \\ f^*\left(\tfrac{\theta+\lambda}{2},\ \imath\right) & \geq \tfrac{2}{\hbar\left(\frac{1}{2}\right)\int_\theta^\lambda \mathfrak{B}(\text{б})d\text{б}}\int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}, \end{aligned}$$

from which, we have

$$\begin{aligned} & \left[f_*\left(\tfrac{\theta+\lambda}{2},\ \imath\right),\ f^*\left(\tfrac{\theta+\lambda}{2},\ \imath\right)\right] \\ \supseteq_I & \tfrac{2}{\hbar\left(\frac{1}{2}\right)\int_\theta^\lambda \mathfrak{B}(\text{б})d\text{б}}\left[\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б},\ \int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}\right], \end{aligned}$$

that is

$$\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \supseteq_{\mathbb{F}} \frac{2}{\hbar\left(\frac{1}{2}\right)\int_\theta^\lambda \mathfrak{B}()d\text{б}} \odot (FA)\int_\theta^\lambda \widetilde{f}(\text{б}) \odot \mathfrak{B}()d\text{б}.$$

This completes the proof. □

Remark 6. *From Theorem 7 and 8, we clearly see that:*
If $\mathfrak{B}(\text{б}) = 1$, *then we acquire the inequality (25).*
Let $\imath = 1$ *and* $\hbar(\kappa) = \frac{1}{\kappa}$. *Then from (45) and (50), we acquire the following inequality, see* [45]:

$$f\left(\frac{\theta+\lambda}{2}\right) \supseteq \frac{1}{\int_\theta^\lambda \mathfrak{B}()d\text{б}}(IA)\int_\theta^\lambda f(\text{б})\mathfrak{B}()d\text{б} \supseteq \frac{f(\theta)+f(\lambda)}{2} \tag{55}$$

If $\widetilde{f}$ is lower Godunova–Levin convex F·N·M on $[\theta, \lambda]$ and $\hbar(\kappa) = \frac{1}{\kappa}$, then we derive the following subsequent inequality, see [33]:

$$\widetilde{f}\left(\frac{\theta+\lambda}{2}\right) \leq_{\mathbb{F}} \frac{1}{\int_\theta^\lambda \mathfrak{B}()d\mathfrak{б}} \odot (FA)\int_\theta^\lambda \widetilde{f}(\mathfrak{б}) \odot \mathfrak{B}()d\mathfrak{б} \leq_{\mathbb{F}} \frac{\widetilde{f}(\theta) \oplus \widetilde{f}(\lambda)}{2} \tag{56}$$

If $\widetilde{f}$ is lower Godunova–Levin convex F·N·M on $[\theta, \lambda]$ with $\imath = 1$ and $\hbar(\kappa) = \frac{1}{\kappa}$, then from (45) and (50) we derive the following subsequent inequality, see [15]:

$$f\left(\frac{\theta+\lambda}{2}\right) \leq_I \frac{1}{\lambda-\theta}(IA)\int_\theta^\lambda f(\mathfrak{б})d\mathfrak{б} \leq_I \frac{f(\theta)+f(\lambda)}{2} \tag{57}$$

If $\widetilde{f}$ is lower Godunova–Levin convex F·N·M on $[\theta, \lambda]$ with $\imath = 1$ and $\hbar(\kappa) = \frac{1}{\kappa}$, then from (45) and (50) we derive the following subsequent inequality, see [15]:

$$f\left(\frac{\theta+\lambda}{2}\right) \leq_I \frac{1}{\int_\theta^\lambda \mathfrak{B}()d\mathfrak{б}}(IA)\int_\theta^\lambda f(\mathfrak{б})\mathfrak{B}()d\mathfrak{б} \leq_I \frac{f(\theta)+f(\lambda)}{2} \tag{58}$$

Let $\hbar(\kappa) = \frac{1}{\kappa}$, and $f_*(\mathfrak{б}, \imath) = f^*(\mathfrak{б}, \imath)$ with $\imath = 1$. Then from (45) and (50), we obtain following classical Fejér inequality.

