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On the Impact of Some Fixed Point Theorems on Dynamic Programming and RLC Circuit Models in $\mathfrak{R}$-Modular b-Metric-like Spaces

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Abstract: In this study, we significantly extend the concept of modular metric-like spaces to introduce the notion of b-metric-like spaces. Furthermore, by incorporating a binary relation $\mathfrak{R}$, we develop the framework of $\mathfrak{R}$-modular b-metric-like spaces. We establish a groundbreaking fixed point theorem for certain extensions of Geraghty-type contraction mappings, incorporating both $\mathcal{Z}$ simulation function and $\mathcal{E}$-type contraction within this innovative structure. Moreover, we present several novel outcomes that stem from our newly defined notations. Afterwards, we introduce an unprecedented concept, the graphical modular b-metric-like space, which is derived from the binary relation $\mathfrak{R}$. Finally, we examine the existence of solutions for a class of functional equations that are pivotal in dynamic programming and in solving initial value problems related to the electric current in an RLC parallel circuit.

Keywords: fixed point; simulation functions; modular metric-like space; dynamic programming; electric circuit equation

MSC: 47H10; 54H25

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Citation: Girgin, E.; Büyükkaya, A.; Kuru, N.K.; Öztürk, M. On the Impact of Some Fixed Point Theorems on Dynamic Programming and RLC Circuit Models in $\mathfrak{R}$-Modular b-Metric-like Spaces. *Axioms* **2024**, *13*, 441. https://doi.org/10.3390/axioms13070441

Academic Editor: Janusz Matkowski

Received: 17 May 2024
Revised: 14 June 2024
Accepted: 18 June 2024
Published: 28 June 2024

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1. Introduction

This study employs the expression $\mathbb{N}$ to refer to a set of all positive natural numbers. In addition, sets of positive and non-negative real numbers are symbolized by $\mathbb{R}^+$ and $\mathbb{R}_0^+$.

There is extensive interest in the domain of metric fixed point theory owing to its compelling structural properties. Within this theoretical framework, Banach contraction mapping, initially introduced by Banach [1] in 1922, occupies a pivotal position due to its foundational significance. This foundational work has catalyzed numerous subsequent research efforts to expand and refine the understanding of this outstanding mapping. Throughout the development of this field, there has been a notable emergence of innovative structures about generalized metric spaces. Among the significant advancements in this domain, the introduction of the b-metric function by Czerwik [2,3] (also independently identified by Bakhtin [4]) stands out as a quintessential natural extension. Introduced in the years 1993 and 1998, respectively, the b-metric function is characterized by the incorporation of a constant $\mathfrak{b} \geq 1$ within its triangular inequality, thus specifying a distinct difference from the conventional metric framework:

- $d(\varphi, \theta) \leq \mathfrak{b}[d(\varphi, \gamma) + d(\gamma, \theta)]$.

Hence, the function $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ is said to be a b-metric function on a non-empty set $\mathcal{U}$. The metric function and b-metric function coincide when $\flat = 1$.

In 2012, Amini–Harandi [5] constituted the topological structure of a new generalized metric space referred to as metric-like space by revising the axiom of metric positivity. Soon after, by considering the notation of the b-metric function, Alghamdi et al. [6] expanded the concept of metric-like to b-metric-like. At almost the same time, Chistyakov's pioneering works, cited as [7–9], significantly advanced the theory of metric spaces in mathematical analysis. Chistyakov developed a foundational framework that thoroughly examined modular metric space by expanding the traditional understanding of metric functions. This unique framework introduced a new perspective to the discussion on metric fixed point theory, distinguished by its unique physical interpretation. Within this scholarly context, Chistyakov's development of the modular metric concept emerged as a seminal contribution, encompassing a more generalized approach to understanding the dynamics of metric spaces. Following this, the idea of modular metric space was expanded to the notion of modular b-metric space designed by Ege and Alaca [10] in 2018. Numerous investigations have been conducted on this new space; for additional information related to modular b-metric spaces, refer to [11–17]. In addition, in 2022, Sharma et al. [18] established the concept of modular metric-like structures.

On the other hand, an alternative approach to generalizing the structure of metric spaces incorporates the utilization of binary relation notation. Consequently, a considerable body of research (see [19–29]) is dedicated to extending the Banach contraction principle to encompass complete metric spaces characterized by a binary relation.

In order to obtain more effective outcomes in fixed point theory, some authors have endeavored to extend the Banach contraction principle by employing auxiliary functions in diverse abstract spaces. These investigations are still ongoing. For instance, in their seminal work, Khojasteh et al. [30] pioneered the concept of simulation functions, presenting them as a novel auxiliary function aimed at extending the applicability of the renowned Banach fixed point theorem. This innovative approach has opened new avenues for research within the domain of mathematical analysis, providing a broader framework for the examination and generalization of Banach's mappings. In 2018, Cho et al. [31] significantly advanced the field of simulation functions by introducing the $\mathcal{Z}$-simulation function as a new structure. This pioneering work not only set a new standard but also catalyzed further research and development within the domain, demonstrating the profound impact of innovative methodologies on the evolution of simulation practices. Very recently, Öztürk et al. [16] (simultaneously in [32]) extended the idea of the $\mathcal{Z}$-simulation function via the class Ψ^*, which is defined as the set of all non-decreasing self-mappings ψ on $[1, +\infty)$ satisfying $\psi^{-1}(\{1\}) = 1$. In another discovery, Jleli and Samet [33] (2014) introduced an innovative concept known as $\mathfrak{L}$-contraction. They further elucidated this concept by presenting and proving a theorem that has since contributed significantly to the field. Besides that, Geraghty [34] introduced a novel class of functions and substantiated the ensuing theorem employing this newfound category. Moreover, Fulga and Proca, in their seminal work, demonstrated an important theorem that represents a significant extension of the Geraghty-type contraction principle, achieved through the application of $\mathcal{E}$-type contraction methods [35]. This contribution not only expands the existing body of knowledge in the field but also opens new avenues for further research into contraction mappings and their applications. Naturally, these notions are not exclusive to these particular functions; the literature contains various ideas.

In this investigation, through utilization of the concept of modular metric-like structures and drawing inspiration from the binary relation $\mathfrak{R}$, we have developed two novel generalized metric spaces, namely the modular b-metric-like space and the $\mathfrak{R}$-modular b-metric-like space, incorporating a new concept. Furthermore, we have formulated and established a new fixed point theorem applicable within this newly defined space. Finally, we have demonstrated the applicability and significance of our findings in the context of functional equations and the equations governing RLC circuits.

2. Preliminaries

In this section, we review some fundamental concepts related to our study.

Firstly, Amini-Harandi [5] introduced the concept of metric-like spaces, also referred to as dislocated spaces, as indicated below:

Definition 1 ([5]). *Consider the function $d_\ell : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ on a non-empty set $\mathcal{U}$ with $\flat \geq 1$. For any $\varphi, \gamma, \theta \in U$, if the circumstances*
(d_ℓ^1) *$d_\ell(\varphi, \gamma) = 0$ implies $\varphi = \gamma$,*
(d_ℓ^2) *$d_\ell(\varphi, \gamma) = d_\ell(\gamma, \varphi)$,*
(d_ℓ^3) *$d_\ell(\varphi, \theta) \leq d_\ell(\varphi, \gamma) + d_\ell(\gamma, \theta)$*
are provided, then d_ℓ is called a metric-like function.

Similarly, in the above definition, if we consider the following instead of $\left(d_\ell^3\right)$, then d_ℓ is referred to as a *b*-metric-like function, which was proposed by Alghamdi et al. [6]:
$\left(d_\ell^{3'}\right)$ $d_\ell(\varphi, \theta) \leq \flat[d_\ell(\varphi, \gamma) + d_\ell(\gamma, \theta)]$.

Besides that, Chistyakov [7,8] has developed the idea of modular metric space, as noted below:

Definition 2 ([7,8]). *Denote the function $\mathfrak{m} : (0, \infty) \times \mathcal{U} \times \mathcal{U} \to [0, \infty]$, where $\mathcal{U}$ represents a non-void set. If the subsequent axioms hold for all $\varphi, \gamma, \theta \in \mathcal{U}$, then $\mathfrak{m}$ is termed a modular metric:*
$(\mathfrak{m}_1)$ *$\mathfrak{m}_\varrho(\varphi, \gamma) = 0$ for all $\varrho > 0$ if and only if $\varphi = \gamma$;*
$(\mathfrak{m}_2)$ *$\mathfrak{m}_\varrho(\varphi, \gamma) = \mathfrak{m}_\varrho(\gamma, \varphi)$ for all $\varrho > 0$;*
$(\mathfrak{m}_3)$ *$\mathfrak{m}_{\varrho+\mu}(\varphi, \gamma) \leq \mathfrak{m}_\varrho(\varphi, \theta) + \mathfrak{m}_\mu(\theta, \gamma)$ for all $\varrho, \mu > 0$.*

These axioms define the function $\mathfrak{m}$ as a modular metric, thereby contributing to the theoretical landscape of metric spaces by expanding the conventional metric definitions to include modular considerations. This exposition not only underscores the mathematical rigor involved in defining such a function but also highlights the potential for innovative applications in fields requiring nuanced measures of distance or similarity.

In the proposed framework, an alternative condition to $(\mathfrak{m}_1)$ is posited as follows:
$(\mathfrak{m}_1{}')$ $\mathfrak{m}_\varrho(\varphi, \varphi) = 0$ for all $\varrho > 0$,
which leads to the conclusion that $\mathfrak{m}$ constitutes a (metric) pseudo-modular on the space $\mathcal{U}$. Furthermore, a modular metric $\mathfrak{m}$ defined on the space $\mathcal{U}$ acquires the designation of being regular upon satisfying the newly introduced condition, which serves as a more generous version of $(\mathfrak{m}_1)$,
$(\mathfrak{m}_1{}'')$ $\varphi = \gamma$ if and only if $\mathfrak{m}_\varrho(\varphi, \varphi) = 0$, for some $\varrho > 0$,
is upheld for some $\varrho > 0$, indicating the modular metric's regularity criteria. Finally, the designation of convex modular is ascribed to $\mathfrak{m}$ if, for any $\varrho, \mu > 0$ and elements $\varphi, \gamma, \theta \in \mathcal{U}$, the following inequality is consistently observed:

$$\mathfrak{m}_{\varrho+\mu}(\varphi, \gamma) \leq \frac{\varrho}{\varrho + \mu}\mathfrak{m}_\varrho(\varphi, \theta) + \frac{\mu}{\varrho + \mu}\mathfrak{m}_\mu(\theta, \gamma).$$

This condition establishes the convexity property of the modular metric in question, thereby contributing to the comprehensive characterization of such metrics in the context of the space $\mathcal{U}$.

On the other hand, the function $\varrho \to \mathfrak{m}_\varrho(\varphi, \gamma)$ is non-increasing on $(0, \infty)$ for any $\varphi, \gamma \in \mathcal{U}$, where $\mathfrak{m}$ is a metric pseudo-modular on the set $\mathcal{U}$. Indeed, for $0 < \mu < \varrho$, it is attested as follows:

$$\mathfrak{m}_\varrho(\varphi, \gamma) \leq \mathfrak{m}_{\varrho-\mu}(\varphi, \varphi) + \mathfrak{m}_\mu(\varphi, \gamma) = \mathfrak{m}_\mu(\varphi, \gamma).$$

If $\mathfrak{m}$ is a modular metric on a set $\mathcal{U}$, then a modular set is identified by

$$\mathcal{U}_{\mathfrak{m}} = \left\{\gamma \in \mathcal{U} : \gamma \overset{\mathfrak{m}}{\sim} \varphi\right\},$$

where $\overset{\mathfrak{m}}{\sim}$ is a binary relation on $\mathcal{U}$ defined by $\varphi \sim \gamma$ if and only if $\lim_{\mu\to\infty} \mathfrak{m}_{\varrho}(\varphi, \gamma) = 0$ for $\varphi, \gamma \in \mathcal{U}$.

