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An Example of a Continuous Field of Roe Algebras

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Abstract: The Roe algebra $C^*(X)$ is a noncommutative C^*-algebra reflecting metric properties of a space X, and it is interesting to understand the correlation between the Roe algebra of X and the (uniform) Roe algebra of its discretization. Here, we perform a minor step in this direction in the simplest non-trivial example, namely $X = \mathbb{R}$, by constructing a continuous field of C^*-algebras over $[0, 1]$, with the fibers over non-zero points constituting the uniform C^*-algebra of the integers, and the fibers over 0 constituting a C^*-algebra related to $\mathbb{R}$.

Keywords: Roe algebra; continuous field of C*-algebras; metric space; discretization

MSC: 46L80

1. Introduction

Roe algebras play an increasingly important role in the index theory of elliptic operators on noncompact manifolds and their generalizations [1–3]. Following the ideology of noncommutative geometry [4], they provide an interplay between metric spaces (e.g., manifolds) and (noncommutative) C^*-algebras, and some geometric properties of spaces can be 'translated' into algebraic properties of the corresponding C^*-algebras. Roe algebras also play an important role in mathematical physics, e.g., for topological insulators [5,6].

Let X be a proper metric measure space, that is, X is a set, which is equipped with a metric d and a measure m, defined on the Borel σ-algebra, which is defined by the topology on X induced by the metric, and all balls are compact. For a Hilbert space H, we write $\mathbb{B}(H)$ (resp., $\mathbb{K}(H)$) for the algebra of all bounded (resp., all compact) operators on H.

Recall the definition of the Roe algebra of X [3]. Let H_X be a Hilbert space with an action of the algebra $C_0(X)$ of continuous functions on X vanishing at infinity (i.e., a $*$-homomorphism $\pi : C_0(X) \to \mathbb{B}(H_X)$). We will assume that

$$\{\pi(f)\xi : f \in C_0(X), \xi \in H_X\} \quad \text{is dense in} \quad H_X \tag{1}$$

and that

$$\pi(f) \in \mathbb{K}(H_X) \quad \text{implies that} \quad f = 0. \tag{2}$$

An operator $T \in \mathbb{B}(H_X)$ is *locally compact* if the operators $T\pi(f)$ and $\pi(f)T$ are compact for any $f \in C_0(X)$. It has *finite propagation* if there exists some $R > 0$ such that $\pi(f)T\pi(g) = 0$ whenever the distance between the supports of $f, g \in C_0(X)$ is greater than R. The *Roe algebra* $C^*(X, H_X)$ is the norm completion of the $*$-algebra of locally compact, finite propagation operators on H_X. As it does not depend on the choice of H_X satisfying (1) and (2) up to a $*$-isomorphism, it is usually denoted by $C^*(X)$. If $X = \mathbb{R}$ with the standard metric and the standard measure (our main example), then we may (and will, for simplicity) take $H_X = L^2(X)$.

When X is discrete, the choice $H_X = l^2(X)$ does not satisfy the condition (2). In order to fix this, one may take $H_X = l^2(X) \otimes H$ for an infinite dimensional Hilbert space H. But there is also another option, namely to still use $H_X = l^2(X)$. The resulting C^*-algebra is called the *uniform Roe algebra* of X and is denoted by $C^*_u(X)$. This C^*-algebra is more

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tractable but has less relations with elliptic theory. In particular, for the uniform Roe algebra of a discrete space D, one can relate its K-theory with the homology of graphs constructed from D [7].

Manifolds and some other spaces X are often endowed with discrete subspaces $D \subset X$ that are ε-dense for some ε, e.g., $\mathbb{Z} \subset \mathbb{R}$; or, more generally, lattices in Lie groups; or, even more generally, Delone sets in metric spaces [8]. Some problems related to X may become simpler when reduced to its discretization $D \subset X$. In particular, it would be interesting to understand the correlation between a Roe-type algebra of X and the uniform Roe algebra of D. Generally, this cannot work, as different discretizations may have different properties, so it makes sense to consider a family of discretizations such that it approximates X. As the first step, here, we consider one of the simplest non-trivial cases, $X = \mathbb{R}$, with the family of discretizations $D_t = t\mathbb{Z}$, $t \in (0, 1]$, and construct a continuous field of C^*-algebras over the segment $[0, 1]$ such that the fiber over 0 is a certain C^*-algebra related to $\mathbb{R}$, while the fiber over any other point is the uniform Roe algebra of $\mathbb{Z}$. Such non-locally trivial continuous fields of C^*-algebras are interesting because they provide relationships between fibers over different points. In particular, they provide a map from the K-theory of the fiber over 0 to the K-theory of the fiber over non-zero points. A similar continuous field with the fibers over 0 considering the algebra of functions on a sphere and the fibers over non-zero points using the algebra of compact operators was used in [9] to give a proof of Bott periodicity in K-theory.

