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A Theory for Interpolation of Metric Spaces

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Abstract: In this work, we develop an interpolation theory for metric spaces inspired by the real method of interpolation. These interpolation spaces preserve Lipschitz operators under certain conditions. We also show that this method, valid in metrics spaces, still holds in normed spaces without any algebraic structure required. Furthermore, this interpolation method for metric spaces when applied to normed spaces is equivalent to the K-method, which has been widely studied in the literature. As an application, we interpolate Fréchet sequence spaces.

Keywords: metric spaces; interpolation spaces; compact operators; Lipschitz operators

MSC: 54E35; 46B70; 46M35

1. Introduction

The theory of the interpolation of Banach spaces was inspired by two specific theorems for function spaces, which are the Riesz–Thorin Theorem and the Marcinkiewicz Theorem; see [1]. The essence of the theory of interpolation of the Banach spaces is to establish conditions and guarantees so that the following holds.

Given Banach spaces $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$, $(Y_0, \|\cdot\|_{Y_0})$, and $(Y_1, \|\cdot\|_{Y_1})$, with $X_0 \cap X_1 \neq \emptyset$ and $Y_0 \cap Y_1 \neq \emptyset$, there are two families of subspaces:

$$\{(X_\alpha, \|\cdot\|_\alpha)\}_{\alpha \in L}$$

and

$$\{(Y_\alpha, \|\cdot\|_\alpha)\}_{\alpha \in L},$$

where L is a set of indices such that the following are true:

- $X_0 \cap X_1 \subseteq X_\alpha \subseteq X_0 + X_1$ *and* $Y_0 \cap Y_1 \subseteq Y_\alpha \subseteq Y_0 + Y_1$, *with continuous inclusions for all* $\alpha \in L$;
- *If* $T : X_0 + X_1 \to Y_0 + Y_1$ *is a linear and continuous operator, where* $T|_{X_0} : X_0 \to Y_0$ *and* $T|_{X_1} : X_1 \to Y_1$ *are also linear and continuous operators, then* $T|_{X_\alpha} : X_\alpha \to Y_\alpha$ *is linear and continuous for all* $\alpha \in L$ *with respect to the norms of these spaces.*

In the early sixties of the last century, Lions, Peetre, Calderón, Gagliardo, Krein, and other authors were investigating the validity of Riesz–Thorin and the Marcinkiewicz Theorems for pairs of Banach spaces more general than function spaces. Two main interpolation methods were developed, the complex method of Calderón [2] and the real method from the work of Lions and Peetre [3]. At the same time, other methods were created, such as the mean method and the trace method.

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These works had a great impact, and in the mid-sixties and early seventies, new works appeared generalizing the real method in several directions. A very general construction for Banach spaces was obtained by Aronszajn and Gagliardo in [4], where they introduced an abstract construction of interpolation spaces encompassing the real and complex methods.

In environments more general than Banach spaces, Peetre, in [5], developed an interpolation theory for normed spaces. The real interpolation method for normed spaces is characterized by two interpolation methods, the J method and the K method, obtained through the functionals J and K. The methods K and J are equivalent in the case of normed spaces in the sense that they generate the same family of interpolated spaces, and these spaces maintain equivalent norms among themselves. Still in this direction, the trace and mean methods are also equivalent to the K method and, consequently, equivalent to each other.

In another direction, Peetre and Sparr in [6] developed an interpolation theory for normed abelian groups. Quasi-Banach spaces are also often considered in interpolation methods. A very general real interpolation theory for Banach spaces was developed in [7].

We also have in Lions-Peetre's paper [3] has a section where the interpolation of locally convex spaces is briefly considered. This calls attention to the possibilities of constructing interpolation theories beyond the normed space framework. The first systematic study in this direction appeared in the paper [8] by P. Krée.

On the other hand, a theory of interpolation spaces for metric spaces has been seldom considered in the literature. In [9], Peetre makes some comments on the interpolation of Lipschits operators, but with few details. A more serious attempt was made by Jan Gustavsson in [10], which is unpublished material as an article. His text has excellent ideas but also unclear points. However, it is the only material we could find on the subject, and it is our main reference and source of inspiration.

Current work follows the suggestive lines of Gustavsson's work [10]. Inspired by the real method, we also introduce the K and J methods for metric spaces. For the J method, we follow a similar approach to Gustavsson's, but for the K case, we introduce a new method. So, even though the statement of the results for the K method is similar to the ones of ref. [10], it must be clear that we are using a different definition. These interpolation methods have the advantage of relying only on the metric structure of the ambient space and do not require any algebraic structure.

The method developed in this work provides a way to interpolate Lipschitz operators between metric spaces, which is a more general case than a linear operator between normed spaces. To build interpolation spaces, we use the technique of completing metric spaces relative to others, using the ideas of Frink [11] to create metrics through distance functions. We also create new interpolation methods for normed spaces.

Among the main results of this work are Proposition 13 on the compactness of Lipschitz operators and all the results of Section 8, where we present a comparative analysis between the usual real interpolation for normed spaces and the interpolation for metric spaces developed here. We prove that the method of interpolation of metric spaces extends the real method of the interpolation of normed spaces. Furthermore, we show that it is possible to increase, in the sense of inclusion, the real normed interpolated space as long as there is a normed space greater than the space sums so that the component spaces are continuously included in this space.

In Sections 2–6, we introduce the theory of interpolation in metric spaces, defining and exploring the basic properties of the introduced methods. Throughout these sections, we seek to prove the properties common to the interpolation of normed spaces, that is, the properties that could be generalized to metric spaces. In Section 2, we provide the definition of the relative completion of a metric space in relation to another metric space, and we prove several properties. In Section 3, we introduce the functionals K_M and J_M, proving several properties that are similar to the ones in the normed case. With these functionals, in Section 4, we define the two interpolation methods for couples of metric spaces, also studying their properties and relations. The behaviors of Lipschitz operators

acting on couples of metric spaces is studied in Section 5. Section 6 is dedicated to the study of the interpolation of compact operators. We show that it is possible to generalize an important compactness result from linear operators between normed spaces to compact Lipschitz operators between metric spaces.

Section 7 makes the connection between the interpolation in normed spaces and in metric spaces. We show that, with a fixed $\theta \in (0,1)$ and $q \in [1,\infty)$, the K-functional of the metric case generates an interpolation space when the metric is given by the norm, which continuously contains the usual interpolation space, provided by the functionals K and J of the normed case. Therefore, metric interpolation, in addition to generalizing the interpolation in normed spaces, also creates a new method of interpolation for normed spaces that does not depend on any algebraic structure.

In the last section we present several examples. Initially, we show that the metric interpolation, in addition to generalizing the interpolation in normed spaces, still works for cases of interpolation of normed spaces that cannot be approached using the usual theory since the field of scalars is neither $\mathbb{R}$ nor $\mathbb{C}$. After, we work with metric Fréchet sequence spaces, obtaining an interesting interpolated space.

2. Relative Completion

In this section, we introduce some important definitions that will be constantly used throughout the rest of this work. For definitions and results of the theory of interpolation in Banach spaces, we suggest [1]. In particular, for the theory of interpolation in normed spaces, the main reference is [9].

Definition 1. *Given a set $M \neq \emptyset$, a metric on M is a function*

$$d : M \times M \to \mathbb{R}$$

$$(x,y) \mapsto d(x,y)$$

such that the following hold:

(i) $d(x,y) = 0 \Leftrightarrow x = y$;
(ii) *If $x \neq y$, then $d(x,y) > 0$;*
(iii) $d(x,y) = d(y,x)$ *for all $x, y \in M$;*
(iv) $d(x,y) \leq d(x,z) + d(z,y)$ *for all $x, y, z \in M$.*

Definition 2. *A metric space is a pair consisting of a non-empty set M and a metric d on M. We denote it by (M,d).*

When there is no possibility of confusion regarding the metric defined on M, we will refer to the metric space (M,d) only by M.

Remark 1. *If a function d satisfies the conditions (i) and (ii), we say that d is a distance function.*

We recall that a Cauchy sequence is a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (M,d) such that for every $\varepsilon > 0$, there exists an integer N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Next, we introduce one of our main definitions, which will be used to extend a metric space.

Let (B, d_B) and (A, d_A) be metric spaces such that $A \subseteq B$, and there is a constant $C > 0$ so that

$$d_B(x,y) \leq C\, d_A(x,y), \quad \text{for all } x, y \in A. \tag{1}$$

Given $x \in B$, if there is a Cauchy sequence (x_n) in (A, d_A) such that

$$d_B(x_n, x) \to 0,$$

we say that (x_n) is an approximation sequence of x in (A, d_A).

Let A_B be the set of all points $x \in B$ such that there is some approximation sequence of x in (A, d_A). We have that $A \subseteq A_B$, because given $x \in A$, the sequence $(x, x, x, \ldots)$ is an approximation sequence of x in (A, d_A). On $A_B \times A_B$, we define the following function:

$$\overline{d_A}(x, y) := \inf \lim_{n\to\infty} d_A(x_n, y_n), \tag{2}$$

where the infimum is taken over all approximation sequences x and y in (A, d_A).

We have the following classical result.

Proposition 1. *If (x_n) and (y_n) are Cauchy sequences in (A, d_A), then the limit $\lim_{n\to\infty} d_A(x_n, y_n)$ exists.*

Proposition 2. *We have that $\overline{d_A}$ is a distance function on A_B.*

Proof. For the positivity, we have that $\overline{d_A}(x, y) \geq 0$ for all $x, y \in A_B$ and $\overline{d_A}(x, x) = 0$. Furthermore, from Proposition 1, we have $\overline{d_A}(x, y) < \infty$. Now, let $x, y \in A_B$ such that $\overline{d_A}(x, y) = 0$. From (1),

$$d_B(x, y) = \inf \lim_{n\to\infty} d_B(x_n, y_n) \leq C \inf \lim_{n\to\infty} d_A(x_n, y_n) = C\,\overline{d_A}(x, y),$$

where the infimum is taken over all approximation sequences x and y in (A, d_A). Then,

$$d_B(x, y) \leq C\,\overline{d_A}(x, y), \tag{3}$$

for all $x, y \in A_B$. Thus, it follows that $d_B(x, y) = 0$, implying that $x = y$.

