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Some New Estimations of Ostrowski-Type Inequalities for Harmonic Fuzzy Number Convexity via Gamma, Beta and Hypergeometric Functions

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Abstract: This paper demonstrates several of Ostrowski-type inequalities for fuzzy number functions and investigates their connections with other inequalities. Specifically, employing the Aumann integral and the Kulisch–Miranker order, as well as the inclusion order on the space of real and compact intervals, we establish various Ostrowski-type inequalities for fuzzy-valued mappings ($F{\cdot}V{\cdot}Ms$). Furthermore, by employing diverse orders, we establish connections with the classical versions of Ostrowski-type inequalities. Additionally, we explore new ideas and results rooted in submodular measures, accompanied by examples and applications to illustrate our findings. Moreover, by using special functions, we have provided some applications of Ostrowski-type inequalities.

Keywords: fuzzy-number mapping; fuzzy harmonic convexity; fuzzy Ostrowski-type inequalities; special functions

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1. Introduction

Interval-valued and fuzzy-valued functions hold a crucial position in numerous academic fields and bear significant mathematical and practical importance. Their distinctive characteristics and properties are integral to the examination of random set analysis, interval differentiation equations, and interval optimization. Specifically, interval-valued functions with integrability and differentiation play a key role in the fields listed above. These functions also hold a key place in fuzzy theory because they make it possible to provide fuzzy-valued functions through a collection of interval-valued functions that make use of the number of levels within a fuzzy interval. This work focuses on Ostrowski-type inequality (see [1]).

The renowned integral inequalities of the Ostrowski, Čebyšev, and Grüss varieties permeate numerous branches of mathematics (for historical context and generalizations, refer to the seminal monograph [2], as well as works [3,4]). The Čebyšev- and Ostrowski-type inequalities, closely intertwined (refer to [5] for elaboration), hold significance in various mathematical applications and have garnered considerable attention from scholars. Ujević [6] derived the subsequent Ostrowski-type inequality. For further information on additional Ostrowski-type inequalities, we direct interested readers to [7–11].

Developing a range of integral inequalities is a contemporary focus. In recent years, significant progress has been made utilizing various integrals, such as the Sugenointegral [12,13], the pseudointegral [14], and the Choquet integral [15], among others. Interval-valued functions [16], as a concept extending beyond traditional functions, have emerged as an important mathematical area, serving as a crucial tool for addressing practical problems, notably in mathematical economics [17]. Recent studies have expanded certain classical integral inequalities to encompass interval-valued functions.

Costa et al. [18] introduced novel interval adaptations of Minkowski and Beckenbach's integral inequalities. They generalized Hermite–Hadamard-, Jensen-, and Ostrowski-type inequalities within this framework [19]. Furthermore, they tackled Hermite–Hadamard and Hermite–Hadamard-type inequalities using interval-valued Riemann–Liouville fractional integrals [20]. Zhao et al. [21–23] investigated Chebyshev-type inequalities, Opial-type integral inequalities, and Jensen and Hermite–Hadamard-type inequalities for interval-valued functions, employing the concepts of gH-differentiability or h-convexity. Budak et al. [24] derived innovative fractional inequalities of the Ostrowski type for interval-valued functions, leveraging the definitions of gH-derivatives. Khan et al. [25] introduced log-h-convex fuzzy-interval-valued functions as a distinct class of convex fuzzy-interval-valued functions, using a fuzzy order relation. This class facilitated the establishment of Jensen and Hermite-Hadamard inequalities.

Incorporating the Ostrowski-type inequality into the realm of fuzzy-valued functions necessitated using the Hukuhara derivative, as illustrated by Anastassiou [26]. Fuzzy-valued functions, also known as functions with interval values, were at the core of Anastassiou's [26] investigation. Interestingly, the fuzzy Ostrowski-type inequalities derived by Anastassiou [26] also expanded their applicability to interval-valued functions. To fully understand the limitations imposed by the concept of the H-derivative on interval-valued functions, it is helpful to examine the works of Bede and Gal [27] and Chalco-Cano et al. [28]. Notably, recent contributions by Chalco-Cano et al. [29] have effectively established an Ostrowski-type inequality specifically tailored to generalized Hukuhara differentiable interval-valued functions. The paramount importance of generalized Hukuhara differentiability as the most comprehensive concept for characterizing the differentiability of interval-valued functions has been emphasized in significant studies by Bede and Gal [27], as well as Chalco-Cano et al. [30]. For more information, see [31–43] and the references therein.

The structure of the study is outlined as follows: In Section 2, pertinent preliminaries are introduced. Section 3 introduces several types of Ostrowski inequalities over harmonic $F{\cdot}V{\cdot}M$. Section 3 delves into a novel estimation of quadrature rules, which includes the special quadrature rule as a special case, building upon the findings. In Section 4, with the help of special functions such as Gamma and Beta functions, some new findings are obtained as applications of Ostrowski-type inequalities.

2. Preliminaries

We let $\mathfrak{R}$ be the set of real numbers. A fuzzy subset A of $\mathfrak{R}$ is characterized by the mapping $\widetilde{\psi}:\mathfrak{R}\to[0,1]$, called the membership function, for each fuzzy set and $\imath\in(0,\ 1]$, then $\imath$-level sets of $\widetilde{\psi}$ are denoted and defined as follows: $\psi_\imath=\{\varkappa\in\mathfrak{R}|\widetilde{\psi}(\varkappa)\geq\imath\}$. If $\imath=0$, then $supp(\widetilde{\psi})=\{\varkappa\in\mathfrak{R}|\widetilde{\psi}(\varkappa)>0\}$ is called the support of $\widetilde{\psi}$. By $[\widetilde{\psi}]^0$, we define the closure of $supp(\widetilde{\psi})$.

Definition 1 ([36])**.** *A fuzzy set is said to be fuzzy number with the following properties:*

a. $\widetilde{\psi}$ *is normal, i.e., there exists* $\varkappa\in\mathfrak{R}$ *such that* $\psi(\varkappa)=1$;
b. $\widetilde{\psi}$ *is upper semi-continuous, i.e., for given* $\varkappa\in\mathfrak{R}$*, there exists* $\varepsilon>0$ *and there exists* $\delta>0$ *such that* $\widetilde{\psi}(\varkappa)-\varphi(\vartheta)<\varepsilon$ *for all* $\vartheta\in\mathfrak{R}$ *with* $|\varkappa-\vartheta|<\delta$;
c. $\widetilde{\psi}$ *is fuzzy convex, i.e.,* $\widetilde{\psi}((1-\tau)\varkappa+\tau\vartheta)\geq min(\widetilde{\psi}(\varkappa),\ \widetilde{\psi}(\vartheta))\ \forall\,\varkappa,\vartheta\in\mathfrak{R},\ \tau\in[0,1]$;
d. $[\widetilde{\psi}]^0$ *is compact.*

$\mathbb{F}_0$ denotes the set of all fuzzy numbers. For a fuzzy number, it is convenient to distinguish the following $\imath$-levels,

$$\psi_\imath = \{\varkappa \in \mathfrak{R} | \widetilde{\psi}(\varkappa) \geq \imath\},$$

From these definitions, we have

$$\psi_\imath = [\psi_*(\imath),\ \psi^*(\imath)],$$

where

$$\psi_*(\imath) = inf\{\varkappa \in \mathfrak{R} | \widetilde{\psi}(\varkappa) \geq \imath\}, \psi^*(\imath) = sup\{\varkappa \in \mathfrak{R} | \widetilde{\psi}(\varkappa) \geq \imath\}.$$

Since each $\rho \in \mathfrak{R}$ is also a fuzzy number, it can be defined as

$$\widetilde{\rho}(\varkappa) = \begin{cases} 1 \text{ if } \varkappa = \rho \\ 0 \text{ if } \varkappa \neq \rho. \end{cases}$$

Thus, a fuzzy number $\widetilde{\psi}$ can be identified by a parametrized pair:

$$\{[\psi_*(\imath), \psi^*(\imath)] : \imath \in [0,\ 1]\}.$$

This leads to the following characterization of a fuzzy number in terms of the two end point functions $\psi_*(\imath)$ and $\psi^*(\imath)$.