$$f\left(\frac{\theta+\lambda}{2}\right) \leq \frac{1}{\int_\theta^\lambda \mathfrak{B}()d\mathfrak{б}}\int_\theta^\lambda f(\mathfrak{б})\mathfrak{B}()d\mathfrak{б} \leq \frac{f(\theta)+f(\lambda)}{2} \tag{59}$$

Example 6. *We consider $\hbar(\kappa) = \frac{1}{\kappa}$, for $\kappa \in [0, 1]$, and the F·N·M $f : [0, 2] \to \mathbb{F}_0$ defined by,*

$$\widetilde{f}(\mathfrak{б})(\theta) = \begin{cases} \frac{\theta-2+\mathfrak{б}^{\frac{1}{2}}}{\frac{3}{2}-2-\mathfrak{б}^{\frac{1}{2}}} & \theta \in \left[2-\mathfrak{б}^{\frac{1}{2}}, \frac{3}{2}\right] \\ \frac{2+\mathfrak{б}^{\frac{1}{2}}-\theta}{2+\mathfrak{б}^{\frac{1}{2}}-\frac{3}{2}} & \theta \in \left(\frac{3}{2}, 2+\mathfrak{б}^{\frac{1}{2}}\right] \\ 0 & \text{otherwise,} \end{cases} \tag{60}$$

Then, for each $\imath \in [0, 1]$, we have $f_\imath(\mathfrak{б}) = \left[(1-\imath)\left(2-\mathfrak{б}^{\frac{1}{2}}\right)+\frac{3}{2}\imath, (1-\imath)\left(2+\mathfrak{б}^{\frac{1}{2}}\right)+\frac{3}{2}\imath\right]$. Since end point mappings $f_*(\mathfrak{б}, \imath)$, $f^*(\mathfrak{б}, \imath)$ are $\hbar$-Godunova–Levin convex mappings for each $\imath \in [0, 1]$, then $\widetilde{f}(\mathfrak{б})$ is U·D-$\hbar$-Godunova–Levin convex F·N·M. If

$$\mathfrak{B}(\mathfrak{б}) = \begin{cases} \sqrt{\mathfrak{б}}, & \sigma \in [0,1], \\ \sqrt{2-\mathfrak{б}}, & \sigma \in (1, 2], \end{cases}$$

then $\mathfrak{B}(2-\mathfrak{б}) = \mathfrak{B}(\mathfrak{б}) \geq 0$, for all $\mathfrak{б} \in [0, 2]$. Since $f_*(\mathfrak{б}, \imath) = (1-\imath)\left(2-\mathfrak{б}^{\frac{1}{2}}\right)+\frac{3}{2}\imath$ and $f^*(\mathfrak{б}, \imath) = (1-\imath)\left(2+\mathfrak{б}^{\frac{1}{2}}\right)+\frac{3}{2}\imath$. Now we compute the following:

$$\begin{aligned}
&\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda [f_*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} = \tfrac{1}{2}\int_0^2 [f_*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} = \tfrac{1}{2}\int_0^1 [f_*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} + \tfrac{1}{2}\int_1^2 f_*(\mathfrak{б}, \imath)\mathfrak{B}(\mathfrak{б})d\mathfrak{б}, \\
&\tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda [f^*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} = \tfrac{1}{2}\int_0^2 [f^*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} = \tfrac{1}{2}\int_0^1 [f^*(\mathfrak{б}, \imath)]\mathfrak{B}(\mathfrak{б})d\mathfrak{б} + \tfrac{1}{2}\int_1^2 f^*(\mathfrak{б}, \imath)\mathfrak{B}(\mathfrak{б})d\mathfrak{б}, \\
&= \tfrac{1}{2}\int_0^1 \left[(1-\imath)\left(2-\mathfrak{б}^{\frac{1}{2}}\right)+\tfrac{3}{2}\imath\right]\left(\sqrt{\mathfrak{б}}\right)d\mathfrak{б} + \tfrac{1}{2}\int_1^2 \left[(1-\imath)\left(2-\mathfrak{б}^{\frac{1}{2}}\right)+\tfrac{3}{2}\imath\right](\sqrt{2-\mathfrak{б}})d\mathfrak{б} \\
&= \tfrac{\imath+5}{12} + \tfrac{1}{24}[3\pi(\imath-1)-4(\imath-4)], \\
&= \tfrac{1}{2}\int_0^1 \left[(1-\imath)\left(2+\mathfrak{б}^{\frac{1}{2}}\right)+\tfrac{3}{2}\imath\right]\left(\sqrt{\mathfrak{б}}\right)d\mathfrak{б} + \tfrac{1}{2}\int_1^2 \left[(1-\imath)\left(2+\mathfrak{б}^{\frac{1}{2}}\right)+\tfrac{3}{2}\imath\right](\sqrt{2-\mathfrak{б}})d\mathfrak{б} \\
&= \tfrac{1}{12}[11-5\imath] + \tfrac{1}{24}[-3\pi\imath-4\imath+3\pi+16].
\end{aligned} \tag{61}$$

