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Sharp Coefficient Bounds for Starlike Functions Associated with Cosine Function †

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† Dedicated to the memory of Professor Lawrence Allen Zalcman (1943–2022).
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Abstract: Let $\mathcal{S}^*_{\cos}$ denote the class of normalized analytic functions f in the open unit disk $\mathbb{D}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \cos z$. In the first result of this article, we find the sharp upper bounds for the initial coefficients a_3, a_4 and a_5 and the sharp upper bound for module of the Hankel determinant $|H_{2,3}(f)|$ for the functions from the class $\mathcal{S}^*_{\cos}$. The next section deals with the sharp upper bounds of the logarithmic coefficients γ_3 and γ_4. Then, in addition, we found the sharp upper bound for $\left|H_{2,2}\left(F_f/2\right)\right|$. To obtain these results we utilized the very useful and appropriate Lemma 2.4 of N.E. Cho et al., which gave a most accurate description for the first five coefficients of the functions from the Carathéodory's functions class, and provided a technique for finding the maximum value of a three-variable function on a closed cuboid. All the maximum found values were checked by using MAPLE™ computer software, and we also found the extremal functions in each case. All of our most recent results are the best ones and give sharp versions of those recently published by Hacet.

Keywords: starlike functions; subordination; cosine functions; initial coefficient bounds; Hankel determinants; logarithmic coefficients

MSC: 30C45; 30C50; 30C80

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all analytic and normalized functions f having the Maclaurin series expansion as follows

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \ z \in \mathbb{D}, \tag{1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ represents the open unit disk in the complex plane $\mathbb{C}$ and we consider a subclass of $\mathcal{S} \subset \mathcal{A}$ containing all the univalent functions of $\mathcal{A}$.

The subclass of $\mathcal{A}$ defined by

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in \mathbb{D} \right\}$$

is called the class of *starlike functions* (with respect to the origin) in $\mathbb{D}$ and it is well-known that $\mathcal{S}^* \subset \mathcal{S}$. Let $\mathcal{B}$ denote the family of functions w, which are analytic in $\mathbb{D}$, such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$. Such functions are referred to as *Schwarz functions*.

Let us consider two analytic functions, f and g in $\mathbb{D}$. We say that f is subordinated to g, written as $f(z) \prec g(z)$ if there exists a Schwarz function w such that $f(z) = g(w(z))$ for

$z \in \mathbb{D}$. If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$, and if g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Based on the geometric properties of the image of $\mathbb{D}$, by some analytic functions, the functions can be categorized from this point of view into different families. Thus, in 1992, Ma and Minda [1] introduced a generalized subclass of $\mathcal{S}^*$ denoted by $\mathcal{S}^*(\phi)$, which is defined in terms of the subordination as follows

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

where ϕ satisfies the conditions $\phi(0) = 1$ and $\operatorname{Re} \phi'(z) > 0$ in $\mathbb{D}$ and ϕ maps the unit disk $\mathbb{D}$ onto a star-shaped domain. Several subclasses of $\mathcal{S}^*$ can be obtained by varying the function ϕ. For example, if we choose $\phi(z) = \dfrac{1+Lz}{1+Mz}$, $-1 \leq M < L \leq 1$, then we get the class $\mathcal{S}^*[L, M] := \mathcal{S}^*\left(\dfrac{1+Lz}{1+Mz}\right)$, which is called the *Janowski starlike functions class* and was investigated in [2]. Sokól and Stankiewicz [3] defined and studied the class $\mathcal{S}^*_L := \mathcal{S}^*\left(\sqrt{1+z}\right)$, where the function $\phi(z) = \sqrt{1+z}$ maps $\mathbb{D}$ onto the image domain bounded by $\left|w^2 - 1\right| < 1$ (the square functions are considered at the main branch, i.e., those branches with $\sqrt{1} = 1$ or $\log 1 = 0$). The class $\mathcal{S}^*_C := \mathcal{S}^*\left(1 + \dfrac{4}{3}z + \dfrac{2}{3}z^2\right)$ was studied by Sharma [4] and is related to the *cardioid function*.

In the case of $\phi(z) = \cosh z$, the class $\mathcal{S}^*_{\cosh}$ was defined and thoroughly examined in [5], whereas the class $\mathcal{S}^*_{\mathrm{e}} := \mathcal{S}^*(\mathrm{e}^z)$ was presented and studied in [6] by Mendiratta et al. For $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*_{\sin}$ was introduced and investigated in [7,8], while if $\phi(z) = 1 + \tanh z$, the class $\mathcal{S}^*_{tanh}$ was defined and studied in [9]. Recently, the class $\mathcal{S}^*_{\cos}$, defined by

$$\mathcal{S}^*_{\cos} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cos z \right\},$$

was introduced and investigated in [10], and determining sharp results connected with those of [11] represents the main goal of the present paper.

The following part of this section is necessary to understand the motivation of this work. In 1916, Bieberbach presented the well-known *Bieberbach conjecture*, which was proved by de Branges [12] in 1985. Prior to de Branges proof of this conjecture, numerous mathematicians exerted considerable effort to prove it, leading to the establishment of coefficient bounds for some remarkable subclasses of the class $\mathcal{S}$. They also developed several new inequalities related to the coefficient bounds for some subclasses of univalent functions, and those related to the *Fekete-Szegő* functional, that is, $\left|a_3 - \lambda a_2^2\right|$ is one of those inequalities. Another coefficient problem closely related to the Fekete–Szegő functional is the *Hankel determinant*, as we will see below.

As we could see in Pommerenke's paper [13], for a function $f \in \mathcal{A}$, the Hankel determinant $H_{m,n}(f)$ is as follows

$$H_{m,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix},$$

where $m,\ n \geq 1$. We remark that

$$H_{2,1}(f) = a_3 - a_2^2, \quad H_{2,2}(f) = a_2 a_4 - a_3^2, \quad H_{2,3}(f) = a_3 a_5 - a_4^2, \tag{2}$$

$$H_{3,1}(f) = 2a_2 a_3 a_4 - a_4^2 - a_3^3 - a_2^2 a_5 + a_3 a_5, \tag{3}$$

where $|H_{2,1}(f)|$ is the classical Fekete–Szegő functional obtained for $\lambda = 1$. Several authors have determined the maximum value of $|H_{2,1}(f)|$ and the upper bound for $|H_{2,2}(f)|$ for

various subclasses of $\mathcal{A}$; see, for example, [14–16]. Many results regarding the second Hankel determinant $H_{2,3}(f)$ could also be found in recent papers like [17–19]. The determinant $H_{2,3}(f)$ has not been studied much in the literature: Babalola [20] studied, for the first time, non-sharp bounds for the determinant $H_{3,1}(f)$ for various subclasses of $\mathcal{S}$, while in 2017, Zaprawa [21] improved the results of Babalola by using a new technique. We mention that the sharp bounds of the modulus of $H_{3,1}(f)$ for the class $\mathcal{S}^*$ were recently obtained by Kowalczyk et al. [22], whereas for the class $\mathcal{C}$ and bounded turning functions, sharp bounds were obtained in [23,24], respectively. For sharp inequalities results for the determinant $H_{3,1}(f)$ for some subclasses of $\mathcal{S}^*$, we refer to [18,25–28].

Very recently, Marimuthu et al. [11] have determined the coefficient bounds, the upper bounds for the second, third, and fourth-order Hankel determinants for the functions of the class $\mathcal{S}^*_{\cos}$, but most of the results presented in this paper are not sharp.

Motivated by the above-mentioned study, in this paper, we have established the sharp results for the upper bounds of the coefficients and the logarithmic coefficients of the functions of the class $\mathcal{S}^*_{\cos}$. We have also developed the sharp upper bounds for the modulus of the second- and third-order Hankel determinants for the functions of this class.

The well-known *Carathéodory class* $\mathcal{P}$ is the family of holomorphic functions h in $\mathbb{D}$, which satisfy the condition $\operatorname{Re} h(z) > 0$, $z \in \mathbb{D}$, with the power series expansion of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \ z \in \mathbb{D}. \tag{4}$$

The study of some coefficient problems in different classes of analytic functions revolves around the idea of expressing function coefficients in a given class by function coefficients that have a positive real part. Thus, the known inequalities for the class $\mathcal{P}$ can be used to study coefficient functionals. We require the following results on the class $\mathcal{P}$ for our next proofs.

We will recall the well-known *Carathéodory lemma* [29] (see also [30] (Corollary 2.3, p. 41), [31] (Carathéodory's Lemma, p. 41)):

Lemma 1. *If $h \in \mathcal{P}$ has the form (4), then*

$$|c_n| \leq 2, \ n \geq 1. \tag{5}$$

The inequality holds for all $n \geq 1$ if and only if $h(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.