Theorem 1 ([36]). *Suppose that* $\psi_*(\imath) : [0,1] \to \mathfrak{R}$ *and* $\psi^*(\imath) : [0,1] \to \mathfrak{R}$ *satisfy the following conditions:*

1. $\psi_*(\imath)$ *is a non-decreasing function.*
2. $\psi^*(\imath)$ *is a non-increasing function.*
3. $\psi_*(1) \leq \psi^*(1)$.
4. $\psi_*(\imath)$ *and* $\psi^*(\imath)$ *are bounded and left continuous on* $(0,\ 1]$ *and right continuous at* $\imath = 0$.
5. *Moreover, if* $\widetilde{\psi} : \mathfrak{R} \to [0,1]$ *is a fuzzy number with parametrization given by* $\{(\psi_*(\imath), \psi^*(\imath)) : \imath \in [0,\ 1]\}$, *then function* $\psi_*(\imath)$ *and* $\psi^*(\imath)$ *find the conditions 1–4.*

We let $\widetilde{\psi}, \widetilde{\phi} \in \mathbb{F}_0$ be represented parametrically $\{(\psi_*(\imath), \psi^*(\imath)) : \imath \in [0,1]\}$ and $\{(\phi_*(\imath), \phi^*(\imath)) : \imath \in [0,1]\}$, respectively. We say that $\widetilde{\psi} \leq_{\mathbb{F}} \widetilde{\phi}$ if for all $\imath \in (0,1]$, $\psi^*(\imath) \leq \phi^*(\imath)$, and $\psi_*(\imath) \leq \phi_*(\imath)$. If $\widetilde{\psi} \leq_{\mathbb{F}} \widetilde{\phi}$, then there exists $\imath \in (0,1]$ such that $\psi^*(\imath) < \phi^*(\imath)$ or $\psi_*(\imath) \leq \phi_*(\imath)$. We say it is comparable if, for any $\widetilde{\psi}, \widetilde{\phi} \in \mathbb{F}_0$, we have $\widetilde{\psi} \leq_{\mathbb{F}} \widetilde{\phi}$ or $\widetilde{\psi} \geq_{\mathbb{F}} \widetilde{\phi}$, otherwise they are non-comparable. We may sometimes write $\widetilde{\psi} \leq_{\mathbb{F}} \widetilde{\phi}$ instead of $\widetilde{\phi} \geq_{\mathbb{F}} \widetilde{\psi}$, and note that we may say that $\mathbb{F}_0$ is a partial ordered set under the relation $\preceq$.

If $\widetilde{\psi}, \widetilde{\phi} \in \mathbb{F}_0$, there exists $\widetilde{\widetilde{\mu}} \in \mathbb{F}_0$ such that $\widetilde{\psi} = \widetilde{\phi} \oplus \widetilde{\mu}$, then, by this result, we have the existence of a generalized Hukuhara ($g\mathcal{H}$) difference of $\widetilde{\psi}$ and $\widetilde{\phi}$, and we say that $\widetilde{\mu}$ is the $g\mathcal{H}$-difference of $\widetilde{\psi}$ and $\widetilde{\phi}$, which is denoted by $\widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi}$ (see [37]). If $g\mathcal{H}$-difference exists, then

$$(\widetilde{\mu})^*(\imath) = \left(\widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi}\right)^*(\imath) = \psi^*(\imath) - \phi^*(\imath),\ (\widetilde{\mu})_*(\imath) = \left(\widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi}\right)_*(\imath) = \psi_*(\imath) - \phi_*(\imath),$$

and

$$\widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi} = \widetilde{\mu} \Leftrightarrow \begin{cases} \widetilde{\mu} = \widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi} \\ or\ \widetilde{\psi} = \widetilde{\phi} \oplus (-1) \odot \widetilde{\mu}. \end{cases}$$

Now, we discuss some properties of fuzzy numbers under addition and scaler multiplication; if $\widetilde{\psi}, \widetilde{\phi} \in \mathbb{F}_0$ and $0 < \rho \in \mathfrak{R}$, then $\widetilde{\psi} \oplus \widetilde{\phi}$ and $\rho \odot \widetilde{\psi}$ can be defined as

$$\widetilde{\psi} \oplus \widetilde{\phi} = \{(\psi_*(\imath) + \phi_*(\imath), \psi^*(\imath) + \phi^*(\imath)) : \imath \in [0,\ 1]\},$$

$$\rho \odot \widetilde{\psi} = \{(\rho\psi_*(\imath), \rho\psi^*(\imath)) : \imath \in [0,\ 1]\}.$$

Remark 1. *Obviously, $\mathbb{F}_0$ is closed under addition and nonnegative scaler multiplication and the above defined properties on $\mathbb{F}_0$ are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number $\rho \in \mathfrak{R}$,*

$$\widetilde{\psi} \oplus \rho = \{(\psi_*(\imath) + \rho, \psi^*(\imath) + \rho) : \imath \in [0,\ 1]\}.$$

It is widely recognized (refer to, for example, [36]) that the space $\mathbb{F}_0$ equipped with the supremum metric, denoted as $\mathcal{D}(\widetilde{\psi},\widetilde{\phi}) = \sup\limits_{0\leq\kappa\leq 1} H\Big([\widetilde{\psi}]^\kappa,\ [\widetilde{\phi}]^\kappa\Big)$, forms a complete metric space with the following properties:

a. $\mathcal{D}(\widetilde{\psi} \oplus \widetilde{\mu}, \widetilde{\phi} \oplus \widetilde{\mu}) = \mathcal{D}(\widetilde{\psi},\widetilde{\phi})$, for all $\widetilde{\psi},\widetilde{\phi},\widetilde{\mu} \in \mathbb{F}_0$;
b. $\mathcal{D}(\rho \odot \widetilde{\psi}, \rho \odot \widetilde{\phi}) = |\rho|\mathcal{D}(\widetilde{\psi},\widetilde{\phi})$, for all $\widetilde{\psi},\widetilde{\phi} \in \mathbb{F}_0$ and $\rho \in \mathfrak{R}$;
c. $\mathcal{D}\Big(\widetilde{\psi} \oplus \widetilde{\phi}, \widetilde{\Theta} \oplus \widetilde{\mu}\Big) = \mathcal{D}(\widetilde{\psi},\Theta) \oplus \mathcal{D}(\widetilde{\phi},\widetilde{\mu})$, for all $\widetilde{\psi},\widetilde{\phi},\Theta,\widetilde{\mu} \in \mathbb{F}_0$;
d. $\mathcal{D}\Big(\widetilde{\psi} \oplus \widetilde{\phi}, \widetilde{0}\Big) \leq \mathcal{D}\Big(\widetilde{\psi},\widetilde{0}\Big) \oplus \mathcal{D}\Big(\widetilde{\phi},\widetilde{0}\Big)$, for all $\widetilde{\psi},\widetilde{\phi} \in \mathbb{F}_0$, where $\widetilde{0}$ is the function $\widetilde{0} : \mathfrak{R} \to [0,1]$ defined by $\widetilde{0}(\varkappa) = 0$ for all $\varkappa \in \mathfrak{R}$;
e. $\mathcal{D}(\widetilde{\psi},\widetilde{\phi}) = \mathcal{D}\Big(\widetilde{\psi} \ominus_{g\mathcal{H}} \widetilde{\phi}, \widetilde{0}\Big)$, for all $\widetilde{\psi},\widetilde{\phi} \in \mathbb{F}_0$.