And

$$\begin{aligned}
&[f_*(\theta,\ \imath)+f_*(\lambda,\ \imath)]\int_0^1 \frac{\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)}{\hbar(\kappa)}d\kappa\\
&=\left[4(1-\imath)-\sqrt{2}(1-\imath)+3\imath\right]\left[\int_0^{\frac{1}{2}}\kappa\sqrt{2\kappa}d\kappa+\int_{\frac{1}{2}}^1\kappa\sqrt{2(1-\kappa)}d\kappa\right]\\
&=\tfrac{1}{3}\Big(4(1-\imath)-\sqrt{2}(1-\imath)+3\imath\Big),\\
&[f^*(\theta,\ \imath)+f^*(\lambda,\ \imath)]\int_0^1 \frac{\mathfrak{B}((1-\kappa)\theta+\kappa\lambda)}{\hbar(\kappa)}d\kappa\\
&=\left[4(1-\imath)+\sqrt{2}(1-\imath)+3\imath\right]\left[\int_0^{\frac{1}{2}}\kappa\sqrt{2\kappa}d\kappa+\int_{\frac{1}{2}}^1\kappa\sqrt{2(1-\kappa)}d\kappa\right]\\
&=\tfrac{1}{3}\Big(4(1-\imath)+\sqrt{2}(1-\imath)+3\imath\Big).
\end{aligned}\tag{62}$$

From (61) and (62), we have

$$\begin{aligned}
&\left[\tfrac{\imath+5}{12}+\tfrac{1}{24}[3\pi(\imath-1)-4(\imath-4)],\ \tfrac{1}{12}[11-5\imath]+\tfrac{1}{24}[-3\pi\imath-4\imath+3\pi+16]\right]\\
&\supseteq_I\left[\tfrac{1}{3}\Big(4(1-\imath)-\sqrt{2}(1-\imath)+3\imath\Big),\ \tfrac{1}{3}\Big(4(1-\imath)+\sqrt{2}(1-\imath)+3\imath\Big)\right],
\end{aligned}$$

for all $\imath\in[0,\ 1]$. *Hence, Theorem 7 is verified.*

For Theorem 8, we have

$$\begin{aligned}
f_*\Big(\tfrac{\theta+\lambda}{2},\ \imath\Big)&=f_*(1,\ \imath)=\tfrac{2+\imath}{2},\\
f^*\Big(\tfrac{\theta+\lambda}{2},\ \imath\Big)&=f^*(1,\ \imath)=\tfrac{3(2-\imath)}{2},
\end{aligned}\tag{63}$$

$$\int_\theta^\lambda \mathfrak{B}(\text{б})d\text{б}=\int_0^1\sqrt{\text{б}}d\text{б}+\int_1^2\sqrt{2-\text{б}}d\text{б}=\frac{4}{3},$$