2. Two Maps

Let $D_t = t\mathbb{Z} \subset \mathbb{R}$. In this section, we construct two maps $\alpha_t : C^*_u(D_t) \to C^*(\mathbb{R})$ and $\beta_t : C^*(\mathbb{R}) \to C^*_u(D_t)$, $t \in (0, 1]$.

Let

$$\varphi_0(x) = \begin{cases} 1 + x, & x \in [-1, 0]; \\ 1 - x, & x \in [0, 1]; \\ 0, & \text{otherwise}, \end{cases}$$

$$\varphi_n(x) = \varphi_0(x - n), \quad \varphi^t_n(x) = \frac{1}{\sqrt{t}} \varphi_n(x/t).$$

Then, $\operatorname{supp} \varphi^t_n = [t(n-1), t(n+1)]$, and $\|\varphi^t_n\|_{L^2} = \sqrt{2/3}$ for any $n \in \mathbb{Z}$ and any $t \in (0, 1]$ (here $\|\cdot\|_{L^2}$ denotes the norm in $L^2(\mathbb{R})$). In particular, φ^t_n and φ^t_m are orthogonal when $|m - n| \geq 2$. Let p_t denote the projection, in $L^2(\mathbb{R})$, onto the closure H_t of the linear span of φ^t_n, $n \in \mathbb{Z}$.

Let $(G_{nm})_{n,m\in\mathbb{Z}}$ be the Gram matrix for $\{\varphi^t_n\}$, $n \in \mathbb{Z}$, $G_{nm} = \langle \varphi_n, \varphi_m \rangle$, (note that G does not depend on t) and let $G \in \mathbb{B}(l^2(\mathbb{Z}))$ be the operator with the Gram matrix with respect to the standard basis of $l^2(\mathbb{Z})$.

Lemma 1. *The operator G is bounded and invertible and has finite propagation.*

Proof. Direct calculation shows that $G_{n,n} = \frac{2}{3}$, $G_{n,n\pm1} = \frac{1}{6}$, and $G_{n,m} = 0$ when $|m - n| \geq 2$. Therefore, $\|G\| \leq \frac{2}{3} + \frac{1}{3}$ and $\|\frac{2}{3} - G\| = \frac{1}{3} < 1$; hence, G is invertible. □

Set $C = G^{-1/2}$. By functional calculus, C can be approximated by polynomials in G; hence, C lies in the norm closure of operators of finite propagation, i.e., $C \in C^*_u(\mathbb{Z})$.

Let $A \in \mathbb{B}(l^2(\mathbb{Z}))$, and let A_{nm} be its matrix elements with respect to the standard basis. Define $\gamma_t(A) \in \mathbb{B}(H_t)$ by $\gamma_t(A)\varphi^t_m = \sum_{n,m\in\mathbb{Z}} A_{nm}\varphi^t_n$. Note that γ_t is a homomorphism but not a $*$-homomorphism.

Lemma 2. *There exist $k_1, k_2 > 0$ such that $k_1\|A\| < \|\gamma_t(A)\| < k_2\|A\|$.*