The next result represents the relations (2.7), (2.8), and (2.9) of Lemma 2.4 from [32]:

Lemma 2 ([32] (Lemma 2.4))**.** *Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk, and $h \in \mathcal{P}$ be given by (4). Then,*

$$\begin{aligned}
c_1 &= 2t_1, \\
c_2 &= 2t_1^2 + 2\left(1 - |t_1|^2\right)t_2, \\
c_3 &= 2t_1^3 + 4\left(1 - |t_1|^2\right)t_1 t_2 - 2\left(1 - |t_1|^2\right)\overline{t_1}t_2^2 + 2\left(1 - |t_1|^2\right)\left(1 - |t_2|^2\right)t_3,
\end{aligned}$$

for some $t_k \in \overline{\mathbb{D}}$ and $k \in \{1, 2, 3\}$.

Note that the extension of this lemma for the coefficients c_4 and c_5 may also be found in Lemma 2.1 of [33].

The next lemma represents the first part of the result from [1] (Remark p. 162).

Lemma 3. *If $h \in \mathcal{P}$ is given by (4), then*

$$\left|c_2 - \nu c_1^2\right| \leq 2 - \nu|c_1|^2,\ 0 < \nu \leq \frac{1}{2}.$$

The main novelty is found in the tools we used, that is, Lemma 2, which offers the best known estimate regarding the first five coefficients of the functions of the class $\mathcal{P}$. Combining with the technique for determining the extremal values of a three-variable function on a compact cuboid, we found the best upper bounds of $|a_3|$, $|a_4|$, $|a_5|$ and of the Hankel determinants $|H_{3,1}(f)|$, $|H_{2,3}(f)|$. In addition, the sharp upper bounds of the logarithmic coefficients $|\gamma_3|$ and $|\gamma_4|$, combined with those of $\left|H_{2,2}\left(F_f/2\right)\right|$, are presented in Section 3.

The main strong point is the fact that all the results are the best possible, and we gave the extremal function where the equalities are obtained. Since Lemma 2 deals only with the first five coefficients of the class $\mathcal{P}$, we believe that the importance of this lemma is remarkable for results involving these coefficients. Unfortunately, it is our opinion that it is hard to find similar results for higher index coefficients because of the computational difficulties, at least in the first steps of the proofs. Thus, the limitations of the below methods include the fact that the order of the Hankel determinant and the indexes of the coefficients or logarithmical coefficients cannot be higher than the present ones. Definitively, our results do not represent general investigation methods for these types of problems.

2. Initial Coefficients Sharp Upper Bounds

The next main results give us the sharp upper bounds for the initial coefficients of the functions from the class $\mathcal{S}^*_{\cos}$.

Theorem 1. *Let $f \in \mathcal{S}^*_{\cos}$ be given by (1). Then,*

$$|a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{2}{27}\sqrt{3}, \quad |a_5| \leq \frac{1}{8},$$

and these bounds are sharp.

Proof. If $f \in \mathcal{S}^*_{\cos}$, then by the definition of subordination there exists a Schwarz function w that is analytic in $\mathbb{D}$ and satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that

$$\frac{zf'(z)}{f(z)} = \cos w(z),\ z \in \mathbb{D}. \tag{6}$$

Therefore, the function h defined by

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots,\ z \in \mathbb{D}, \tag{7}$$

which has the property $h \in \mathcal{P}$.

Using the relations (6) and (7) and by equating the first four coefficients, we get

$$a_2 = 0, \quad a_3 = -\frac{1}{16}c_1^2, \quad a_4 = \frac{1}{24}c_1^3 - \frac{1}{12}c_1 c_2, \tag{8}$$

$$a_5 = -\frac{1}{48}c_1^4 - \frac{1}{32}c_2^2 + \frac{3}{32}c_1^2 c_2 - \frac{1}{16}c_1 c_3. \tag{9}$$

(i) From the second relation of (8), according to the inequality (5), it follows that

$$|a_3| \leq \frac{1}{4},$$

and this inequality is attained for the function f_* from Remark 2.2 of [11]. Thus, the above upper bound is sharp, which is the best possible.

(ii) To find the upper bound of $|a_4|$, we see that the third equality of (8) could be written in the form

$$a_4 = -\frac{c_1}{12}\left(c_2 - \frac{c_1^2}{2}\right),$$

and using Lemma 3 for $\nu = \frac{1}{2}$ we obtain

$$|a_4| \leq \frac{|c_1|}{12}\left(2 - \frac{|c_1|^2}{2}\right).$$

Denoting $c := |c_1|$, from Lemma 1 we have $0 \leq c \leq 2$, hence

$$|a_4| \leq \frac{c}{12}\left(2 - \frac{c^2}{2}\right) =: F(c),\ c \in [0,2]. \tag{10}$$

It is easy to check that the function F attained the maximum value at $c = \frac{2}{\sqrt{3}}$; then, according to the above inequality, we get

$$|a_4| \leq F\left(\frac{2}{\sqrt{3}}\right) = \frac{2}{27}\sqrt{3}.$$

To prove the sharpness of this upper bound, let $\widehat{t} = \frac{2}{\sqrt{3}}$, $h_1(z) = \frac{1-z^2}{1-\widehat{t}z+z^2}$ and

$$w_1(z) = \frac{h_1(z)-1}{h_1(z)+1} = \frac{z(3z-\sqrt{3})}{\sqrt{3}z-3},\ z \in \mathbb{D},$$

thus, $w_1(0) = 0$. To show that $|w_1(z)| < 1$ in $\mathbb{D}$, we remark that the function w_1 could be written in the form $w_1(z) = z\varphi(z)$, where

$$\varphi(z) = \frac{3z-\sqrt{3}}{\sqrt{3}z-3},\ z \in \mathbb{D}.$$

Since $|\varphi(z)| < 1$, $z \in \mathbb{D}$, if and only if $\operatorname{Re} H(z) > 0$ for all $z \in \mathbb{D}$, where

$$H(z) := \frac{1-\varphi(z)}{1+\varphi(z)} = \frac{\left(\sqrt{3}-3\right)(z+1)}{\left(\sqrt{3}+3\right)(z-1)},\ z \in \mathbb{D},$$

is a circular transform. It is easy to check that

$$H(0) = \frac{3-\sqrt{3}}{\sqrt{3}+3} > 0,\quad H(-1) = 0,\quad H(i) = \frac{\left(3-\sqrt{3}\right)i}{\sqrt{3}+3},\quad H(-i) = \frac{\left(\sqrt{3}-3\right)i}{\sqrt{3}+3}.$$

Using the fact that every circular transform maps the circles (in the large sense, circles or lines) of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ into circles of $\mathbb{C}_\infty$, from the above values of H it follows that $\operatorname{Re} H(z) > 0$ for all $z \in \mathbb{D}$, which implies $|\varphi(z)| < 1$ in $\mathbb{D}$, hence

$$|w_1(z)| = |z|\,|\varphi(z)| < 1,\ z \in \mathbb{D}.$$

Therefore, the function

$$f_1(z) := z\exp\left(\int_0^z \frac{\cos w_1(t)-1}{t}\,dt\right) = z - \frac{1}{12}z^3 + \frac{2}{27}\sqrt{3}z^4 + \ldots,\ z \in \mathbb{D}, \tag{11}$$

belongs to the class $\mathcal{S}^*_{\cos}$; hence, $|a_4| = \frac{2}{27}\sqrt{3}$ for the above function f_1, which proves the sharpness of the second inequality of this theorem.

(iii) To determine the upper bound of $|a_5|$, by using the relation (9) combined with Lemma 2, we obtain

$$a_5 = \frac{1}{96}\left[4t_1^4 + 24|t_1|^2t_2^2\left(1-|t_1|^2\right) - 12t_2^2\left(1-|t_1|^2\right)^2 - 24t_1\left(1-|t_1|^2\right)\left(1-|t_2|^2\right)t_3\right].$$

Setting $x := |t_1| \in [0,1]$, $y := |t_2| \in [0,1]$ and $u := |t_3| \in [0,1]$, from the triangle inequality we obtain

$$\begin{aligned}|a_5| &\le \frac{1}{24}\left[x^4 + 6x^2\left(1-x^2\right)y^2 + 3\left(1-x^2\right)^2y^2 + 6x\left(1-x^2\right)\left(1-y^2\right)u\right]\\ &= \frac{1}{24}F(x,y,u),\ x,y,u \in [0,1],\end{aligned} \tag{12}$$

where

$$F(x,y,u) := x^4 + 6x^2\left(1-x^2\right)y^2 + 3\left(1-x^2\right)^2y^2 + 6x\left(1-x^2\right)\left(1-y^2\right)u. \tag{13}$$

Denoting by $\Omega := [0,1]\times[0,1]\times[0,1]$ the closed unit cuboid, we will find the maximum value of F in Ω.