Definition 2 ([36])**.** *A F·V·M $\widetilde{\mathcal{P}} : [\theta,\ \lambda] \to \mathbb{F}_0$ is said to be $\mathcal{D}$-continuous, i.e., for a given $\varkappa_0 \in \mathfrak{R}$, there exists $\varepsilon > 0$ and there exists $\delta(\varepsilon, \varkappa_0) = \delta > 0$ such that $\mathcal{D}\Big(\widetilde{\mathcal{P}}(\vartheta), \widetilde{\mathcal{P}}(\varkappa_0)\Big) < \varepsilon$ for all $\vartheta \in \mathfrak{R}$ with $|\vartheta - \varkappa_0| < \delta$.*

Definition 3 ([40])**.** *The mapping $\widetilde{\mathcal{P}} : [\theta,\ \lambda] \to \mathbb{F}_0$ is called F·V·M. For each $\imath \in [0,\ 1]$, we denote $\Big[\widetilde{\mathcal{P}}(\varkappa)\Big]^\imath = \mathcal{P}_\imath(\varkappa) = [\mathcal{P}_*(\varkappa,\imath),\ \mathcal{P}^*(\varkappa,\imath)]$. Thus, a fuzzy mapping $\widetilde{\mathcal{P}}$ can be identified by parametrized triples:*

$$\Big[\widetilde{\mathcal{P}}(\varkappa)\Big]^\imath = \Big\{\Big(\widetilde{\mathcal{P}}_*(\varkappa,\ \imath), \widetilde{\mathcal{P}}^*(\varkappa,\ \imath)\Big) : \imath \in [0,\ 1]\Big\}.$$

Definition 4 ([28])**.** *Let $L = (m,\ n)$ and $\varkappa \in L$. Then, F·V·M $\widetilde{\mathcal{P}} : (m,\ n) \to \mathbb{F}_0$ is said to be a generalized Hukuhara differentiable (in short, $g\mathcal{H}$-differentiable) at $\varkappa$ if there exists the element $\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa) \in \mathbb{F}_0$ such that for all $0 < \tau$, sufficiently small, there exist $\widetilde{\mathcal{P}}(\varkappa + \tau) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa)$, $\widetilde{\mathcal{P}}(\varkappa) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa - \tau)$ and the limits (in the metric $\mathcal{D}$)*

$$\lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa+\tau) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa)}{\tau} = \lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa-\tau)}{\tau} = \widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa),$$

or

$$\lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa+\tau)}{-\tau} = \lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa-\tau) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa)}{-\tau} = \widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa),$$

or

$$\lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa+\tau) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa)}{\tau} = \lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa-\tau) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa)}{-\tau} = \widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa),$$

or

$$\lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa+\tau)}{-\tau} = \lim_{\tau\to 0^+} \frac{\widetilde{\mathcal{P}}(\varkappa) \ominus_{g\mathcal{H}} \widetilde{\mathcal{P}}(\varkappa-\tau)}{\tau} = \widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa),$$

By examining the collection $\mathcal{B}(\widetilde{\mathcal{P}}, \mathbb{F}_0)$ comprising all bounded fuzzy-valued functions $\widetilde{\mathcal{P}} : \nabla \to \mathbb{F}_0$, and since $(\mathbb{F}_0,\ \oplus,\ \odot)$ constitutes a quasilinear space, we can consequently establish a quasilinear space structure on $\mathcal{B}(\mathcal{P}, \mathfrak{R}_I)$, where the quasinorm $\|\cdot\|$ is provided (refer to [39]) as

$$\|\widetilde{\mathcal{P}}\| = \sup_{0\leq\kappa\leq 1}\{\|\mathcal{P}_\imath\|\} = \sup_{\varkappa\in\nabla} \mathcal{D}(\widetilde{\mathcal{P}}(\varkappa), \widetilde{0}).$$

Definition 5 ([40]). *Let $\widetilde{\mathcal{P}} : [\theta, \lambda] \subset \mathfrak{R} \to \mathbb{F}_0$ be an F·V·M. Then, the fuzzy integral of $\widetilde{\mathcal{P}}$ over $[\theta, \lambda]$, denoted by $(FA)\int_\theta^\lambda \widetilde{\mathcal{P}}(\varkappa)d\varkappa$, is given level-wise by*

$$\left[(FA)\int_\theta^\lambda \widetilde{\mathcal{P}}(\varkappa)d\varkappa\right]^\imath = (IA)\int_\theta^\lambda \mathcal{P}_\imath(\varkappa)d\varkappa = \left\{\int_\theta^\lambda \mathcal{P}(\varkappa,\imath)d\varkappa : \mathcal{P}(\varkappa,\imath) \in \mathcal{R}_{([\theta,\, \lambda],\, \imath)}\right\}.$$

for all $\imath \in (0, 1]$, where $\mathcal{R}_{([\theta,\, \lambda],\, \imath)}$ denotes the collection of Lebesgue-integrable mappings of *I-V-Ms*. The *F·V·M* $\widetilde{\mathcal{P}}$ is *FA*-integrable over $[\theta, \lambda]$ if $(FA)\int_\theta^\lambda \widetilde{\mathcal{P}}(\varkappa)d\varkappa \in \mathbb{F}_0$. Note that, if $\mathcal{P}_*(\varkappa,\imath)$, $\mathcal{P}^*(\varkappa,\imath)$ are Lebesgue-integrable, then $\mathcal{P}$ is fuzzy Aumann-integrable mapping over $[\theta, \lambda]$.

Theorem 2 ([40]). *Let $\widetilde{\mathcal{P}} : [\theta, \lambda] \subset \mathfrak{R} \to \mathbb{F}_0$ be an F·V·M, and its I-V-Ms are classified according to their $\imath$-levels $\mathcal{P}_\imath : [\theta, \lambda] \subset \mathfrak{R} \to \mathcal{L}_C$ which are given by $\mathcal{P}_\imath(\varkappa) = [\mathcal{P}_*(\varkappa,\imath), \mathcal{P}^*(\varkappa,\imath)]$ for all $\varkappa \in [\theta, \lambda]$ and for all $\imath \in (0, 1]$. Then, $\widetilde{\mathcal{P}}$ is FA-integrable over $[\theta, \lambda]$ if, and only if, $\mathcal{P}_*(\varkappa,\imath)$ and $\mathcal{P}^*(\varkappa,\imath)$ are both A-integrable over $[\theta, \lambda]$. Moreover, if $\widetilde{\mathcal{P}}$ is FA-integrable over $[\theta, \lambda]$, then*

$$\left[(FA)\int_\theta^\lambda \widetilde{\mathcal{P}}(\varkappa)d\varkappa\right]^\imath = \left[(A)\int_\theta^\lambda \mathcal{P}_*(\varkappa,\imath)d\varkappa,\ (A)\int_\theta^\lambda \mathcal{P}^*(\varkappa,\imath)d\varkappa\right]$$
$$= (IA)\int_\theta^\lambda \mathcal{P}_\imath(\varkappa)d\varkappa.$$

Definition 6 ([34]). *The set $\Lambda = [\theta, \lambda]$ is said to be a harmonically (H) convex set, if, for all б, ꝟ $\in \Lambda$, $\kappa \in [0, 1]$, we have*

$$\frac{бꝟ}{\kappa б + (1-\kappa)ꝟ} \in \Lambda.$$

Definition 7 ([34]). *The relation $\mathcal{P} : [\theta, \lambda] \to \mathfrak{R}$ is named an H-convex mapping on $[\theta, \lambda]$ if*

$$\mathcal{P}\left(\frac{бꝟ}{\kappa б + (1-\kappa)ꝟ}\right) \le (1-\kappa)\mathcal{P}(б) + \kappa\mathcal{P}(ꝟ), \tag{1}$$

for all б, ꝟ $\in [\theta, \lambda]$, $\kappa \in [0, 1]$, where $\mathcal{P}(б) \ge 0$ for all б $\in [\theta, \lambda]$. If Expression (1) is inverted, then $\mathcal{P}$ is named H-concave mapping on $[\theta, \lambda]$, such that

$$\mathcal{P}\left(\frac{бꝟ}{\kappa б + (1-\kappa)ꝟ}\right) \ge (1-\kappa)\mathcal{P}(б) + \kappa\mathcal{P}(ꝟ). \tag{2}$$

Definition 8 ([35]). *The F·V·M $\widetilde{\mathcal{P}} : [\theta, \lambda] \to \mathbb{F}_0$ is named a harmonically (H) convex F·V·M on $[\theta, \lambda]$ if*

$$\widetilde{\mathcal{P}}\left(\frac{бꝟ}{\kappa б + (1-\kappa)ꝟ}\right) \le_{\mathbb{F}} \hbar(1-\kappa) \odot \widetilde{\mathcal{P}}(б) \oplus \hbar(\kappa) \odot \widetilde{\mathcal{P}}(ꝟ), \tag{3}$$

for all б, ꝟ $\in [\theta, \lambda]$, $\kappa \in [0, 1]$, given $\widetilde{\mathcal{P}}(б) \ge_{\mathbb{F}} \widetilde{0}$ for all б belongs to $[\theta, \lambda]$ and $\hbar : [0,1] \subseteq [\theta, \lambda] \to \mathfrak{R}$ such that $\hbar \not\equiv 0$. In the event that Expression (3) is reversed, $\mathcal{P}$ is denoted as an H-concave F·V·M on $[\theta, \lambda]$.