Proof. Let S denote the right shift on $l^2(\mathbb{Z})$, $x = \sum_{i\in\mathbb{Z}} x_i\varphi_i^t$. Then,

$$\begin{aligned}\|\gamma_t(A)x\|^2 &= \sum_{i,j,k,l\in\mathbb{Z}} \bar{x}_i x_j \bar{A}_{ki} A_{lj}\langle\varphi_k^t, \varphi_l^t\rangle \\ &= \sum_{i,j,k\in\mathbb{Z}} \bar{x}_i x_j \bar{A}_{ki} A_{kj} + \frac{1}{6}\sum_{i,j,k,l\in\mathbb{Z}} \bar{x}_i x_j \bar{A}_{ki} A_{k\pm1,j} \\ &= \frac{2}{3}\|A\tilde{x}\| + \frac{1}{6}\langle A\tilde{x}, (S+S^*)A\tilde{x}\rangle,\end{aligned}$$

where $\tilde{x} \in l^2(\mathbb{Z})$ has coordinates x_i with respect to the standard basis of $l^2(\mathbb{Z})$. As $|\frac{1}{6}\langle A\tilde{x}, (S+S^*A\tilde{x}\rangle| \le \frac{1}{3}\|A\|\|\tilde{x}\|$, the conclusion follows. □

Set $\psi_n^t = \gamma_t(C)\varphi_n^t = \sum_{m\in\mathbb{Z}} C_{mn}\varphi_m^t$. Then, $\langle\psi_n^t, \psi_m^t\rangle = \langle\gamma_t(G^{-1})\varphi_n, \varphi_m\rangle = \delta_{n,m}$; hence, $\{\psi_n^t\}_{n\in\mathbb{Z}}$ is an orthonormal system. The invertibility of C implies that the closures of the linear spans of $\{\varphi_n^t\}_{n\in\mathbb{Z}}$ and $\{\psi_n^t\}_{n\in\mathbb{Z}}$ coincide. The advantage of this orthonormal system with respect to the system obtained from $\{\varphi_n^t\}_{n\in\mathbb{Z}}$ by Gram–Schmidt orthogonalization is that it is obtained from the original non-orthogonal system by an operator from $C_u^*(\mathbb{Z})$.

Define a map $\alpha_t : C_u^*(\mathbb{Z}) \to C^*(\mathbb{R})$. Let $T \in C_u^*(\mathbb{Z})$, $T = (T_{nm})_{n,m\in\mathbb{Z}}$. Set

$$\alpha_t(T)(f) = \sum_{n,m\in\mathbb{Z}} T_{nm}\psi_n^t\langle\psi_m^t, f\rangle, \quad f \in L^2(\mathbb{R}).$$

Let $U_t : l^2(\mathbb{Z}) \to L^2(\mathbb{R})$ be the isometry defined by $U_t(\delta_n) = \psi_n^t$. Then, it is easy to see that $\alpha_t(T) = U_t T U_t^*$. Hence, α_t is a $*$-homomorphism, in particular, it is isometric. As T is bounded, $\alpha_t(T)$ is bounded as well.

As $\gamma_t(C)$ can be considered as the transition matrix from the basis $\{\varphi_n^t\}$ to the basis $\{\psi_n^t\}$, we can write $\alpha_t(T) = \gamma_t(C)^{-1}\gamma_t(T)\gamma_t(C)$.

What remains to be checked is that $\alpha_t(T) \in C^*(\mathbb{R}) \subset \mathbb{B}(L^2(\mathbb{R}))$. To this end, consider one more basis for H_t. By constructing C, for any $\varepsilon > 0$, there exists an operator $C_\varepsilon \in \mathbb{B}(l^2(\mathbb{Z}))$ of finite propagation M_ε such that $\|C - C_\varepsilon\| < \varepsilon$. Set $\psi_n^{t,\varepsilon} = \gamma_t(C_\varepsilon)\varphi_n^t$. Set $\tilde{T}_\varepsilon = \gamma_t(C_\varepsilon)^{-1}\gamma(T)\gamma(C_\varepsilon)$.

Lemma 3. *For sufficiently small ε, there exists $K > 0$ such that $\|\alpha_t(T) - \tilde{T}_\varepsilon\| < K\varepsilon$ for any $t \in (0, 1]$.*

Proof. One should take ε small enough to provide invertibility of $\gamma(C_\varepsilon)$. Then,

$$\begin{aligned}\|\alpha_t(T) - \tilde{T}_\varepsilon\| &\le \|\gamma_t(C)^{-1} - \gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C)\| \\ &\quad + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C) - \gamma_t(C_\varepsilon)\| \\ &\le \|\gamma_t(C - C_\varepsilon)\| \cdot \|\gamma_t(C)^{-1}\| \cdot \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(C)\| \\ &\quad + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C - C_\varepsilon)\| \\ &< k_2\varepsilon(\|\gamma_t(C)^{-1}\| \cdot \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(C)\| + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\|).\end{aligned}$$