I. First, let us consider that (x,y,u) belongs to the interior of Ω, denoted by $\operatorname{int}\Omega$. Differentiating (13) with respect to u, we obtain

$$\frac{\partial F(x,y,u)}{\partial u} = 6x\left(1-x^2\right)\left(1-y^2\right) \neq 0,\ (x,y,u) \in \operatorname{int}\Omega,$$

therefore, the function F has no extremal values in $\operatorname{int}\Omega$.

II. Next, we will discuss the existence of the maximum value for F on the open six faces of Ω, as follows.

(i) On the face $x = 0$, the next inequality holds

$$F(0,y,u) = 3y^2 < 3,\ y,u \in (0,1). \tag{14}$$

(ii) On the face $x = 1$, we have the equality

$$F(1,y,u) = 1,\ y,u \in (0,1). \tag{15}$$

(iii) On the plane $y = 0$, let us denote

$$\kappa_1(x,u) := F(x,0,u) = x^4 + 6\left(1-x^2\right)xu,\ x,u \in (0,1), \tag{16}$$

and because

$$\frac{\partial \kappa_1(x,u)}{\partial u} = 6x\left(1-x^2\right) \neq 0,\ x \in (0,1),$$

it follows that the function κ_1 has no extremal values on $(0,1)\times(0,1)$.

(iv) On the open face $y = 1$, the function F reduces to

$$s_1(x) := F(x,1,u) = x^4 + 6x^2\left(1-x^2\right) + 3\left(1-x^2\right)^2,\ x \in (0,1). \tag{17}$$

Since $s_1'(x) = -8x^3 < 0$, $x \in (0,1)$, therefore s_1 is a strictly decreasing function on $(0,1)$,

$$F(x,1,u) < 3,\ x,u \in (0,1). \tag{18}$$

(v) For the open face $u = 0$, we get

$$\kappa_2(x,y) := F(x,y,0) = x^4 + 6x^2\left(1-x^2\right)y^2 + 3\left(1-x^2\right)^2 y^2,\ x,y \in (0,1),$$

therefore,

$$\frac{\partial \kappa_2(x,y)}{\partial x} = -4x^3\left(3y^2-1\right),\quad \frac{\partial \kappa_2(x,y)}{\partial y} = -6(x-1)(x+1)y\left(x^2+1\right),$$

hence the system of equations $\frac{\partial \kappa_2(x,y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x,y)}{\partial y} = 0$ has no solutions in $(0,1) \times (0,1)$.

(vi) On the open face $u = 1$, the function F becomes

$$\kappa_3(x,y) := x^4 + 6x^2\left(1-x^2\right)y^2 + 3y^2\left(1-x^2\right)^2 + 6x\left(1-x^2\right)\left(1-y^2\right),\ x,y \in (0,1),$$

and it is easy to check that the system of equations $\frac{\partial \kappa_3(x,y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x,y)}{\partial y} = 0$ has no solutions in $(0,1) \times (0,1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω.

(i) From (13), we obtain

$$F(x,0,0) = x^4 \le 1,\ x \in [0,1].$$

(ii) Also, from (16) we get $s_2(x) := F(x,0,1) = x^4 + 6x\left(1-x^2\right)$. We may easily see that the zero of s_2' in $[0,1]$ given by $x_0 = 0.621924\ldots$ satisfies $s_2''(x_0) < 0$, consequently

$$F(x,0,1) \le F(x_0,0,1) = 2.437828\ldots,\ x \in [0,1].$$

(iii) Putting $x = 0$ in (16) we have

$$F(0,0,u) = 0,\ u \in [0,1].$$

(iv) Since (17) is independent of u, similar to the inequality (18), we deduce

$$F(x,1,1) = F(x,1,0) \le 3,\ x \in [0,1],$$

and putting $x = 0$ in (17), we obtain

$$F(0,1,u) = 3,\ u \in [0,1].$$

(v) The function given by (15) is independent of the variables y and u, thus,

$$F(1,y,0) = F(1,y,1) = F(1,0,u) = F(1,1,u) = 1,\ y,u \in [0,1].$$

(vi) Since the function defined by (14) is independent of the variable u, we have

$$F(0,y,0) = F(0,y,1) \le 3,\ y \in [0,1].$$

For all the above reasons, we conclude that

$$\max\{F(x,y,u) : (x,y,u) \in \Omega\} = F(0,1,u) = 3,$$

and, according to (12), we finally obtain that $|a_5| \leq \frac{1}{8}$. This upper bound for $|a_5|$ is sharp for the function

$$f_2(z) := z \exp\left(\int_0^z \frac{\cos(t^2) - 1}{t} dt\right) = z - \frac{1}{8}z^5 + \ldots, \ z \in \mathbb{D}, \tag{19}$$

which completes our proof. □

Remark 1. *1. In [11] it was proved that* $|a_4| \leq \frac{1}{3}$ *and* $|a_5| \leq \frac{11}{24}$*, and the authors gave in Remark 2.2 a function of the class* $\mathcal{S}^*_{\cos}$*, for which* $|a_4| = 0$ *and* $|a_5| = \frac{1}{24}$*. In the above theorem, all the results are sharp and we gave the best upper bounds.*

2. The maximum value of the function F defined by (13) could be easily found using the MAPLE™ 2016 software with the code

```
[> maximize(F,x=0..1,y=0..1,u=0..1,location=true),
```

which gives the same result as above.

In the following two theorems, we determined the sharp upper bounds for the Hankel determinants $|H_{3,1}|$ and $|H_{2,3}|$, respectively, over the class $\mathcal{S}^*_{\cos}$.

Theorem 2. *If* $f \in \mathcal{S}^*_{\cos}$ *is given by (1), then*

$$|H_{3,1}(f)| \leq -\frac{3115}{164{,}268} + \frac{671}{657{,}072}\sqrt{1342} = 0.018446\ldots,$$

and this result is sharp.

Proof. Using the relations (8) and (9) in (3), we obtain

$$H_{3,1}(f) = -\frac{7}{36{,}864}c_1^6 + \frac{5}{4608}c_1^4c_2 + \frac{1}{256}c_1^3c_3 - \frac{23}{4608}c_1^2c_2^2,$$

and replacing all the variables of the above relation with those of Lemma 2, it follows that

$$H_{3,1}(f) = \frac{1}{576}\Big[3t_1^6 - 36t_1^2|t_1|^2t_2^2\left(1-|t_1|^2\right) - 46t_1^2t_2^2\left(1-|t_1|^2\right)^2 + 36t_1^3\left(1-|t_1|^2\right)\left(1-|t_2|^2\right)t_3\Big].$$

If we set $x := |t_1| \in [0,1]$, $y := |t_2| \in [0,1]$ and $u := |t_3| \in [0,1]$, using the above relation and the triangle inequality, we get

$$\begin{aligned}|H_{3,1}(f)| &\leq \frac{1}{576}\left[3x^6 + 36x^4\left(1-x^2\right)y^2 + 46x^2\left(1-x^2\right)^2y^2 + 36x^3\left(1-x^2\right)\left(1-y^2\right)u\right] \\ &= \frac{1}{576}F(x,y,u), \ x,y,u \in [0,1],\end{aligned} \tag{20}$$

where

$$F(x,y,u) := 3x^6 + 36x^4\left(1-x^2\right)y^2 + 46x^2\left(1-x^2\right)^2y^2 + 36x^3\left(1-x^2\right)\left(1-y^2\right)u. \tag{21}$$

With the same notations and method as in the proof of Theorem 1, next we will find the maximum value of F in Ω.

I. If we consider that $(x, y, u) \in \operatorname{int}\Omega$, differentiating (21) with respect to u, we obtain

$$\frac{\partial F(x,y,u)}{\partial u} = 36x^3\Big(1-x^2\Big)\Big(1-y^2\Big) \neq 0,\ (x,y,u) \in \operatorname{int}\Omega,$$

hence, it follows that the function F has no maximum value in $\operatorname{int}\Omega$.

II. In the sequel, we will study the existence of the maximum value of F in the interior of six faces of Ω.

(i) On the face $x = 0$, we have

$$F(0,y,u) = 0,\ y,u \in (0,1). \tag{22}$$

(ii) On the face $x = 1$, we get

$$F(1,y,u) = 3,\ y,u \in (0,1). \tag{23}$$

(iii) On the plane $y = 0$, the function F can be written as

$$\kappa_1(x,u) := F(x,0,u) = 3x^6 + 36x^3\Big(1-x^2\Big)u,\ x,u \in (0,1). \tag{24}$$

Since

$$\frac{\partial \kappa_1(x,u)}{\partial u} = 36x^3\Big(1-x^2\Big) \neq 0,\ x \in (0,1),$$

it implies that the function κ_1 has no maximum points in the face of Ω.