The symmetry follows from the symmetry of d_A. Therefore, the function $\overline{d_A}$ is a distance function and will be called in this case the distance function associated to the metric d_A. □

Definition 3. *Given a non-empty set A and $x, y \in A$, we define a linking sequence from x to y in A to any $(n+1)$-tuple $(x_0, x_1, x_2, \ldots, x_n)$ of points in A such that $x_0 = x$ and $x_n = y$. Let us denote a sequence of the type $(x, x_1, x_2, \ldots, x_{n-1}, y)$ by $(x_n^{x,y})$, which links x to y in A, and the set of all linking sequences from x to y in A by $[x, y]_A$.*

Remark 2. *In particular, n-tuples with repeated elements, and even those with all the same elements, are linking sequences. For the case where the linking sequences are of type $(x, x, x, \ldots, x)$ they will be linking sequences from x to x in A. We also have that $(x, x_1, x_2, \ldots, x_{n-1}, x)$ is a linking sequence from x to x in the set A.*

We now define the function d in $A_B \times A_B$, given by

$$d(x, y) := \inf_{(x_n^{x,y})} \sum_{k=0}^{n-1} \overline{d_A}(x_k, x_{k+1}), \tag{4}$$

where the infimum is taken over all sequences $(x_n^{x,y}) \in [x, y]_{A_B}$.

Remark 3. *Definition 6 appears in [10], but the function d in (4) is original from the current work.*

Lemma 1. *The function d is a metric on A_B.*

Proof. For the positivity, we have $d(x, y) \geq 0$ for all $x, y \in A_B$. We also have $d(x, x) = 0$, since, if we take the linking sequence from x to x in A_B given by $x_0 = x$ and $x_1 = x$, we have $d(x_0, x_1) \leq \overline{d_A}(x_0, x_1) = \overline{d_A}(x, x) = 0$. Furthermore, with any $x, y \in A_B$, we have $d(x, y) \leq \overline{d_A}(x, y) < \infty$ just by taking the linking sequence of x to y in A_B given by $x_0 = x$ and $x_1 = y$.

Now, let $x, y \in A_B$ be such that $d(x,y) = 0$ and consider any linking sequence from x to y in A_B. We have from inequality (3) that

$$d_B(x_k, x_{k+1}) \leq C\,\overline{d_A}(x_k, x_{k+1}),$$

for all $k = 0, 1, \ldots, n-1$. Through summing over k,

$$d_B(x,y) \leq \sum_{k=0}^{n-1} d_B(x_k, x_{k+1}) \leq C \sum_{k=0}^{n-1} \overline{d_A}(x_k, x_{k+1}).$$

In taking the infimum in the definition of d,

$$d_B(x,y) \leq C\,d(x,y), \tag{5}$$

for all $x, y \in A_B$. Then, it follows that $x = y$.

For the symmetry, note that given $x, y \in A_B$, if $x_0 = x, x_1, x_2, \ldots, x_n = y$ is a linking sequence from x to y in A_B, then $y_0 = x_n = y, y_1 = x_{n-1}, y_2 = x_{n-2}, \ldots, y_n = x_0 = x$ is a linking sequence from y to x in A_B, and

$$\sum_{k=0}^{n-1} \overline{d_A}(x_k, x_{k+1}) = \sum_{j=0}^{n-1} \overline{d_A}(y_j, y_{j+1}).$$

Therefore, there is a bijection between the linking sequences from x to y in A_B and the linking sequences from y to x in A_B, where the corresponding sequences in this bijection have the same sum as above. Thus, $d(x,y) = d(y,x)$ for all $x, y \in A_B$.

For the triangular inequality, let $x, y, z \in A_B$ and $\varepsilon > 0$. So there are linking sequences $(x_{n_\varepsilon}^{x,y})$ and $(y_{m_\varepsilon}^{y,z})$ from x to y in A_B and from y to z in A_B, respectively, such that

$$\sum_{i=0}^{n_\varepsilon - 1} \overline{d_A}(x_i, x_{i+1}) < d(x,y) + \varepsilon$$

and

$$\sum_{j=0}^{m_\varepsilon - 1} \overline{d_A}(y_j, y_{j+1}) < d(y,z) + \varepsilon.$$

The sequence given by $z_k = x_k$, for $0 \leq k \leq n_\varepsilon$ and $z_{n_\varepsilon + r} = y_r$ for $0 \leq r \leq m_\varepsilon$, is a linking sequence from x to z in A_B, and

$$\begin{aligned} d(x,z) &\leq \sum_{k=0}^{n_\varepsilon + m_\varepsilon - 1} \overline{d_A}(z_k, z_{k+1}) \\ &= \sum_{i=0}^{n_\varepsilon - 1} \overline{d_A}(x_i, x_{i+1}) + \sum_{j=0}^{m_\varepsilon - 1} \overline{d_A}(y_j, y_{j+1}) \\ &< d(x,y) + d(y,z) + 2\,\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $d(x,z) \leq d(x,y) + d(y,z)$. Therefore, d is a metric on A_B. □

Remark 4. *Given $x, y \in A_B$, taking the linking sequence (x_0, x_1) with $x_0 = x$ and $x_1 = y$, we obtain*

$$d(x,y) \leq \overline{d_A}(x,y). \tag{6}$$

From (5), we obtain that the inclusion of A_B in B is continuous in relation to the metrics defined in these spaces.

Definition 4. *The space (A_B, d), as constructed above, is called the relative completion of (A, d_A) in the space (B, d_B).*

Lemma 2. *If (A_B, d) is the relative completion of (A, d_A) with respect to the space (B, d_B), then $A \subseteq A_B$, and the inclusion is continuous, with respect to the metrics defined in those spaces.*

Proof. Given $x, y \in A$, consider the approximate sequences of x in (A, d_A) and of y in (A, d_A), respectively, given by $(x, x, x, \ldots)$ and $(y, y, y, \ldots)$. We have, by equality (2) that

$$\overline{d_A}(x, y) \leq d_A(x, y), \tag{7}$$

for all $x, y \in A$. Putting together (6) and (7), we obtain

$$d(x, y) \leq \overline{d_A}(x, y) \leq d_A(x, y), \tag{8}$$

for all $x, y \in A$. □

Relative Completion Properties

We present in this subsection the main properties of the relative completion.

Lemma 3. *If (A_B, d) is the relative completion of (A, d_A) in the space (B, d_B), then A is dense in (A_B, d).*

Proof. Let $x \in A_B$. Then, there is a Cauchy sequence (x_n) in (A, d_A) so that $d_B(x_n, x) \to 0$. For each $k \in \mathbb{N}$, we have from (6) that

$$d(x, x_k) \leq \overline{d_A}(x, x_k) \leq \lim_{n\to\infty} d_A(x_n, x_k).$$

Taking the limit with $k \to \infty$, we have $d(x, x_k) \to 0$, since (x_n) is a Cauchy sequence in (A, d_A). □

Proposition 3. *If (B, d_B) is a complete metric space and (A_B, d) is the relative completion of (A, d_A) in the space (B, d_B), then (A_B, d) is complete.*

Proof. Let $(x^N)_{N\in\mathbb{N}}$ be a Cauchy sequence in (A_B, d). For fixed N, and for every approximation sequence $(x_n^N)_{n\in\mathbb{N}}$ of x^N in (A, d_A), there is $n_N \in \mathbb{N}$ so that, for all $m, n \geq n_N$, one has

$$d_A(x_n^N, x_m^N) \leq \frac{1}{N}. \tag{9}$$

Given $\varepsilon > 0$, since (x^N) is a Cauchy sequence, there exists $N_0 \in \mathbb{N}$ such that $N_0 \geq 1/\varepsilon$, and

$$d(x^N, x^M) \leq \varepsilon\,, \tag{10}$$

for all $M, N \geq N_0$. For fixed $M, N \geq N_0$, there are approximation sequences (x_n^N) from x^N and (x_n^M) from x^M, respectively, so that

$$d(x^N, x^M) \leq \lim_{n\to\infty} d_A(x_n^M, x_n^N) < d(x^M, x^N) + \varepsilon \leq 2\varepsilon.$$

Since $\lim_{n\to\infty} d_A(x_n^M, x_n^N) \leq 2\varepsilon$, there is $n_\varepsilon \in \mathbb{N}$ such that

$$d_A(x_n^M, x_n^N) \leq 2\varepsilon, \quad \text{for all } n \geq n_\varepsilon. \tag{11}$$

For all $N \in \mathbb{N}$, let $y_N = x_{n_N}^N$, where (x_n^N) and (x_n^M) satisfy (11).

Now, if $n \geq \max\{n_\varepsilon, n_N, n_M\}$, we have

$$\begin{aligned} d_A(y_N, y_M) &\leq d_A(y_N, x_n^N) + d_A(x_n^N, x_n^M) + d_A(x_n^M, y_M) \\ &= d_A(x_{n_N}^N, x_n^N) + d_A(x_n^N, x_n^M) + d_A(x_n^M, x_{n_M}^M) \\ &\leq \frac{1}{N} + 2\varepsilon + \frac{1}{M} \leq 4\varepsilon. \end{aligned}$$

Thus, $(y_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in (A, d_A). Now, by the continuity of inclusion, $A_B \subset B$, (y^N) is a Cauchy sequence in (B, d_B), which is complete. Then, there is $x \in B$ such that $d_B(y_N, x) \to 0$. So, $x \in A_B$. Let us show that $d(x^N, x) \to 0$. We have, for all $N \geq N_0$,

$$d(x^N, x) \leq \lim_{j\to\infty} d_A(x_j^N, y_j) = \lim_{j\to\infty} d_A(x_j^N, x_{n_j}^j).$$

If $N \geq \max\{N_0, n_\varepsilon\}$, we have $d(x^N, x) \leq 2\varepsilon$ and, therefore, $d(x^N, x) \to 0$, completing the proof. □

We also need the following technical lemma.

Lemma 4. *Let (B, d_B) be a metric space and $A \subseteq B$. Let m_1 and m_2 be metrics on A with*

$$d_B(x, y) \leq \mu\, m_1(x, y) \leq \nu\, m_2(x, y),$$

for all $x, y \in A$, where μ and ν are positive constants. If A_B^1 and A_B^2 are the relative completions of (A, m_1) and (A, m_2), respectively, in (B, d_B), then $A_B^2 \subseteq A_B^1$ with continuous inclusion. We also have that

$$\mu\, d_1(x, y) \leq \nu\, d_2(x, y),$$

for all $x, y \in A_B^2$, where d_i is the metric of the completion of (A, m_i) in (B, d_B), with $i = 0, 1$.