Definition 3 ([7,8]). *Let $\mathfrak{m}$ be a pseudo-modular on $\mathcal{U}$. Then, the following sets are mentioned as modular space around $\varphi_0 \in \mathcal{U}$:*

- $\mathcal{U}_{\mathfrak{m}} = \mathcal{U}_{\mathfrak{m}}(\varphi_0) = \left\{\varphi \in \mathcal{U} : \mathfrak{m}_{\varrho}(\varphi, \varphi_0) \to 0 \text{ as } \varrho \to \infty\right\}$;
- $\mathcal{U}^*_{\mathfrak{m}} = \mathcal{U}^*_{\mathfrak{m}}(\varphi_0) = \left\{\varphi \in \mathcal{U} : \text{there exists } \varrho = \varrho(\varphi) > 0 \text{ such that } \mathfrak{m}_{\varrho}(\varphi, \varphi_0) < \infty\right\}$.

Definition 4 ([9]). *Let $\mathfrak{m}$ be a modular on $\mathcal{U}$. Then:*

- *a sequence of elements $\{\varphi_n\} = \{\varphi_n\}_{n=1}^{\infty}$ from $\mathcal{U}_{\mathfrak{m}}$ or $\mathcal{U}^*_{\mathfrak{m}}$ is said to be $\mathfrak{m}$-convergent to an element $\varphi \in \mathcal{U}$ if there exists a number $\varrho > 0$, possibly depending on $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ, such that $\lim_{n\to\infty} \mathfrak{m}_{\varrho}(\varphi_n, \varphi) = 0$. Then, any such element φ is called a modular limit of the sequence $\{\varphi_n\}$.*
- *a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{U}^*_{\mathfrak{m}}$ is said to be $\mathfrak{m}$-Cauchy if there exists a number $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\mathfrak{m}_{\varrho}(x_n, x_m) \to 0$ as $n, m \to \infty$.*
- *the modular space $\mathcal{U}^*_{\mathfrak{m}}$ is said to be $\mathfrak{m}$-complete if each $\mathfrak{m}$-Cauchy sequence from $\mathcal{U}^*_{\mathfrak{m}}$ is $\mathfrak{m}$-convergent in the following (more precise) sense: if $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{U}^*_{\mathfrak{m}}$ and there exists a $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\lim_{n,m\to\infty} \mathfrak{m}_{\varrho}(\varphi_n, \varphi_m) = 0$, then there exists an $\varphi \in \mathcal{U}^*_{\mathfrak{m}}$ such that $\lim_{n\to\infty} \mathfrak{m}_{\varrho}(\varphi_n, \varphi) = 0$.*

In 2018, Ege and Alaca [10] introduced the notion of modular b-metric space by only revising the condition $(\mathfrak{m}_3)$ of Definition 2 with the following one:

$(\mathfrak{m}'_3)$ $\mathfrak{m}_{\varrho+\mu}(\varphi, \gamma) \leq \flat\left[\mathfrak{m}_{\varrho}(\varphi, \theta) + \mathfrak{m}_{\mu}(\theta, \gamma)\right]$ for all $\varrho, \mu > 0$.

A modular metric space can be achieved from a modular b-metric space in the case of $\flat = 1$. Further, the reader is referred to [11–17] for additional information related to the concept of modular b-metric.

Besides this, in 2022, the idea of modular metric-like space was introduced by Sharma et al. [18] by changing the axiom $(\mathfrak{m}_1)$ of Definition 2 to the one below:

$\left(\mathfrak{m}^L_1\right)$ $\mathfrak{m}_{\varrho}(\varphi, \gamma) = 0$ for all $\varrho > 0$ implies $\varphi = \gamma$.

The subsequent examples illustrate instances of a modular metric-like space, denoted as $(\mathcal{U}, \mathfrak{m})$.

Example 1 ([18]). *Let $\mathcal{U} = \mathbb{R}$. Then, the mappings $\mathfrak{m}^j_{\varrho} : \mathcal{U}_{\mathfrak{m}} \times \mathcal{U}_{\mathfrak{m}} \to \mathbb{R}^+ (j = 1, 2, 3)$ defined by*

$$\mathfrak{m}^1_{\varrho}(\varphi, \gamma) = \frac{|\varphi| + |\gamma| + a}{\varrho},$$

$$\mathfrak{m}^2_{\varrho}(\varphi, \gamma) = \frac{|\varphi - b| + |\gamma - b|}{\varrho},$$

$$\mathfrak{m}^3_{\varrho}(\varphi, \gamma) = \frac{\varphi^2 + \gamma^2}{\varrho}$$

are modular metric-like on $\mathcal{U}$, where $a \geq 1$ and $b \in \mathbb{R}$.

In what follows, we sum up the concept of binary relation and its properties.

Definition 5 ([19,20]). *Let $\mathcal{U}$ be a non-empty set, and the non-empty binary relation $\mathfrak{R}$ be a subset of $\mathcal{U}^2$. If any two elements $\varphi, \gamma \in \mathcal{U}$ are related with $\mathfrak{R}$, then it is written as $(\varphi, \gamma) \in \mathfrak{R}$ or $\varphi \mathfrak{R} \gamma$. Furthermore, the inverse of $\mathfrak{R}$ is denoted by $\mathfrak{R}^{-1}$ and defined as*

$$\mathfrak{R}^{-1} = \{(\varphi, \gamma) \in \mathcal{U} \times \mathcal{U} : (\gamma, \varphi) \in \mathfrak{R}\}$$

and $\mathfrak{S} = \mathfrak{R} \cup \mathfrak{R}^{-1}$. We assert that for all $\varphi, \gamma, \theta \in \mathcal{U}$:

- *$\mathfrak{R}$ is reflexive if $\varphi \mathfrak{R} \varphi$;*
- *$\mathfrak{R}$ is symmetric if $\varphi \mathfrak{R} \gamma$ implies $\gamma \mathfrak{R} \varphi$;*
- *$\mathfrak{R}$ is antisymmetric if $\varphi \mathfrak{R} \gamma$ and $\gamma \mathfrak{R} \varphi$ implies $\varphi = \gamma$;*
- *$\mathfrak{R}$ is transitive if $\varphi \mathfrak{R} \gamma$ and $\gamma \mathfrak{R} \theta$ implies $\varphi \mathfrak{R} \theta$.*

Definition 6 ([21]). *Let $\mathcal{U}$ be a non-empty set and $\mathfrak{R}$ be a binary relation on $\mathcal{U}$.*

- *A sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is called an $\mathfrak{R}$-sequence if $\varphi_n \mathfrak{R} \varphi_{n+1}$, for all $n \in \mathbb{N}$.*
- *A mapping $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ is $\mathfrak{R}$-preserving if $\varphi \mathfrak{R} \gamma$ implies $\mathcal{T}\varphi \mathfrak{R} \mathcal{T}\gamma$.*

Definition 7 ([21]). *Let $(\mathcal{U}, d)$ be a metric space and $\mathfrak{R}$ be a binary relation on $\mathcal{U}$. Then, the triple $(\mathcal{U}, d, \mathfrak{R})$ is called an $\mathfrak{R}$-metric space.*

Definition 8 ([22]). *A mapping $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ is $\mathfrak{R}$-continuous at $\varphi_0 \in \mathcal{U}$ if for each $\mathfrak{R}$-sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{U}$, $\mathcal{T}(\varphi_n) \to \mathcal{T}(\varphi_0)$ when $\varphi_n \to \varphi_0$. So, $\mathcal{T}$ is $\mathfrak{R}$-continuous on $\mathcal{U}$ provided that $\mathcal{T}$ is $\mathfrak{R}$-continuous at each $\varphi_0 \in \mathcal{U}$.*

Khalehoghli et al. [22] presented a theorem that builds upon and extends the foundational results attributed to Banach.

Theorem 1 ([22]). *Let $\mathcal{T}$ be an $\mathfrak{R}$-preserving and $\mathfrak{R}$-continuous mapping on an $\mathfrak{R}$-complete $\mathfrak{R}$-metric space with $\varphi_0 \in \mathcal{U}$ such that $\varphi_0 \mathfrak{R} \gamma$ for each $\gamma \in \mathcal{U}$. If the inequality*

$$d(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq r d(\varphi, \gamma), \text{ where } r \in (0, 1)$$

is fulfilled for all $\varphi, \gamma \in \mathcal{U}$ with $\varphi \mathfrak{R} \gamma$, then $\mathcal{T}$ is named as $\mathfrak{R}$-contraction and has a unique fixed point.

The following definition was introduced by Kolman et al. [23].

Definition 9 ([23]). *Consider that $\mathfrak{R}$ is a binary relation on $\mathcal{U}$. A path in $\mathfrak{R}$ from φ to γ is a sequence $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_n\} \subseteq \mathcal{U}$ such that:*

(i) $\varphi_0 = \varphi$ and $\varphi_n = \gamma$;
(ii) $(\varphi_j, \varphi_{j+1}) \in \mathfrak{R}$ for all $j = \{0, 1, 2, \cdots\cdots, n-1\}$.

The set of all paths from φ to γ in $\mathfrak{R}$ is indicated as $\Gamma(\varphi, \gamma, \mathfrak{R})$. The path of length n involves $n+1$ items of $\mathcal{U}$, which are not always distinct.

For more results involving this concept, see [24–29].

As an auxiliary function, Khojasteh et al. [30] constitute the concept of simulation functions, as indicated below:

Definition 10 ([30]). *A function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function if the statements*

(ξ_1) $\xi(0, 0) = 0$,
(ξ_2) $\xi(\hat{c}, \hat{q}) < \hat{q} - \hat{c}$ for all $\hat{c}, \hat{q} > 0$,
(ξ_3) if $\{\hat{c}_n\}_{n \in \mathbb{N}}$, $\{\hat{q}_n\}_{n \in \mathbb{N}}$ are sequences in $(0, \infty)$ such that $\lim_{n \to \infty} \hat{c}_n = \lim_{n \to \infty} \hat{q}_n > 0$, then

$$\limsup_{n \to \infty} \xi(\hat{c}_n, \hat{q}_n) < 0$$

are provided. Further, from (ξ_2)*, it follows that* $\xi(\hat{c}, \hat{q}) < 0$ *for all* $\hat{c} \geq \hat{q} > 0$.

The symbol $\mathcal{Z}$ accurately denotes the comprehensive set of all simulation functions.

Definition 11 ([30]). *A mapping* $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ *on a metric space* $(\mathcal{U}, d)$ *is called a* $\mathcal{Z}$*-contraction with respect to* $\xi \in \mathcal{Z}$ *if the inequality*

$$\xi(d(\mathcal{T}\varphi, \mathcal{T}\gamma), d(\varphi, \gamma)) \geq 0$$

is fulfilled for all $\varphi, \gamma \in \mathcal{U}$.

In 2018, Cho et al. [31] introduced $\mathcal{Z}$-simulation function in the sense of Khojasteh as a natural extension of the simulation function, as noted below:

Definition 12 ([31]). *Consider* $\zeta : [1, \infty) \times [1, \infty) \to \mathbb{R}$ *as a mapping that fulfills the following assertions:*

(ζ_1) $\zeta(1, 1) = 1$;
(ζ_2) $\zeta(\hat{c}, \hat{q}) < \hat{q}/\hat{c}$, for all $\hat{c}, \hat{q} > 1$;
(ζ_3) *for all sequences* $\{\hat{c}_n\}, \{\hat{q}_n\} \subset (1, \infty)$ *with* $\hat{c}_n \leq \hat{q}_n$, for all $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} \hat{c}_n = \lim_{n\to\infty} \hat{q}_n > 1 \quad \text{implies} \quad \limsup_{n\to\infty} \zeta(\hat{c}_n, \hat{q}_n) < 1.$$

In the context of the present discourse, the function denoted by ζ is characterized as a $\mathcal{Z}$-simulation function. This notation delineates a specific subset within the mathematical framework, where the set $\mathcal{Z}$ is comprehensively defined as the family encompassing all such mappings. It merits emphasis that the relationship $\zeta(\hat{c}, \hat{c}) < 1$, for all $\hat{c} > 1$ holds universally. This inequality is pivotal, suggesting that for any value of $\hat{c}$ greater than 1, the evaluation of the ζ function yields a result less than 1, thereby underscoring a fundamental characteristic of the $\mathcal{Z}$-simulation function within the specified mathematical framework.