(iv) On $y = 1$ the function F reduces to

$$s_1(x) := F(x,1,u) = 3x^6 + 36x^4\Big(1-x^2\Big) + 46x^2\Big(1-x^2\Big)^2,\ x \in (0,1). \tag{25}$$

Since $s_1'(x) = 78x^5 - 224x^3 + 92x,\ x \in (0,1)$, it follows that $s_1'(x) = 0$ has the zero $x_0 = \dfrac{\sqrt{2184 - 39\sqrt{1342}}}{39} = 0.704685\ldots \in (0,1)$ that satisfy the inequality $s_1''(x_0) < 0$. Therefore, we obtain

$$F(x,1,u) \leq -\frac{49{,}840}{4563} + \frac{2684}{4563}\sqrt{1342} = 10.625427\ldots,\ x,u \in (0,1). \tag{26}$$

(v) On the face $u = 0$, the function F becomes

$$\kappa_2(x,y) := F(x,y,0) = 3x^6 + 36x^4(1-x^2)y^2 + 46x^2(1-x^2)^2y^2,\ x,y \in (0,1),$$

thus,

$$\frac{\partial \kappa_2(x,y)}{\partial x} = \Big(60y^2+18\Big)x^5 - 224x^3y^2 + 92x^2, \quad \frac{\partial \kappa_2(x,y)}{\partial y} = 4x^2y\Big(5x^4 - 28x^2 + 23\Big).$$

It is easy to see that the system of equations $\dfrac{\partial \kappa_2(x,y)}{\partial x} = 0$ and $\dfrac{\partial \kappa_2(x,y)}{\partial y} = 0$ has no solutions in $(0,1)\times(0,1)$.

(vi) On $u = 1$ the function F takes the form

$$\kappa_3(x,y) = 3x^6 + 36x^4(1-x^2)y^2 + 46x^2(1-x^2)^2y^2 + 36x^3(1-x^2)(1-y^2),\ x,y \in (0,1).$$

Similarly, the system of equations $\dfrac{\partial \kappa_3(x,y)}{\partial x} = 0$ and $\dfrac{\partial \kappa_3(x,y)}{\partial y} = 0$ has no solutions in $(0,1)\times(0,1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω.

(i) From (24) we obtain

$$F(x,0,0) = 3x^6 \leq 3,\ x \in [0,1].$$

(ii) The relation (24) at $u = 1$ becomes $s_2(x) := F(x,0,1) = 3x^6 + 36x^3(1-x^2)$, $x \in [0,1]$. The solution of $s_2'(x) = 18x^2\left(x^3 - 10x^2 + 6\right) = 0$ is $x_0 = 0.807920\ldots \in [0,1]$ and satisfies $s_2''(x_0) < 0$, hence

$$F(x,0,1) \leq F(x_0,0,1) = 7.427102\ldots,\ x \in [0,1].$$

(iii) Putting $x = 0$ in (24), we have

$$F(0,0,u) = 0,\ u \in [0,1]$$

(iv) Since (25) is independent of u, similarly as above, we obtain

$$F(x,1,1) = F(x,1,0) \leq -\frac{49{,}840}{4563} + \frac{2684}{4563}\sqrt{1342},\ x \in [0,1],$$

while if we take $x = 0$ in (25) it follows

$$F(0,1,u) = 0,\ u \in [0,1].$$

(v) The relation (23) is independent of the variables $y, u \in [0,1]$, hence

$$F(1,y,0) = F(1,y,1) = F(1,0,u) = F(1,1,u) = 3,\ y,u \in [0,1].$$

(vi) Finally, since (22) is independent of the variable $u \in [0,1]$, we have

$$F(0,y,0) = F(0,y,1) = 0,\ y \in [0,1].$$

All the inequalities we obtained above show that

$$\max\{F(x,y,u) : (x,y,u) \in \Omega\} = F\left(\frac{\sqrt{2184 - 39\sqrt{1342}}}{39}, 1, u\right)$$
$$= -\frac{49{,}840}{4563} + \frac{2684}{4563}\sqrt{1342} = 10.625427\ldots,$$

and using (20), our inequality is proved.

For proving the sharpness of this inequality, let us consider $\widetilde{t} = \dfrac{\sqrt{2184 - 39\sqrt{1342}}}{39}$, $h_3(z) = \dfrac{1 + (1+i)\widetilde{t}z + iz^2}{1 - (1-i)\widetilde{t}z - iz^2}$ and

$$w_3(z) = \frac{h_2(z) - 1}{h_2(z) + 1} = \frac{z\left(\sqrt{2184 - 39\sqrt{1342}} + 39iz\right)}{39 + \sqrt{2184 - 39\sqrt{1342}}iz},\ z \in \mathbb{D}.$$

We can see that $w_3(0) = 0$, and let us write the function w_3 as $w_3(z) = z\psi(z)$, where

$$\psi(z) = \frac{\sqrt{2184 - 39\sqrt{1342}} + 39iz}{39 + \sqrt{2184 - 39\sqrt{1342}}iz},\ z \in \mathbb{D}.$$

For the same reasons regarding the circular transforms as in the proof of the sharpness of Theorem 1 item (ii), we will show that $|\psi(z)| < 1$ in $\mathbb{D}$ by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1-\psi(z)}{1+\psi(z)} = \frac{\left(\sqrt{2184-39\sqrt{1342}}-39\right)(z+i)}{\left(\sqrt{2184-39\sqrt{1342}}+39\right)(-z+i)},\ z \in \mathbb{D},$$

is a circular transform. Since

$$H(0) = \frac{-\sqrt{2184-39\sqrt{1342}}+39}{\sqrt{2184-39\sqrt{1342}}+39} = 0.173236\ldots > 0, \quad H(-i) = 0,$$
$$H(-1) = \frac{-\left(\sqrt{2184-39\sqrt{1342}}-39\right)i}{\sqrt{2184-39\sqrt{1342}}+39}, \quad H(1) = \frac{\left(\sqrt{2184-39\sqrt{1342}}-39\right)i}{\sqrt{2184-39\sqrt{1342}}+39},$$

as in the above-mentioned proof, these values of H lead us to $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, which implies $|\psi(z)| < 1$ in $\mathbb{D}$; therefore,

$$|w_3(z)| = |z|\,|\psi(z)| < 1,\ z \in \mathbb{D}.$$

Thus, the function

$$f_3(z) = z\exp\left(\int_0^z \frac{\cos w_3(t)-1}{t}dt\right) = z + \left(-\frac{14}{39}+\frac{\sqrt{1342}}{156}\right)z^3$$
$$-\frac{1}{4563}\left(-17+\sqrt{1342}\right)\sqrt{2184-39\sqrt{1342}}z^4 + \left(\frac{23{,}135}{36{,}504}-\frac{163\sqrt{1342}}{9126}\right)z^5+\ldots,\ z \in \mathbb{D},$$

belongs to the class $\mathcal{S}^*_{\cos}$, with the initial coefficients

$$a_2 = 0, \quad a_3 = -\frac{14}{39}+\frac{\sqrt{1342}}{156},$$
$$a_4 = -\frac{1}{4563}\left(\left(-17+\sqrt{1342}\right)\sqrt{2184-39\sqrt{1342}}\right)i, \quad a_5 = \frac{23{,}135}{36{,}504}-\frac{163\sqrt{1342}}{9126}.$$

From the relation (3), we get $|H_{3,1}(f_3)| = -\dfrac{3115}{164{,}268}+\dfrac{671}{657{,}072}\sqrt{1342} = 0.018446\ldots$ and the proof is complete. □

Remark 2. *In [11], it is proved that* $|H_{3,1}(f)| \le \dfrac{139}{576} = 0.241319\ldots$ *for all* $f \in \mathcal{S}^*_{\cos}$, *but that result was not the best possible. If we compare the upper bounds of* $|H_{3,1}(f)|$ *for* $f \in \mathcal{S}^*_{\cos}$, *obtained here with those of [11], the result of Theorem 2 is a significant improvement of the previous one. Moreover, the inequality obtained in above theorem is sharp, thus the found upper bound for* $|H_{3,1}(f)|$ *if* $f \in \mathcal{S}^*_{\cos}$ *cannot be improved.*

Theorem 3. *If* $f \in \mathcal{S}^*_{\cos}$ *has the form* (1), *then*

$$|H_{2,3}(f)| \le \frac{77}{15{,}552}+\frac{29}{15{,}552}\sqrt{58} = 0.019152\ldots,$$

and the result is sharp.