Proof. Let $x \in A_B^2$. Then, there is a Cauchy sequence (x_n) in (A, m_2) such that $d_B(x_n, x) \to 0$. As $\mu\, m_1 \leq \nu\, m_2$, we have that (x_n) is a Cauchy sequence in (A, m_1). So, $x \in A_B^1$. We also have

$$\mu\, \overline{m_1}(x, y) \leq \inf \lim_{n\to\infty} \mu m_1(x_n, y_n) \leq \nu \inf \lim_{n\to\infty} m_2(x_n, y_n) = \nu\, \overline{m_2}(x, y),$$

where the infimum is taken over all approximation sequences x and y in (A, m_2). Then, it follows that $\mu\, d_1(x, y) \leq \nu\, \overline{m_2}(x, y)$, for all $x, y \in A_B^2$.

Now, let $x_0 = x, x_1, \ldots, x_n = y$ be a linking sequence from x to y in A_B^2. We have

$$\mu\, d_1(x, y) \leq \mu \sum_{k=0}^{n-1} d_1(x_k, x_{k+1}) \leq \nu \sum_{k=0}^{n-1} \overline{m_2}(x_k, x_{k+1}).$$

Taking the infimum over all linking sequences from x to y in A_B^2, we have $\mu\, d_1(x, y) \leq \nu\, d_2(x, y)$ for all $x, y \in A_B^2$. □

3. Compatible Metric Space Couple

Here, we begin to introduce the interpolation of metric spaces.

Definition 5. *Let X_0 and X_1 be two sets such that $X_0 \cap X_1 \neq \emptyset$, and both are subsets of a set X. Also, assume that d_0, d_1, and d_X are metrics on the sets X_0, X_1, and X, respectively, such that there are positive constants C_0 and C_1 satisfying*

$$d_X(x, y) \leq C_i\, d_i(x, y), . \tag{12}$$

for all $x, y \in X_i$, $i \in \{0, 1\}$. The pair (X_0, X_1) is called a compatible metric space couple, or a pair of compatible metric spaces, and it is denoted by $\overrightarrow{X}$ or $(X_0, X_1)_X$.

Remark 5. *It is always possible to obtain a compatible pair from two metric spaces X_0 and X_1, as we can see in the following.*

Let (X_0, d_0) and (X_1, d_1) be metric spaces with $X_0 \cap X_1 \neq \emptyset$. Consider $X = X_0 \times X_1$, and over X, consider the function d_X given by

$$d_X((x_0, x_1); (y_0, y_1)) := d_0(x_0, y_0) + d_1(x_1, y_1)\,.$$

We have the following results.

Proposition 4. *The function d_X is a metric over X.*

Proof. The triangle inequality is the only metric property that is not obvious. To prove it, let $(x_0, x_1), (y_0, y_1), (z_0, z_1) \in X$. We have

$$\begin{aligned} d_X((x_0, x_1); (y_0, y_1)) &= d_0(x_0, y_0) + d_1(x_1, y_1) \\ &\leq [d_0(x_0, z_0) + d_0(z_0, y_0)] + [d_1(x_1, z_1) + d_1(z_1, y_1)] \\ &= [d_0(x_0, z_0) + d_1(x_1, z_1)] + [d_0(z_0, y_0) + d_1(z_1, y_1)] \\ &= d_X((x_0, x_1); (z_0, z_1)) + d_X((z_0, z_1); (y_0, y_1)). \end{aligned}$$

□

Proposition 5. *Let $x_0 \in X_0$. Then, there exists an isometry between (X_1, d_1) and $\{x_0\} \times X_1$ with the metric induced by (X, d_X).*

Proof. Consider the function

$$f : X_1 \to \{x_0\} \times X_1$$

given by $f(x) = (x_0, x) \in \{x_0\} \times X_1$. It is clear that f is a bijection. Let us prove that f is an isometry. Indeed, let $x, y \in X_1$. Then,

$$d_X(f(x), f(y)) = d_X((x_0, x); (x_0, y)) = d_0(x_0, x_0) + d_1(x, y) = d_1(x, y)\,.$$

□

By Proposition 5, we can identify (X_1, d_1) with $(\{x_0\} \times X_1, d_X)$, and analogously, if $x_1 \in X_1$, there exists an isometry between (X_0, d_0) and $(X_0 \times \{x_1\}, d_X)$. This allows us to write, by identification,

$$X_k \subset X,$$

with

$$d_X(x, y) \leq d_k(x, y), \quad \forall x, y \in X_k, \quad k = 0, 1\,.$$

Given a metric space M, we denote the space $M \times M$ by M^2. Let $(X_0, X_1)_X$ be a compatible metric space couple and $t > 0$. For $(x, y) \in X_0^2 \cup X_1^2$, let $h_t : X_0^2 \cup X_1^2 \to \mathbb{R}^+$ be the function defined by

$$h_t(x, y) = \begin{cases} \min\{d_0(x, y), t\, d_1(x, y)\}, \text{ if } x, y \in X_0 \cap X_1; \\ d_0(x, y), \text{ if } (x, y) \in X_0^2 \setminus X_1^2; \\ t\, d_1(x, y), \text{ if } (x, y) \in X_1^2 \setminus X_0^2. \end{cases}$$

Definition 6. *Let X_0 and X_1 be sets such that $X_0 \cap X_1 \neq \emptyset$. Given $x, y \in X_0 \cup X_1$, an admissible linking sequence from x to y in $X_0 \cup X_1$ is a finite sequence of points $x_0, x_1, \ldots, x_n \in X_0 \cup X_1$ that satisfies the following:*

- *$x_0 = x$ and $x_n = y$;*
- *For each $k \in \{0, 1, \ldots, n-1\}$, we have $(x_k, x_{k+1}) \in X_0^2 \cup X_1^2$.*

Let us denote an admissible linking sequence $(x, x_1, x_2, \ldots, x_{n-1}, y)$ *from* x *to* y *in* $X_0 \cup X_1$ *by* $(x_n^{x,y})_{adm}$.

Remark 6. *Note that given any* $x, y \in X_0 \cup X_1$, *there is always an admissible linking sequence from* x *to* y *in* $X_0 \cup X_1$, *because if* $x, y \in X_i$, *then* $x_0 = x$ *and* $x_1 = y$ *is an admissible linking sequence from* x *to* y *in* $X_0 \cup X_1$, *for* $i = 0, 1$. *And when* $x \in X_i \setminus X_{1-i}$ *and* $y \in X_{1-i} \setminus X_i$, *since* $X_0 \cap X_1 \neq \emptyset$, *in choosing an element* $z \in X_0 \cap X_1$, *the sequence* $x_0 = x$, $x_1 = z$, *and* $x_2 = y$ *is an admissible linking sequence from* x *to* y *in* $X_0 \cup X_1$, *for* $i = 0, 1$.

3.1. The Functional K_M

In this section and in the next we introduce functionals analogous to the functionals K and J defined for normed spaces. These new functionals are suitable for the environment of metric spaces as they fulfill the same function as the functionals of normed spaces, without, however, depending on the sum or the scalar product.

Definition 7. *Let* $(X_0, X_1)_X$ *be a compatible metric space couple and* $t > 0$. *Given* $x, y \in X_0 \cup X_1$, *the functional* K_M *is defined by*

$$K_M(t, x, y) := \inf \sum_{k=0}^{n-1} h_t(x_k, x_{k+1}), \tag{13}$$

where the infimum is taken over all admissible linking sequences from x *to* y *in the set* $X_0 \cup X_1$.

Remark 7. *Using the function* h_t *in the definition of the functional* K_M, *we will obtain an interpolation method that is fundamentally different from the K method developed in Gustavsson's work, ref. [10], because changing the functional K changes the interpolated space. Gustavsson's K functional is finer in the sense that* $K_G(t, x, y) \leq K_M(t, x, y)$, *where* K_G *is the functional K defined in Gustavsson's original text. This inequality stems from the fact that the definition of the* K_G *functional admits more sequences than the definition of the* K_M *functional.*

The main advantage of the current work's K method over Gustavsson's K method is that his method is not necessarily equivalent to the usual K method for normed spaces, whereas the method developed in this text for metric spaces generalizes the usual K method.

Proposition 6. *If* $(X_0, X_1)_X$ *is a compatible metric space couple, then the functional* $K_M(1, \cdot, \cdot)$ *is a metric on* $X_0 \cup X_1$.

Proof. Given $x, y \in X_0 \cup X_1$, if $(x_n^{x,y})_{adm}$ is an admissible linking sequence of x in y, then

$$K_M(1, x, y) \leq \sum_{k=0}^{n-1} h_1(x_k, x_{k+1}) < \infty.$$

For the positivity, let $x \in X_0 \cup X_1$. So, $x_0 = x$ and $x_1 = x$ is an admissible linking sequence from x to x in $X_0 \cup X_1$. Then, we have $0 \leq K_M(1, x, x) \leq h_1(x, x) = 0$. From this, it follows that

$$K_M(1, x, x) = 0,$$

for all $x \in X_0 \cup X_1$. Now, let $x, y \in X_0 \cup X_1$ such that $K_M(1, x, y) = 0$. Let $(x_n^{x,y})_{adm}$ be an admissible linking sequence from x to y in $X_0 \cup X_1$. We have, for $C = \max\{C_0, C_1\}$,

$$d_X(x, y) \leq \sum_{k=0}^{n-1} d_X(x_k, x_{k+1}) \leq C \sum_{k=0}^{n-1} h_1(x_k, x_{k+1}).$$

Taking the infimum over all admissible linking sequences from x to y in $X_0 \cup X_1$, we have

$$d_X(x, y) \leq C\, K_M(1, x, y), \tag{14}$$

for all $x, y \in X_0 \cup X_1$. Since d_X is a metric, then $x = y$.