Example 2 ([31]). *The set* $\mathcal{Z}$ *includes the functions* $\zeta_1, \zeta_2, \zeta_3 : [1, \infty) \times [1, \infty) \to \mathbb{R}$*, which are listed below.*

(1) $\zeta_1(\hat{c}, \hat{q}) = \hat{q}^{\alpha} \Big/ \hat{c}$, for all $\hat{c}, \hat{q} \geq 1$, *where* $\alpha \in (0, 1)$;

(2) $\zeta_2(\hat{c}, \hat{q}) = \hat{q}/\hat{c}\phi(\hat{q})$, for all $\hat{c}, \hat{q} \geq 1$, *where* ϕ *is a non-decreasing and lower semi-continuous self-mapping on* $[1, \infty)$ *such that* $\phi^{-1}(\{1\}) = 1$;

(3)

$$\zeta_3(\hat{c}, \hat{q}) = \begin{cases} 1, & \text{if } (\hat{c}, \hat{q}) = (1, 1), \\ \frac{\hat{q}}{2^{\hat{c}}}, & \text{if } \hat{q} < \hat{c}, \\ \frac{\hat{q}^{\alpha}}{\hat{c}}, & \text{otherwise,} \end{cases}$$

for all $\hat{c}, \hat{q} \geq 1$, *where* $\alpha \in (0, 1)$.

Let the class Ψ^* consist of all non-decreasing self-mappings ψ on the interval $[1, +\infty)$ that satisfy $\psi^{-1}(\{1\}) = 1$. By using the class Ψ^*, Öztürk et al. [16] (also in [32]) expanded on the concept of $\mathcal{Z}$-simulation function, as indicated below:

Definition 13 ([16]). *Let* $\hat{Z}$ *be the class of all mappings* $\eta : [1, \infty)^2 \to \mathbb{R}$*. Let a function* $\psi \in \Psi^*$ *and a coefficient* $\lambda \geq 1$ *exist such that*

(η_1) $\eta(1, 1) = 1$;
(η_2) $\eta(\hat{c}, \hat{q}) < \frac{\psi(\hat{q})}{\psi(\hat{c})}$, for all $\hat{c}, \hat{q} > 1$;
(η_2') $\eta(\hat{c}, \hat{q}) < \frac{\psi(\hat{q})}{\psi(\flat^{\lambda}\hat{c})}$, for all $\hat{c}, \hat{q} > 1$;

(η_3) *for all sequences* $\{\hat{c}_n\}, \{\hat{q}_n\} \subset (1,\infty)$ *with* $\hat{c}_n \leq \hat{q}_n$, for all $n = 1, 2, 3, ...$

$$\lim_{n\to\infty} \hat{c}_n = \lim_{n\to\infty} \hat{q}_n > 1 \quad \text{implies} \quad \limsup_{n\to\infty} \eta(\hat{c}_n, \hat{q}_n) < 1.$$

If η *provides* (η_2)-(η_3), *it is referred to as a generalized* Ψ^**-simulation function. Likewise, if* η *fulfills only the conditions* (η_2')-(η_3), *then* η *is called a generalized* $\Psi^*\flat$*-simulation function. Furthermore, note that if the axioms* (η_1)-(η_2)-(η_3) *are met by selecting* $\psi(\hat{c}) = \hat{c}$ *for all* $\hat{c} \geq 1$, *then the function* η *reduces to* $\mathcal{Z}$*-simulation function as defined in [31].*

Example 3 ([16]). *Consider the functions* $\eta_a, \eta_b, \eta_c, \eta_d : [1,\infty)^2 \to \mathbb{R}$ *as illustrated below:*

(1) $\eta_a(\hat{c}, \hat{q}) = \frac{\alpha\psi(\hat{q})}{\psi(\flat\hat{c})}$, for all $\hat{c}, \hat{q} \geq 1; \alpha \in (0,1)$;

(2) $\eta_b(\hat{c}, \hat{q}) = \frac{\psi(\hat{q})}{\psi(\flat^\alpha \hat{c})\phi(\hat{q})}$, for all $\hat{c}, \hat{q} \geq 1$ *and a coefficient* $\alpha \geq 1$, *where* ϕ *is a non-decreasing and lower semi-continuous self-mapping on* $[1,\infty)$ *such that* $\phi^{-1}(\{1\}) = 1$;

(3) $\eta_c(\hat{c}, \hat{q}) = \frac{\phi(\hat{q})}{\psi(\flat^\alpha \hat{c})}$, for all $\hat{c}, \hat{q} \geq 1$ *and a coefficient* $\alpha \geq 1$, *where* $\phi : [1,\infty) \to [1,\infty)$ *is a continuous function such that* $\phi(\hat{q}) < \psi(\hat{q})$ *for all* $\hat{q} > 0$.

(4)

$$\eta_d(\hat{c}, \hat{q}) = \begin{cases} 1, & \text{if } (\hat{c}, \hat{q}) = (1,1), \\ \frac{\psi(\hat{q})}{k\psi(\hat{c})}, & \text{if } \hat{q} < \hat{c}, \\ \frac{[\psi(\hat{q})]^p}{\psi(\hat{c})}, & \text{otherwise,} \end{cases}$$

for all $\hat{c}, \hat{q} \geq 1$, *where* $k \geq 1$ *and* $p \in (0,1)$.

The functions $\eta_a, \eta_b, \eta_c, \eta_d$ *belong to the class of generalized* $\Psi^*\flat$*-simulation functions. These functions also belong to the class of generalized* Ψ^**-simulation functions for* $\flat = 1$.

In 2014, the concept of $\mathfrak{L}$-contraction was introduced by Jleli and Samet [33], as follows:

Theorem 2 ([33]). *On a complete metric space* $(\mathcal{U}, d)$, $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ *is said to be* $\mathfrak{L}$*-contraction; that is,* $k \in (0,1)$ *exists such that the inequality*

$$d(\mathcal{T}\varphi, \mathcal{T}\gamma) \neq 0 \quad \text{implies} \quad \mathfrak{L}(d(\mathcal{T}\varphi, \mathcal{T}\gamma)) \leq [\mathfrak{L}(d(\varphi, \gamma))]^k$$

is provided for all $\varphi, \gamma \in \mathcal{U}$, *where* $\mathfrak{L} : (0,\infty) \to (1,\infty)$ *obey the following circumstances:*

$(\mathfrak{L}_1)$ $\mathfrak{L}$ *is non-decreasing;*

$(\mathfrak{L}_2)$ *for each sequence* $\{\hat{c}_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \mathfrak{L}(\hat{c}_n) = 1$ *if and only if* $\lim_{n\to\infty} \hat{c}_n = 0^+$;

$(\mathfrak{L}_3)$ $r \in (0,1)$ *and* $\gamma \in (0,\infty]$ *exist such that* $\lim_{\hat{c}\to 0^+} \frac{\mathfrak{L}(\hat{c})-1}{\hat{c}^r} = \gamma$.

Thereupon, the mapping $\mathcal{T}$ *owns a unique fixed point in* $(\mathcal{U}, d)$.

Define $\Theta = \{\mathfrak{L} : (0,\infty) \to (1,\infty) : \ \mathfrak{L} \text{ holds } (\mathfrak{L}_1)\}$.

Moreover, the following theorem was established by Geraghty [34] by using a novel class of functions. It is referred to as a Geraghty-type contraction.

Theorem 3 ([34]). *Consider the mapping* $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ *on a complete metric space* $(\mathcal{U}, d)$ *and the function* $\mathcal{B} : [0,\infty) \to [0,1)$ *which satisfies* $\lim_{n\to\infty} \mathcal{B}(\hat{q}_n) = 1$ implies $\lim_{n\to\infty} \hat{q}_n = 0$. *If the inequality, which is known as a Geraghty-type contraction,*

$$d(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq \mathcal{B}(d(\varphi, \gamma))d(\varphi, \gamma)$$

is fulfilled for all $\varphi, \gamma \in \mathcal{U}$, *then* $\mathcal{T}$ *has a unique fixed point in* $(\mathcal{U}, d)$.

Moreover, we denote the family of all the functions $\mathcal{B}$ by $\mathfrak{B}$.

In [36], the function $\mathcal{B}$ is modified using a suitable constant $\flat \geq 1$, which is verified in the b-metric function, as follows:

$$\mathfrak{B}_{\flat} = \left\{ \mathcal{B} : [0,\infty) \to \left[0, \frac{1}{\flat}\right) \middle| \lim_{n\to\infty} \mathcal{B}(\hat{q}_n) = \frac{1}{\flat} \quad \text{implies} \quad \lim_{n\to\infty} \hat{q}_n = 0 \right\}.$$

Fulga and Proca [35] presented the following theorem, which is an extension of the Geraghty-type contraction principle:

Theorem 4 ([35]). *Let $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ be a mapping on complete metric space $(\mathcal{U}, d)$. If there exists $\mathcal{B} \in \mathfrak{B}$ satisfying*

$$d(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq \mathcal{B}(\mathcal{E}(\varphi,\gamma))\mathcal{E}(\varphi,\gamma),$$

where

$$\mathcal{E}(\varphi,\gamma) = d(\varphi,\gamma) + |d(\varphi,\mathcal{T}\varphi) - d(\gamma,\mathcal{T}\gamma)|$$

then the mapping $\mathcal{T}$ owns a unique fixed point in $(\mathcal{U}, d)$.

Remark 1. *As a result of $\mathcal{B} : [0,\infty) \to \left[0, \frac{1}{\flat}\right)$, we derive*

$$d(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq \mathcal{B}(\mathcal{E}(\varphi,\gamma))\mathcal{E}(\varphi,\gamma) < \mathcal{E}(\varphi,\gamma).$$

3. Modular and $\mathfrak{R}$-Modular b-Metric-like Spaces

In the forthcoming section, we embark on an exploration of the concept of modular b-metric-like spaces, dissecting the intricate spaces delineated by this function alongside a thorough examination of their topological attributes. Subsequently, the discourse continues to uncover a novel structure, cleverly built by the merger of the modular b-metric-like framework and the auxiliary binary relation $\mathfrak{R}$. The clarification of key fundamental topological concepts further enhances this section of the research.

3.1. Modular b-Metric-like Spaces and Some Topological Properties

We briefly define a new concept referred to as modular b-metric-like space, as follows.

Definition 14. *Let $\mathcal{U}$ be a non-empty set, and $\flat \geq 1$ be a real number. A mapping $\varpi^* : (0,\infty) \times \mathcal{U} \times \mathcal{U} \to [0,\infty]$ is called a modular b-metric-like, if the circumstances*
$(\varpi_1^*)\varpi_\varrho^*(\varphi,\gamma) = 0$ for all $\varrho > 0$ implies $\varphi = \gamma$,
$(\varpi_2^*)\,\varpi_\varrho^*(\varphi,\gamma) = \varpi_\varrho^*(\gamma,\varphi)$ for all $\varrho > 0$,
$(\varpi_3^*)\,\varpi_{\varrho+\mu}^*(\varphi,\gamma) \leq \flat\left[\varpi_\varrho^*(\varphi,\theta) + \varpi_\mu^*(\theta,\gamma)\right]$ for all $\varrho,\mu > 0$
are provided for all $\varphi,\gamma,\theta \in \mathcal{U}$. Hence, $(\mathcal{U},\varpi^*) = \mathcal{U}_{\varpi^*}$ is called a modular b-metric-like space.

Note that a modular b-metric-like space reduces to a modular metric-like space in the case of $\flat = 1$. We say that a modular b-metric-like space is a natural extension of modular metric-like space.

Modular b-metric-like function ϖ^* on $\mathcal{U}_{\varpi^*}$ will also generate a topology τ_{ϖ^*} on $\mathcal{U}_{\varpi^*}$ whose base is the family of open ϖ^*-ball, $\{B_{\varpi^*}(\varphi,\varepsilon) : \varphi \in \mathcal{U} \text{ and } \varepsilon > 0\}$, where

- $\mathcal{B}_{\varpi^*}(\varphi,\varepsilon) = \left\{\gamma \in \mathcal{U}_{\varpi^*} : \left|\varpi_\varrho^*(\varphi,\gamma) - \varpi_\varrho^*(\varphi,\varphi)\right| < \varepsilon\right\}$

for all $\varphi \in \mathcal{U}_{\varpi^*}$, $0 < \varepsilon < \infty$ and $\varrho > 0$.