Proof. Replacing in (2) the values of a_2, a_3, a_2, and a_5 given by (8) and (9), we obtain

$$H_{2,3}(f) = -\frac{1}{2304}c_1^6+\frac{5}{4608}c_1^4c_2+\frac{1}{256}c_1^3c_3-\frac{23}{4608}c_1^2c_2^2.$$

According to Lemma 2, from the above relation we deduce that

$$H_{2,3}(f) = \frac{1}{288}\Big[-3t_1^6 - 18t_1^2|t_1|^2t_2^2\Big(1-|t_1|^2\Big) - 23t_1^2t_2^2\Big(1-|t_1|^2\Big)^2 + 18t_1^3\Big(1-|t_1|^2\Big)\Big(1-|t_2|^2\Big)t_3\Big].$$

Setting $x := |t_1| \in [0,1]$, $y := |t_2| \in [0,1]$ and $u := |t_3| \in [0,1]$, then, using the triangle's inequality, the above relation leads us to

$$\begin{aligned}|H_{2,3}(f)| &\leq \frac{1}{288}\Big[3x^6 + 18x^4\Big(1-x^2\Big)y^2 + 23x^2\Big(1-x^2\Big)^2y^2 + 18x^3\Big(1-x^2\Big)\Big(1-y^2\Big)u\Big] \\ &= \frac{1}{288}F(x,y,u),\ x,y,u \in [0,1],\end{aligned} \tag{27}$$

where

$$F(x,y,u) := 3x^6 + 18x^4\Big(1-x^2\Big)y^2 + 23x^2\Big(1-x^2\Big)^2y^2 + 18x^3\Big(1-x^2\Big)\Big(1-y^2\Big)u. \tag{28}$$

With the same notations as those in the proofs of the two previous theorems, we have to find the maximum value of F on $\operatorname{int}\Omega$, on the six faces, and on the twelve edges of Ω.

I. First, we consider the arbitrary interior point $(x,y,u) \in \operatorname{int}\Omega$. Differentiating (28) with respect to u, we obtain

$$\frac{\partial F(x,y,u)}{\partial u} = 18x^3\Big(1-x^2\Big)\Big(1-y^2\Big) \neq 0,\ (x,y,u) \in \operatorname{int}\Omega,$$

therefore, we have no maximum value of F in $\operatorname{int}\Omega$.

II. Next, we will study the existence of the maximum value of the function F in the interior of six faces of Ω.

(i) On the face $x = 0$, the function F reduces to

$$F(0,y,u) = 0,\ y,u \in (0,1). \tag{29}$$

(ii) On the face $x = 1$, it takes the form

$$F(1,y,u) = 3,\ y,u \in (0,1). \tag{30}$$

(iii) On $y = 0$, the function F can be written as

$$\kappa_1(x,u) := F(x,0,u) = 3x^6 + 18x^3\Big(1-x^2\Big)u,\ x,u \in (0,1). \tag{31}$$

Since

$$\frac{\partial \kappa_1(x,u)}{\partial u} = 18x^3\Big(1-x^2\Big) \neq 0,\ x \in (0,1),$$

it follows that κ_1 has no maximum point in this face of Ω.

(iv) On $y = 1$, the function F reduces to

$$s_1(x) := F(x,1,u) = 3x^6 + 18x^4\Big(1-x^2\Big) + 23x^2\Big(1-x^2\Big)^2,\ x \in (0,1), \tag{32}$$

thus,

$$s_1'(x) = 48x^5 - 112x^3 + 46x,\ x \in (0,1).$$

Hence $s_1'(x) = 0$ has the zero $x_0 = \dfrac{\sqrt{42-3\sqrt{58}}}{6} = 0.729396\ldots \in (0,1)$ and $s_1''(x_0) < 0$. Therefore,

$$F(x,1,u) \le F(x_0,1,u) = \frac{77}{54} + \frac{29}{54}\sqrt{58} = 5.515878\ldots,\ x,u \in (0,1). \tag{33}$$

(v) On the face $u = 0$, the function F becomes

$$\kappa_2(x,y) := F(x,y,0) = 3x^6 + 18x^4\left(1-x^2\right)y^2 + 23x^2\left(1-x^2\right)^2y^2,\ x,y \in (0,1).$$

Therefore,

$$\frac{\partial\kappa_2(x,y)}{\partial x} = 2x\left(15x^4y^2 + 9x^4 - 56x^2y^2 + 23y^2\right),\quad \frac{\partial\kappa_2(x,y)}{\partial y} = 2x^2y\left(5x^4 - 28x^2 + 23\right),$$

and the system of equations $\dfrac{\partial\kappa_2(x,y)}{\partial x} = 0$ and $\dfrac{\partial\kappa_2(x,y)}{\partial y} = 0$ has no solutions in $(0,1)\times(0,1)$.

(vi) On $u = 1$, the function F takes the form

$$\kappa_3(x,y) = 3x^6 + 18x^4\left(1-x^2\right)y^2 + 23x^2\left(1-x^2\right)^2y^2 + 18x^3\left(1-x^2\right)\left(1-y^2\right),\ x,y \in (0,1),$$

hence,

$$\begin{aligned}\frac{\partial\kappa_3(x,y)}{\partial x} &= 6\left(5y^2+3\right)x^5 - 90\left(1-y^2\right)x^4 - 112x^3y^2 + 54\left(1-y^2\right)x^2 + 46xy^2,\\ \frac{\partial\kappa_3(x,y)}{\partial y} &= 10(x-1)^2(x+1)y\left(x+\frac{23}{5}\right)x^2.\end{aligned}$$

Therefore, the system of equations $\dfrac{\partial\kappa_3(x,y)}{\partial x} = 0$ and $\dfrac{\partial\kappa_3(x,y)}{\partial y} = 0$ has no solutions in $(0,1)\times(0,1)$.

III. Now we investigate the maximum of F on the edges of Ω.

(i) From (31), we obtain

$$F(x,0,0) = 3x^6 \le 3,\ x \in [0,1].$$

(ii) Also, from (31) at $u = 1$ we get $s_2(x) := F(x,0,1) = 3x^6 + 18x^3\left(1-x^2\right)$. The solution in $[0,1]$ of the equation $s_2'(x) = 18x^5 - 90x^4 + 54x^2 = 0$ for which $s_2''(x) < 0$ is $x_0 = 0.850256\ldots \in [0,1]$, thus

$$F(x,0,1) \le F(x_0,0,1) = 4.199005\ldots,\ x \in [0,1].$$

(iii) Putting $x = 0$ in (31) we have

$$F(0,0,u) = 0,\ u \in [0,1].$$

(iv) Since (32) is independent of u, according to (33), we obtain

$$F(x,1,1) = F(x,1,0) \le \frac{77}{54} + \frac{29}{54}\sqrt{58},\ x \in [0,1].$$

(v) If we take $x = 0$ in (32), we get

$$F(0,1,u) = 0,\ u \in [0,1].$$

Using the fact that the relation (30) is independent of the variables y and u, we deduce

$$F(1,y,0) = F(1,y,1) = F(1,0,u) = F(1,1,u) = 3,\ y,u \in [0,1].$$

(vi) Since the relation (29) is independent of the variable u, we have

$$F(0,y,0) = F(0,y,1) = 0,\ y \in [0,1].$$

Consequently, for the above reasons, we conclude that

$$\max\{F(x,y,u) : (x,y,u) \in \Omega\} = F\left(\frac{\sqrt{42-3\sqrt{58}}}{6}, 1, u\right)$$
$$= \frac{77}{54} + \frac{29}{54}\sqrt{58} = 5.515878\ldots,$$

and, combining with (27), it follows that

$$|H_{2,3}(f)| \leq \frac{77}{15{,}552} + \frac{29}{15{,}552}\sqrt{58} = 0.019152\ldots.$$

To prove the sharpness of the above result, let us consider $t^* = \frac{1}{3}\sqrt{42-3\sqrt{58}}$, $h_4(z) = \frac{1+t^*z+z^2}{1-z^2}$ and

$$w_4(z) = \frac{h_4(z)-1}{h_4(z)+1} = \frac{z\left(\sqrt{42-3\sqrt{58}}+6z\right)}{6+\sqrt{42-3\sqrt{58}}z},\ z \in \mathbb{D}.$$

First, $w_4(0) = 0$ and we will denote $w_4(z) = z\chi(z)$, where

$$\chi(z) = \frac{\sqrt{42-3\sqrt{58}}+6z}{6+\sqrt{42-3\sqrt{58}}z},\ z \in \mathbb{D}.$$