For the symmetry, if $(x_n^{x,y})_{adm}$ is an admissible linking sequence from x to y in $X_0 \cup X_1$, then

$$y_0 = x_n = y, y_1 = x_{n-1}, y_2 = x_{n-2}, \ldots, y_n = x_0 = x \in X_0 \cup X_1$$

is an admissible linking sequence from y to x such that

$$\sum_{i=0}^{n-1} h_1(x_i, x_{i+1}) = \sum_{j=0}^{n-1} h_1(y_j, y_{j+1}).$$

Thus, there is a bijection between the admissible linking sequences from x to y in $X_0 \cup X_1$ and the admissible linking sequences from y to x in $X_0 \cup X_1$. Furthermore, the sums of the images of each pair by the function h_1 of sequences corresponding by such a bijection are equal. Thus,

$$K_M(1, x, y) \leq \sum_{i=0}^{n-1} h_1(x_i, x_{i+1}) = \sum_{j=0}^{n-1} h_1(y_j, y_{j+1}),$$

for every admissible linking sequence $(y_n^{y,x})_{adm}$ from y to x in $X_0 \cup X_1$. From this, it follows that $K_M(1, x, y) \leq K_M(1, y, x)$. We also have

$$K_M(1, y, x) \leq \sum_{j=0}^{n-1} h_1(y_j, y_{j+1}) = \sum_{i=0}^{n-1} h_1(x_i, x_{i+1}),$$

for every admissible linking sequence $(x_n^{x,y})_{adm}$ from x to y in $X_0 \cup X_1$. Thus, $K_M(1, y, x) \leq K_M(1, x, y)$. And consequently, $K_M(1, x, y) = K_M(1, y, x)$.

For the triangular inequality, let $x, y, z \in X_0 \cup X_1$ and $\varepsilon > 0$. So there are $a_0 = x, a_1, \ldots, a_n = z$ and $b_0 = z, b_1, \ldots, b_m = y$ admissible linking sequences from x to z and from z to y, respectively, in $X_0 \cup X_1$ such that

$$\sum_{i=0}^{n-1} h_1(a_i, a_{i+1}) < K_M(1, x, z) + \varepsilon$$

and

$$\sum_{j=0}^{m-1} h_1(b_j, b_{j+1}) < K_M(1, z, y) + \varepsilon.$$

Define $c_k = a_k$, for $0 \leq k \leq n$, and $c_{n+r} = b_r$, for $0 \leq r \leq m$. Note that $(c_{m+n+1}^{x,y})_{adm}$ is an admissible linking sequence from x to y in $X_0 \cup X_1$ and that

$$\begin{aligned} K_M(1, x, y) &\leq \sum_{k=0}^{m+n-1} h_1(c_k, c_{k+1}) = \\ &= \sum_{i=0}^{n-1} h_1(a_i, a_{i+1}) + \sum_{j=0}^{m-1} h_1(b_j, b_{j+1}) \\ &< K_M(1, x, z) + K_M(1, z, y) + 2\,\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$K_M(1, x, y) \leq K_M(1, x, z) + K_M(1, z, y)\,,$$

which completes the proof. □

Lemma 5. *If $(X_0, X_1)_X$ is a compatible metric spaces couple, then*

$$K_M(1, x, y) \leq d_i(x, y),$$

for all $x, y \in X_i$, with $i \in \{0, 1\}$.

Proof. Let $x, y \in X_i$, with $i = 0, 1$. So, for the admissible linking sequence $x_0 = x$ and $x_1 = y$ from x to y in $X_0 \cup X_1$, we have

$$K_M(1, x, y) \leq h_1(x, y) \leq d_i(x, y).$$

□

Henceforth, unless otherwise noted and where there is no possibility of confusion, (X_0, d_0) and (X_1, d_1) will be considered a pair of compatible metric spaces with respect to the space (X, d_X).

Lemma 6. *If* $0 < a < b$, *then*

$$K_M(a, x, y) \leq K_M(b, x, y)\,,$$

for all $x, y \in X_0 \cup X_1$.

Proof. It suffices to note that given an admissible linking sequence $(x_n^{x,y})_{adm}$ from x to y in $X_0 \cup X_1$, we have

$$\sum_{k=0}^{n-1} h_a(x_k, x_{k+1}) \leq \sum_{k=0}^{n-1} h_b(x_k, x_{k+1}).$$

This result follows from taking the infimum over all admissible linking sequences from x to y in $X_0 \cup X_1$. □

Lemma 7. *If* $a, b \in (0, \infty)$ *and* $x, y \in X_0 \cup X_1$, *then*

$$K_M(a, x, y) \leq \max\left\{1, \frac{a}{b}\right\} K_M(b, x, y)\,. \tag{15}$$

Proof. If $a < b$, we have $\max\left\{1, \frac{a}{b}\right\} = 1$. Then,

$$\max\left\{1, \frac{a}{b}\right\} K_M(b, x, y) = K_M(b, x, y) \geq K_M(a, x, y),$$

from Lemma 6.

If $a > b$, $\max\left\{1, \frac{a}{b}\right\} = \frac{a}{b}$, and given any admissible linking sequence $(x_n^{x,y})_{adm}$ from x to y in $X_0 \cup X_1$,

$$\frac{a}{b} \sum_{k=0}^{n-1} h_b(x_{k+1}, x_k) \geq \sum_{k=0}^{n-1} h_a(x_{k+1}, x_k),$$

since $h_a(x_k, x_{k+1}) \leq \frac{a}{b}\, h_b(x_k, x_{k+1})$ for all $k = 0, 1, 2, \ldots, n-1$.

The result follows from taking the infimum over all admissible linking sequences from x to y in $X_0 \cup X_1$. □

Using Proposition 23 and Lemmas 24 to 26, we obtain the following result.

Theorem 1. *If* $t \in (0, \infty)$, *then the function* $K_M(t, \cdot, \cdot)$ *is a metric on* $X_0 \cup X_1$.

Proof. Let $t > 0$. By Lemma 7, with $a = 1$ and $b = t$, we have

$$\min\{1,\, t\}\, K_M(1, x, y) \leq K_M(t, x, y), \tag{16}$$

since $(\min\{1, t\})^{-1} = \max\{1, 1/t\}$. From this follows the positivity of $K_M(t, \cdot, \cdot)$, since $K_M(1, \cdot, \cdot)$ is a metric in $X_0 \cup X_1$.

Also from Lemma 7, with $a = t$ and $b = 1$, we have

$$K_M(t,x,y) \leq \max\{1,t\}\, K_M(1,x,y) < \infty,$$

for all $t > 0$ and $x, y \in X_0 \cup X_1$.

For symmetry, let $(x_n^{x,y})_{adm}$ be an admissible linking sequence from x to y in $X_0 \cup X_1$. Consider the admissible linking sequence from y to x in $X_0 \cup X_1$ given by $y_k = x_{n-k}$. We have

$$\sum_{k=0}^{n-1} h_t(x_k, x_{k+1}) = \sum_{j=0}^{n-1} h_t(y_j, y_{j+1}).$$

Taking the infimum and noting that this relationship between admissible linking sequences is one-to-one, we have the equality $K_M(t,x,y) = K_M(t,y,x)$.

For the triangular inequality, let $x, y, z \in X_0 \cup X_1$ and $\varepsilon > 0$. Then, there are admissible linking sequences $(x_n^{x,z})_{adm}$ from x to z in $X_0 \cup X_1$ and $(y_m^{z,y})_{adm}$ from z to y in $X_0 \cup X_1$ such that

$$\sum_{i=0}^{n-1} h_t(x_i, x_{i+1}) < K_M(t,x,z) + \varepsilon$$

and

$$\sum_{j=0}^{m-1} h_t(y_j, y_{j+1}) < K_M(t,z,y) + \varepsilon\,.$$

Consider the sequence given by $z_k = x_k$ if $0 \leq k \leq n$ and $z_{k+j} = y_j$ if $0 \leq j \leq m$. Note that $(z_{m+n}^{x,y})_{adm}$ is an admissible linking sequence from x to y in $X_0 \cup X_1$. So, we can write

$$\begin{aligned} K_M(t,x,y) \leq & \sum_{i=0}^{m+n-1} h_t(z_i, z_{i+1}) = \\ & = \sum_{k=0}^{n-1} h_t(x_k, x_{k+1}) + \sum_{j=0}^{m-1} h_t(y_j, y_{j+1}) < \\ & < K_M(t,x,z) + K_M(t,z,y) + 2\,\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$K_M(t,x,y) \leq K_M(t,x,z) + K_M(t,z,y).$$

□

3.2. The Functional J_M

If $x, y \in X_0 \cap X_1$ and $t > 0$, we define the functional as follows:

$$J_M(t,x,y) := \max\{d_0(x,y), t\, d_1(x,y)\}\,.$$

Proposition 7. *$J_M(t, \cdot\,, \cdot\,)$ is a metric on $X_0 \cap X_1$.*

Proof. We have $0 \leq J_M(t,x,y) < \infty$ for all $x, y \in X_0 \cap X_1$ and also $J_M(t,x,x) = 0$ for any $x \in X_0 \cap X_1$. Now, let $x, y \in X_0 \cap X_1$ such that $J_M(t,x,y) = 0$. Then, $0 = J_M(t,x,y) = \max\{d_0(x,y), t\, d_1(x,y)\}$, that is,

$$d_0(x,y) = t\, d_1(x,y) = 0 \Rightarrow x = y.$$

The symmetry is immediate and follows from the symmetry of d_0 and d_1. For the triangular inequality, given $x, y, z \in X_0 \cap X_1$,

$$J_M(t,x,z) + J_M(t,z,y) \geq d_0(x,z) + d_0(z,y) \geq d_0(x,y)$$

and

$$J_M(t,x,z) + J_M(t,z,y) \geq t\, d_1(x,z) + t\, d_1(z,y) \geq t\, d_1(x,y).$$

Thus, $J_M(t,x,z) + J_M(t,z,y) \geq \max\{d_0(x,y), t\, d_1(x,y)\} = J_M(t,x,y)$. □

Lemma 8. *If $a, b \in (0,\infty)$ and $x, y \in X_0 \cap X_1$, then*

$$J_M(a,x,y) \leq \max\left\{1, \frac{a}{b}\right\} J_M(b,x,y). \tag{17}$$

Proof. Let $a < b$. Then, $\max\left\{1, \frac{a}{b}\right\} = 1$. Since J_M is not decreasing in t, we have

$$J_M(a,x,y) \leq 1\, J_M(b,x,y) = \max\left\{1, \frac{a}{b}\right\} J_M(b,x,y).$$