Definition 15. *Let $\mathcal{U}_{\varpi^*}$ be a modular b-metric-like space and $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}_{\varpi^*}$.*

(a) The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is called ϖ^*-convergent to $\varphi \in \mathcal{U}_{\varpi^*}$ if there exists a number $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\varpi_\varrho^*(\varphi_n,\varphi) \to \varpi_\varrho^*(\varphi,\varphi)$ as $n \to \infty$;

(b) The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is called ϖ^*-Cauchy sequence in $\mathcal{U}_{\varpi^*}$ if if there exists a number $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\lim_{n,m\to\infty} \varpi_\varrho^*(\varphi_n,\varphi_m)$ exists and is finite;

(*c*) *the space* $\mathcal{U}_{\varpi^*}$ *is said to be* ϖ^**-complete if every* ϖ^**-Cauchy sequence* $\{\varphi_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{U}_{\varpi^*}$ *is* ϖ^**-convergent with respect to* τ_{ϖ^*} *such that*

$$\lim_{m,n\to\infty} \varpi^*_\varrho(\varphi_m,\varphi_n) = \lim_{n\to\infty} \varpi^*_\varrho(\varphi_n,\varphi) = \varpi^*_\varrho(\varphi,\varphi);$$

(*d*) *The mapping* $\mathcal{T}:\mathcal{U}_{\varpi^*}\to\mathcal{U}_{\varpi^*}$ *is continuous if the following limit exists (finite):*

$$\lim_{n\to\infty} \varpi^*_\varrho(\varphi_n,\varphi) = \varpi^*_\varrho(\mathcal{T}\varphi,\varphi).$$

Example 4. *Let* $(\mathcal{U},\mathfrak{m})$ *be a modular metric-like space and* $\varpi^*_\varrho(\varphi,\gamma) = \left[\mathfrak{m}_\varrho(\varphi,\gamma)\right]^p$*, where* $p>1$ *is a real number. Hence,* $(\mathcal{U},\varpi^*)$ *is a modular b-metric-like space with* $\mathfrak{b}=2^{p-1}$*. Indeed, if we consider the inequality*

$$(a+b)^s \le 2^{s-1}(a^s+b^s)$$

for $a,b\in\mathbb{R}^+$*, then the proof can be easily completed.*

The following examples can be presented considering Example 1 and the above.

Example 5. *Let* $\mathcal{U}=\mathbb{R}$ *and the mappings* $\varpi^*_{\varrho j}:\mathcal{U}^*_\varpi\times\mathcal{U}^*_\varpi\to\mathbb{R}^+(j=1,2,3)$ *defined by*

$$\varpi^*_{\varrho 1}(\varphi,\gamma) = \frac{(|\varphi|+|\gamma|+a)^p}{\varrho},$$

$$\varpi^*_{\varrho 2}(\varphi,\gamma) = \frac{(|\varphi-b|+|\gamma-b|)^p}{\varrho},$$

$$\varpi^*_{\varrho 3}(\varphi,\gamma) = \frac{\left(\varphi^2+\gamma^2\right)^p}{\varrho}$$

be modular b-metric-like on $\mathcal{U}$*, where* $a\ge 1$ *and* $b\in\mathbb{R}$*.*

Example 6. *Let* $\mathcal{U}=\mathbb{R}$*. Then, the mapping* $\varpi^*_\varrho:\mathcal{U}^*_\varpi\times\mathcal{U}^*_\varpi\to\mathbb{R}^+$ *defined by*

$$\varpi^*_\varrho(\varphi,\gamma) = \frac{|\varphi-\gamma|^2}{\varrho} + \max\{\varphi,\gamma\} \tag{1}$$

is a modular b-metric-like function on $\mathcal{U}$ *with* $\mathfrak{b}=2$*. The representation provided in Figure 1 illustrates the graph of Equation (1) under the condition that* $\varrho=1$ *within the interval* $[0,1]\subseteq\mathbb{R}$*.*

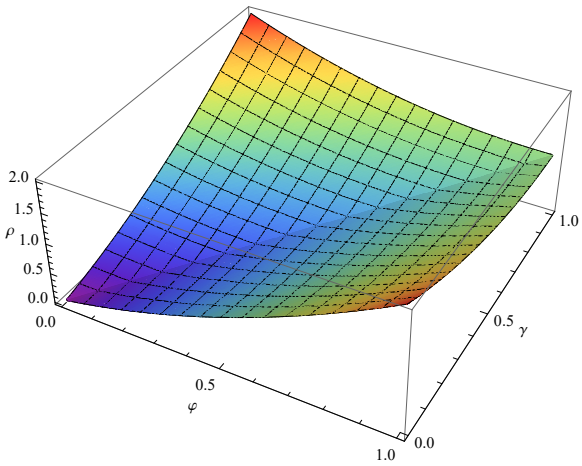


Figure 1. Three-dimensional representation of the equality (1).

Also, if $\varrho=0.1$*, then* ϖ^*_ϱ *equals "10" at the top.*

3.2. $\mathfrak{R}$-Modular b-Metric-like Spaces and Some Topological Properties

In the subsequent discourse, we endeavor to broaden the conceptual framework by introducing a novel notion wherein the binary relation, denoted as $\mathfrak{R}$, is further delineated and recognized within the context of $\mathfrak{R}$-modular *b*-metric-like spaces. This exploration aims to enrich the theoretical landscape and provide a foundation for advanced investigations in this domain.

Definition 16. *Let $\mathcal{U}$ be a non-empty set and $\mathfrak{R}$ be a reflexive binary relation on $\mathcal{U}$, denoted as $(\mathcal{U}, \mathfrak{R})$. A mapping $\omega^{\mathfrak{R}} : (0,\infty) \times \mathcal{U} \times \mathcal{U} \to [0,\infty]$ is called $\mathfrak{R}$-modular b-metric-like if the following statements hold: for all $\varphi, \gamma, \theta \in \mathcal{U}$ with $\varphi \mathfrak{S} \gamma$:*
$(\omega_1^{\mathfrak{R}})$ $\omega_\varrho^{\mathfrak{R}}(\varphi,\gamma) = 0$ *for all* $\varrho > 0$ implies $\varphi = \gamma$;
$(\omega_2^{\mathfrak{R}})$ $\omega_\varrho^{\mathfrak{R}}(\varphi,\gamma) = \omega_\varrho^{\mathfrak{R}}(\gamma,\varphi)$ *for all* $\varrho > 0$;
$(\omega_3^{\mathfrak{R}})$ $\omega_{\varrho+\mu}^{\mathfrak{R}}(\varphi,\gamma) \leq \flat\left[\omega_\varrho^{\mathfrak{R}}(\varphi,\theta) + \omega_\mu^{\mathfrak{R}}(\theta,\gamma)\right]$ *for all* $\varrho, \mu > 0$.
The triple $(\mathcal{U}_{\omega^{\mathfrak{R}}}, \mathfrak{R}, \flat) = \mathcal{U}_{\omega^{\mathfrak{R}}}$ is called an $\mathfrak{R}$-modular b-metric-like space.

Remark 2. *Let $\mathcal{U}$ be a set with a reflexive binary relation $\mathfrak{R}$. Consider $\omega^{\mathfrak{R}} : (0,\infty) \times \mathcal{U} \times \mathcal{U} \to [0,\infty]$ such that $(\omega_1^{\mathfrak{R}})$-$(\omega_3^{\mathfrak{R}})$ hold only for elements that are comparable under the reflexive binary relation $\mathfrak{R}$.*

It is feasible within the parameters of metric space theory that an entity defined as an $\mathfrak{R}$-modular b-metric-like structure may not necessarily conform to the characteristics of a modular b-metric-like configuration. However, it is imperative to note that the inverse proposition holds validity.

In what follows, we present that $\mathfrak{R}$-modular *b*-metric-like space with $\varrho \geq 1$ needs to be a modular *b*-metric-like space with $\varrho \geq 1$.

Example 7. *Let $\mathcal{U} = \{-2, -1, 0, 1\}$ and the binary relation $\mathfrak{R}$ be defined by $\varphi \mathfrak{R} \gamma$ if and only if $\varphi = \gamma$ or $\varphi, \gamma > 0$. Then, it is evident to conclude that*

$$\omega_\varrho^{\mathfrak{R}}(\varphi,\gamma) = (\max\{\varphi,\gamma\})^2$$

is an $\mathfrak{R}$-modular b-metric-like on $\mathcal{U}$ with $\varrho = 2$. But, for $\varphi = -2$ and $\gamma = 0$, the condition of $(\omega_1^{\mathfrak{R}})$ is not met, i.e., $\omega_\varrho^{\mathfrak{R}}(\varphi,\gamma) = (\max\{-2,0\})^2 = 0$ implies $\varphi \neq \gamma$, which means that it is not a modular b-metric-like on $\mathcal{U}$.

Definition 17. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an $\mathfrak{R}$-sequence in $\mathcal{U}_{\omega^{\mathfrak{R}}}$, that is, $\varphi_n \mathfrak{S} \varphi_{n+1}$ for each $n \in \mathbb{N}$. Then:*
(a) *the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is called $\omega^{\mathfrak{R}}-$convergent for some $\varphi \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ if if there exists a number $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\lim_{n\to\infty} \omega_\varrho^{\mathfrak{R}}(\varphi_n,\varphi) = \omega_\varrho^{\mathfrak{R}}(\varphi,\varphi)$ and $\varphi_n \mathfrak{R} \varphi$ for each $n \geq p$ with $p \in \mathbb{N}$;*
(b) *the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is called $\omega^{\mathfrak{R}}-$Cauchy sequence (or $\mathfrak{R}$-Cauchy) in $\mathcal{U}_{\omega^{\mathfrak{R}}}$ if there exists a number $\varrho = \varrho(\{\varphi_n\}) > 0$ such that $\lim_{n,m\to\infty} \omega_\varrho^{\mathfrak{R}}(\varphi_n,\varphi_m)$ exists and is finite;*
(c) *the space $\mathcal{U}_{\omega^{\mathfrak{R}}}$ is $\mathfrak{R}$-complete if for every $\mathfrak{R}$-Cauchy $\mathfrak{R}$-sequence in $\mathcal{U}_{\omega^{\mathfrak{R}}}$, there is $\varphi \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ with $\lim_{m,n\to\infty} \omega_\varrho^{\mathfrak{R}}(\varphi_m,\varphi_n) = \lim_{n\to\infty} \omega_\varrho^{\mathfrak{R}}(\varphi_n,\varphi) = \omega_\varrho^{\mathfrak{R}}(\varphi,\varphi)$ and $\varphi_n \mathfrak{R} \varphi$ for each $n \geq p$ with $p \in \mathbb{N}$.*

We assume that $\mathfrak{R}$ has a transitivity property throughout the study.

Lemma 1. *Let $\mathcal{T} : \mathcal{U}_{\omega^{\mathfrak{R}}} \to \mathcal{U}_{\omega^{\mathfrak{R}}}$ be a mapping and $\mathfrak{R}$ be a binary relation on $\mathcal{U}_{\omega^{\mathfrak{R}}}$. Also, there exists $\varphi_0 \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ be such that $\varphi_0 \mathfrak{R} \mathcal{T} \varphi_0$. Define an $\mathfrak{R}$-sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ by $\varphi_n = \mathcal{T}\varphi_{n-1} = \mathcal{T}^n \varphi_0$. Then, we have $\varphi_n \mathfrak{R} \varphi_m$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since $\mathcal{T}$ is $\mathfrak{R}$-preserving and $\varphi_0 \mathfrak{R} \mathcal{T} \varphi_0$, we conclude that $\varphi_1 \mathfrak{R} \varphi_2 = \mathcal{T}\varphi_0 \mathfrak{R} \mathcal{T}\varphi_1$. By continuing this way, we obtain $\varphi_n \mathfrak{R} \varphi_{n+1}$ for each $n \geq 0$. Presume that $\varphi_n \mathfrak{R} \varphi_m$ and prove

that $\varphi_n \mathfrak{R} \varphi_{m+1}$, where $m > n$. Since $\mathfrak{R}$ has a transitive property and $\varphi_m \mathfrak{R} \varphi_{m+1}$, we yield that $\varphi_n \mathfrak{R} \varphi_{m+1}$. Thus, we achieve that $\varphi_n \mathfrak{R} \varphi_m$ for all $m, n \in \mathbb{N}$ with $n < m$. □

4. Some Fixed Point Results

In the subsequent analysis, we establish the validity of a fixed point theorem within the framework of $\mathfrak{R}$-modular b-metric-like spaces.