$$\begin{aligned}
\tfrac{2}{\hbar(\frac{1}{2})\int_\theta^\lambda\mathfrak{B}(\text{б})d\text{б}}\int_\theta^\lambda f_*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}&=\tfrac{3}{4}\Big[\tfrac{13-\imath}{6}+\tfrac{\pi(\imath-1)}{4}\Big],\\
\tfrac{2}{\hbar(\frac{1}{2})\int_\theta^\lambda\mathfrak{B}(\text{б})d\text{б}}\int_\theta^\lambda f^*(\text{б},\imath)\mathfrak{B}(\text{б})d\text{б}&=\tfrac{3}{24}\Big[[11-5\imath]+\tfrac{1}{2}[-3\pi\imath-4\imath+3\pi+16]\Big].
\end{aligned}\tag{64}$$

From (63) and (64), we have

$$\left[\frac{2+\imath}{2},\ \frac{3(2-\imath)}{2}\right]\supseteq_I\left[\frac{3}{4}\left[\frac{13-\imath}{6}+\frac{\pi(\imath-1)}{4}\right],\ \frac{3}{24}\left[[11-5\imath]+\frac{1}{2}[-3\pi\imath-4\imath+3\pi+16]\right]\right].$$

Hence, Theorem 8 has been verified.

4. Fuzzy Version of Ostrowski's Type Inequality via $U\cdot D$-ħ-Godunova–Levin $F\cdot N\cdot M$s

Here, an Ostrowski-type inequality was formulated in conjunction with several illustrations for Godunova-Levin functions within a broader category.

First, recalling some basic notations that will be helpful in this section such that:

Gamma and Beta functions are respectively characterized as

$$\Gamma(\dagger)=\int_0^\infty \kappa^{\dagger-1}e^{-\kappa}d\kappa$$

for $\mathcal{R}(\dagger)>0$

$$(\dagger,)=\int_0^1\kappa^{\dagger-1}(1-\kappa)^{-1}d\kappa=\frac{\Gamma(\dagger)\Gamma()}{\Gamma(\dagger+)}$$

for $\mathcal{R}(\dagger)>0,\mathcal{R}(\text{ƶ})>0$.

The integral representation of the hypergeometric function is

$$ {}_2F_1(\dagger,;c;б) = \frac{1}{(,c-)}\int_0^1 \kappa^{-1}(1-\kappa)^{c---1}(1-x\kappa)^{-\dagger}d\kappa$$

for $|x|<1$, $\mathcal{R}(c)>0$, $\mathcal{R}(\text{ƶ})>0$.

The subsequent lemma aids in achieving our goal.

Lemma 1. *Let* $f_*(.,\rangle), f^*(.,\rangle) : [\theta,\ \lambda] \to \mathfrak{N}^+$ *be a differentiable function on* I_0 *with* (θ,λ), *where* $\rangle \in [0,\ 1]$. *If* $\widetilde{f}$ *is integrable over* $[\theta,\ \lambda]$, *then*

$$\begin{array}{l}\left[f_*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi, f^*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right]\\ =\frac{(б-\theta)^2}{\lambda-\theta}\int_0^1 \kappa\left[{f_*}'(\kappa б+(1-\kappa)\theta,\imath), {f^*}'(\kappa б+(1-\kappa)\theta,\imath)\ \right]d\kappa\\ +\frac{(\lambda-б)^2}{\lambda-\theta}\int_0^1 \kappa\left[{f_*}'(\kappa\lambda+(1-\kappa)б,\imath), {f^*}'(\kappa\lambda+(1-\kappa)б,\imath)\ \right]d\kappa.\end{array} \tag{65}$$

Proof. Integration by parts finalizes the proof. □

Now, employing Lemma 1, we derive the principal outcomes.