Note that Definition 8 is helpful in proving the upcoming results.

In 1938, Ostrowski [1] explored the following compelling integral inequalities:

Let $\mathcal{P} : I = [\theta,\ \lambda] \to \mathfrak{R}$ be a differentiable function on I_0 with (θ, λ). If $\mathcal{P} \in \mathcal{L}[\theta,\ \lambda]$ and $|\mathcal{P}'(\varkappa)| \leq M$, for all $\varkappa \in [\theta,\ \lambda]$, then

$$\left|\mathcal{P}(\varkappa) - \frac{1}{\lambda-\theta}\int_{\theta}^{\lambda} \mathcal{P}(u)du\right| \leq M(\lambda-\theta)\left[\frac{1}{4} - \frac{\left(\varkappa - \frac{\theta+\lambda}{2}\right)^2}{(\lambda-\theta)^2}\right].$$

On the other hand, Ujević [6] derived the subsequent Ostrowski-type inequality:

$$\left|\mathcal{P}(\varkappa) - \frac{1}{\lambda-\theta}\int_{\theta}^{\lambda} \mathcal{P}(u)du - \frac{\mathcal{P}(\lambda)-\mathcal{P}(\theta)}{\lambda-\theta}\left(\varkappa - \frac{\theta+\lambda}{2}\right)\right| \leq \frac{\sqrt{(\lambda-\theta)\|\mathcal{P}'\|_2^2 - (\mathcal{P}(\lambda)-\mathcal{P}(\theta))^2}}{2\sqrt{3}}.$$

where $\mathcal{P} : [\theta,\ \lambda] \to \mathfrak{R}$ is a differentiable function with $\mathcal{P}' \in L_2[\theta,\ \lambda]$ and $\frac{1}{2\sqrt{3}}$ is the best possible value.

Note that, with the support of Gamma and Beta functions, some new findings are obtained as applications of Ostrowski-type inequalities:

Gamma and Beta functions are respectively characterized as

$$\Gamma(y) = \int_0^{\infty} \kappa^{y-1} e^{-\kappa} d\kappa, \tag{4}$$

for $\mathcal{R}(y) > 0$

$$ß(y, z) = \int_0^1 \kappa^{y-1}(1-\kappa)^{z-1} d\kappa = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}, \tag{5}$$

for $\mathcal{R}(y) > 0, \mathcal{R}(z) > 0$.

The integral representation of the hypergeometric function is

$${}_2F_1(y, z; c; \varkappa) = \frac{1}{ß(z, c-z)}\int_0^1 \kappa^{z-1}(1-\kappa)^{c-z-1}(1-x\kappa)^{-y} d\kappa, \tag{6}$$

for $|x| < 1, \mathcal{R}(c) > 0, \mathcal{R}(z) > 0$.

3. Main Results

In this section, we introduce novel Ostrowski-type inequalities for gH-differentiable fuzzy-valued functions. Some generalized forms of classical Ostrowski-type inequalities are also obtained that can be seen as applications. Additionally, some new exceptional cases are also discussed by using Gamma and Beta functions. Firstly, we start with the following identity:

Lemma 1. *Let* $\mathcal{P}_*(.,\imath), \mathcal{P}^*(.,\imath) : [\theta,\ \lambda] \to \mathfrak{R}$ *be two* $g\mathcal{H}$*-differentiable functions on* I_0 *with* (θ, λ)*, where* $\imath \in [0,1]$*. If* $\mathcal{P}$ *is integrable over* $[\theta,\ \lambda]$*, then*

$$\left[\mathcal{P}_*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_{\theta}^{\lambda} \mathcal{P}_*(u,\imath)du, \mathcal{P}^*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_{\theta}^{\lambda} \mathcal{P}^*(u,\imath)du\right]$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\left[\mathcal{P}_*{}'\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right), \mathcal{P}^{*\prime}\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right)\right]d\kappa$$

$$+(\lambda-\varkappa)^2\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\left[\mathcal{P}_*{}'\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right), \mathcal{P}^{*\prime}\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right)\right]d\kappa.$$

Proof. Integration by parts finalizes the proof. □

Theorem 3. *Let* $\widetilde{\mathcal{P}} : [\theta,\ \lambda] \to \mathbb{F}_0$ *be a* $g\mathcal{H}$*-differentiable* $F \cdot V \cdot M$ *on* I_0 *with* $\varkappa \in (\theta, \lambda)$*, where* $\iota \in [0,1]$*. If* $\widetilde{\mathcal{P}}'_{g\mathcal{H}}$ *is* $\mathcal{D}$*-continuous as well as* $\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}\right\|$ *is the harmonic convex* $F \cdot V \cdot M$*, then for* $q \geq 1$*, we have*

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda-\theta} \odot \left\{ \left((\varkappa-\theta)^2 \odot [\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\theta)\|^q \odot \psi_1^\circ(\theta,\varkappa;q;\kappa) \oplus \|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa)\|^q \odot \psi_2^\circ(\theta,\varkappa;q;\kappa)]^{\frac{1}{q}} \oplus (\lambda-\varkappa)^2 \odot \left[\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\lambda)\|^q \odot \psi_3^\circ(\lambda,\varkappa;q;\kappa) \oplus \|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa)\|^q \odot \psi_4^\circ(\lambda,\varkappa;q;\kappa)\right]^{\frac{1}{q}} \right) \right\},$$

where

$$\begin{aligned}\psi_1^\circ(\theta,\varkappa;q;\kappa) &= \int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa \\ &= \tfrac{ß(1+q,1)}{\varkappa^2}{}_2F_1\left(2q, 1+q; 2+q; 1-\tfrac{\theta}{\varkappa}\right) \\ &\quad -\tfrac{ß(2+q,1)}{\varkappa^2}{}_2F_1\left(2q, 2+q; 3+q; 1-\tfrac{\theta}{\varkappa}\right),\end{aligned} \tag{7}$$

$$\begin{aligned}\psi_2^\circ(\theta,\varkappa;q;\kappa) &= \int_0^1 \tfrac{\kappa^{1+q}}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa, \\ &= \tfrac{ß(2+q,1)}{\varkappa^2}{}_2F_1\left(2q, 2+q; 3+q; 1-\tfrac{\theta}{\varkappa}\right),\end{aligned} \tag{8}$$

$$\begin{aligned}\psi_3^\circ(\lambda,\varkappa;q;\kappa) &= \int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa, \\ &= \tfrac{ß(1+q,1)}{\varkappa^2}{}_2F_1\left(2q, 1+q; 2+q; 1-\tfrac{\lambda}{\varkappa}\right) \\ &\quad -\tfrac{ß(2+q,1)}{\varkappa^2}{}_2F_1\left(2q, 2+q; 3+q; 1-\tfrac{\lambda}{\varkappa}\right),\end{aligned} \tag{9}$$

$$\begin{aligned}\psi_4^\circ(\lambda,\varkappa;q;\kappa) &= \int_0^1 \tfrac{\kappa^{1+q}}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa, \\ &= \tfrac{ß(2+q,1)}{\varkappa^2}{}_2F_1\left(2q, 2+q; 3+q; 1-\tfrac{\lambda}{\varkappa}\right),\end{aligned} \tag{10}$$

for all $\varkappa \in [\theta,\ \lambda]$.