Theorem 5. *Let $\mathcal{U}_{\omega^{\mathfrak{R}}}$ be an $\mathfrak{R}$-complete $\mathfrak{R}$-modular b-metric-like space with a constant $\flat \geq 1$ and $\varphi_0 \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ be such that $\varphi_0 \mathfrak{R} \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\mathfrak{R}}}$. Let $\mathcal{T} : \mathcal{U}_{\omega^{\mathfrak{R}}} \to \mathcal{U}_{\omega^{\mathfrak{R}}}$ be an $\mathfrak{R}$-preserving mapping and there exists a generalized $\Psi^* \flat$-simulation function with respect to η, $\mathfrak{L} \in \Theta$ and $\mathcal{B} \in \mathfrak{B}_\flat$ satisfying the following inequality for all $\varphi, \gamma \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ with $\varphi \mathfrak{R} \gamma$, $\varphi \mathfrak{R} \mathcal{T} \varphi$, $\gamma \mathfrak{R} \mathcal{T} \gamma$, $\varphi \mathfrak{R} \mathcal{T} \gamma$ and $\gamma \mathfrak{R} \mathcal{T} \varphi$:*

$$\eta\Big(\mathfrak{L}\Big(\flat \omega_\varrho^{\mathfrak{R}}(\mathcal{T}\varphi, \mathcal{T}\gamma)\Big), \mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi, \gamma))\mathcal{E}(\varphi, \gamma) + k\mathcal{N}(\varphi, \gamma))\Big) \geq 1, \tag{2}$$

where

$$\mathcal{E}(\varphi, \gamma) = \omega_\varrho^{\mathfrak{R}}(\varphi, \gamma) + \Big|\omega_\varrho^{\mathfrak{R}}(\varphi, \mathcal{T}\varphi) - \omega_\varrho^{\mathfrak{R}}(\gamma, \mathcal{T}\gamma)\Big| \tag{3}$$

and $k \geq 0$ with

$$\mathcal{N}(\varphi, \gamma) = \min\Big\{\omega_\varrho^{\mathfrak{R}}(\varphi, \mathcal{T}\varphi), \omega_\varrho^{\mathfrak{R}}(\gamma, \mathcal{T}\gamma), \omega_\varrho^{\mathfrak{R}}(\varphi, \mathcal{T}\gamma), \omega_\varrho^{\mathfrak{R}}(\gamma, \mathcal{T}\varphi)\Big\}.$$

Then $\mathcal{T}$ owns a fixed point $\varphi^* \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ with $\omega_\varrho^{\mathfrak{R}}(\varphi^*, \varphi^*) = 0$.

Proof. As $\varphi_0 \in \mathcal{U}_{\omega^{\mathfrak{R}}}$ is such that $\varphi_0 \mathfrak{R} \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\mathfrak{R}}}$, then by using the $\mathfrak{R}$-preserving nature of $\mathcal{T}$, we construct an $\mathfrak{R}$-sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ such that $\varphi_n = \mathcal{T}\varphi_{n-1} = \mathcal{T}^n \varphi_0$ and $\varphi_{n-1} \mathfrak{R} \varphi_n$ for each $n \in \mathbb{N}$. We consider $\varphi_n \neq \varphi_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Hence, from (2), we obtain

$$1 \leq \eta\Big(\mathfrak{L}\Big(\flat \omega_\varrho^{\mathfrak{R}}(\mathcal{T}\varphi_{n-1}, \mathcal{T}\varphi_n)\Big), \mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1}, \varphi_n))\mathcal{E}(\varphi_{n-1}, \varphi_n) + k\mathcal{N}(\varphi_{n-1}, \varphi_n))\Big) \tag{4}$$

and, by (η_2'),

$$1 < \frac{\psi(\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1}, \varphi_n))\mathcal{E}(\varphi_{n-1}, \varphi_n) + k\mathcal{N}(\varphi_{n-1}, \varphi_n)))}{\psi\Big(\flat^\lambda \mathfrak{L}\Big(\flat \omega_\varrho^{\mathfrak{R}}(\mathcal{T}\varphi_{n-1}, \mathcal{T}\varphi_n)\Big)\Big)}$$

that is,

$$\psi\Big(\flat^\lambda \mathfrak{L}\Big(\flat \omega_\varrho^{\mathfrak{R}}(\varphi_n, \varphi_{n+1})\Big)\Big) < \psi(\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1}, \varphi_n))\mathcal{E}(\varphi_{n-1}, \varphi_n) + k\mathcal{N}(\varphi_{n-1}, \varphi_n))).$$

Due to features of the function ψ, the above inequality becomes

$$\flat^\lambda \mathfrak{L}\Big(\flat \omega_\varrho^{\mathfrak{R}}(\varphi_n, \varphi_{n+1})\Big) < \mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1}, \varphi_n))\mathcal{E}(\varphi_{n-1}, \varphi_n) + k\mathcal{N}(\varphi_{n-1}, \varphi_n)).$$

As $\mathfrak{L} \in \Theta$, we derive that

$$\flat^{\lambda+1} \omega_\varrho^{\mathfrak{R}}(\varphi_n, \varphi_{n+1}) < \mathcal{B}(\mathcal{E}(\varphi_{n-1}, \varphi_n))\mathcal{E}(\varphi_{n-1}, \varphi_n) + k\mathcal{N}(\varphi_{n-1}, \varphi_n), \tag{5}$$

where

$$\begin{aligned}\mathcal{E}(\varphi_{n-1}, \varphi_n) &= \omega_\varrho^{\mathfrak{R}}(\varphi_{n-1}, \varphi_n) + \Big|\omega_\varrho^{\mathfrak{R}}(\varphi_{n-1}, \mathcal{T}\varphi_{n-1}) - \omega_\varrho^{\mathfrak{R}}(\varphi_n, \mathcal{T}\varphi_n)\Big| \\ &= \omega_\varrho^{\mathfrak{R}}(\varphi_{n-1}, \varphi_n) + \Big|\omega_\varrho^{\mathfrak{R}}(\varphi_{n-1}, \varphi_n) - \omega_\varrho^{\mathfrak{R}}(\varphi_n, \varphi_{n+1})\Big|\end{aligned}$$

and

$$\mathcal{N}(\varphi_{n-1},\varphi_n)=\min\Big\{\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\mathcal{T}\varphi_{n-1}),\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n,\mathcal{T}\varphi_n),\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\mathcal{T}\varphi_n),\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n,\mathcal{T}\varphi_{n-1})\Big\}$$

$$=\min\Big\{\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_n),\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n,\varphi_{n+1}),\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_{n+1}),0\Big\}=0.$$

Now, if we denote $\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_n)$ by χ_n, then (5) turns into

$$\mathfrak{b}^{\lambda+1}\chi_{n+1}<\mathcal{B}(\chi_n+|\chi_n-\chi_{n+1}|)(\chi_n+|\chi_n-\chi_{n+1}|). \tag{6}$$

Presume that there exists $n>0$ such that $\chi_n\leq\chi_{n+1}$. Hence, from (6), we obtain

$$\mathfrak{b}^{\lambda+1}\chi_{n+1}<\mathcal{B}(\chi_{n+1})\chi_{n+1}<\chi_{n+1},$$

which causes a contradiction. Thereby, for all $n>0$, we achieve $\chi_n>\chi_{n+1}$ such that $\mathcal{E}(\varphi_{n-1},\varphi_n)=2\chi_n-\chi_{n+1}$. Consequently, $\chi_n=\Big\{\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_n)\Big\}_{n\in\mathbb{N}}$ is a non-decreasing sequence and there exists a real number $r\geq 0$ such that $\lim_{n\to\infty}\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_n)=r$.

Now, we will demonstrate that $r=0$. Presume that r is not equal to zero, which gives us two cases to consider.

Case (1): Assume that $\mathfrak{b}>1$. Then, (5) becomes

$$\mathfrak{b}^{\lambda+1}\chi_{n+1}<\mathcal{B}(2\chi_n-\chi_{n+1})(2\chi_n-\chi_{n+1})<(2\chi_n-\chi_{n+1}).$$

In the above, if we consider the limit to be $n\to\infty$, we gain $\mathfrak{b}^{\lambda+1}r<r$, which defies our assumption.

Case (2): Assume that $\mathfrak{b}=1$. Again, from (4), we have

$$1\leq\eta(\mathfrak{L}(\chi_{n+1}),\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1},\varphi_n))\mathcal{E}(\varphi_{n-1},\varphi_n)+k\mathcal{N}(\varphi_{n-1},\varphi_n))), \tag{7}$$

where $\mathcal{E}(\varphi_{n-1},\varphi_n)$ and $\mathcal{N}(\varphi_{n-1},\varphi_n)$ as evaluated above. Consider

$$\hat{c}_n=\mathfrak{L}(\chi_{n+1})$$

and

$$\hat{q}_n=\mathfrak{L}(\mathcal{B}(2\chi_n-\chi_{n+1})(2\chi_n-\chi_{n+1})).$$

Then, the limits of $\{\hat{c}_n\}_{n\in\mathbb{N}}$ and $\{\hat{q}_n\}_{n\in\mathbb{N}}$ tend to $\mathfrak{L}(r)$. Indeed, considering (7) and (η_2), we obtain

$$\begin{aligned}\mathfrak{L}(\chi_{n+1})=\hat{c}_n&\leq\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_{n-1},\varphi_n))\mathcal{E}(\varphi_{n-1},\varphi_n))\\&=\hat{q}_n\\&=\mathfrak{L}(\mathcal{B}(2\chi_n-\chi_{n+1})(2\chi_n-\chi_{n+1}))\\&<\mathfrak{L}(2\chi_n-\chi_{n+1}).\end{aligned}$$

We take the limit in the above

$$\mathfrak{L}(r)=\lim_{n\to\infty}\mathfrak{L}(\chi_{n+1})\leq\lim_{n\to\infty}\hat{q}_n\leq\lim_{n\to\infty}\mathfrak{L}(2\chi_n-\chi_{n+1})=\mathfrak{L}(r),$$

which yields $\lim_{n\to\infty}\hat{c}_n=\lim_{n\to\infty}\hat{q}_n=\mathfrak{L}(r)>1$. Hence, from (η_3), we obtain that

$$\lim_{n\to\infty}\sup\eta(\hat{c}_n,\hat{q}_n)<1;$$

however, that is contradictory. Thereupon, in the two cases, we obtain $r=0$, that is,

$$\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n-1},\varphi_n)\to 0\quad\text{as}\quad \mathrm{n}\to\infty. \tag{8}$$

Now, we prove that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an $\mathfrak{R}$-Cauchy $\mathfrak{R}$-sequence. But in contrast, by considering Definition 17 (b), $\varepsilon > 0$ exists such that two sequences $\{m_k\}$ and $\{n_k\}$ can be constructed of positive integers satisfying $n_k > m_k \geq k$ such that $\omega^{\mathfrak{R}}_{2\varrho}(\varphi_{m_k}, \varphi_{n_k}) \geq \varepsilon$ for some $\varrho = \varrho(\{\varphi_n\}) > 0$. Note that $\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k}, \varphi_{n_k}) \geq \omega^{\mathfrak{R}}_{2\varrho}(\varphi_{m_k}, \varphi_{n_k})$, which yields

$$\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k}, \varphi_{n_k}) \geq \varepsilon. \tag{9}$$

Also, let n_k be the smallest index for some $\varrho = \varrho(\{\varphi_n\}) > 0$ satisfying the above condition such that

$$\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k}, \varphi_{n_k-1}) < \varepsilon. \tag{10}$$

By using (10) and $(\omega^{\mathfrak{R}}_3)$, we have

$$\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \varphi_{n_k-1}) \leq \flat\omega^{\mathfrak{R}}_{\frac{\varrho}{2}}(\varphi_{m_k-1}, \varphi_{m_k}) + \flat\omega^{\mathfrak{R}}_{\frac{\varrho}{2}}(\varphi_{m_k}, \varphi_{n_k-1}).$$