□

Lemma 4. *$\alpha_t(T) \in C^*(\mathbb{R})$ for any $T \in C_u^*(\mathbb{Z})$.*

Proof. As $T \in C_u^*(\mathbb{Z})$, it can be approximated by finite propagation operators T^N, $N \in \mathbb{N}$, with propagation N. This means that the matrix of T^N has the band structure ($T_{nm}^N = 0$ when $|m - n| > c$ for some $c > 0$). Then we may write T^N as a matrix with $2N + 1$ diagonals: $T^N\delta_n = \sum_{k=-N}^N \lambda_{n,k}\delta_{n+k}$, where the numbers $\lambda_{n,k}$ are uniformly bounded by $\|T\|$.

As $\alpha_t(T^N)$ can be approximated by operators of the form $\tilde{T}_\varepsilon^N$, it suffices to show that $\tilde{T}_\varepsilon^N \in C^*(\mathbb{R})$.

Let $f \in C_0(\mathbb{R})$ have compact support, say $[a,b] \subset \mathbb{R}$. Then,

$$\begin{aligned} \tilde{T}_\varepsilon^N \pi(f)(g) &= \sum_{n,m\in\mathbb{Z}} T_{nm}^N \langle \psi_m^{t,\varepsilon}, fg\rangle \psi_n^{t,\varepsilon} \\ &= \sum_{n,m\in\mathbb{Z}} T_{nm}^N \langle \gamma_t(C\varepsilon)\varphi_m^t, fg\rangle \gamma_t(C_\varepsilon)\varphi_n^t \\ &= \sum_{n\in\mathbb{Z}} \sum_{k=-N}^{N} \lambda_{n,k} \langle \gamma_t(C_\varepsilon)\varphi_{n+k}^t, fg\rangle \gamma_t(C_\varepsilon)\varphi_n^t. \end{aligned} \tag{3}$$

As $\mathrm{supp}(fg) \subset [a,b]$ and as propagation of $C_\varepsilon \leq M_\varepsilon$, we have

$$\mathrm{supp}(\gamma_t(C_\varepsilon)\varphi_n^t) \subset [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)].$$

Therefore, $\langle \gamma_t(C_\varepsilon)\varphi_{n+k}^t, fg\rangle \neq 0$ only when $[a,b] \cap [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)] \neq \emptyset$; thus, the sum (3) contains only a finite number of non-zero summands, i.e., $\mathrm{Ran}\, \tilde{T}_\varepsilon^N \pi(f)$ is finite-dimensional. Similarly, $\mathrm{Ran}\, \pi(f)\tilde{T}_\varepsilon^N$ is finite-dimensional. Thus, $\alpha_t(T)\pi(f)$ and $\pi(f)\alpha_t(T)$ are compact. The approximation of functions in $C_0(\mathbb{R})$ by functions f with finite support proves that $\alpha_t(T)$ is locally compact.

Similarly, one can show that $\alpha_t(T)$ is of finite propagation. Indeed, let $f, g \in C_0(\mathbb{R})$ such that the distance between their supports be greater than R. Then,

$$\pi(f)\alpha_t(T_\varepsilon^N)\pi(g)(h) = \sum_{n\in\mathbb{Z}} \sum_{k=-N}^{N} \lambda_{n,k} \langle \gamma_t(C_\varepsilon)\varphi_{n+k}^t, gh\rangle f\gamma_t(C_\varepsilon)\varphi_n^t.$$

We have $\gamma_t(C_\varepsilon)\varphi_{n+k}^t, gh\rangle = 0$ when $\mathrm{supp}\, g \cap [t(n-k-1-N), t(n+k+1+N)] = \emptyset$, while $f\gamma_t(C_\varepsilon)\varphi_n^t = 0$ when $\mathrm{supp}\, f \cap [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)] = \emptyset$, so if R is sufficiently great, then their product vanishes. □