Theorem 9. *Let* $\widetilde{f} : [\theta,\ \lambda] \to \mathbb{F}_0^+$ *be a differentiable function on* I_0 *with* $б \in (\theta,\lambda)$, *where* $\rangle \in [0,\ 1]$ *and let* $\widetilde{f}$ *be a integrable over* $[\theta,\ \lambda]$. *If* $\left|\widetilde{f}'\right|$ *is* ħ*-Godunova–Levin* $F{\cdot}N{\cdot}M$*s, with* $\left|\widetilde{f}'\right| \geq_{\mathbb{F}} \widetilde{0}$. *If* $M=[m_1,m_2]$, *then*

$$\begin{array}{l}\left[\left|f_*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi\right|, \left|f^*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right|\right]\\ \supseteq_I \frac{M\left[(б-\theta)^2+(\lambda-б)^2\right]}{\lambda-\theta}\int_0^1\left[\frac{1}{(\kappa)\hbar(\kappa)}+\frac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa,\end{array} \tag{66}$$

for all $б \in [\theta,\ \lambda]$, *where* $m_2 \geq \left|{f_*}'(б,\imath)\right|$ *and* $\left|{f^*}'(б,\imath)\right| \geq m_1$.

Proof. In accordance with Lemma 1 and $|f'|$ is ħ-Godunova–Levin $F{\cdot}N{\cdot}M$, for $\imath \in [0,\ 1]$ we have

$$\left|f_*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi\right| = \frac{(б-\theta)^2}{\lambda-\theta}\int_0^1 \kappa {f_*}'(\kappa б+(1-\kappa)\theta,\imath)d\kappa+\frac{(\lambda-б)^2}{\lambda-\theta}\int_0^1 \kappa {f_*}'(\kappa\lambda+(1-\kappa)б,\imath)d\kappa$$

and

$$\left|f^*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right| = \frac{(б-\theta)^2}{\lambda-\theta}\int_0^1 \kappa {f^*}'(\kappa б+(1-\kappa)\theta,\imath)d\kappa+\frac{(\lambda-б)^2}{\lambda-\theta}\int_0^1 \kappa {f^*}'(\kappa\lambda+(1-\kappa)б,\imath)d\kappa$$

From above equations, we have

$$\left|f_*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi\right| \leq \frac{(б-\theta)^2}{\lambda-\theta}\int_0^1 \kappa\left[\frac{\left|{f_*}'(б,\imath)\right|}{\hbar(\kappa)}+\frac{\left|{f_*}'(\theta,\imath)\right|}{\hbar(1-\kappa)}\right]d\kappa+\frac{(\lambda-б)^2}{\lambda-\theta}\int_0^1 \kappa\left[\frac{\left|{f_*}'(\lambda,\imath)\right|}{\hbar(\kappa)}+\frac{\left|{f_*}'(б,\imath)\right|}{\hbar(1-\kappa)}\right]d\kappa,$$

and

$$\begin{array}{l}\left|f^*(б,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right|\\ \geq \frac{(б-\theta)^2}{\lambda-\theta}\int_0^1 \kappa\left[\frac{\left|{f^*}'(б,\imath)\right|}{\hbar(\kappa)}+\frac{\left|{f^*}'(\theta,\imath)\right|}{\hbar(1-\kappa)}\right]d\kappa+\frac{(\lambda-б)^2}{\lambda-\theta}\int_0^1 \kappa\left[\frac{\left|{f^*}'(\lambda,\imath)\right|}{\hbar(\kappa)}+\frac{\left|{f^*}'(б,\imath)\right|}{\hbar(1-\kappa)}\right]d\kappa.\end{array}$$

As a result, we obtain

$$\begin{aligned}
&\left|f_*(\mathfrak{б},\imath)-\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi\right|\\
&\leq \tfrac{(\mathfrak{б}-\theta)^2}{\lambda-\theta}\int_0^1\left[\tfrac{|f_*'(\mathfrak{б},\imath)|}{(\kappa)\hbar(\kappa)}+\tfrac{|f_*'(\theta,\imath)|}{(\kappa)\hbar(1-\kappa)}\right]d\kappa+\tfrac{(\lambda-\mathfrak{б})^2}{\lambda-\theta}\int_0^1\left[\tfrac{|f_*'(\lambda,\imath)|}{(\kappa)\hbar(\kappa)}+\tfrac{|f_*'(\mathfrak{б},\imath)|}{(\kappa)\hbar(1-\kappa)}\right]d\kappa,\\
&\leq \tfrac{M(\mathfrak{б}-\theta)^2}{\lambda-\theta}\int_0^1\left[\tfrac{1}{(\kappa)\hbar(\kappa)}+\tfrac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa+\tfrac{M(\lambda-\mathfrak{б})^2}{\lambda-\theta}\int_0^1\left[\tfrac{1}{(\kappa)\hbar(\kappa)}+\tfrac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa,
\end{aligned}$$