So, considering (8) and (10), we obtain

$$\limsup_{k\to\infty} \omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \varphi_{n_k-1}) \leq \flat\varepsilon. \tag{11}$$

On the other hand, from Lemma 1, we have $\varphi_{m_k-1}\mathfrak{R}\varphi_{n_k-1}$. Thereupon, from (2), we achieve

$$1 \leq \eta\left(\begin{array}{l} \flat\mathfrak{L}\Big(\omega^{\mathfrak{R}}_{\varrho}\big(\mathcal{T}\varphi_{m_k-1}, \mathcal{T}\varphi_{n_k-1}\big)\Big), \\ \\ \mathfrak{L}\big(\mathcal{B}\big(\mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big)\big)\mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big)\big) + k\mathcal{N}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) \end{array}\right), \tag{12}$$

where

$$\begin{aligned} &\mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) \\ &\quad = \omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \varphi_{n_k-1}) + \Big|\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \mathcal{T}\varphi_{m_k-1}) - \omega^{\mathfrak{R}}_{\varrho}(\varphi_{n_k-1}, \mathcal{T}\varphi_{n_k-1})\Big| \\ &\quad = \omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \varphi_{n_k-1}) + \Big|\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k-1}, \varphi_{m_k}) - \omega^{\mathfrak{R}}_{\varrho}(\varphi_{n_k-1}, \varphi_{n_k})\Big|. \end{aligned}$$

Using (8) and (11), we derive the following by taking the limit in the above as $k \to \infty$:

$$\limsup_{k\to\infty} \mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) \leq \flat\varepsilon. \tag{13}$$

Also, we have

$$\mathcal{N}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) = \min\left\{\begin{array}{l} \omega^{\mathfrak{R}}_{\varrho}\big(\varphi_{m_k-1}, \mathcal{T}\varphi_{m_k-1}\big), \omega^{\mathfrak{R}}_{\varrho}\big(\varphi_{n_k-1}, \mathcal{T}\varphi_{n_k-1}\big) \\ \\ \omega^{\mathfrak{R}}_{\varrho}\big(\varphi_{m_k-1}, \mathcal{T}\varphi_{n_k-1}\big), \omega^{\mathfrak{R}}_{\varrho}\big(\varphi_{n_k-1}, \mathcal{T}\varphi_{m_k-1}\big) \end{array}\right\}.$$

Thus, by (8), we achieve

$$\limsup_{k\to\infty} \mathcal{N}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) = 0. \tag{14}$$

Consequently, by contemplating $(\eta_2{}')$, $\mathfrak{L} \in \Theta$, $\mathcal{B} \in \mathfrak{B}_\flat$ and the function ψ, the inequality (12) becomes

$$\flat^{\lambda+1}\omega^{\mathfrak{R}}_{\varrho}(\varphi_{m_k}, \varphi_{n_k}) < \mathcal{B}\big(\mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big)\big)\mathcal{E}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big) + k\mathcal{N}\big(\varphi_{m_k-1}, \varphi_{n_k-1}\big).$$

Also, by taking the limit and considering (9), (13), and (14), the above expression turns into $\flat^{\lambda+1}\varepsilon < \frac{1}{\flat}\flat\varepsilon$. But this causes a contradiction. Hence, we obtain that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an $\mathfrak{R}$-Cauchy $\mathfrak{R}$-sequence and by the completeness of $\mathcal{U}_{\varpi^{\mathfrak{R}}}$, there exists $\varphi^* \in \mathcal{U}_{\varpi^{\mathfrak{R}}}$ such that

$$\lim_{n\to\infty} \varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \varphi^*) = \lim_{n,m\to\infty} \varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \varphi_m) = \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \varphi^*)$$

and $\varphi_n \mathfrak{R} \varphi^*$ for each $n \geq p$ (for some value of p). Thereby, for each $n \geq p$, we have $\varphi_n \mathfrak{R} \varphi^*$ and

$$\lim_{n\to\infty} \varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \varphi^*) = \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \varphi^*) = 0. \tag{15}$$

It will be demonstrated that φ^* is a fixed point that belongs to $\mathcal{T}$. Presume that $\varphi^* \neq \mathcal{T}\varphi^*$, so $\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \varphi^*) > 0$. Hence, from (2), we obtain

$$1 \leq \eta\Big(\flat\mathfrak{L}\Big(\varpi_{\varrho}^{\mathfrak{R}}(\mathcal{T}\varphi_n, \mathcal{T}\varphi^*)\Big), \mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi_n, \varphi^*))\mathcal{E}(\varphi_n, \varphi^*)) + k\mathcal{N}(\varphi_n, \varphi^*)\Big).$$

Likewise, by using (η_2'), $\mathfrak{L} \in \Theta$ and non-decreasing of the function ψ, the above inequality becomes

$$\flat^{\lambda+1}\varpi_{\varrho}^{\mathfrak{R}}(\varphi_{n+1}, \mathcal{T}\varphi^*) < \mathcal{B}(\mathcal{E}(\varphi_n, \varphi^*))\mathcal{E}(\varphi_n, \varphi^*) + k\mathcal{N}(\varphi_n, \varphi^*), \tag{16}$$

where

$$\mathcal{E}(\varphi_n, \varphi^*) = \varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \varphi^*) + \Big|\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \mathcal{T}\varphi_n) - \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*)\Big|,$$

which implies, by (15), $\mathcal{E}(\varphi_n, \varphi^*) \to \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*)$ as $n \to \infty$. Also,

$$\mathcal{N}(\varphi_n, \varphi^*) = \min\Big\{\varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \mathcal{T}\varphi_n), \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*), \varpi_{\varrho}^{\mathfrak{R}}(\varphi_n, \mathcal{T}\varphi^*), \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi_n)\Big\}.$$

Thus, considering (15), we achieve $\mathcal{N}(\varphi_n, \varphi^*) \to 0$ as $n \to \infty$.

Consequently, we take the limit as $n \to \infty$ in (16) and consider (15) together with the obtained ones; the inequality (16) turns into

$$\flat^{\lambda+1}\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*) < \mathcal{B}\Big(\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*)\Big)\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*) < \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*).$$

But this contradicts our assumption. Thereupon, we obtain $\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \mathcal{T}\varphi^*) = 0$, which implies $\varphi^* = \mathcal{T}\varphi^*$. □

Moreover, the uniqueness of the fixed point can be established by incorporating the property $(\mathcal{P})$ into the premises of Theorem 5.

$(\mathcal{P})$ For each fixed point φ^* and γ^* of $\mathcal{T}$ we have $\varphi^* \mathfrak{R} \gamma^*$ or $\gamma^* \mathfrak{R} \varphi^*$.

Indeed, from $(\mathcal{P})$, we have $\mathcal{T}\varphi^* = \varphi^* \neq \gamma^* = \mathcal{T}\gamma^*$ with $\varphi^* \mathfrak{R} \gamma^*$. Then, from (2), we obtain

$$1 \leq \eta\Big(\flat\mathfrak{L}\Big(\varpi_{\varrho}^{\mathfrak{R}}(\mathcal{T}\varphi^*, \mathcal{T}\gamma^*)\Big), \mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi^*, \gamma^*))\mathcal{E}(\varphi^*, \gamma^*)) + k\mathcal{N}(\varphi^*, \gamma^*)\Big)$$

and, similarly, inequality reduces

$$\flat^{\lambda+1}\varpi_{\varrho}^{\mathfrak{R}}(\mathcal{T}\varphi^*, \mathcal{T}\gamma^*) < \mathcal{B}(\mathcal{E}(\varphi^*, \gamma^*))\mathcal{E}(\varphi^*, \gamma^*) + k\mathcal{N}(\varphi^*, \gamma^*). \tag{17}$$

Thereupon, it is obvious that $\mathcal{E}(\varphi^*, \gamma^*) = \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*)$ and $\mathcal{N}(\varphi^*, \gamma^*) = 0$. Therefore, we obtain

$$\flat^{\lambda+1}\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*) < \mathcal{B}\Big(\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*)\Big)\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*) < \varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*),$$

which is a contradiction in the case of $\varpi_{\varrho}^{\mathfrak{R}}(\varphi^*, \gamma^*) \neq 0$. Thereby, we obtain $\varphi^* = \gamma^*$.

We detail several findings derived from applying Theorem 5. The results are elucidated as follows.

Corollary 1. *Let $\mathcal{U}_{\omega^{\Re}}$ be an $\Re$-complete $\Re$-modular b-metric-like space with a constant $\flat \geq 1$ and $\varphi_0 \in \mathcal{U}$ be such that $\varphi_0 \Re \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\Re}}$. Let $\mathcal{T} : \mathcal{U}_{\omega^{\Re}} \to \mathcal{U}_{\omega^{\Re}}$ be an $\Re$-preserving mapping and there exist $\psi \in \Psi^*$, $\mathfrak{L} \in \Theta$ and $\mathcal{B} \in \mathfrak{B}_\flat$ satisfying the following inequality for all $\varphi, \gamma \in \mathcal{U}_{\omega^{\Re}}$ with $\varphi \Re \gamma$, $\varphi \Re \mathcal{T}\varphi$ and $\gamma \Re \mathcal{T}\gamma$:*

$$\psi\left(\flat^{\lambda}\mathfrak{L}\left(\flat\omega_{\varrho}^{\Re}(\mathcal{T}\varphi, \mathcal{T}\gamma)\right)\right) \leq \psi(\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi, \gamma))\mathcal{E}(\varphi, \gamma))),$$

where $\mathcal{E}(\varphi, \gamma)$ defined as in (3). Then, $\mathcal{T}$ has a fixed point $\varphi^* \in \mathcal{U}_{\omega^{\Re}}$ with $\omega_{\varrho}^{\Re}(\varphi^*, \varphi^*) = 0$.

Proof. Taking into account the conditions set forth by $(\eta_2{}')$ and setting $k = 0$, we are able to achieve the intended result successfully. □

Corollary 2. *Let $\mathcal{U}_{\omega^{\Re}}$ be an $\Re$-complete $\Re$-modular b-metric-like space with $\flat \geq 1$ and $\varphi_0 \in \mathcal{U}$ be such that $\varphi_0 \Re \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\Re}}$. Let $\mathcal{T} : \mathcal{U}_{\omega^{\Re}} \to \mathcal{U}_{\omega^{\Re}}$ be an $\Re$-preserving mapping and there exist $\psi \in \Psi^*$, $\mathfrak{L} \in \Theta$ and $\mathcal{B} \in \mathfrak{B}_\flat$ satisfying the following inequality for all $\varphi, \gamma \in \mathcal{U}_{\omega^{\Re}}$ with $\varphi \Re \gamma$, $\varphi \Re \mathcal{T}\varphi$ and $\gamma \Re \mathcal{T}\gamma$:*

$$\psi\left(\flat^{\lambda}\mathfrak{L}\left(\flat\omega_{\varrho}^{\Re}(\mathcal{T}\varphi, \mathcal{T}\gamma)\right)\right) \leq \phi(\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi, \gamma))\mathcal{E}(\varphi, \gamma))),$$

where $\mathcal{E}(\varphi, \gamma)$ defined as in (3) and $\phi : [1, \infty) \to [1, \infty)$ is a continuous mapping, which has the property $\phi(\hat{q}) < \psi(\hat{q})$ for all $\hat{q} > 0$. Then, $\mathcal{T}$ has a fixed point $\varphi^* \in \mathcal{U}_{\omega^{\Re}}$ with $\omega_{\varrho}^{\Re}(\varphi^*, \varphi^*) = 0$.