Now, if $b < a$, one has $\max\left\{1, \frac{a}{b}\right\} = \frac{a}{b}$. Then,

$$\begin{aligned}\max\left\{1, \frac{a}{b}\right\} J_M(b,x,y) = \frac{a}{b}\, J_M(b,x,y) &= \max\left\{\frac{a}{b} d_0(x,y), \frac{a}{b} b\, d_1(x,y)\right\} \\ &\geq \max\left\{d_0(x,y), a\ d_1(x,y)\right\} = J_M(a,x,y).\end{aligned}$$

□

Lemma 9. *If $x, y \in X_0 \cap X_1$ and $a, b \in (0,\infty)$, then*

$$K_M(a,x,y) \leq \min\left\{1, \frac{a}{b}\right\} J_M(b,x,y). \tag{18}$$

Proof. The sequence $x_0 = x$ and $x_1 = y$ links x to y in $X_0 \cup X_1$ and is admissible. Soon,

$$\begin{aligned}K_M(a,x,y) &\leq h_a(x,y) \leq d_0(x,y) \\ &\leq J_M(b,x,y) = \max\{d_0(x,y), b\, d_1(x,y)\},\end{aligned}$$

and

$$\begin{aligned}K_M(a,x,y) &\leq h_a(x,y) \leq a\, d_1(x,y) \\ &\leq \max\left\{\frac{a}{b}\, d_0(x,y), a\ d_1(x,y)\right\} \\ &= \frac{a}{b}\, J_M(b,x,y).\end{aligned}$$

Thus,

$$K_M(a,x,y) \leq \min\left\{1, \frac{a}{b}\right\} J_M(b,x,y).$$

□

4. Interpolation Spaces

In this section we make a connection between relative completion and the K_M and J_M functionals defined in the previous section. Such a connection will be made through the functionals $\Phi_{\theta,q}$ and $\Gamma_{\theta,q}$ defined below, which are the same as the functionals of the normed case.

Definition 8. *Given* $0 < \theta < 1$, $q \in [1,\infty]$, *and a real measurable function* ϕ, *let* $\Phi_{\theta,q}$ *be the functional*

$$\Phi_{\theta,q}(\phi) := \begin{cases} \left[\int_0^\infty [t^{-\theta}|\phi(t)|]^q \frac{dt}{t}\right]^{\frac{1}{q}}, & \text{if } q < \infty \\ \sup\limits_{t>0} \{t^{-\theta}|\phi(t)|\}, & \text{if } q = \infty\,. \end{cases}$$

We will also need the following result, which we adapted from [1].

Lemma 10. *If* $f : (0,\infty) \to [0,\infty)$ *is a measurable function and* $s > 0$, *then*

$$\Phi_{\theta,q}(f(t/s)) = s^{-\theta}\,\Phi_{\theta,q}(f(t))\,.$$

Proof. Let $q < \infty$. We have that, with $s > 0$

$$\begin{aligned} \Phi_{\theta,q}(f(t/s)) &= (\int_0^\infty (t^{-\theta} f(t/s))^q dt/t)^{1/q} \\ &= s^{-\theta}(\int_0^\infty ((t/s)^{-\theta} f(t/s))^q d(t/s)/(t/s))^{1/q} \\ &= s^{-\theta}\Phi_{\theta,q}(f(t))\,. \end{aligned}$$

□

Definition 9. *Given* $0 < \theta < 1$, $q \in [1,\infty]$, *and a sequence* $c = (c_k)_{k\in\mathbb{Z}}$, *let* $\Gamma_{\theta,q}(c)$ *be the functional*

$$\Gamma_{\theta,q}(c) := \begin{cases} \left(\sum\limits_{k=-\infty}^{\infty} [2^{-k\theta}|c_k|]^q\right)^{\frac{1}{q}}, & \text{if } q < \infty \\ \sup\limits_{k\in\mathbb{Z}} [2^{-k\theta}|c_k|], & \text{if } q = \infty\,. \end{cases}$$

The set of sequences $c = (c_k)_{k\in\mathbb{Z}}$ *such that* $\Gamma_{\theta,q}(c) < \infty$ *is a normed vector space that we denote by* $\lambda_{\theta,q}$, *with norm* $\| (c_k) \|_{\lambda_{\theta,q}} = \Gamma_{\theta,q}((c_k))$.

Our objective is to build metric interpolation spaces using the relative completion of the intersection of convenient metrics adapted to the spaces in question.

Theorem 2. *If* $\theta \in (0,1)$ *and* $q \in [1,\infty]$, *then*

$$\beta_{\theta,q}(x,y) := \Phi_{\theta,q}(K_M(\,\cdot\,,x,y))\,,$$

is a metric on $X_0 \cap X_1$, *where* $\Phi_{\theta,q}$ *is given in Definition 8.*

Proof. Let $q < \infty$. We have $\beta_{\theta,q}(x,y) \geq 0$ for all $x, y \in X_0 \cap X_1$ and $\beta_{\theta,q}(x,x) = 0$ for all $x \in X_0 \cap X_1$. Furthermore, by Lemma 9, with $a = t$ and $b = 1$, we have

$$K_M(t,x,y) \leq \min\{1,t\} J_M(1,x,y),$$

for all $x, y \in X_0 \cap X_1$, and $t > 0$. Applying the functional $\Phi_{\theta,q}$:

$$\beta_{\theta,q}(x,y) \leq M_{\theta,q}\, J_M(1,x,y), \tag{19}$$

where

$$M_{\theta,q} = \Phi_{\theta,q}(\min\{1,\cdot\}) \in \mathbb{R}_+. \tag{20}$$

Thus, if $x, y \in X_0 \cap X_1$, then $\beta_{\theta,q}(x,y) < \infty$.

Now, let $x, y \in X_0 \cap X_1$ be such that $\beta_{\theta,q}(x,y) = 0$. Putting together inequalities (14) and (16), and applying the functional $\Phi_{\theta,q}$, we obtain

$$C^{-1} M_{\theta,q}\, d_X(x,y) \leq \beta_{\theta,q}(x,y). \tag{21}$$

Since d_X is a metric, then $x = y$.

The symmetry follows from the symmetry of $K_M(\cdot,\cdot,\cdot)$ in the second and third inputs.

For the triangular inequality, given $x, y, z \in X_0 \cap X_1$, for all $t > 0$, we have

$$K_M(t,x,y) \leq K_M(t,x,z) + K_M(t,z,y).$$

In applying the functional $\Phi_{\theta,q}$ in the above inequality,

$$\Phi_{\theta,q}(K_M(\cdot,x,y)) \leq \Phi_{\theta,q}(K_M(\cdot,x,z) + K_M(\cdot,z,y)),$$

that is,

$$\begin{aligned}
\left[\int_0^\infty [t^{-\theta} K_M(t,x,y)]^q \frac{dt}{t}\right]^{1/q} &\leq \left[\int_0^\infty [t^{-\theta}\,(K_M(t,x,z) + K_M(t,z,y))]^q \frac{dt}{t}\right]^{1/q} \\
&= \left[\int_0^\infty [t^{-\theta-1/q} K_M(t,x,z) + t^{-\theta-1/q} K_M(t,z,y)]^q dt\right]^{1/q} \\
&\overset{(*)}{\leq} \left[\int_0^\infty [t^{-\theta} K_M(t,x,z)]^q \frac{dt}{t}\right]^{1/q} + \\
&+ \left[\int_0^\infty [t^{-\theta} K_M(t,z,y)]^q \frac{dt}{t}\right]^{1/q},
\end{aligned}$$

where the inequality in $(*)$ is due to Minkowski's inequality applied to the functions $f(t) = t^{-\theta-1/q} K_M(t,x,z)$ and $g(t) = t^{-\theta-1/q} K_M(t,z,y)$. So,

$$\beta_{\theta,q}(x,y) \leq \beta_{\theta,q}(x,z) + \beta_{\theta,q}(z,y),$$

as we want. Therefore, $\beta_{\theta,q}$ is a metric on $X_0 \cap X_1$.

If $q = \infty$, we have $\beta_{\theta,\infty}(x,y) = \sup_{t>0} t^{-\theta} K(t,x,y) \geq 0$ and $\beta_{\theta,\infty}(x,x) = 0$. Let $x, y \in X_0 \cap X_1$ such that $\beta_{\theta,\infty}(x,y) = 0$. Again, from inequalities (14) and (16),

$$t^{-\theta} \min\{1,t\}\, d_X(x,y) \leq C\, t^{-\theta} K(t,x,y).$$

In taking the supremum over all $t > 0$,

$$d_X(x,y) \leq C\, \beta_{\theta,\infty}(x,y) = 0.$$

Since d_X is a metric, $x = y$. Symmetry is immediate.