and

$$\begin{aligned}
&\left|f^*(\mathfrak{б},\imath)-\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right|\\
&\geq \tfrac{(\mathfrak{б}-\theta)^2}{\lambda-\theta}\int_0^1\left[\tfrac{|f^{*\prime}(\mathfrak{б},\imath)|}{(\kappa)\hbar(\kappa)}+\tfrac{|f^{*\prime}(\theta,\imath)|}{(\kappa)\hbar(1-\kappa)}\right]d\kappa+\tfrac{(\lambda-\mathfrak{б})^2}{\lambda-\theta}\int_0^1\left[\tfrac{|f^{*\prime}(\lambda,\imath)|}{(\kappa)\hbar(\kappa)}+\tfrac{|f^{*\prime}(\mathfrak{б},\imath)|}{(\kappa)\hbar(1-\kappa)}\right]d\kappa,\\
&\geq \tfrac{M(\mathfrak{б}-\theta)^2}{\lambda-\theta}\int_0^1\left[\tfrac{1}{(\kappa)\hbar(\kappa)}+\tfrac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa+\tfrac{M(\lambda-\mathfrak{б})^2}{\lambda-\theta}\int_0^1\left[\tfrac{1}{(\kappa)\hbar(\kappa)}+\tfrac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa.
\end{aligned}$$

That is,

$$\left[\left|f_*(\mathfrak{б},\imath)-\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f_*(\varphi,\imath)d\varphi\right|,\left|f^*(\mathfrak{б},\imath)-\tfrac{1}{\lambda-\theta}\int_\theta^\lambda f^*(\varphi,\imath)d\varphi\right|\right]\supseteq_I \tfrac{M\left[(\mathfrak{б}-\theta)^2+(\lambda-\mathfrak{б})^2\right]}{\lambda-\theta}\int_0^1\left[\tfrac{1}{(\kappa)\hbar(\kappa)}+\tfrac{1}{(\kappa)\hbar(1-\kappa)}\right]d\kappa,$$

this completes the proof. □

5. Conclusions

Our principal objective entails applying classical integral operators and $U{\cdot}D$-relations to recent findings elucidated by the authors of the referenced works [39,41,47]. Various forms of $U{\cdot}D$-relations, and classical integral operators, along with fuzzy Aumman's integral operators, have been applied to derive several novel outcomes. Our primary contributions encompass the establishment of fuzzy integral inequalities of the H.H., Fejér, Pachpatte and Ostrowski's inequalities via newly defined class $U{\cdot}D$-ħ-Godunova–Levin convex $F{\cdot}N{\cdot}M$s. Furthermore, we provided initial clarifications on concepts related to up and down, and pseudo order relations to underscore their differences and offer commentary on key revelations. The outcomes of this research are poised to significantly influence the realms of inequality and optimization theory.

Author Contributions: Conceptualization, A.A. and A.F.A.; validation, S.A. and A.F.A.; formal analysis, S.A. and A.F.A.; investigation, A.A. and M.V.C.; resources, A.A. and M.V.C.; writing—original draft, A.A. and M.V.C.; writing—review and editing, A.A., A.F.A. and S.A.; visualization, M.V.C. and A.A.; supervision, M.V.C. and A.A.; project administration, M.V.C. and A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Taif University, Saudi Arabia, project No (TU-DSPP-2024-87).

Data Availability Statement: There is no data availability statement to be declared.

Conflicts of Interest: The authors claim to have no conflicts of interest.

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