Proof. In selecting $\eta(\hat{c}, \hat{q}) = \frac{\phi(\hat{q})}{\psi\left(\flat^{\lambda}\hat{c}\right)}$ for all $\hat{c}, \hat{q} > 1$ with $k = 0$, we draw upon the principles outlined in Theorem 5. This approach underscores an analytical framework where the relationship between the variables $\hat{c}$ and $\hat{q}$ is mediated through the functions ϕ and ψ, respectively. The decision to employ this specific form of η is substantiated by the comprehensive analysis presented in the theorem, which elucidates the conditions under which such a formulation yields optimal outcomes. □

Corollary 3. *Let $\mathcal{U}_{\omega^{\Re}}$ be an $\Re$-complete $\Re$-modular b-metric-like space with a constant $\flat \geq 1$ and $\varphi_0 \in \mathcal{U}$ be such that $\varphi_0 \Re \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\Re}}$. Let $\mathcal{T} : \mathcal{U}_{\omega^{\Re}} \to \mathcal{U}_{\omega^{\Re}}$ be an $\Re$-preserving mapping satisfying the following inequality for all $\varphi, \gamma \in \mathcal{U}_{\omega^{\Re}}$ with $\varphi \Re \gamma$, $\varphi \Re \mathcal{T}\varphi$ and $\gamma \Re \mathcal{T}\gamma$:*

$$\omega_{\varrho}^{\Re}(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq \delta\mathcal{E}(\varphi, \gamma),$$

where $\mathcal{E}(\varphi, \gamma)$ defined as in (3) and $\delta = \frac{1}{\flat^{\lambda+1}}$. Then, $\mathcal{T}$ has a fixed point $\varphi^* \in \mathcal{U}_{\omega^{\Re}}$ with $\omega_{\varrho}^{\Re}(\varphi^*, \varphi^*) = 0$.

Proof. Taking into consideration Remark 1, and incorporating $\psi \in \Psi^*$ alongside $\mathfrak{L} \in \Theta$, the subsequent proof is derived regarding Corollary 1. □

Upon choosing $\flat = 1$, a spectrum of outcomes is ushered into $\Re$-modular metric-like spaces. Notably, the adoption of $\flat = 1$, as delineated in Corollary 1, coupled with the contemplation of ψ belonging to Ψ^* and $\mathfrak{L}$ residing within Θ, culminates in the derivation of a novel result. This result embodies an intricate analysis of Theorem 3.6 as expounded in [18], enriching the field discourse.

Corollary 4. *Let $\mathcal{U}_{\omega^{\Re}}$ be an $\Re$-complete $\Re$-modular metric-like space and $\varphi_0 \in \mathcal{U}$ be such that $\varphi_0 \Re \gamma$ for each $\gamma \in \mathcal{U}_{\omega^{\Re}}$. Let $\mathcal{T} : \mathcal{U}_{\omega^{\Re}} \to \mathcal{U}_{\omega^{\Re}}$ be an $\Re$-preserving mapping and there exists $\mathcal{B} \in \mathfrak{B}$ satisfying the following inequality for all $\varphi, \gamma \in \mathcal{U}_{\omega^{\Re}}$ with $\varphi \Re \gamma$, $\varphi \Re \mathcal{T}\varphi$ and $\gamma \Re \mathcal{T}\gamma$:*

$$\omega_{\varrho}^{\Re}(\mathcal{T}\varphi, \mathcal{T}\gamma) \leq \mathcal{B}(\mathcal{E}(\varphi, \gamma))\mathcal{E}(\varphi, \gamma),$$

where $\mathcal{E}(\varphi, \gamma)$ defined as in (3). Then, $\mathcal{T}$ has a fixed point $\varphi^* \in \mathcal{U}_{\omega^{\Re}}$ with $\omega_{\varrho}^{\Re}(\varphi^*, \varphi^*) = 0$.

Building upon the foundational aspects delineated in Theorem 5, it is pertinent to note that the implications derived therefrom maintain their validity within the framework of modular *b*-metric-like spaces as well. In the subsequent discourse, we endeavor to articulate our principal theorem contextualized within this specific space.

Theorem 6. *Let $\mathcal{U}_{\omega^*}$ be an ω^*-complete modular b-metric-like space with a constant $\flat \geq 1$ and $\mathcal{T} : \mathcal{U}_{\omega^*} \to \mathcal{U}_{\omega^*}$ be a self-mapping. Then, there exists a generalized $\Psi^*\flat$-simulation function with respect to η, $\mathfrak{L} \in \Theta$ and $\mathcal{B} \in \mathfrak{B}_\flat$ such that the inequality*

$$\eta\Big(\mathfrak{L}\Big(\flat\omega^*_\varrho(\mathcal{T}\varphi,\mathcal{T}\gamma)\Big),\mathfrak{L}(\mathcal{B}(\mathcal{E}(\varphi,\gamma))\mathcal{E}(\varphi,\gamma)+k\mathcal{N}(\varphi,\gamma))\Big)\geq 1$$

is provided for all $\varphi,\gamma \in \mathcal{U}_{\omega^*}$, where

$$\mathcal{E}(\varphi,\gamma)=\omega^*_\varrho(\varphi,\gamma)+\Big|\omega^*_\varrho(\varphi,\mathcal{T}\varphi)-\omega^*_\varrho(\gamma,\mathcal{T}\gamma)\Big|$$

and $k \geq 0$ with

$$\mathcal{N}(\varphi,\gamma)=\min\Big\{\omega^*_\varrho(\varphi,\mathcal{T}\varphi),\omega^*_\varrho(\gamma,\mathcal{T}\gamma),\omega^*_\varrho(\varphi,\mathcal{T}\gamma),\omega^*_\varrho(\gamma,\mathcal{T}\varphi)\Big\}.$$

So, $\mathcal{T}$ owns a fixed point $\varphi^* \in \mathcal{U}_{\omega^*}$ with $\omega^*_\varrho(\varphi^*,\varphi^*)=0$.

Proof. The demonstration proceeds analogously to the argument presented in the proof of Theorem 5. □

In addition to the preceding theorem, various implications can be elucidated within the framework of modular *b*-metric-like spaces and modular metric-like spaces. For instance, Corollary 4 can be straightforwardly extrapolated from Theorem 3.6 as delineated in [18], albeit within the context of modular metric-like spaces. This underscored our findings as offering a more comprehensive generalization than the extant results documented within the scholarly literature.

5. Applications

5.1. On Dynamic Programming

Let Σ and Y be two Banach spaces such that $\beth \subseteq \Sigma$ and $\Xi \subseteq$ Y, where $\beth$ and Ξ denote state space and decision space, respectively. Then, two bounded functions $f : \beth \times \Xi \to \mathbb{R}$ and $G : \beth \times \Xi \times \mathbb{R} \to \mathbb{R}$ exist, and also, $\xi : \beth \times \Xi \to \beth$ such that

$$q(\varphi)=\sup_{\gamma\in\Xi}\{f(\varphi,\gamma)+G(\varphi,\gamma,q(\xi(\varphi,\gamma)))\},\ \varphi\in\beth.$$

Let $\mathcal{U}_{\omega^{\Re}}=B(\beth)$ denote the space of all bounded real-valued functions on $\beth$. Consider the metric defined by

$$\omega^{\Re}_\varrho(\varsigma,\kappa)=e^{-\varrho}\|\varsigma-\kappa\|^2+\|\varsigma\|+\|\kappa\|,$$

for all $\varsigma,\kappa\in\Sigma$ and for all $\varrho>0$, where $\|\varsigma\|=\sup_{\varphi\in\beth}|\varsigma(\varphi)|$. Also, the relation on $\mathcal{U}$ is defined by $\varsigma\Re\kappa$ if and only if $\varsigma=\kappa$ or $\varsigma,\kappa\geq 0$. Evidently, $(\mathcal{U}_{\omega^{\Re}},\Re,2)$ is an $\Re$-complete $\Re$-modular *b*-metric-like space. Moreover, let the mapping $\mathcal{T}:\mathcal{U}_{\omega^{\Re}}\to\mathcal{U}_{\omega^{\Re}}$ be given by

$$\mathcal{T}\varsigma(\varphi)=\sup_{\gamma\in\Xi}\{f(\varphi,\gamma)+G(\varphi,\gamma,\varsigma(\xi(\varphi,\gamma)))\},\tag{18}$$

where $\varphi\in\beth$ and $\varsigma\in\mathcal{U}_{\omega^{\Re}}$. If the functions f and G are bounded, then Σ and Y are well-defined.

Theorem 7. *Let $\mathcal{T}:\mathcal{U}_{\omega^{\Re}}\to\mathcal{U}_{\omega^{\Re}}$ be an operator as defined in (18) and presume that the below statements are met:*

(*i*) *f and G are bounded;*
(*ii*) *for all* $\varsigma, \kappa \in \mathcal{U}_{\omega^{\mathfrak{R}}}$, for all $\varphi \in \beth$ *and* $\gamma \in \Xi$, *there exists* $\delta \in (0,1)$ *such that*

$$|G(\varphi,\gamma,\varsigma(\varphi)) - G(\varphi,\gamma,\kappa(\varphi))| < \delta^{1/2}|\varsigma(\varphi) - \kappa(\varphi)|;$$

(*iii*) *for every* $(\varphi,\gamma) \in \beth \times \Xi$, *we have*

$$G(\varphi,\gamma,\varsigma(\varphi)) \leq \varsigma(\varphi) - f(\varphi,\gamma)$$

where $f : \beth \times \Xi \to \mathbb{R}$;
(*iv*) *for all* $\varsigma \in \mathcal{U}_{\omega^{\mathfrak{R}}}$, $|\sup \varsigma(\varphi)| \leq \delta|\varsigma(\varphi)|$, *where* $\delta \in (0,1)$.
The function Equation (18) *has a bounded solution; that is,* $\mathcal{T}$ *has a fixed point.*

Proof. Let $\epsilon \in \mathbb{R}^+$ be arbitrary, $\varphi \in \beth$ and $\varsigma \in \mathcal{U}_{\omega^{\mathfrak{R}}}$. Assume that $\mathcal{T}\varsigma \neq \varsigma$. Then, $\gamma_1, \gamma_2 \in \Xi$ exist such that

$$\mathcal{T}\varsigma(\varphi) < f(\varphi,\gamma_1) + G(\varphi,\gamma_1,\varsigma(\xi(\varphi,\gamma_1))) + \epsilon, \tag{19}$$

$$\mathcal{T}\kappa(\varphi) < f(\varphi,\gamma_2) + G(\varphi,\gamma_2,\kappa(\xi(\varphi,\gamma_1))) + \epsilon, \tag{20}$$

$$\mathcal{T}\varsigma(\varphi) \geq f(\varphi,\gamma_2) + G(\varphi,\gamma_2,\varsigma(\xi(\varphi,\gamma_2))), \tag{21}$$

$$\mathcal{T}\kappa(\varphi) \geq f(\varphi,\gamma_1) + G(\varphi,\gamma_1,\kappa(\xi(\varphi,\gamma_1))). \tag{22}$$

Then, from (19) and (22), we yield that

$$\begin{aligned}\mathcal{T}\varsigma(\varphi) - \mathcal{T}\kappa(\varphi) &< G(\varphi,\gamma_1,\varsigma(\xi(\varphi,\gamma_1))) - G(\varphi,\gamma_1,\kappa(\xi(\varphi,\gamma_1))) + \epsilon \\ &\leq |G(\varphi,\gamma_1,\varsigma(\xi(\varphi,\gamma_1))) - G(\varphi,\gamma_1,\kappa(\xi(\varphi,\gamma_1)))| + \epsilon \\ &< \delta^{1/2}|\varsigma(\varphi) - \kappa(\varphi)| + \epsilon.\end{aligned}$$

Likewise, using (20) and (21), we obtain

$$\begin{aligned}\mathcal{T}\kappa(\varphi) - \mathcal{T}\varsigma(\varphi) &< G(\varphi,\gamma_2,\kappa(\xi(\varphi,\gamma_2))) - G(\varphi,\gamma_2,\varsigma(\xi(\varphi,\gamma_2))) + \varepsilon \\ &\leq |G(\varphi,\gamma_2,\kappa(\xi(\varphi,\gamma_2))) - G(\varphi,\gamma_2,\varsigma(\xi(\varphi,\gamma_2)))| + \epsilon \\ &< \delta^{1/2}|\varsigma(\varphi) - \kappa(\varphi)| + \epsilon.\end{aligned}$$

Hence, by considering the above inequalities, we conclude that

$$|\mathcal{T}\varsigma(\varphi) - \mathcal{T}\kappa(\varphi)| < \delta^{1/2}|\varsigma(\varphi) - \kappa(\varphi)| + \epsilon,$$

and, for an arbitrary ϵ, we obtain

$$|\mathcal{T}\varsigma(\varphi) - \mathcal{T}\kappa(\varphi)| \leq \delta^{1/2}|\varsigma(\varphi) - \kappa(\varphi)|,$$

or, equivalently,

$$\|\mathcal{T}\varsigma(\varphi) - \mathcal{T}\kappa(\varphi)\| \leq \delta^{1/2}\|\varsigma(\varphi) - \kappa(\varphi)\|. \tag{23}$$