Proof. In accordance with Lemma 1, we have

$$\mathcal{P}_*(\varkappa,\iota) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\iota)du$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\iota\right)d\kappa + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\iota\right)d\kappa,$$

and

$$\mathcal{P}^*(\varkappa,\iota) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\iota)du$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\iota\right)d\kappa + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\iota\right)d\kappa.$$

Using the power mean inequality and given that $|\mathcal{P}'|$ is harmonic convex $F{\cdot}V{\cdot}M$, for $\iota \in [0,\ 1]$, then we have

$$\left|\mathcal{P}_*(\varkappa,\iota) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\iota)du\right|$$

$$\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 1d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa^q}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\theta,\iota)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}} + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 1d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa^q}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\lambda,\iota)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}},$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right|$$

$$\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 1d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \tfrac{\kappa^q}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q+\kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}+$$
$$\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 1d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \tfrac{\kappa^q}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q+\kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}.$$

As a result, we obtain

$$\left|\mathcal{P}_*(\varkappa,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right|$$

$$\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}+$$
$$\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}},$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath)-\frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right|$$

$$\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}+$$
$$\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\hbar(1-\kappa)d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}},$$

That is,

$$\mathcal{D}\left(\tfrac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \tfrac{\widetilde{\mathcal{P}}(u)}{u^2}du,\widetilde{\mathcal{P}}(\varkappa)\right)\leq_{\mathbb{F}}\tfrac{\theta\lambda}{\lambda-\theta}\odot\left\{\left((\varkappa-\theta)^2\odot[\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\theta)\right\|^q\odot\psi_1^\circ(\theta,\varkappa;q;\hbar)\oplus\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa)\right\|^q\odot\right.\right.$$
$$\left.\left.\psi_2^\circ(\theta,\varkappa;q;\hbar)]^{\frac{1}{q}}\oplus(\lambda-\varkappa)^2\odot\left[\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\lambda)\right\|^q\odot\psi_3^\circ(\lambda,\varkappa;q;\hbar)\oplus\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}(\varkappa)\right\|^q\odot\psi_4^\circ(\lambda,\varkappa;q;\hbar)\right]^{\frac{1}{q}}\right)\right\},$$

hence the required result. □

The subsequent outcome integrates the suitable rendition for powers of the absolute value of the initial derivative:

Theorem 4. *Let* $\widetilde{\mathcal{P}}:[\theta,\ \lambda]\to\mathbb{F}_0$ *be a* $g\mathcal{H}$*-differentiable* $F\cdot V\cdot M$ *on* I_0 *with* $\varkappa\in(\theta,\lambda)$*. Let* $\widetilde{\mathcal{P}}'_{g\mathcal{H}}$ *be* $\mathcal{D}$*-continuous as well as* $\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}\right\|$ *be a harmonic convex* $F\cdot V\cdot M$ *with* $\left|\widetilde{\mathcal{P}}'(\varkappa)\right|\geq_{\mathbb{F}}\widetilde{0}$*, where* $\imath\in[0,1]$*. Then, for* $q\geq 1$*, we have*

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du,\widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}}\tfrac{\theta\lambda}{\lambda-\theta}\odot\left[\Phi^{1-\frac{1}{q}}(\theta,\varkappa)(\varkappa-\theta)^2\odot\left\{\left\|\widetilde{\mathcal{P}}'(\theta)\right\|^q\odot\psi_1^\circ(\theta,\varkappa;1;\kappa)\oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\odot\psi_2^\circ(\theta,\varkappa;1;\kappa)\right\}^{\frac{1}{q}}\oplus\right.$$
$$\left.\Phi^{1-\frac{1}{q}}(\lambda,\varkappa)(\lambda-\varkappa)^2\odot\left\{\left\|\widetilde{\mathcal{P}}'(\lambda)\right\|^q\odot\psi_3^\circ(\lambda,\varkappa;1;\kappa)\oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\odot\psi_4^\circ(\lambda,\varkappa;1;\kappa)\right\}^{\frac{1}{q}}\right],$$

where

$$\Phi(\theta,\varkappa)=\frac{1}{\varkappa-\theta}\left\{\frac{1}{\theta}-\frac{\ln\varkappa-\ln\theta}{\varkappa-\theta}\right\},$$
$$\Phi(\lambda,\varkappa)=\frac{1}{\lambda-\varkappa}\left\{\frac{\ln\lambda-\ln\varkappa}{\lambda-\varkappa}-\frac{1}{\lambda}\right\}.$$

for all $\varkappa \in [\theta, \lambda]$, *and* $\psi_1^{\circ}(\theta,\varkappa;q;\kappa)$, $\psi_2^{\circ}(\theta,\varkappa;q;\kappa)$, $\psi_3^{\circ}(\lambda,\varkappa;q;\kappa)$, *and* $\psi_4^{\circ}(\lambda,\varkappa;q;\kappa)$ *can be obtained as (7), (8), (9) and (10), respectively.*

Proof. In accordance with Lemma 1 and $|\mathcal{P}'|^q$ as harmonic convex $F{\cdot}V{\cdot}M$, for $\imath \in [0, 1]$, we have

$$\mathcal{P}_*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right)d\kappa + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right)d\kappa,$$

and

$$\mathcal{P}^*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right)d\kappa + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right)d\kappa.$$

Since $|\mathcal{P}'|^q$ is harmonic convex $F{\cdot}V{\cdot}M$, then by power mean inequality, from the above equations, we have

$$\begin{aligned}
&\left|\mathcal{P}_*(\varkappa,\imath) - \tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right| \\
&\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\textstyle\int_0^1 \tfrac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}d\kappa\right)^{1-\frac{1}{q}}\left[\textstyle\int_0^1 \tfrac{\kappa^q}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left((1-\kappa)\left|\mathcal{P}_*{}'(\theta,\imath)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right)d\kappa\right]^{\frac{1}{q}} + \\
&\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\textstyle\int_0^1 \tfrac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}d\kappa\right)^{1-\frac{1}{q}}\left[\textstyle\int_0^1 \tfrac{\kappa^q}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left((1-\kappa)\left|\mathcal{P}_*{}'(\lambda,\imath)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right)d\kappa\right]^{\frac{1}{q}}, \\
&= \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\tfrac{1}{\varkappa-\theta}\left\{\tfrac{1}{\theta} - \tfrac{\ln\varkappa-\ln\theta}{\varkappa-\theta}\right\}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{P}_*{}'(\theta,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\theta+(1-\kappa)\varkappa)^2}d\kappa\right) \\
&+\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\tfrac{1}{\lambda-\varkappa}\left(\tfrac{1}{\lambda-\varkappa}\left\{\tfrac{\ln\lambda-\ln\varkappa}{\lambda-\varkappa} - \tfrac{1}{\lambda}\right\}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{P}_*{}'(\lambda,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\lambda+(1-\kappa)\varkappa)^2}d\kappa\right),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\left|\mathcal{P}^*(\varkappa,\imath) - \tfrac{1}{\lambda-\theta}\textstyle\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right| \\
&\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\textstyle\int_0^1 \tfrac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}d\kappa\right)^{1-\frac{1}{q}}\left[\textstyle\int_0^1 \tfrac{\kappa^q}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left((1-\kappa)\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right)d\kappa\right]^{\frac{1}{q}} + \\
&\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\textstyle\int_0^1 \tfrac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}d\kappa\right)^{1-\frac{1}{q}}\left[\textstyle\int_0^1 \tfrac{\kappa^q}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left((1-\kappa)\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right)d\kappa\right]^{\frac{1}{q}}, \\
&= \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\tfrac{1}{\varkappa-\theta}\left\{\tfrac{1}{\theta} - \tfrac{\ln\varkappa-\ln\theta}{\varkappa-\theta}\right\}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\theta+(1-\kappa)\varkappa)^2}d\kappa\right) \\
&+\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\tfrac{1}{\lambda-\varkappa}\left\{\tfrac{\ln\lambda-\ln\varkappa}{\lambda-\varkappa} - \tfrac{1}{\lambda}\right\}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^q(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\textstyle\int_0^1 \tfrac{\kappa^{q+1}}{(\kappa\lambda+(1-\kappa)\varkappa)^2}d\kappa\right).
\end{aligned} \tag{12}$$