The second map, $\beta_t : C^*(\mathbb{R}) \to C_u^*(\mathbb{Z})$, goes in the opposite direction and is not a homomorphism (but linear and even completely positive). In fact, it extends to a completely positive map from a greater C^*-algebra $C_p^*(\mathbb{R}) \supset C^*(\mathbb{R})$, which is the norm closure of all bounded operators of finite propagation without the requirement of local compactness. For $S \in C_p^*(\mathbb{R})$, set $(\beta_t(S))_{nm} = \langle \psi_n^t, S\psi_m^t\rangle$. Then, the operator $\beta_t(S)$ can be written as $\beta_t(S)(\delta_m) = \sum_{n\in\mathbb{Z}} \langle \psi_n^t, S\psi_m^t\rangle \delta_n$. Recall that we denote by $U_t : l^2(\mathbb{Z}) \to L^2(\mathbb{R})$ the isometry that maps the standard basis $\{\delta_n\}_{n\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ to the basis $\{\psi_n^t\}_{n\in\mathbb{Z}}$ of $H_t \subset L^2(\mathbb{R})$. Then, $\beta_t(S) = U_t^* S U$. In particular, this implies that $\beta_t(S)$ is bounded for any bounded operator S.

Lemma 5. *Let $S \in C_p^*(\mathbb{R})$. Then $\beta_t(S) \in C_u^*(\mathbb{Z})$ for any $t > 0$.*

Proof. It suffices to show that $\beta_t(S) \in C_u^*(\mathbb{Z})$ for operators of finite propagation. For an operator S of finite propagation, set $\tilde{S} = U_t^* D^* S D U_t$, where $D = C_\varepsilon C^{-1}$. As $\|1 - D\| < \varepsilon\|C^{-1}\|$, $\beta_t(S)$ can be approximated by operators of the form $\tilde{S}$. Let us show that $\tilde{S}$ has finite propagation, which means, for discrete spaces, that the matrix of this operator is a band matrix. We have

$$\tilde{S}_{nm} = \langle \psi_n^t, D^* S D \psi_m^t\rangle = \langle D\psi_n^t, SD\psi_m^t\rangle = \langle \psi_n^{t,\varepsilon}, S\psi_m^{t,\varepsilon}\rangle.$$

As $\mathrm{supp}\, \psi_n^{t,\varepsilon} \in [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)]$ and as S has finite propagation, $\tilde{S}_{nm} = 0$ when $|n-m|$ is sufficiently great. □

Note that $\beta_t \circ \alpha_t(S) = p_t S|_{H_t}$; in particular, this means that $p_t S|_{H_t}$ is locally compact for any $S \in C_p^*(\mathbb{R})$.

3. The Fiber over 0

Let $L^\infty_0(\mathbb{R})$ denote the norm closure of $\cup_N L^\infty([-N,N]) \subset L^\infty(\mathbb{R})$. The group $\mathbb{R}$ acts on $L^\infty_0(\mathbb{R})$ by translations. Set $A_0 = L^\infty_0(\mathbb{R}) \rtimes \mathbb{R}$.

Lemma 6. *$A_0 \subset C^*_p(\mathbb{R})$.*

Proof. Let $f \in L^\infty([-N,N])$, and let $g \in C_0(\mathbb{R})$ be a continuous function with compact support. The linear combinations of operators of the form $S_{f,g}$, where

$$S_{f,g}(u)(x) = \int_{\mathbb{R}} f(x)g(y)u(x-y)\,dy,$$

are dense in $L^\infty_0(\mathbb{R}) \rtimes \mathbb{R}$, so it suffices to show that $S_{f,g} \in C^*_p(\mathbb{R})$. Let $\mathrm{supp}(g) \subset [-M,M]$ and let $\varphi, \psi \in C_0(\mathbb{R})$ have supports at the distance greater than L. Then,

$$(\pi(\varphi)S_{f,g}\pi(\psi)(u))(x) = \varphi(x)\int_{\mathbb{R}} f(x)g(y)\psi(x-y)u(x-y)\,dy = 0$$

if $L > M$. □

Recall that C is the transition matrix that maps φ^t_n to ψ^t_n, i.e., $\psi^t_n = \sum_{m\in\mathbb{Z}} C_{mn}\varphi^t_m$. We define C by $C = G^{-1/2}$, where G is the Gram matrix for $\{\varphi^t_n\}_{n\in\mathbb{N}}$. We need the following technical result:

Lemma 7. *The series $\sum_{n\in\mathbb{Z}} |C_{nm}|$ and $\sum_{m\in\mathbb{Z}} |C_{nm}|$ converge. The sums $\sum_{n\in\mathbb{Z}} |C_{nm}|$ (resp., $\sum_{m\in\mathbb{Z}} |C_{nm}|$) are bounded uniformly with respect to m (resp., to n).*

Proof. When working with matrices with the same entries along any diagonal, it is convenient to identify $l^2(\mathbb{Z})$ with the square-integrable functions on the circle, and the basis $\{\delta_n\}_{n\in\mathbb{Z}}$ with the basis $\{e^{inx}\}$. Under this identification, the matrix $B_{nm} = b_{n-m}$ can be identified with the operator of multiplication using the function $\sum_{n\in\mathbb{N}} b_n e^{inx}$. Thus, the Gram matrix G corresponds to the invertible function $\frac{2}{3} + \frac{1}{3}\cos x$, and the matrix C corresponds to the function $(\frac{2}{3} + \frac{1}{3}\cos x)^{-1/2}$. As this function is smooth, its Fourier coefficients a_n, $n = 0, 1, \ldots$ are of rapid decay, i.e., $a_n = o(n^{-k})$ for any $k \in \mathbb{N}$. Therefore, the series $\sum_{n\in\mathbb{N}} |a_n|$ is convergent. As $C_{nm} = a_{|n-m|}$, the series $\sum_{n\in\mathbb{Z}} |C_{nm}|$ and $\sum_{m\in\mathbb{Z}} |C_{nm}|$ converge. Uniform boundedness is obvious. □

Denote the map $t \mapsto \beta_t(S)$ by $\beta^S : (0,\infty) \to C^*_u(\mathbb{Z})$.

Theorem 1. *The map β^S is norm-continuous on $(0,\infty)$ for any $S \in A_0$.*

Proof. Note that the linear combinations of operators $S_{f,g}$, $S_{f,g}(u)(x) = \int f(x)g(y)u(x-y)\,dy$, with $f \in L^\infty(\mathbb{R})$, $g \in C_0(\mathbb{R})$ of finite support are dense in $A_0 = L^\infty_0(\mathbb{R}) \rtimes \mathbb{R}$, so it suffices to show continuity of the map $t \mapsto \beta_t(S)$ for $S = S_{f,g}$ for f and g with compact support.

Let $\|S_{f,g}\| = 1$, $\mathrm{supp}(f), \mathrm{supp}(g) \subset [-N,N]$, $a = \sum_{n\in\mathbb{Z}} a_n\delta_n \in l^2(\mathbb{Z})$, $\|a\| = 1$. Then,

$$\begin{aligned}
\|(\beta_t(S_{f,g}) - \beta_{t_0}(S_{f,g}))a\|^2 &= \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}(\langle\psi^t_n, S_{f,g}\psi^t_m\rangle - \langle\psi^{t_0}_n, S_{f,g}\psi^{t_0}_m\rangle)a_m\Big)^2 \\
&\le \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\langle\psi^t_n - \psi^{t_0}_n, S_{f,g}\psi^t_m\rangle a_m\Big)^2 \qquad (4) \\
&+ \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}(\langle\psi^{t_0}_n, S_{f,g}(\psi^t_m - \psi^{t_0}_m)\rangle a_m\Big)^2. \qquad (5)
\end{aligned}$$

We shall estimate the first summand (4). The second summand (5) can be estimated in the same way (or, passing to the adjoint of $S_{f,g}$).

Recall that $\psi_n^t = \sum_{k\in\mathbb{Z}} C_{kn}\varphi_k^t$. Then

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\langle\psi_n^t-\psi_n^{t_0},S_{f,g}\psi_m^t\rangle a_m\Big)^2=\sum_{n\in\mathbb{Z}}\Big(\sum_{m,k,l\in\mathbb{Z}}C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m\Big)^2. \tag{6}$$

Let $t\in[\frac{t_0}{2},2t_0]$. As the supports of f and g lie in $[-N,N]$, $S_{f,g}\varphi_l^t=0$ for $|l|>(N+2)/t$; hence, the sum over l is finite, over $|l|\le 2(N+2)/t_0$. Also, the support of $S_{f,g}\varphi_l^t$ lies in $[(l-1)t-N,(l+1)t+N)]$; hence, there are only finitely many k such that $\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle\neq 0$. In other words, the sum in (6) can be written as follows:

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{|k|,|l|\le M}\sum_{m\in\mathbb{Z}}C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m\Big)^2 \tag{7}$$

for some M.