Using the same property of the circular transforms as in the proof of the sharpness of Theorem 1 item (ii) and Theorem 2, we will show that $|\chi(z)| < 1$ in $\mathbb{D}$ by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1-\chi(z)}{1+\chi(z)} = \frac{\left(\sqrt{42-3\sqrt{58}}-6\right)(z-1)}{\left(\sqrt{42-3\sqrt{58}}+6\right)(z+1)},$$

is a circular transform. Computing the below values

$$H(0) = -\frac{\sqrt{42-3\sqrt{58}}-6}{\sqrt{42-3\sqrt{58}}+6} = 0.156472\ldots > 0, \quad H(1) = 0,$$
$$H(i) = \frac{\left(\sqrt{42-3\sqrt{58}}-6\right)i}{\sqrt{42-3\sqrt{58}}+6}, \quad H(-i) = -\frac{\left(\sqrt{42-3\sqrt{58}}-6\right)i}{\sqrt{42-3\sqrt{58}}+6},$$

for similar reasons as in the above-mentioned proofs, these values of H impliy $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, which yields $|\chi(z)| < 1$ in $\mathbb{D}$, therefore

$$|w_4(z)| = |z|\,|\chi(z)| < 1,\ z \in \mathbb{D}.$$

Consequently, the function

$$f_4(z) = z\exp\left(\int_0^z \frac{\cos w_4(t)-1}{t}dt\right) = z + \left(-\frac{7}{24}+\frac{1}{48}\sqrt{58}\right)z^3 - \frac{1}{216}\left(-2+\sqrt{58}\right)\sqrt{42-3\sqrt{58}}z^4 + \left(-\frac{7}{54}+\frac{5}{216}\sqrt{58}\right)z^5+\ldots,\ z\in\mathbb{D},$$

belongs to the class $\mathcal{S}^*_{\cos}$. In the above power series expansion, we have $a_3 = -\frac{7}{24}+\frac{1}{48}\sqrt{58}$, $a_4 = -\frac{1}{216}\left(-2+\sqrt{58}\right)\sqrt{42-3\sqrt{58}}$ and $a_5 = -\frac{7}{54}+\frac{5}{216}\sqrt{58}$, hence

$$|H_{2,3}(f_4)| = \left|a_3a_5 - a_4^2\right| = \frac{77}{15,552}+\frac{29}{15,552}\sqrt{58},$$

which completes our proof. □

Remark 3. *The maximum values of the functions F defined by (21) and (28) could be also found by using the MAPLE™ computer software codes like in the Remark 1 item 2, and we obtain the same values as in both of the above two theorems.*

3. Logarithmic Coefficients Sharp Upper Bounds

The logarithmic coefficients $\gamma_n := \gamma_n(f)$, $n\in\mathbb{N}$ for the function $f\in\mathcal{S}$ are defined by

$$F_f(z) := \log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n,\ z\in\mathbb{D}. \tag{34}$$

Since the function $\phi(z) := \cos z$ has a positive real part in $\mathbb{D}$, and moreover

$$\mathrm{Re}(\cos z) > \frac{1}{2},\ z\in\mathbb{D},$$

it follows that $\mathcal{S}^*_{\cos}\subset\mathcal{S}^*\subset\mathcal{S}$ (see [11], p. 610). Therefore, it is possible to define the logarithmic coefficients for the functions $f\in\mathcal{S}^*_{\cos}$.

In this section, we give the sharp upper bounds estimates for the third and fourth logarithmic coefficients of the functions that belong to the class $\mathcal{S}^*_{\cos}$.

Theorem 4. *If $f\in\mathcal{S}^*_{\cos}$ is given by (1), then*

$$|\gamma_3|\le\frac{\sqrt{3}}{27},\quad |\gamma_4|\le\frac{1}{16}.$$

These bounds are sharp.

Proof. If $f\in\mathcal{S}^*_{\cos}$ has the form (1), then

$$\log\frac{f(z)}{z} = a_2z + \left(-\frac{a_2^2}{2}+a_3\right)z^2 + \left(-a_2a_3+a_4+\frac{a_2^3}{3}\right)z^3+\ldots,\ z\in\mathbb{D}. \tag{35}$$

And, equating the first four coefficients of (34) with those of (35), we get

$$\gamma_1 = \frac{a_2}{2},\quad \gamma_2 = \frac{1}{4}\left(2a_3-a_2^2\right),\quad \gamma_3 = \frac{1}{6}\left(a_2^3-3a_2a_3+3a_4\right), \tag{36}$$

$$\gamma_4 = \frac{1}{8}\left(-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2+4a_5\right). \tag{37}$$

With the same notation as in the proof of Theorem 1, replacing in (36) and (37) the values of a_2, a_3, a_4 and a_5 from the relations (8) and (9), we obtain

$$\gamma_1 = 0, \quad \gamma_2 = -\frac{1}{32}c_1^2, \quad \gamma_3 = \frac{1}{48}c_1^3 - \frac{1}{24}c_1c_2, \tag{38}$$

$$\gamma_4 = -\frac{35}{3072}c_1^4 - \frac{1}{64}c_2^2 + \frac{3}{64}c_1^2c_2 - \frac{1}{32}c_1c_3 \tag{39}$$

For the upper bound of $|\gamma_3|$, using (38), we write

$$\gamma_3 = -\frac{c_1}{24}\left(c_2 - \frac{c_1^2}{2}\right),$$

and according to Lemma 3 for $\nu = \frac{1}{2}$, we obtain

$$|\gamma_3| \leq \frac{|c_1|}{24}\left(2 - \frac{|c_1^2|}{2}\right).$$

Denoting $c := |c_1|$, from Lemma 1, we have

$$|\gamma_3| \leq \frac{c}{24}\left(2 - \frac{c^2}{2}\right) = F(c),\ c \in [0,2].$$

Using the result we got for the computation of the maximum F given by (10) we get $|\gamma_3| \leq \frac{\sqrt{3}}{27}$.

To prove the sharpness of this bound, let us consider the function f_1 given by (11), were $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{12}$ and $a_4 = \frac{2}{27}\sqrt{3}$. Therefore, for this function, by using the last of the relations from (36), we obtain $\gamma_3 = \frac{\sqrt{3}}{27}$.

To find the upper bound of $|\gamma_4|$, from (37) combined with Lemma 2, we can write

$$\gamma_4 = \frac{1}{192}\left[t_1^4 + 24|t_1|^2t_2^2\left(1-|t_1|^2\right) - 12t_2^2\left(1-|t_1|^2\right)^2 - 24t_1\left(1-|t_1|^2\right)\left(1-|t_2|^2\right)t_3\right],$$

and setting $x := |t_1| \in [0,1]$, $y := |t_2| \in [0,1]$ and $u := |t_3| \in [0,1]$, using the triangle's inequality, we obtain

$$\begin{aligned}|\gamma_4| &\leq \frac{1}{192}\left[x^4 + 24x^2\left(1-x^2\right)y^2 + 12\left(1-x^2\right)^2y^2 + 24x\left(1-x^2\right)\left(1-y^2\right)u\right] \\ &= \frac{1}{192}F(x,y,u), \quad x,y,u \in [0,1],\end{aligned} \tag{40}$$

where

$$F(x,y,u) := x^4 + 24x^2\left(1-x^2\right)y^2 + 12\left(1-x^2\right)^2y^2 + 24x\left(1-x^2\right)\left(1-y^2\right)u. \tag{41}$$

Using the notations and the technique from the proofs of the previous theorems, we will determine the maximum of F on Ω as follows.

I. In the points $(x,y,u) \in \operatorname{int}\Omega$, differentiating (41) with respect to u, we obtain

$$\frac{\partial F(x,y,u)}{\partial u} = 24x\left(1-x^2\right)\left(1-y^2\right) \neq 0,\ (x,y,u) \in \operatorname{int}\Omega,$$

therefore, the function F does not attain its maximum value in $\operatorname{int}\Omega$.

II. In the next items, we will discuss the existence of the maximum value of F in the interior of six faces of Ω.

(i) On the face $x = 0$, we get

$$F(0, y, u) = 12y^2 \leq 12,\ y, u \in (0, 1). \tag{42}$$

(ii) On the face $x = 1$, the function F takes the form

$$F(1, y, u) = 1,\ y, u \in (0, 1). \tag{43}$$

(iii) On $y = 0$, the function can be written as

$$\kappa_1(x, u) := F(x, 0, u) = x^4 + 24x\left(1 - x^2\right)u,\ x, u \in (0, 1), \tag{44}$$

and because

$$\frac{\partial \kappa_1(x, u)}{\partial u} = 24x\left(1 - x^2\right) \neq 0,\ x \in (0, 1),$$

it implies that the function κ_1 has no maximum in $(0, 1) \times (0, 1)$.