As a result, from (11) and (12), we obtain

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\begin{aligned}
&\leq_{\mathbb{F}} \tfrac{\theta\lambda}{\lambda-\theta}\odot\left[\Phi^{1-\frac{1}{q}}(\theta,\varkappa)(\varkappa-\theta)^2\odot\left\{\|\widetilde{\mathcal{P}}'(\theta)\|^q\odot\psi_1(\theta,\varkappa;q;\hbar)\oplus\|\widetilde{\mathcal{P}}'(\varkappa)\|^q\odot\psi_2(\theta,\varkappa;q;\hbar)\right\}\oplus\right. \\
&\left.\Phi^{1-\frac{1}{q}}(\lambda,\varkappa)(\lambda-\varkappa)^2\odot\left\{\|\widetilde{\mathcal{P}}'(\lambda)\|^q\odot\psi_3(\lambda,\varkappa;q;\hbar)\oplus\|\widetilde{\mathcal{P}}'(\varkappa)\|^q\odot\psi_4(\lambda,\varkappa;q;\hbar)\right\}\right],
\end{aligned}$$

hence the required result. □

Further generalized versions of Theorems 3 and 4 are provided below:

Theorem 5. *Let $\widetilde{\mathcal{P}} : [\theta,\ \lambda] \to \mathbb{F}_0$ be a $g\mathcal{H}$-differentiable $F \cdot V \cdot M$ on I_0 with $\varkappa \in (\theta, \lambda)$, where $\imath \in [0,1]$. If $\widetilde{\mathcal{P}}'_{g\mathcal{H}}$ is $\mathcal{D}$-continuous and $\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}\|$ is harmonic convex $F \cdot V \cdot M$, then for $q \geq 1$, we have*

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta} \odot (FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \odot \left[\left\{(\varkappa-\theta)^2 \odot \left(\|\widetilde{\mathcal{P}}'(\theta)\|^q \odot \psi_1^\circ(\theta,\varkappa;q;\kappa) \oplus \|\widetilde{\mathcal{P}}'(\varkappa)\|^q \odot \psi_2^\circ(\theta,\varkappa;q;\kappa)\right)^{\frac{1}{q}} \oplus (\lambda-\varkappa)^2 \odot \left[\|\widetilde{\mathcal{P}}'(\lambda)\|^q \odot \psi_3^\circ(\lambda,\varkappa;q;\kappa) \oplus \|\widetilde{\mathcal{P}}'(\varkappa)\|^q \odot \psi_4^\circ(\lambda,\varkappa;q;\kappa)\right]^{\frac{1}{q}}\right\}\right],$$

where

$$\psi_1^\circ(\theta,\varkappa;q;\kappa) = \int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa$$
$$= \frac{1}{2\varkappa^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{\theta}{\varkappa}\right)$$
$$-\frac{1}{12\varkappa^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{\theta}{\varkappa}\right),$$

$$\psi_2^\circ(\theta,\varkappa;q;\kappa) = \int_0^1 \frac{\kappa^2}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa,$$
$$= \frac{1}{12\varkappa^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{\theta}{\varkappa}\right),$$

$$\psi_3^\circ(\lambda,\varkappa;q;\kappa) = \int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa,$$
$$= \frac{1}{2\varkappa^{2q}}\,{}_2F_1\left(2q,2;3;1-\frac{\lambda}{\varkappa}\right)$$
$$-\frac{1}{12\varkappa^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{\lambda}{\varkappa}\right),$$

$$\psi_4^\circ(\lambda,\varkappa;q;\kappa) = \int_0^1 \frac{\kappa^2}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa,$$
$$= \frac{1}{12\varkappa^{2q}}\,{}_2F_1\left(2q,3;4;1-\frac{\lambda}{\varkappa}\right),$$

for all $\varkappa \in [\theta,\ \lambda]$.

Proof. In accordance with Lemma 1, we have

$$\mathcal{P}_*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du \tag{13}$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right)d\kappa + (\lambda-\varkappa)^2\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}_*{}'\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right)d\kappa, \tag{14}$$

and

$$\mathcal{P}^*(\varkappa,\imath) - \frac{\theta\lambda}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du \tag{15}$$

$$= \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa},\imath\right)d\kappa + (\lambda-\varkappa)^2\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^2}\mathcal{P}^{*\prime}\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa},\imath\right)d\kappa. \tag{16}$$

Using the power mean inequality and given that $|\mathcal{P}'|$ is harmonic convex $F{\cdot}V{\cdot}M$, for $\imath \in [0,\ 1]$, then we have

$$\left|\mathcal{P}_*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right| \tag{17}$$

$$\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \kappa d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\theta,\imath)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}} + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \kappa d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\lambda,\imath)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}, \tag{18}$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right| \tag{19}$$

$$\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \kappa d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}} + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \kappa d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}. \tag{20}$$

As a result, we obtain

$$\left|\mathcal{P}_*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right| \tag{21}$$

$$\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa^2}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}} + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa^2}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}, \tag{22}$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right| \tag{23}$$

$$\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa^2}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}} + \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa + \left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa^2}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}. \tag{24}$$

That is,

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda-\theta}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\odot\left\{\left((\varkappa-\theta)^2\odot\left[\left\|\widetilde{\mathcal{P}}'(\theta)\right\|^q\odot\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa \oplus \left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\odot \int_0^1 \frac{\kappa^2}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}} \oplus (\lambda-\varkappa)^2\odot\left[\left\|\widetilde{\mathcal{P}}'(\lambda)\right\|^q\odot\int_0^1 \frac{\kappa(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa \oplus \left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\odot\int_0^1 \frac{\kappa^2}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}\right)\right\},$$

hence the required result. □

Theorem 6. *Let $\widetilde{\mathcal{P}} : [\theta,\ \lambda] \to \mathbb{F}_0$ be a $g\mathcal{H}$-differentiable $F \cdot V \cdot M$ on I_0 with $\varkappa \in (\theta, \lambda)$, where $\imath \in [0,1]$. If $\widetilde{\mathcal{P}}'_{g\mathcal{H}}$ is $\mathcal{D}$-continuous and $\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}\right\|$ is harmonic convex $F \cdot V \cdot M$, then for $q \geq 1$, we have*

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right) \tag{25}$$

$$\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \odot\left[\left\{(\varkappa-\theta)^{2} \odot\left(\left\|\widetilde{\mathcal{P}}'(\theta)\right\|^{q} \odot \psi_{1}^{\circ}(\theta, \varkappa; q; \kappa) \oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^{q} \odot \psi_{2}^{\circ}(\theta, \varkappa; q; \kappa)\right)^{\frac{1}{q}} \oplus(\lambda-\varkappa)^{2} \odot\left[\left\|\widetilde{\mathcal{P}}'(\lambda)\right\|^{q} \odot \psi_{3}^{\circ}(\theta, \varkappa; q; \kappa) \oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^{q} \odot \psi_{4}^{\circ}(\theta, \varkappa; q; \kappa)\right]^{\frac{1}{q}}\right\}\right], \tag{26}$$

where

$$\begin{aligned}
\psi_{1}^{\circ}(\theta, \varkappa; q; \kappa) &= \int_{0}^{1} \frac{1-\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}} d\kappa \\
&= \frac{1}{\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 1; 2; 1-\frac{\theta}{\varkappa}\right) \\
&\quad -\frac{1}{2\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 2; 3; 1-\frac{\theta}{\varkappa}\right), \\
\psi_{2}^{\circ}(\theta, \varkappa; q; \kappa) &= \int_{0}^{1} \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}} d\kappa, \\
&= \frac{1}{2\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 2; 3; 1-\frac{\theta}{\varkappa}\right), \\
\psi_{3}^{\circ}(\lambda, \varkappa; q; \kappa) &= \int_{0}^{1} \frac{1-\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}} d\kappa, \\
&= \frac{1}{2\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 1; 2; 1-\frac{\lambda}{\varkappa}\right) \\
&\quad -\frac{1}{2\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 2; 3; 1-\frac{\lambda}{\varkappa}\right), \\
\psi_{4}^{\circ}(\lambda, \varkappa; q; \kappa) &= \int_{0}^{1} \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}} d\kappa, \\
&= \frac{1}{2\varkappa^{2q}}\, {}_{2}F_{1}\left(2q, 2; 3; 1-\frac{\lambda}{\varkappa}\right),
\end{aligned}$$

for all $\varkappa \in [\theta,\ \lambda]$.