Let $x, y, z \in X_0 \cap X_1$. We have

$$t^{-\theta} K(t,x,y) \leq t^{-\theta} K(t,x,z) + t^{-\theta} K(t,z,y),$$

for all $t > 0$. In taking the supreme,

$$\beta_{\theta,\infty}(x,y) \leq \beta_{\theta,\infty}(x,z) + \beta_{\theta,\infty}(z,y).$$

□

Proposition 8 (Discretization of $\beta_{\theta,q}$)**.** *If $\theta \in (0,1)$, $q \in [1,\infty]$, and $x, y \in X_0 \cap X_1$, then $\| (K_M(2^i,x,y))_{i\in\mathbb{Z}} \|_{\lambda_{\theta,q}}$ is a metric equivalent to $\beta_{\theta,q}(x,y)$.*

Proof. Let $q < \infty$. For all $i \in \mathbb{Z}$, we have

$$K_M(2^{i+1}, x, y) \leq \max\{1, 2^{i+1}/2^i\} K_M(2^i, x, y) = 2K_M(2^i, x, y).$$

Then, for every $t > 0$, there is $i \in \mathbb{Z}$ such that $2^i \leq t \leq 2^{i+1}$. Thus,

$$\beta^q_{\theta,q}(x, y) = \sum_{i\in\mathbb{Z}} \int_{2^i}^{2^{i+1}} [t^{-\theta} K_M(t, x, y)]^q \frac{dt}{t}.$$

Taking into account that $2^{-i\theta} \geq t^{-\theta} \geq 2^{-i\theta}2^{-\theta}$ and that $K_M(\cdot, x, y)$ is non-decreasing, we have

$$[2^{-\theta}2^{-i\theta} K_M(2^i, x, y)]^q \leq [t^{-\theta} K_M(t, x, y)]^q \leq [2^{-i\theta+1} K_M(2^i, x, y)]^q.$$

Integrating each member with respect to $\frac{dt}{t}$ in the interval $[2^i, 2^{i+1}]$, we have

$$\begin{aligned}\int_{2^i}^{2^{i+1}} [2^{-\theta}2^{-i\theta} K_M(2^i, x, y)]^q \frac{dt}{t} &\leq \int_{2^i}^{2^{i+1}} [t^{-\theta} K_M(t, x, y)]^q \frac{dt}{t} \\ &\leq \int_{2^i}^{2^{i+1}} [2^{-i\theta+1} K_M(2^i, x, y)]^q \frac{dt}{t}.\end{aligned}$$

And we obtain

$$\begin{aligned}2^{-\theta q} \ln 2 [2^{-i\theta} K_M(2^i, x, y)]^q &\leq \int_{2^i}^{2^{i+1}} [t^{-\theta} K_M(t, x, y)]^q \frac{dt}{t} \\ &\leq 2^q \ln 2 [2^{-i\theta} K_M(2^i, x, y)]^q .\end{aligned}$$

Through summing in $i \in \mathbb{Z}$,

$$\begin{aligned}2^{-\theta} (\ln 2)^{1/q} \parallel (K_M(2^i, x, y)) \parallel_{\lambda_{\theta,q}} &\leq \beta_{\theta,q}(x, y) \\ &\leq 2(\ln 2)^{1/q} \parallel (K_M(2^i, x, y)) \parallel_{\lambda_{\theta,q}} .\end{aligned}$$

If $q = \infty$, since for all $i \in \mathbb{Z}$ we have

$$K_M(2^{i+1}, x, y) \leq \max\{1, 2^{i+1}/2^i\} K_M(2^i, x, y) = 2K_M(2^i, x, y) ,$$

and for each $t > 0$, there exists $i \in \mathbb{Z}$ such that $2^i \leq t \leq 2^{i+1}$, we have $2^{-i\theta} \geq t^{-\theta} \geq 2^{-i\theta}2^{-\theta}$, and since $K_M(\cdot, x, y)$ is not decreasing, we obtain

$$2^{-\theta}2^{-i\theta} K_M(2^i, x, y) \leq t^{-\theta} K_M(t, x, y) \leq 2^{-i\theta+1} K_M(2^i, x, y),$$

for $2^i \leq t \leq 2^{i+1}$. For fixed $t > 0$, for all $i \in \mathbb{Z}$ such that $2^i \leq t \leq 2^{i+1}$, one has

$$2^{-\theta}\, 2^{-j\theta}\, K_M(2^j, x, y) \leq t^{-\theta}\, K_M(t, x, y) \leq \beta_{\theta,\infty}(x, y).$$

Since $t > 0$ is arbitrary, it holds for all $j \in \mathbb{Z}$. Soon,

$$2^{-\theta} \parallel (K(2^j, x, y))_{j\in\mathbb{Z}} \parallel_{\lambda_{\theta,\infty}} \leq \beta_{\theta,\infty}(x, y).$$

By similar reasoning, we have

$$\beta_{\theta,\infty}(x, y) \leq 2 \parallel (K_M(2^j, x, y))_{j\in\mathbb{Z}} \parallel_{\lambda_{\theta,\infty}} .$$

□

Definition 10. *Given $x, y \in X_0 \cap X_1$, let $(x_n^{x,y})$ be a linking sequence from x to y in $X_0 \cap X_1$ and $(k_j)_{j=0}^{n-1}$ be an n-tuple of integers, both fixed. We define*

$$c_i = c_i((x_n^{x,y}),(k_j)) = \begin{cases} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1}) \\ 0, \qquad\qquad \text{if } k_j \neq i, \quad \text{for all } j \in \{0,1,\ldots,n-1\}. \end{cases}$$

A sequence $c = (c_i)_{i\in\mathbb{Z}}$ as above is called a derived sequence from x to y, associated with the linking sequence $(x_n^{x,y})$ in $X_0 \cap X_1$ and the sequence (k_j).

Remark 8. *The definition of the sequence c given in [10], used to define the J method, is not the same one of Definition 10. The zero c_i elements in c, such that there is no term in the n-tuple equal to the index i, was introduced in the current work. Therefore, we cannot compare the J method developed by Gustavsson and the one developed in this text.*

Example 1. *Let $x_0 = x, x_1, x_2, x_3, x_4 = y \in X_0 \cap X_1$ be a linking sequence from x to y and $(-5, 2, -5, 7)$ be a sequence of integers with four terms. For $i = 0$, we have*

$$c_0 = \begin{cases} \sum_{k_j=0} J_M(2^0, x_j, x_{j+1}) \\ 0, \qquad\qquad \text{se } k_j \neq i, \quad \forall\, j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since no term of the sequence $(-5, 2, -5, 7)$ is equal to zero, it follows that $c_0 = 0$. For $i = -5$,

$$c_{-5} = \begin{cases} \sum_{k_j=-5} J_M(2^{-5}, x_j, x_{j+1}) \\ 0, \qquad\qquad \text{se } k_j \neq i, \quad \forall\, j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since -5 occurs in the zeroth and second positions of the sequence $(-5, 2, -5, 7)$, it follows that

$$c_{-5} = J_M(2^{-5}, x_0, x_1) + J_M(2^{-5}, x_2, x_3).$$

The sequence (c_i) for this case is $c_{-5} = J_M(2^{-5}, x_0, x_1) + J_M(2^{-5}, x_2, x_3)$, $c_2 = J_M(2^2, x_1, x_2)$, $c_7 = J_M(2^7, x_3, x_4)$, and all other terms are zero.

Proposition 9. *If $x, y \in X_0 \cap X_1$, $\theta \in (0,1)$, $q \in [1,\infty)$, and $t > 0$, then for every derived sequence (c_i) from x to y in $X_0 \cap X_1$, we have*

$$K_M(t,x,y) \leq \gamma_{\theta,q,t}\, \Gamma_{\theta,q}((c_i)_{i\in\mathbb{Z}}),$$

where $\gamma_{\theta,q,t} > 0$ is a constant that depends only on θ, q, and t.

Proof. Let $x, y \in X_0 \cap X_1$, $(x_n^{x,y})$ be a linking sequence from x to y in $X_0 \cap X_1$ and $(k_0, k_1, \ldots, k_{n-1})$ be an n-tuple of integers. So, for every $t > 0$,

$$\begin{aligned} K_M(t,x,y) &\leq \sum_{j=0}^{n-1} K_M(t, x_j, x_{j+1}) \\ &\leq \sum_{j=0}^{n-1} \min\{1, t/2^{k_j}\} J_M(2^{k_j}, x_j, x_{j+1}) \\ &= \sum_{i\in\mathbb{Z}} \min\{1, t/2^i\}\, c_i\,. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \min\{1, t/2^i\}c_i &= \sum_{i\in\mathbb{Z}} (2^{i\theta}\min\{1, t/2^i\})(2^{-i\theta}c_i)\\ &\leq \left[\sum_{i\in\mathbb{Z}} [2^{i\theta}\min\{1, t/2^i\}]^p\right]^{\frac{1}{p}} \left[\sum_{i\in\mathbb{Z}} [2^{-i\theta}c_i]^q\right]^{\frac{1}{q}}\\ &= \gamma_{\theta,q,t}\,\Gamma_{\theta,q}(c),\end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\gamma_{\theta,q,t} = \left[\sum_{i\in\mathbb{Z}} [2^{i\theta}\min\{1, t/2^i\}]^{\frac{q}{q-1}}\right]^{\frac{q-1}{q}} = \left(\frac{t^p 2^{N_t(\theta-1)p}}{1-2^{(\theta-1)p}} + \frac{2^{(N_t-1)\theta p}}{2^{\theta p}-1}\right)^{1/p},$$

where N_t is the smallest integer such that $t \leq 2^{N_t}$. □

Theorem 3. *If $\theta \in (0,1)$ and $q \in [1,\infty)$, then*

$$\alpha_{\theta,q}(x,y) := \inf_{(c_i)_{i\in\mathbb{Z}}} \Gamma_{\theta,q}((c_i)_{i\in\mathbb{Z}})$$

is a metric over $X_0 \cap X_1$, where the infimum is taken over all derived sequences (c_i) obtained by all linking sequences from x to y and all sequences $(k_j)_{j=0}^{n-1}$, as in the Definition 10.

Proof. For each $x \in X_0 \cap X_1$, we have that $z_0 = x = z_1$ is a linking sequence from x to x in $X_0 \cap X_1$. For the set $\{k_0 = 0\}$, we have

$$c_i = \sum_{0=i} J_M(2^0, z_j, z_{j+1}).$$

This means that only the term c_0 cannot be null. However, by choosing this linking sequence, we have $c_0 = J_M(2^0, x, x) = 0$. Therefore, $\alpha_{\theta,q}(x,x) = 0$.

It is also immediate that $\alpha_{\theta,q}(x,y) \in \mathbb{R}$ for all $x, y \in X_0 \cap X_1$, because for the sequence $z_0 = x$ and $z_1 = y$ and $\{k_0 = 0\}$, we have $c_0 = J_M(1, x, y) \in \mathbb{R}$ and $c_i = 0$ for all $i \in \mathbb{Z} \setminus \{0\}$.

From Proposition 9, taking the infimum over all sequences derived from x to y, we obtain

$$K_M(t,x,y) \leq \gamma_{\theta,q,t}\,\alpha_{\theta,q}(x,y), \tag{22}$$

where $\gamma_{\theta,q,t} > 0$ is a constant that only depends on θ, q, and t. This inequality guarantees the positivity of $\alpha_{\theta,q}$ in $X_0 \cap X_1$.