(iv) On $y = 1$, the function F reduces to

$$s_1(x) := F(x, 1, u) = -11x^4 + 12,\ x \in (0, 1). \tag{45}$$

Since $s_1'(x) = -44x^3 < 0$, $x \in (0, 1)$, the function s_1 is strictly decreasing on $(0, 1)$, hence

$$F(x, 1, u) \leq F(0, 1, u) = 12,\ x, u \in (0, 1). \tag{46}$$

(v) On the face $u = 0$, the function F will have the form

$$\kappa_2(x, y) := F(x, y, 0) = x^4 + 24x^2\left(1 - x^2\right)y^2 + 12\left(1 - x^2\right)^2 y^2,\ x, y \in (0, 1).$$

Therefore,

$$\frac{\partial \kappa_2(x, y)}{\partial x} = -4x^3\left(12y^2 - 1\right),\quad \frac{\partial \kappa_2(x, y)}{\partial y} = -24y(x - 1)(x + 1)\left(x^2 + 1\right),$$

thus, the system of equations $\frac{\partial \kappa_2(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

(vi) On $u = 1$, the function F will be

$$\kappa_3(x, y) := -12x^4y^2 + 24x^3y^2 + x^4 - 24x^3 - 24xy^2 + 12y^2 + 24x,\ x, y \in (0, 1).$$

Similarly, it is easy to check that the system of equations $\frac{\partial \kappa_3(x, y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x, y)}{\partial y} = 0$ has no solutions in $(0, 1) \times (0, 1)$.

III. Now we will investigate the existence of the maximum of F on the edges of Ω.

(i) From (44) at $u = 0$, we have

$$F(x, 0, 0) = x^4 \leq 1,\ x \in [0, 1].$$

(ii) Using (44) at $u = 1$, we get $s_2(x) := F(x, 0, 1) = x\left(x^3 - 24x^2 + 24\right)$, hence, $s_2'(x) = 4x^3 - 72x^2 + 24$. The solution of $s_2'(x) = 0$ in $[0, 1]$ is $x_0 = 0.587000\ldots \in [0, 1]$ and $s_2''(x_0) < 0$; it follows that

$$F(x, 0, 1) \leq F(x_0, 0, 1) = 9.352439\ldots,\ x \in [0, 1].$$

(iii) Putting $x = 0$ in (44), we get

$$F(0,0,u) = 0,\ u \in [0,1],$$

(iv) Since (45) is independent of u, we obtain

$$F(x,1,1) = F(x,1,0) \leq 12,\ x \in [0,1],$$

while for $x = 0$ in (45), we get

$$F(0,1,u) = 12.$$

(v) The relation (43) is independent with respect to the variables y and u, thus

$$F(1,y,0) = F(1,y,1) = F(1,0,u) = F(1,1,u) = 1,\ y,u \in [0,1].$$

(vi) Finally, since (42) is independent of the variable u, we have

$$F(0,y,0) = F(0,y,1) \leq 1,\ z \in [0,1].$$

The above computations lead to

$$\max\{F(x,y,u) : (x,y,u) \in \Omega\} = F(0,1,u) = 12,$$

and from (40) we conclude that $|\gamma_4| \leq \frac{1}{16}$.

For proving the sharpness of the above inequality, we consider the function f_2 given by (19). In this case, $a_1 = 1$, $a_2 = a_3 = a_4 = 0$ and $a_5 = -\frac{1}{8}$, and from (37) we get $|\gamma_4| = \frac{1}{16}$, which completes our proof. □

Remark 4. *In [11] it is proved that* $|\gamma_3| \leq \frac{1}{6}$ *and* $|\gamma_4| \leq \frac{85}{384}$, *while our above results are sharp and give the best upper bounds for these coefficients.*

Theorem 5. *Let* $f \in \mathcal{S}^*_{\cos}$ *be given by* (1). *Then, for the function* F_f *given by* (34), *the next inequality holds*

$$\left|H_{2,2}\left(F_f/2\right)\right| \leq -\frac{10{,}577}{1{,}028{,}376} + \frac{1549}{4{,}113{,}504}\sqrt{1549} = 0.015678\ldots,$$

and this result is sharp.

Proof. Replacing the values of (38) and (39) in the relation (2), we obtain

$$H_{2,2}\left(F_f/2\right) = \gamma_2\gamma_4 - \gamma_3^2 = -\frac{23}{294{,}912}c_1^6 + \frac{5}{18{,}432}c_1^4c_2 + \frac{1}{1024}c_1^3c_3 - \frac{23}{18{,}432}c_1^2c_2^2.$$

Using the Lemma 2, we obtain

$$H_{2,2}\left(F_f/2\right) = \frac{1}{4608}\Big[-3t_1^6 - 72t_1^2|t_1|^2t_2^2\left(1-|t_1|^2\right) - 92t_1^2t_2^2\left(1-|t_1|^2\right)^2 + 72t_1^3\left(1-|t_1|^2\right)\left(1-|t_2|^2\right)t_3\Big],$$

and denoting $x := |t_1| \in [0,1]$, $y := |t_2| \in [0,1]$, $u := |t_3| \in [0,1]$, by using the triangle's inequality, the above relation leads us to

$$\left|H_{2,2}\left(F_f/2\right)\right| \leq \frac{1}{4608}\left[3x^6 + 72x^4\left(1-x^2\right)y^2 + 92x^2\left(1-x^2\right)^2y^2 + 72x^3\left(1-x^2\right)\left(1-y^2\right)u\right] = \frac{1}{4608}F(x,y,u),\ x,y,u \in [0,1], \tag{47}$$

where

$$F(x,y,u) := 3x^6 + 72x^4\left(1-x^2\right)y^2 + 92x^2\left(1-x^2\right)^2y^2 + 72x^3\left(1-x^2\right)\left(1-y^2\right)u. \tag{48}$$

With the notations used in the proofs of the previous three theorems, we will determine the maximum value of F in Ω.

I. For all the interior points $(x,y,u) \in \operatorname{int}\Omega$, differentiating (48) with respect to u, we obtain

$$\frac{\partial F(x,y,u)}{\partial u} = 72x^3\left(1-x^2\right)\left(1-y^2\right) \neq 0,\ (x,y,u) \in \operatorname{int}\Omega,$$

hence, the function does not get its maximum value in $\operatorname{int}\Omega$.

II. Next, we will study whether it is possible to obtain the maximum value of F in the interior of six faces of Ω.

(i) On the face $x = 0$, we have

$$F(0,y,u) = 0,\ y,u \in (0,1). \tag{49}$$

(ii) On the face $x = 1$, the function F takes the form

$$F(1,y,u) = 3,\ y,u \in (0,1). \tag{50}$$

(iii) On $y = 0$, the function F can be written as

$$\kappa_1(x,u) := F(x,0,u) = 3x^6 + 72x^3\left(1-x^2\right)u,\ x,u \in (0,1). \tag{51}$$

Since

$$\frac{\partial \kappa_1(x,u)}{\partial u} = 72x^3\left(1-x^2\right) \neq 0,\ x \in (0,1),$$

it follows that the function κ_1 has no maximum value in $(0,1)\times(0,1)$.

(iv) On $y = 1$, the function F becomes

$$s_1(x) := F(x,1,u) = 3x^6 + 72x^4\left(1-x^2\right) + 92x^2\left(1-x^2\right)^2,\ x \in (0,1). \tag{52}$$

Since $s_1'(x) = 138x^5 - 448x^3 + 184x = 0$ has on $(0,1)$ the root $x_0 = \dfrac{\sqrt{7728 - 138\sqrt{1549}}}{69} = 0.694547\ldots \in (0,1)$ and $s_1''(x_0) < 0$, we deduce that

$$F(x,1,u) \leq F(x_0,1,u) = -\frac{676{,}928}{14{,}283} + \frac{24{,}784}{14{,}283}\sqrt{1549} = 20.899268\ldots,\ x \in (0,1). \tag{53}$$

(v) On the face $u = 0$, the function F reduces to

$$\kappa_2(x,y) := F(x,y,0) = 3x^6 + 72x^4\left(1-x^2\right)y^2 + 92x^2\left(1-x^2\right)^2y^2,\ x,y \in (0,1),$$

therefore,

$$\frac{\partial \kappa_2(x,y)}{\partial x} = 2x\left(60x^4y^2 + 9x^4 - 224x^2y^2 + 92y^2\right),\quad \frac{\partial \kappa_2}{\partial y} = 8x^2\left(x^2-1\right)y\left(5x^2-23\right).$$

Thus, the system of equations $\frac{\partial \kappa_2(x,y)}{\partial x} = 0$ and $\frac{\partial \kappa_2(x,y)}{\partial y} = 0$ has no solutions in $(0,1) \times (0,1)$.