Proof. In accordance with Lemma 1, we have

$$\mathcal{P}_{*}(\varkappa, \imath)-\frac{\theta\lambda}{\lambda-\theta} \int_{\theta}^{\lambda} \mathcal{P}_{*}(u, \imath) du \tag{27}$$

$$=\frac{\theta\lambda(\varkappa-\theta)^{2}}{\lambda-\theta} \int_{0}^{1} \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2}} \mathcal{P}_{*}{}'\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa}, \imath\right) d\kappa+\frac{\theta\lambda(\lambda-\varkappa)^{2}}{\lambda-\theta} \int_{0}^{1} \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2}} \mathcal{P}_{*}{}'\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa}, \imath\right) d\kappa, \tag{28}$$

and

$$\mathcal{P}^{*}(\varkappa, \imath)-\frac{\theta\lambda}{\lambda-\theta} \int_{\theta}^{\lambda} \mathcal{P}^{*}(u, \imath) du \tag{29}$$

$$=\frac{\theta\lambda(\varkappa-\theta)^{2}}{\lambda-\theta} \int_{0}^{1} \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2}} \mathcal{P}^{*\prime}\left(\frac{\theta\varkappa}{\kappa\theta+(1-\kappa)\varkappa}, \imath\right) d\kappa+\frac{\theta\lambda(\lambda-\varkappa)^{2}}{\lambda-\theta} \int_{0}^{1} \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2}} \mathcal{P}^{*\prime}\left(\frac{\lambda\varkappa}{\kappa\lambda+(1-\kappa)\varkappa}, \imath\right) d\kappa. \tag{30}$$

Using the power mean inequality and given that $|\mathcal{P}'|$ is harmonic convex $F{\cdot}V{\cdot}M$, for $\imath \in [0,\ 1]$, then we have

$$\left|\mathcal{P}_*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right| \tag{31}$$

$$\begin{aligned}\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \kappa^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\theta,\imath)\right|^q+\kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}+ \\ \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \kappa^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}_*{}'(\lambda,\imath)\right|^q+\kappa\left|\mathcal{P}_*{}'(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}},\end{aligned} \tag{32}$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right| \tag{33}$$

$$\begin{aligned}\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \kappa^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q+\kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}+ \\ \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \kappa^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q+\kappa\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\right]d\kappa\right)^{\frac{1}{q}}.\end{aligned} \tag{34}$$

As a result, we obtain

$$\left|\mathcal{P}_*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}_*(u,\imath)du\right| \tag{35}$$

$$\begin{aligned}\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \frac{(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}+ \\ \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \frac{(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}},\end{aligned} \tag{36}$$

and

$$\left|\mathcal{P}^*(\varkappa,\imath) - \frac{1}{\lambda-\theta}\int_\theta^\lambda \mathcal{P}^*(u,\imath)du\right| \tag{37}$$

$$\begin{aligned}\leq \frac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{P}^{*\prime}(\theta,\imath)\right|^q\int_0^1 \frac{(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}+ \\ \frac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{P}^{*\prime}(\lambda,\imath)\right|^q\int_0^1 \frac{(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa+\left|\mathcal{P}^{*\prime}(\varkappa,\imath)\right|^q\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}.\end{aligned} \tag{38}$$

That is,

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda-\theta}\odot(FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\begin{aligned}\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda-\theta}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ &\odot\Bigg\{\Bigg((\varkappa-\theta)^2 \\ &\odot\left[\left\|\widetilde{\mathcal{P}}'(\theta)\right\|^q\odot\int_0^1 \frac{(1-\kappa)}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\right. \\ &\left.\odot\int_0^1 \frac{\kappa}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}} \\ &\oplus(\lambda-\varkappa)^2\odot\left[\left\|\widetilde{\mathcal{P}}'(\lambda)\right\|^q\odot\int_0^1 \frac{(1-\kappa)}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right. \\ &\left.\oplus\left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q\odot\int_0^1 \frac{\kappa}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right]^{\frac{1}{q}}\Bigg)\Bigg\},\end{aligned}$$

hence the required result. □

In the upcoming result, it is evident that the inequalities derived in Theorem 7 surpass those in inequality Theorem 6.

Theorem 7. *Let $\widetilde{\mathcal{P}} : [\theta, \lambda] \to \mathbb{F}_0$ be a $g\mathcal{H}$-differentiable $F \cdot V \cdot M$ on I_0 with $\varkappa \in (\theta, \lambda)$, where $\iota \in [0,1]$. If $\widetilde{\mathcal{P}}'_{g\mathcal{H}}$ is $\mathcal{D}$-continuous and $\left\|\widetilde{\mathcal{P}}'_{g\mathcal{H}}\right\|$ is harmonic convex $F \cdot V \cdot M$, then for $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda - \theta} \odot (FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}} \frac{\theta\lambda}{\lambda - \theta} \odot \left\{(\varkappa - \theta)^2(\mathcal{E}_1(\theta, \varkappa; p))^{\frac{1}{p}} \odot \left[\frac{\left\|\widetilde{\mathcal{P}}'(\theta)\right\|^q \oplus \left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q}{2}\right]^{\frac{1}{q}} \oplus (\lambda - \varkappa)^2(\mathcal{E}_2(\lambda, \varkappa; p))^{\frac{1}{p}} \odot \left[\frac{\left\|\widetilde{\mathcal{P}}'(\lambda)\right\|^q \oplus \left\|\widetilde{\mathcal{P}}'(\varkappa)\right\|^q}{2}\right]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{E}_1(\theta, \varkappa; p) = \int_0^1 \frac{\kappa^p}{(\kappa\theta + (1-\kappa)\varkappa)^{2p}}d\kappa,$$

$$= \frac{ß(p+1,1)}{\varkappa^{2p}}{}_2F_1\left(2p, 1+p; 2+p; 1 - \frac{\theta}{\varkappa}\right),$$

$$\mathcal{E}_2(\lambda, \varkappa; p) = \int_0^1 \frac{\kappa^p}{(\kappa\lambda + (1-\kappa)\varkappa)^{2p}}d\kappa,$$

$$= \frac{ß(p+1,1)}{\varkappa^{2p}}{}_2F_1\left(2p, 1+p; 2+p; 1 - \frac{\lambda}{\varkappa}\right).$$