For the symmetry of $\alpha_{\theta,q}$, let $(x_n^{x,y})$ be a linking sequence from x to y in $X_0 \cap X_1$ and $(k_0, k_1, \ldots, k_{n-1})$ be an n-tuple of integers. Consider the sequence $(y_n^{y,x})$ given by $y_j = x_{n-j}$ and $(\tau_0, \tau_1, \ldots, \tau_{n-1})$ be an n-tuple of integers given by $\tau_r = k_{n-1-r}$. If $(c_i)_{i\in\mathbb{Z}}$ is the derived sequence from x to y associated with the linking sequence $(x_n^{x,y})$ and the n-tuple $(k_0, k_1, \ldots, k_{n-1})$ and $(d_j)_{j\in\mathbb{Z}}$ is the derived sequence from y to x associated with the linking sequence $(y_n^{y,x})$ and n-tuple $(\tau_0, \tau_1, \ldots, \tau_{n-1})$, we have

$$\Gamma_{\theta,q}((c_i)_{i\in\mathbb{Z}}) = \Gamma_{\theta,q}((d_j)_{j\in\mathbb{Z}}).$$

Thus, if (c_i) is any derived sequence from x to y, we have

$$\alpha_{\theta,q}(y,x) \leq \alpha_{\theta,q}(x,y).$$

Conversely, if (d_j) is any derived sequence from y to any x, we have

$$\alpha_{\theta,q}(x,y) \leq \alpha_{\theta,q}(y,x).$$

Thus, it follows that $\alpha_{\theta,q}(x,y) = \alpha_{\theta,q}(y,x)$.

For the triangular inequality, let $x, y, z \in X_0 \cap X_1$ and $\varepsilon > 0$. So, there are linking sequences $(a^{x,z}_{n_\varepsilon})$ and $(b^{z,y}_{m_\varepsilon})$ from x to z and from z to y, respectively, both in $X_0 \cap X_1$, and finite integer sequences $(k_0, k_1, \ldots, k_{n_\varepsilon - 1})$ and $(l_0, l_1, \ldots, l_{m_\varepsilon - 1})$, such that if (c^ε_i) is the derived sequence from x to z associated with the linking sequence $(a^{x,z}_{n_\varepsilon})$ and the n_ε-tuple $(k_0, k_1, \ldots, k_{n_\varepsilon - 1})$ and (d^ε_j) is the derived sequence from z to y associated with the linking sequence $(b^{x,z}_{m_\varepsilon})$ and the m_ε-tuple $(l_0, l_1, \ldots, l_{m_\varepsilon - 1})$, then we have

$$\Gamma_{\theta,q}((c_i)) < \alpha_{\theta,q}(x,z) + \varepsilon$$

and

$$\Gamma_{\theta,q}((d_j)) < \alpha_{\theta,q}(z,y) + \varepsilon.$$

We define the linking sequence $(e^{x,y}_s)$ from x to y in $X_0 \cap X_1$ given by $e_s = a_s$ if $0 \le s \le n_\varepsilon$, and $e_{n_\varepsilon + r} = b_r$ if $0 \le r \le m_\varepsilon$, as well as an $(m_\varepsilon + n_\varepsilon)$-tuple $(\tau_0, \tau_1, \tau_2, \ldots, \tau_{m_\varepsilon + n_\varepsilon - 1})$ given by $\tau_j = k_j$ if $0 \le j \le n_\varepsilon - 1$ and $\tau_{n_\varepsilon + i} = l_i$, if $0 \le i \le m_\varepsilon - 1$. If (f_k) is the derived sequence from x to y associated with the linking sequence $(e^{x,y}_s)$ in $X_0 \cap X_1$ and with $(m_\varepsilon + n_\varepsilon)$-tuple $(\tau_0, \tau_1, \ldots, \tau_{m_\varepsilon + n_\varepsilon - 1})$, we have

$$\begin{aligned} \alpha_{\theta,q}(x,y) &\le \Gamma_{\theta,q}((f_i)) \\ &\overset{(2)}{\Gamma}_{\theta,q}((a_i)) + \Gamma_{\theta,q}((b_i)) \\ &\alpha_{\theta,q}(x,z) + \alpha_{\theta,q}(z,y) + 2\,\varepsilon\,, \end{aligned}$$

where (2) is due to the Minkowski inequality for the sequence spaces ℓ_q. Since $\varepsilon > 0$ is arbitrary,

$$\alpha_{\theta,q}(x,y) \le \alpha_{\theta,q}(x,z) + \alpha_{\theta,q}(z,y).$$

□

Remark 9. *Inequality (22) tells us that $(X_0 \cap X_1, \alpha_{\theta,q})$ is continuously contained in (X, d_X). This follows from inequalities (14) and (16) .*

Interpolation Spaces

Here, inspired by the normed case and [10], we introduce two interpolated metric spaces.

Definition 11 (The K_M-space)**.** *Given a metric couple $(X_0, X_1)_X$, $\theta \in (0,1)$ and $q \in [1, \infty]$, we define the K_M-space as the relative completion of $(X_0 \cap X_1, \beta_{\theta,q})$ in the space (X, d_X). We will denote this space by $\overrightarrow{X}^K_{\theta,q}$ and the metric of this space by $\mathcal{D}_{\theta,q}$.*

Definition 12 (The J_M-space)**.** *Given a metric couple $(X_0, X_1)_X$, $\theta \in (0,1)$ and $q \in [1, \infty]$, we define the J_M-space as the relative completion of $(X_0 \cap X_1, \alpha_{\theta,q})$ in the space (X, d_X). We will denote this space by $\overrightarrow{X}^J_{\theta,q}$ and the metric of this space by $d_{\theta,q}$.*

The interpolation of metric spaces generates spaces with basic properties that are analogous to those of the normed case.

Definition 13. *Let $(X_0, X_1)_X$ be a pair of compatible metric spaces and E be a subset of X. We say that (E, d_E) is an intermediate metric space for the pair $(X_0, X_1)_X$ if*

$$X_0 \cap X_1 \subseteq E \subseteq X$$

and the inclusions are continuous, considering $X_0 \cap X_1$ endowed with the metric $J_M(1, \cdot, \cdot)$.

Lemma 11. *If $x, y \in X_0 \cap X_1$ and $\overrightarrow{X}^J_{\theta,q}$ is the J-space endowed with the metric $d_{\theta,q}$, then*

$$d_{\theta,q}(x,y) \le \alpha_{\theta,q}(x,y) \le 2^{-m\theta}\, J_M(2^m,x,y),$$

for all $m \in \mathbb{Z}$.

Proof. Given $x, y \in X_0 \cap X_1$, let $x_0 = x$, $x_1 = y$, and $m \in \mathbb{Z}$ be given. So, by definition, if $q < \infty$,

$$\begin{aligned}\alpha^q_{\theta,q}(x,y) &\le \sum_{i\in\mathbb{Z}}\left[2^{-i\theta}\sum_{m=i} J_M(2^i,x,y)\right]^q \\ &= [2^{-m\theta}\, J_M(2^m,x,y)]^q, \quad \forall\lambda \in \mathbb{Z}.\end{aligned}$$

From inequality (8),

$$\begin{aligned}d^q_{\theta,q}(x,y) &\le \alpha^q_{\theta,q,\overrightarrow{X}}(x,y) \\ &\le \sum_{i\in\mathbb{Z}}\left[2^{-i\theta}\sum_{m=i} J_M(2^i,x,y)\right]^q \\ &= [2^{-m\theta}\, J_M(2^m,x,y)]^q,\end{aligned}$$

for all $m \in \mathbb{Z}$. □

Proposition 10. *If* $E = \overrightarrow{X}^K_{\theta,q}$ *or* $E = \overrightarrow{X}^J_{\theta,q}$, *there are* $\mu, \eta > 0$ *such that*

$$d_X(x,y) \le \mu\, d_E(x,y) \tag{23}$$

for all $x, y \in E$, *and*

$$d_E(x,y) \le \eta\, J(1,x,y), \tag{24}$$

for all $x, y \in X_0 \cap X_1$.

Proof. These statements follow directly from inequality (5) and from Lemmas 9 and 11 and remain valid if $q = \infty$. □

Remark 10. *From Lemma 3, we have that* $X_0 \cap X_1$ *is dense in* $(\overrightarrow{X}^K_{\theta,q}, \mathcal{D}_{\theta,q})$ *and in* $(\overrightarrow{X}^J_{\theta,q}, d_{\theta,q})$. *And by Proposition 3, we have that* $(\overrightarrow{X}^K_{\theta,q}, \mathcal{D}_{\theta,q})$ *and* $(\overrightarrow{X}^J_{\theta,q}, d_{\theta,q})$ *are complete whenever* (X, d_X) *is complete.*

As a consequence of Proposition 10, we have that $\overrightarrow{X}^K_{\theta,q}$ and $\overrightarrow{X}^J_{\theta,q}$ are intermediate spaces.

Proposition 11. *If* $E = \overrightarrow{X}^K_{\theta,q}$ *or* $E = \overrightarrow{X}^J_{\theta,q}$, *then it holds that*

$$d_E(x,y) \le Ct^{-\theta} J_M(t,x,y),$$

for all $x, y \in X_0 \cap X_1$.

Proof. If $E = \overrightarrow{X}^K_{\theta,q}$, since $K_M(s,x,y) \le \min\{1, s/t\} J_M(t,x,y)$, for all $x, y \in X_0 \cap X_1$, then by applying $\Phi_{\theta,q}$, in relation to s, we have

$$\beta_{\theta,q}(x,y) \le t^{-\theta} M_{\theta,q} J_M(t,x,y).$$

In $X_0 \cap X_1$ we have $\beta_{\theta,q} \ge \mathcal{D}_{\theta,q}$. So,

$$\mathcal{D}_{\theta,q}(x,y) \le t^{-\theta} M_{\theta,q} J_M(t,x,y)$$

and the result follows taking $C = M_{\theta,q}$.