Furthermore, we have

$$|\mathcal{T}\varsigma(\varphi)| = \left|\sup_{\gamma_1 \in \Xi}\{f(\varphi,\gamma_1) + G(\varphi,\gamma_1,\varsigma(\xi(\varphi,\gamma_1)))\}\right| \leq \delta|\varsigma(\varphi)|$$

and hence,

$$\|\mathcal{T}\varsigma(\varphi)\| \leq \delta\|\varsigma(\varphi)\|. \tag{24}$$

So, from (23) and (24), we derive

$$\begin{aligned}\varpi_{\varrho}^{\Re}(\mathcal{T}\varsigma(\varphi),\mathcal{T}\kappa(\varphi)) &= e^{-\varrho}\|\mathcal{T}\varsigma(\varphi)-\mathcal{T}\kappa(\varphi)\|^2+\|\mathcal{T}\varsigma(\varphi)\|+\|\mathcal{T}\kappa(\varphi)\|\\ &\leq e^{-\varrho}\delta\|\varsigma(\varphi)-\kappa(\varphi)\|^2+\delta\|\varsigma(\varphi)\|+\delta\|\kappa(\varphi)\|\\ &= \delta\left[e^{-\varrho}\|\varsigma(\varphi)-\kappa(\varphi)\|^2+\|\varsigma(\varphi)\|+\|\kappa(\varphi)\|\right]\\ &= \delta\varpi_{\varrho}^{\Re}(\varsigma(\varphi),\kappa(\varphi)).\end{aligned} \tag{25}$$

Consequently, from the inequality (25), we deduce

$$\begin{aligned}\varpi_{\varrho}^{\Re}(\mathcal{T}\varsigma(\varphi),\mathcal{T}\kappa(\varphi)) &\leq \delta\varpi_{\varrho}^{\Re}(\varsigma(\varphi),\kappa(\varphi))\\ &\leq \delta\left[\varpi_{\rho}^{\Re}(\varsigma(\varphi),\kappa(\varphi))+\left|\varpi_{\rho}^{\Re}(\varsigma(\varphi),\mathcal{T}\varsigma(\varphi))-\varpi_{\rho}^{\Re}(\kappa(\varphi),\mathcal{T}\kappa(\varphi))\right|\right],\\ &= \delta\mathcal{E}(\varsigma(\varphi),\kappa(\varphi)),\end{aligned}$$

which means that by taking $\delta=\frac{1}{\rho^{\lambda+1}}\in(0,1)$, all the conditions of Corollary 3 are satisfied. Thus, we gain that $\mathcal{T}$ has a fixed point, i.e., the functional Equation (18) has a bounded solution. □

5.2. RLC Circuit Model

Photovoltaic is the direct conversion of light into atomic-level electricity. These materials exhibit a property known as the photoelectric effect, which causes them to absorb photons of light and release electrons. When these free electrons are captured, a current can be used as electricity. The first photovoltaic module was described as a solar cell but was too costly to be widely used. With the technology brought by space exploration, costs have been reduced, and solar panels as an alternative energy source have become an important energy source in today's world. By acquiring a fundamental comprehension of the process by which light is converted into electrical energy, it becomes possible to construct a mathematical model of the flow of electric current inside a parallel RLC circuit, commonly referred to as a tuning circuit; see [37].

In Figure 2, V is the voltage of the power source, I is the current in the circuit, R is the resistance of the resistor, L is the inductance of the inductor, and C is the capacitance of the capacitor. Of course, such problems can be mathematically modelled as initial value problems for the second-order ordinary differential equations of the form:

$$\begin{cases}\frac{d^2u}{d\mathfrak{t}^2}+\frac{\mathrm{R}}{\mathrm{L}}\frac{du}{d\mathfrak{t}}=\Im(\mathfrak{t},u(\mathfrak{t})),\ \mathfrak{t}\in[0,1],\\ u(0)=u'(0)=0,\end{cases} \tag{26}$$

where $\Im:[0,1]\times\mathbb{R}^+\to\mathbb{R}$ is a continuous function. In this section, we prove the existence of the solution to the RLC differential Equation (26). The problem (26) is equivalent to the following integral equation:

$$u(\mathfrak{t})=\int_0^1 \mathfrak{G}(\mathfrak{t},\vartheta)\Im(\vartheta,u(\vartheta))d\vartheta,\ \mathfrak{t}\in[0,1], \tag{27}$$

where $\mathfrak{G}$ is Green's function defined by

$$\mathfrak{G}(\mathfrak{t},\vartheta)=\begin{cases}(\mathfrak{t}-\vartheta)e^{\tau(\mathfrak{t}-\vartheta)}, & \text{if } 0\leq\vartheta\leq\mathfrak{t}\leq 1\\ 0, & \text{if } 0\leq\vartheta\leq\mathfrak{t}\leq 1\end{cases}$$

where $\tau > 0$ is a constant, calculated in terms of R and L, mentioned in (26).

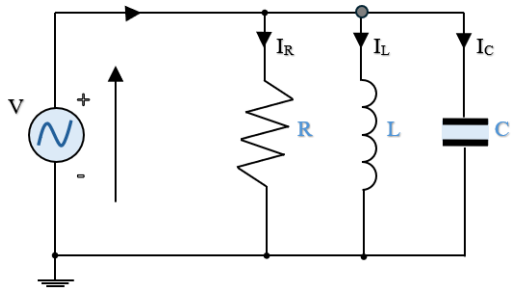


Figure 2. RLC parallel circuit [38].

Let $\mathcal{U} = (C[0,1], \mathbb{R}^+)$ be the space of all continuous functions defined on $[0,1]$. We endowed $\mathcal{U}$ with the modular b-metric like function $\omega^{\mathfrak{R}} : (0,\infty) \times \mathcal{U} \times \mathcal{U} \to [0,\infty]$ defined by

$$\omega^{\mathfrak{R}}_{\varrho}(\varphi,\gamma) = \frac{1}{\varrho} \sup_{\mathfrak{t}\in[0,1]} (|\varphi(\mathfrak{t})| + |\gamma(\mathfrak{t})|)^p, \quad \text{for all } \varrho > 0.$$

Also, consider $\mathfrak{R} = \{(u,v) \in \mathcal{U} \times \mathcal{U} : u(\mathfrak{t}) \geq v(\mathfrak{t}), \mathfrak{t} \in [0,1]\}$. Evidently, we conclude that $\mathcal{U}_{\omega^{\mathfrak{R}}}$ is an $\mathfrak{R}$-complete $\mathfrak{R}$-modular b-metric-like space with the constant $\flat = 2^{p-1}$.

Theorem 8. *Presume that the below statements are satisfied:*

(i) $\mathfrak{G}(\mathfrak{t},\vartheta) : [0,1] \times [0,1] \to [0,\infty)$ *is a continuous map;*

(ii) *a map* $\Im : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ *fulfills*

$$|\Im(\mathfrak{t}, u(\mathfrak{t}))| + |\Im(\mathfrak{t}, v(\mathfrak{t}))| \leq \delta^{\frac{1}{p}}(|u(\mathfrak{t})| + |v(\mathfrak{t})|),$$

where $\delta \in (0,1)$ *and for all* $u, v \in \mathcal{U}_{\omega^{\mathfrak{R}}}$.

Then, the integral Equation (27) *has a solution.*

Proof. Define $\mathcal{T} : \mathcal{U}_{\omega^{\mathfrak{R}}} \to \mathcal{U}_{\omega^{\mathfrak{R}}}$ by

$$\mathcal{T}u(\mathfrak{t}) = \int_0^1 \mathfrak{G}(\mathfrak{t},\vartheta)\Im(\vartheta, u(\vartheta))d\vartheta$$

for all $\mathfrak{t} \in [0,1]$. Note that the existence of a solution to the Equation (27) is equivalent to the existence of a fixed point of the mapping $\mathcal{T}$. For $u(\mathfrak{t}) \geq v(\mathfrak{t})$, we have

$$\begin{aligned}
\varpi_{\varrho}^{\Re}(Tu(\mathfrak{t}), Tv(\mathfrak{t})) &= \tfrac{1}{\rho}(|Tu(\mathfrak{t})| + |Tv(\mathfrak{t})|)^{p} \\
&= \tfrac{1}{\rho}\left(\left|\int_0^1 \mathfrak{G}(\mathfrak{t},\vartheta)\Im(\vartheta,u(\vartheta))d\vartheta\right| + \left|\int_0^1 \mathfrak{G}(\mathfrak{t},\vartheta)\Im(\vartheta,u(\vartheta))d\vartheta\right|\right)^{p} \\
&\leq \tfrac{1}{\rho}\left(\delta^{\frac{1}{p}} \sup_{\mathfrak{t}\in[0,1]} |u(\mathfrak{t})| + |v(\mathfrak{t})|\right)^{p}\left(\int_0^1 \mathfrak{G}(\mathfrak{t},\vartheta)d\vartheta\right)^{p} \\
&= \delta\tfrac{1}{\rho}\left(\sup_{\mathfrak{t}\in[0,1]} |u(\mathfrak{t})| + |v(\mathfrak{t})|\right)^{p}\left(\int_0^1 (\mathfrak{t}-\vartheta)e^{\tau(\mathfrak{t}-\vartheta)}d\vartheta - \tfrac{\vartheta^2}{2}e^{\tau(\mathfrak{t}-\vartheta)}\tau\left(\tfrac{-\vartheta^2}{2}\right)\right)^{p} \\
&= \delta\tfrac{1}{\rho}\left(\sup_{\mathfrak{t}\in[0,1]} |u(\mathfrak{t})| + |v(\mathfrak{t})|\right)^{p}\left(1-\tau e^{-\tau\vartheta} - e^{-\tau\vartheta}\right)^{p} \\
&\leq \delta\varpi_{\varrho}^{aR}(u(\mathfrak{t}), v(\mathfrak{t})),
\end{aligned}$$

where $\left(1-\tau e^{-\tau\vartheta} - e^{-\tau\vartheta}\right)^{p} \leq 1$. Also, we achieve

$$\begin{aligned}
\varpi_{\varrho}^{aR}(Tu(\mathfrak{t}), Tv(\mathfrak{t})) &\leq \delta\left[\varpi_{\varrho}^{aR}(u(\mathfrak{t}), v(\mathfrak{t})) + \left|\varpi_{\varrho}^{aR}(u(\mathfrak{t}), Tu(\mathfrak{t})) - \varpi_{\varrho}^{aR}(v(\mathfrak{t}), Tv(\mathfrak{t}))\right|\right] \\
&= \delta\mathcal{E}(u(\mathfrak{t}), v(\mathfrak{t})),
\end{aligned}$$

which explains that all the conditions of Corollary 3 hold. Consequently, the initial value problem (26) that emerges in an electric circuit has a solution. □

6. Conclusions

In conclusion, our work introduces the concept of a new, generalized metric space, which we have termed a modular *b*-metric-like space. This development extends and enhances the idea of modular metric-like spaces initially proposed by Sharma et al. [18]. Furthermore, our primary theorem significantly builds upon and contributes to the foundational results established by Geraghty [34], Fulga et al. [35], Karapınar et al. [39], and Aydi et al. [40], thereby broadening the scope and applicability of these seminal works within the realm of metric space theory.

Author Contributions: Conceptualization, E.G., A.B. and M.Ö.; Methodology, E.G., A.B. and M.Ö.; Formal analysis, E.G., A.B., N.K.K. and M.Ö.; Investigation, E.G., A.B., N.K.K. and M.Ö.; Data curation, E.G. and A.B.; Writing—original draft preparation, E.G., A.B., N.K.K. and M.Ö.; Writing—review and editing, E.G., A.B., N.K.K. and M.Ö.; Supervision, M.Ö. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors express their gratitude towards the anonymous reviewers for their insightful recommendations, which significantly contributed to the enhancement of the study.

Conflicts of Interest: The authors declare no conflicts of interest.

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