For any $\varepsilon>0$ there exists $\delta>0$ such that $\|\varphi_k^t-\varphi_k^{t_0}\|_{L^2}<\frac{\varepsilon}{M^2}$ for any $|k|\le M$ when $|t-t_0|<\delta$.

Fix k and l, and estimate

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m\Big)^2\\&\le\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}|C_{kn}C_{lm}|\frac{\varepsilon}{M^2}\|S_{f,g}\|\|\varphi_l^t\|_{L^2}\|a\|\Big)^2=\sum_{n\in\mathbb{Z}}|C_{kn}|\Big(\sum_{m\in\mathbb{Z}}|C_{lm}|\Big)^2\frac{\varepsilon^2}{M^4}.\end{aligned}$$

By Lemma 7, the series $\sum_{m\in\mathbb{Z}}|C_{lm}|$ converges and hence is bounded by some L, hence

$$\sum_{n\in\mathbb{Z}}|C_{kn}|\Big(\sum_{m\in\mathbb{Z}}|C_{lm}|\Big)^2\frac{\varepsilon^2}{M^4}\le L^3\frac{\varepsilon^2}{M^4}.$$

For the sake of brevity, set $x_{nmkl}=C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m$. We show that

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}x_{nmkl}\Big)^2\le L^3\varepsilon^2/M^2$$

for any k,l.

Coming back to (7) and using $2xx'\le x^2+(x')^2$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\Big(\sum_{|k|,|l|\le M}\sum_{m\in\mathbb{Z}}x_{nmkl}\Big)^2&=\sum_{n,m,m',k,k',l,l'}x_{nmkl}x_{nm'k'l'}\\&\le M^2\sum_{n,m,m',k,l}x_{nmkl}x_{nm'kl}\\&=M^2\sum_{k,l}\sum_n\Big(\sum_m x_{nmkl}\Big)^2\\&\le M^4\cdot L^3\varepsilon^2/M^4=L^3\varepsilon^2.\end{aligned}$$

Thus, for $|t-t_0|<\delta$, we have $\|\beta_t(S_{f,g})-\beta_{t_0}(S_{f,g})\|^2<2L^3\varepsilon^2$ which proves continuity. □

4. Continuous Field of Roe Algebras

Continuous fields of C^*-algebras (aka bundles of C^*-algebras or $C(T)$-C^*-algebras) were introduced by Fell [10] and Dixmier ([11], Section 10). Recall that a continuous field of C^*-algebras over a locally compact Hausdorff space T is a triple $(T,A,\pi_t:A\to A_t)$, where A and A_t, $t\in A$ are C^*-algebras; the $*$-homomorphisms π_t are surjective; the family $\{\pi_t\}_{t\in T}$ is faithful; and the map $t\mapsto\|\pi_t(a)\|$ is continuous for any $a\in A$.

Set $T = [0,1]$, $A_t = C_u^*(\mathbb{Z})$ for $t \neq 0$. The fiber A_0 over 0 was defined in the previous section. Set

$$A = C_0((0,1]; C_u^*(\mathbb{Z})) + \{\beta^S : S \in A_0\} \subset \prod_{t\in T} A_t.$$

Lemma 8. *The set A is norm-closed.*

Proof. First, let us show that

$$\sup_{t\in(0,1]} \|\beta_t(S)\| = \|S\| = \lim_{t\to 0} \|\beta_t(S)\|. \tag{8}$$