Proof. In accordance with Lemma 1, using Hölder's inequality and given that $|\mathcal{P}'|$ is harmonic convex $F{\cdot}V{\cdot}M$, for $\iota \in [0,\ 1]$, then we have

$$\left|\mathcal{P}_*(\varkappa, \iota) - \frac{1}{\lambda - \theta}\int_\theta^\lambda \mathcal{P}_*(u, \iota)du\right|$$

$$\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \tfrac{\kappa^p}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\kappa)\left|\mathcal{P}_*{}'(\theta,\iota)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}} +$$
$$\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \tfrac{\kappa^p}{(\kappa\lambda+(1-\kappa)\varkappa)^{2q}}d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\kappa)\left|\mathcal{P}_*{}'(\lambda,\iota)\right|^q + \kappa\left|\mathcal{P}_*{}'(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}},$$

and

$$\left|\mathcal{P}^*(\varkappa, \iota) - \frac{1}{\lambda - \theta}\int_\theta^\lambda \mathcal{P}^*(u, \iota)du\right|$$

$$\leq \tfrac{\theta\lambda(\varkappa-\theta)^2}{\lambda-\theta}\left(\int_0^1 \tfrac{\kappa^p}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\theta,\iota)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}} +$$
$$\tfrac{\theta\lambda(\lambda-\varkappa)^2}{\lambda-\theta}\left(\int_0^1 \tfrac{\kappa^p}{(\kappa\theta+(1-\kappa)\varkappa)^{2q}}d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\kappa)\left|\mathcal{P}^{*\prime}(\lambda,\iota)\right|^q + \kappa\left|\mathcal{P}^{*\prime}(\varkappa,\iota)\right|^q\right]d\kappa\right)^{\frac{1}{q}}.$$

As a result, we obtain

$$\mathcal{D}\left(\frac{\theta\lambda}{\lambda - \theta} \odot (FA)\int_\theta^\lambda \frac{\widetilde{\mathcal{P}}(u)}{u^2}du, \widetilde{\mathcal{P}}(\varkappa)\right)$$

$$\leq_{\mathbb{F}} \tfrac{\theta\lambda}{\lambda-\theta} \odot \left\{\left((\varkappa - \theta)^2(\mathcal{E}_1(\theta, \varkappa; p))^{\frac{1}{p}} \odot \left[\tfrac{\|\widetilde{\mathcal{P}}'(\theta)\|^q \oplus \|\widetilde{\mathcal{P}}'(\varkappa)\|^q}{2}\right]^{\frac{1}{q}} \oplus (\lambda -\right.\right.$$
$$\left.\left.\varkappa)^2(\mathcal{E}_2(\lambda, \varkappa; p))^{\frac{1}{p}} \odot \left[\tfrac{\|\widetilde{\mathcal{P}}'(\lambda)\|^q \oplus \|\widetilde{\mathcal{P}}'(\varkappa)\|^q}{2}\right]^{\frac{1}{q}}\right)\right\},$$

hence the required result. □

4. Applications

In this section, we discuss some applications of Ostrowski-type inequalities for fuzzy number functions over harmonic convexity. We start the with following proposition as follows:

Proposition 1. *If $\varkappa = \frac{\theta+\lambda}{2}$, and $\mathcal{P}_\imath(\varkappa) = C\varkappa$, as stated in Theorem 3, then for $\lambda > \theta > 0$, the following holds:*

$$\mathcal{D}\Big(A(\theta,\lambda),\ G^2(\theta,\lambda)L^{-1}(\theta,\lambda)\Big)$$

$$\leq_I \frac{\theta\lambda(\lambda-\theta)^{\frac{2-q}{q}}C}{2^{\frac{2}{q}-2}(\theta+\lambda)^2}\Bigg\{\Big[ß(1+q,1)_2F_1\Big(2q,1+q;2+q;\tfrac{\lambda-\theta}{\lambda+\theta}\Big) - ß(2+q,1)_2F_1\Big(2q,2+q;3+q;\tfrac{\lambda-\theta}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}} + \Big[ß(1+q,1)_2F_1\Big(2q,1+q;2+q;\tfrac{\theta-\lambda}{\lambda+\theta}\Big) - ß(2+q,1)_2F_1\Big(2q,2+q;3+q;\tfrac{\theta-\lambda}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}}\Bigg\},$$

where

$$C = [c_*(\imath), c^*(\imath)].$$

Proposition 2. *If $\varkappa = \frac{\theta+\lambda}{2}$, and $\mathcal{P}_\imath(\varkappa) = C\varkappa$, as stated in Theorem 4, then for $\lambda > \theta > 0$, the following holds:*

$$\mathcal{D}\Big(A(\theta,\lambda),\ G^2(\theta,\lambda)L^{-1}(\theta,\lambda)\Big)$$

$$\leq_I \frac{\theta\lambda(\lambda-\theta)^{\frac{3-2q}{q}}C}{2^{\frac{2}{q}-1}(\theta+\lambda)^{\frac{2}{q}}}\Bigg\{\Big(\tfrac{1}{\theta} - \tfrac{2}{\lambda-\theta}\ln\Big(\tfrac{\theta+\lambda}{2\theta}\Big)\Big)^{1-\frac{1}{q}}\Big[{}_2F_1\Big(2q,1+q;2+q;\tfrac{\lambda-\theta}{\lambda+\theta}\Big) - \tfrac{1}{3}{}_2F_1\Big(2q,2+q;3+q;\tfrac{\lambda-\theta}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}} + \Big(\tfrac{1}{\lambda-\theta}\ln\Big(\tfrac{2\lambda}{\theta+\lambda}\Big) - \tfrac{1}{\lambda}\Big)^{1-\frac{1}{q}}\Big[{}_2F_1\Big(2q,1+q;2+q;\tfrac{\theta-\lambda}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}}\Bigg\}.$$

Proposition 3. *If $\varkappa = \frac{\theta+\lambda}{2}$, and $\mathcal{P}_\imath(\varkappa) = C\varkappa$. as stated in Theorem 5, then for $\lambda > \theta > 0$, the following holds:*

$$\mathcal{D}\Big(A(\theta,\lambda),\ G^2(\theta,\lambda)L^{-1}(\theta,\lambda)\Big)$$

$$\leq_I \frac{\theta\lambda(\lambda-\theta)^{\frac{2-q}{q}}C}{2^{\frac{2-q}{q}}(\theta+\lambda)^2}\Bigg\{\Big[{}_2F_1\Big(2q,2;3;\frac{\lambda-\theta}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}} + \Big[{}_2F_1\Big(2q,2;3;\frac{\theta-\lambda}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}}\Bigg\}.$$

Proposition 4. *If $\varkappa = \frac{\theta+\lambda}{2}$, and $\mathcal{P}_\imath(\varkappa) = C\varkappa$. as stated in Theorem 6, then for $\lambda > \theta > 0$, the following holds:*

$$\mathcal{D}\Big(A(\theta,\lambda),\ G^2(\theta,\lambda)L^{-1}(\theta,\lambda)\Big)$$

$$\leq_I \frac{\theta\lambda(\lambda-\theta)^{\frac{2-q}{q}}C}{2^{\frac{2(1-q)}{q}}(\theta+\lambda)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg\{\Big[{}_2F_1\Big(2q,1;2;\frac{\lambda-\theta}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}} + \Big[{}_2F_1\Big(2q,1;2;\frac{\theta-\lambda}{\lambda+\theta}\Big)\Big]^{\frac{1}{q}}\Bigg\}.$$

Proposition 5. *If $\varkappa = \frac{\theta+\lambda}{2}$, and $\mathcal{P}_\imath(\varkappa) = C\varkappa$. as stated in Theorem 7, then for $\lambda > \theta > 0$, the following holds:*

$$\mathcal{D}\Big(A(\theta,\lambda),\ G^2(\theta,\lambda)L^{-1}(\theta,\lambda)\Big)$$

$$\leq_I \frac{\theta\lambda(\lambda-\theta)^{\frac{2-q}{q}}C}{2^{\frac{2}{q}-2q}(\theta+\lambda)^{2q-2}}\Bigg\{\Big[{}_2F_1\Big(2p,p+1;p+2;\frac{\lambda-\theta}{\lambda+\theta}\Big)\Big]^{\frac{1}{p}}\Bigg\}.$$

5. Conclusions

In this study, utilizing the power mean integral inequality and an enhanced version of it, we establish inequalities for fuzzy-valued mappings whose derivatives at certain powers exhibit convexity in absolute value. Through this approach, we derive a new integral identity for differentiable functions. Numerical experiments demonstrate that the enhanced power mean integral inequality offers a more effective approach compared to the standard power mean integral inequality. Some exceptional cases have been acquired that can be considered as applications of the main results. Additionally, we present applications concerning special means of real numbers and obtain error estimates for the midpoint formula.

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