(vi) On $u = 1$, the function F takes the form

$$\kappa_3(x,y) := F(x,y,1) = 3x^6 + 72x^4\left(1-x^2\right)y^2 + 92x^2\left(1-x^2\right)^2y^2 + 72x^3\left(1-x^2\right)\left(1-y^2\right),\ x,y \in (0,1).$$

We get

$$\frac{\partial \kappa_3(x,y)}{\partial x} = \left(120y^2+18\right)x^5 + \left(360y^2-360\right)x^4 - 448x^3y^2 + \left(-216y^2+216\right)x^2 + 184xy^2,$$

and

$$\frac{\partial \kappa_3}{\partial y} = 8yx^2(5x+23)(x+1)(x-1)^2,$$

hence, the system of equations $\frac{\partial \kappa_3(x,y)}{\partial x} = 0$ and $\frac{\partial \kappa_3(x,y)}{\partial y} = 0$ has no solutions in $(0,1) \times (0,1)$.

III. Now we will investigate the maximum of F on the edges of Ω.

(i) From (51) for $u = 0$ we have

$$F(x,0,0) = 3x^6 \leq 3,\ x \in [0,1].$$

(ii) From (51) for $u = 1$, let $s_2(x) := F(x,0,1) = 3x^6 - 72x^5 + 72x^3$. The solutions in $[0,1]$ of the equation $s_2'(x) = 18x^5 - 360x^4 + 216x^2 = 0$ are $x_0 = 0.790371\ldots \in [0,1]$ and $x_1 = 0$. Since $s_2''(x_0) < 0$ and $s_2(0) = 0$ we deduce that

$$F(x,0,1) \leq F(x_0,0,1) = 14.073287\ldots,\ x \in [0,1].$$

(iii) Putting $x = 0$ in (51) we have

$$F(0,0,u) = 0,\ u \in [0,1].$$

(iv) Since the relation (52) is independent on u, from (53) we obtain

$$F(x,1,1) = F(x,1,0) \leq -\frac{676{,}928}{14{,}283} + \frac{24{,}784}{14{,}283}\sqrt{1549},\ x \in [0,1].$$

(v) If we take $x = 0$ in (52), we obtain

$$F(0,1,u) = 0,\ u \in [0,1]$$

(vi) The relation (50) is independent of the variables y and u, hence

$$F(1,y,0) = F(1,y,1) = F(1,0,u) = F(1,1,u) = 3,\ y,u \in [0,1].$$

Similarly, since the Formula (49) is independent of the variable u, we have

$$F(0,y,0) = F(0,y,1) = 0,\ y \in [0,1].$$

For the above reasons, it follows that

$$\max\{F(x,y,u) : (x,y,u) \in \Omega\} = F(x_0,1,u) = -\frac{676{,}928}{14{,}283} + \frac{24{,}784}{14{,}283}\sqrt{1549},$$

and using (47), we conclude that

$$\left|H_{2,2}\left(F_f/2\right)\right| \leq -\frac{10{,}577}{1{,}028{,}376} + \frac{1549}{4{,}113{,}504}\sqrt{1549} = 0.015678\ldots.$$

To prove the sharpness of the above inequality, we denote $t_{\triangle} = \dfrac{2}{69}\sqrt{7728 - 138\sqrt{1549}}$ and $h_5(z) = \dfrac{1 + t_{\triangle} z + z^2}{1 - z^2}$ such that

$$w_5(z) = \frac{h_5(z) - 1}{h_5(z) + 1} = \frac{z(\sqrt{7728 - 138\sqrt{1549}} + 69z)}{69 + \sqrt{7728 - 138\sqrt{1549}}z}, \ z \in \mathbb{D}.$$

It is easy to see that $w_7(0) = 0$, and to prove that $|w_7(z)| < 1$ in $\mathbb{D}$, we will define the function w_5 as $w_5(z) = z\rho(z)$, where

$$\rho(z) = \frac{\sqrt{7728 - 138\sqrt{1549}} + 69z}{69 + \sqrt{7728 - 138\sqrt{1549}}z}, \ z \in \mathbb{D}.$$

According to the same reasons regarding the circular transforms as in the proofs of the sharpness of Theorem 1 item (ii) and Theorem 2, we will show that $|\rho(z)| < 1$, $z \in \mathbb{D}$, by proving that $\operatorname{Re} H(z) > 0$, $z \in \mathbb{D}$, where

$$H(z) := \frac{1 - \rho(z)}{1 + \rho(z)} = \frac{\left(\sqrt{7728 - 138\sqrt{1549}} - 69\right)(z - 1)}{\left(\sqrt{7728 - 138\sqrt{1549}} + 69\right)(z + 1)}, \ z \in \mathbb{D},$$

is a circular transform. It is easy to compute

$$H(0) = \frac{\sqrt{7728 - 138\sqrt{1549}} - 69}{\sqrt{7728 - 138\sqrt{1549}} + 69} = 0.1802561912\ldots > 0, \quad H(1) = 0,$$
$$H(i) = \frac{\left(\sqrt{7728 - 138\sqrt{1549}} - 69\right)i}{\sqrt{7728 - 138\sqrt{1549}} + 69}, \quad H(-i) = -\frac{\left(\sqrt{7728 - 138\sqrt{1549}} - 69\right)i}{\sqrt{7728 - 138\sqrt{1549}} + 69},$$

and these values of the circular transform H imply that $\operatorname{Re} H(z) > 0$ for all $z \in \mathbb{D}$, which leads us to $|\rho(z)| < 1$ in $\mathbb{D}$, hence

$$|w_5(z)| = |z|\,|\rho(z)| < 1, \ z \in \mathbb{D}.$$

Consequently, the function

$$\begin{aligned} f_5(z) = z\exp\left(\int_0^z \frac{\cos w_5(t) - 1}{t}dt\right) &= z + \left(-\frac{28}{69} + \frac{1}{138}\sqrt{1549}\right)z^3 \\ &\quad - \frac{1}{14{,}283}\left(-43 + 2\sqrt{1549}\right)\sqrt{7728 - 138\sqrt{1549}}\,z^4 \\ &\quad + \left(-\frac{71{,}467}{114{,}264} + \frac{241}{14{,}283}\sqrt{1549}\right)z^5 + \ldots, \ z \in \mathbb{D}, \end{aligned}$$

belongs to the class $\mathcal{S}^*_{\cos}$. The initial coefficients of f_5 are

$$a_3 = -\frac{28}{69} + \frac{1}{138}\sqrt{1549}, \quad a_4 = -\frac{1}{14{,}283}\left(-43 + 2\sqrt{1549}\right)\sqrt{7728 - 138\sqrt{1549}},$$
$$a_5 = -\frac{71{,}467}{114{,}264} + \frac{241}{14{,}283}\sqrt{1549},$$

and from (36) and (37), we obtain

$$\gamma_2 = -\frac{14}{69} + \frac{1}{276}\sqrt{1549}, \quad \gamma_3 = -\frac{(-43+2\sqrt{1549})\sqrt{7728-138\sqrt{1549}}}{28{,}566},$$
$$\gamma_4 = -\frac{42{,}761}{114{,}264} + \frac{283}{28{,}566}\sqrt{1549}.$$

Hence,

$$\left|H_{2,2}\left(F_{f_5}/2\right)\right| = \left|\gamma_2\gamma_4 - \gamma_3^2\right| = -\frac{10{,}577}{1{,}028{,}376} + \frac{1549}{4{,}113{,}504}\sqrt{1549},$$

which proves the sharpness of our estimation. □

Remark 5. *As we already mentioned in Remark 3, the same maximum values of the functions F defined by (41) and (48) could also be found by using the MAPLE™ computer software codes as in Remark 1 item 2.*

4. Conclusions

In this article, we have obtained the sharp coefficient bounds for the starlike functions that are connected to the cosine function, and we have also obtained the sharp coefficient bounds for the logarithmic coefficients of such functions.

The main tools of our results were those of N.E. Cho et al. [32], which seem to be a very efficient for the estimation of the first coefficients of Carathéodory's function. In addition, we have determined the sharp bound for the fifth coefficient. The technique of this paper can be used to determine the sharp upper bound of the initial coefficients for a variety of classes of analytic functions.

Moreover, we have used these results to determine the sharp upper bounds for the Hankel determinants up to order three, and we emphasize that all the results we obtained are the best possible, so they cannot be improved.

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