Consider the projections $p_t = U_t U_t^*$ in $L^2(\mathbb{R})$. Note that $\|U_t^* S U_t\| = \|p_t S p_t\|$. Let $f \in C_0(\mathbb{R})$ be a Lipschitz function with finite support $[a,b]$, and let L be the Lipschitz constant for f. Let $g_t = \sum_{n\in\mathbb{Z}} \sqrt{t} f(tn)\varphi_n^t$ be a piecewise linear function such that $g_t(tn) = f(tn)$ for any $n \in \mathbb{N}$. Then, $|f(x) - g_t(x)|_{L^2} \leq Lt$ for any $x \in \mathbb{R}$; hence,

$$\|f - g_t\|_{L^2} \leq Lt\sqrt{b-a+2}.$$

As g_t lies in the linear span of the functions φ_n^t, $n \in \mathbb{N}$; thus, we have $g_t = p_t g_t$. As $\|f - p_t f\|_{L^2} \leq \|f - p_t g_t\|_{L^2}$, and we have $\lim_{t\to 0}(f - p_t f) = 0$. As Lipschitz functions with finite support are dense in $L^2(\mathbb{R})$, we conclude that the $*$-strong limit of p_t is the identity operator. Note also that

$$\varphi_n^{2t} = \sqrt{2}\varphi_{2n}^t + \frac{\sqrt{2}}{2}\varphi_{2n-1}^t + \frac{\sqrt{2}}{2}\varphi_{2n+1}^t,$$

and hence the linear span of $\{\varphi_n^t\}$ lies in the linear span of $\{\varphi_n^{t/2}\}$; therefore, $p_t \leq p_{t/2}$ for any $t \in (0,1]$, and the sequence $\|p_{t/2^k} S p_{t/2^k}\|$ is increasing.

Consider the norm closure $\overline{A}$ of A. Then, $I = C_0((0,1]; C_u^*(\mathbb{Z}))$ is a closed ideal in $\overline{A}$. Let $\{f_n + \beta^{S_n}\}$ be a Cauchy sequence in A. Passing to the quotient C^*-algebra $\overline{A}/I$, the sequence $\{f_n + \beta^{S_n} + I\} = \{\beta^{S_n} + I\}$ is also a Cauchy sequence, as the quotient $*$-homomorphisms have norm 1 ([11], Section 1.8). Note that

$$\begin{aligned} \|\beta^S + I\| &= \inf_{f\in I} \|f + \beta^S\| \\ &\geq \lim_{t\to 0} \|f(t) + \beta_t(S)\| \\ &= \lim_{t\to 0} \|\beta_t(S)\| = \|S\|, \end{aligned}$$

and hence $\{S_n\}$ is also a Cauchy sequence. As A_0 is norm-closed, it has a limit in A_0. But then $\{f_n\}$ is also a Cauchy sequence, and as I is norm-closed, its limit lies in I. Thus, $\{f_n + \beta^{S_n}\}$ converges in A. □

Define $\pi_t : A \to A_t$ by $\pi_t(f + \beta^S) = f(t) + \beta_t(S)$ for $t > 0$, and $\pi_0(f + \beta^S) = S$. These maps are well defined as $f_1 + \beta^{S_1} = f_2 + \beta^{S_2}$ implies that $f_1 = f_2$ and $S_1 = S_2$.

Theorem 2. *The triple $(T, A, \pi_t : A \to A_t)$ is a continuous field of C^*-algebras.*

Proof. Each π_t is clearly surjective. If $\pi_t(f_1 + \beta^{S_1}) = \pi_t(f_2 + \beta^{S_2})$ for any $t \in T$, then, taking $t = 0$, we conclude that $S_1 = S_2$. Then, we see that $f_1(t) = f_2(t)$ for any $t \in (0,1]$; hence, $f_1 = f_2$. Finally, we have to check that the map $t \mapsto \pi_t(a)$ is continuous. Let $a = f + \beta^S$. Continuity at $t > 0$ follows from continuity of f (by definition) and continuity of β^S (by Theorem 1). Continuity at 0 follows from (8). □

Remark 1. *For $X = \mathbb{R}$, we used the fact that $\mathbb{R}$ is scaleable. General manifolds do not have this property, so our approach cannot be used for them. In [12], we developed another approach, which*

works for general manifolds, but the continuous field of C^*-algebras is over the space $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ instead of $[0,1]$, which is not so useful, as continuity is meaningful only at 0. It would be interesting to combine the two approaches. It would also be interesting to evaluate the class of the generator of $K_1(C^*(\mathbb{R}))$ under the map β_t for sufficiently small t.

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