If $E = \overrightarrow{X}^{J}_{\theta,q}$, given $t > 0$, there exists $\lambda_0 \in \mathbb{Z}$ such that $2^{\lambda_0} \leq t \leq 2^{\lambda_0+1}$. Since $J_M(\cdot,\cdot,\cdot)$ is non-decreasing on its first entry, it follows that

$$J_M(2^{\lambda_0}, x, y) \leq J_M(t, x, y) \leq J_M(2^{\lambda_0+1}, x, y).$$

From $2^{\lambda_0} \leq t \leq 2^{\lambda_0+1}$, we have

$$t \leq 2^{\lambda_0+1} \Rightarrow 2^{-\theta\lambda_0} \leq 2^{\theta} t^{-\theta}.$$

Putting together these inequalities,

$$(2^{\lambda_0})^{-\theta} J_M(2^{\lambda_0}, x, y) \leq 2^{\theta} t^{-\theta} J_M(t, x, y).$$

From Lemma 11, the result follows, with $C = 2^{\theta}$ □

Corollary 1. *Proposition 11 is equivalent to*

$$d_E(x,y) \leq C(d_0(x,y))^{1-\theta}(d_1(x,y))^{\theta}, \tag{25}$$

for all $x, y \in X_0 \cap X_1$, *where* $E = \overrightarrow{X}^{J}_{\theta,q}$ *or* $E = \overrightarrow{X}^{K}_{\theta,q}$.

Proof. If we assume (25), it is enough consider $t = d_0(x,y)/d_1(x,y)$. For the contrary equivalence, for all $t > 0$, we have that $d_0(x,y) \leq J_M(t,x,y)$ and $d_1(x,y) \leq J_M(t,x,y)/t$. Thus,

$$\begin{aligned} d_E(x,y) &\leq C(J_M(t,x,y))^{1-\theta}((J_M(t,x,y)/t)^{\theta} \\ &= Ct^{-\theta} J_M(t,x,y). \end{aligned}$$

□

Now, one of our main results follows.

Theorem 4. *We have that*

$$\overrightarrow{X}^{J}_{\theta,q} \subseteq \overrightarrow{X}^{K}_{\theta,q}$$

with

$$\mathcal{D}_{\theta,q}(x,y) \leq 2\,\gamma_{\theta,\infty,1}\, d_{\theta,q}(x,y),$$

for all $x, y \in \overrightarrow{X}^{J}_{\theta,q}$.

Proof. It is enough to prove that

$$\beta_{\theta,q}(x,y) \leq c\,\alpha_{\theta,q}, \quad x, y \in X_0 \cap X_1, \tag{26}$$

for some constant $c > 0$. For this, let $x, y \in X_0 \cap X_1$, $(x_n^{x,y})$ be a linking sequence from x to y in $X_0 \cap X_1$ and $(k_0, k_1, \ldots, k_{n-1})$ be an n-tuple of integers. As it was shown in Lemma 9, we have

$$\begin{aligned} K_M(t,x,y) &\leq \sum_{j=0}^{n-1} K_M(t, x_j, x_{j+1}) \\ &\sum_{j=0}^{n-1} \min\{1, t/2^{k_j}\} J_M(2^{k_j}, x_j, x_{j+1}) \\ &\sum_{i\in\mathbb{Z}} \min\{1, t/2^{i}\}\, c_i, \end{aligned}$$

for all $t > 0$. In particular, if $t = 2^j$,

$$K_M(2^j, x, y) \leq \sum_{i\in\mathbb{Z}} \min\{1, 2^{j-i}\}\, c_i.$$

If $i = k + j$, we have

$$\begin{aligned} K_M(2^j, x, y) &\leq \sum_{i\in\mathbb{Z}} \min\{1, 2^{j-i}\}\, c_i \\ &= \sum_{k\in\mathbb{Z}} \min\{1, 2^{-k}\}\, c_{k+j} \\ &= \sum_{k\in\mathbb{Z}} [2^{k\theta} \min\{1, 2^{-k}\}]\, [2^{-k\theta}\, c_{k+j}]. \end{aligned}$$

Now, we apply the functional $\Gamma\theta, q$ with respect to the variable j:

$$\Gamma_{\theta,q}(K_M(2^j, x, y)_{j\in\mathbb{Z}}) \leq \Gamma_{\theta,q}((c_{k+j})_{j\in\mathbb{Z}})) \sum_{k\in\mathbb{Z}} 2^{k\theta}\, \min\{1, 2^{-k}\}\,.$$

From Proposition 8, we may write

$$\beta_{\theta,q}(x, y) \leq 2\, \gamma_{\theta,\infty,1}\, \alpha_{\theta,q}(x, y).$$

Now, we apply Lemma 4 for $A = X_0 \cap X_1$, $m_1 = \beta_{\theta,q}$, $m_2 = \alpha_{\theta,q}$, and $B = X$ with $d_X = d_B$. □

This inclusion is valid for the interpolation of normed spaces. However, in the normed case, we have the fundamental lemma of interpolation, which guarantees the opposite inclusion. For the metric case, we do not know whether there is an analogue to the fundamental lemma.

Proposition 12. *Given a pair of compatible metric spaces $(X_0, X_1)_X$, $\theta \in (0, 1)$ and $q \in [1, \infty]$, if the inclusion of $(X_0 \cup X_1, d_X)$ in $(X_0 \cup X_1, K(1, \cdot, \cdot))$ is continuous, then*

$$K_M(t, x, y) \leq C t^\theta d_E(x, y),$$

for all $x, y \in E \cap (X_0 \cup X_1)$, where $E = \overrightarrow{X}^K_{\theta,q}$ or $E = \overrightarrow{X}^J_{\theta,q}$.

Proof. Let $E = \overrightarrow{X}^K_{\theta,q}$. We know that $K_M(t, a, b) \leq \max\{1, t/s\} K_M(s, a, b)$ for all $a, b \in X_0 \cup X_1$. So,

$$\min\{1, s/t\} K_M(t, a, b) \leq K_M(s, a, b).$$

Applying $\Phi_{\theta,q}$ in the variable s and using Lemma 10, we have

$$t^{-\theta} M_{\theta,q} K_M(t, a, b) \leq \beta_{\theta,q}(a, b),$$

or

$$K_M(t, a, b) \leq M^{-1}_{\theta,q} t^\theta \beta_{\theta,q}(a, b),$$

for all $a, b \in X_0 \cap X_1$. Now, let $x, y \in \overrightarrow{X}^K_{\theta,q} \cap (X_0 \cup X_1)$. Then, there are Cauchy sequences (x_n) and (y_n) in $(X_0 \cap X_1, \beta_{\theta,q})$ such that $\lim_{n\to\infty} d_X(x_n, x) = \lim_{n\to\infty} d_X(y_n, y) = 0$. So,

$$K_M(t, x_n, y_n) \leq M^{-1}_{\theta,q} t^\theta \beta_{\theta,q}(x_n, y_n),$$

for all $n \in \mathbb{N}$. This inequality is valid for any approximating sequences of x and y in $(X_0 \cap X_1, \beta_{\theta,q})$. So,

$$\inf \lim_{n\to\infty} K_M(t, x_n, y_n) \leq \inf \lim_{n\to\infty} M^{-1}_{\theta,q}\, t^\theta\, \beta_{\theta,q}(x_n, y_n),$$

where the infimum is taken over all approximating sequences of x and y in $(X_0 \cap X_1, \beta_{\theta,q})$. From the continuous inclusion of the hypothesis, it follows that $K_M(t,x,y) \leq M_{\theta,q}^{-1}\, t^\theta\, \overline{\beta_{\theta,q}}(x,y)$. If $x_0 = x, x_1, \ldots, x_n = y$ is a linking sequence from x to y in $\overrightarrow{X}^K_{\theta,q}$, we have

$$K_M(t,x,y) \leq \sum_{k=0}^{n-1} K_M(t,x_k,x_{k+1}) \leq M_{\theta,q}^{-1}\, t^\theta \sum_{k=0}^{n-1} \overline{\beta_{\theta,q}}(x_k,x_{k+1}).$$

Taking the infimum over all linking sequences from x to y in $\overrightarrow{X}^K_{\theta,q}$, we have

$$K_M(t,x,y) \leq M_{\theta,q}^{-1}\, t^\theta\, \mathcal{D}_{\theta,q}(x,y).$$

Now, if $E = \overrightarrow{X}^J_{\theta,q}$, the proof follows from $K_M(t,x,y) \leq Ct^\theta$ and the previous theorem, since

$$\mathcal{D}_{\theta,q}(x,y) \leq 2\,\gamma_{\theta,\infty,1}\, d_{\theta,q}(x,y),$$

for all $x, y \in \overrightarrow{X}^J_{\theta,q}$. □

Definition 14. *We say that $(X_0,X_1)_X$ satisfies the condition $\mathcal{H}$ if for all $x,y \in X_0 \cap X_1$, there is a linking sequence $(x_n^{x,y})$, from x to y in $X_0 \cap X_1$ and a n-tuple of integers $(k_0,k_1,\ldots,k_{n-1})$, such that for some constant γ, the following holds:*

$$c_i \leq \gamma\, K_M(2^i,x,y), \tag{27}$$

with $c = (c_i)$ being the derived sequence from x to y in $X_0 \cap X_1$ associated with the linking sequence $(x_n^{x,y})$ and the n-tuple of integers above.

It is immediate that for pairs of metric spaces satisfying the condition $\mathcal{H}$ we have

$$\overrightarrow{X}^J_{\theta,q} = \overrightarrow{X}^K_{\theta,q},$$

with equivalent metrics.

Spaces where some kind of theorem as the fundamental lemma of interpolation for normed spaces holds necessarily satisfy the condition $\mathcal{H}$.

Theorem 5. *Let $(X_0,X_1)_X$ be a pair of compatible metric spaces. Then, we have the following:*

(a) If $(X_0,X_1)^K_{\theta,q}$ denotes the K_M interpolated space of $(X_0,X_1)_X$, then

$$(X_0,X_1)^K_{\theta,q} = (X_1,X_0)^K_{1-\theta,q},$$

with equal metrics.

(b) If $q \leq r$, then $\overrightarrow{X}^K_{\theta,q} \subseteq \overrightarrow{X}^K_{\theta,r}$ and

$$\mathcal{D}_{\theta,q}(x,y) \leq C'\, \mathcal{D}_{\theta,r}(x,y),$$

for all $x, y \in \overrightarrow{X}^K_{\theta,r}$.

(c) If $X_1 \subseteq X_0$ (continuous Lipschitz inclusion) and $X = X_0$ with $d_X = d_0$, then $\overrightarrow{X}^K_{\theta_1,q} \subseteq \overrightarrow{X}^K_{\theta_0,q}$ for all $q \in [1,\infty]$ and $0 < \theta_0 < \theta_1 < 1$.

Proof. (a) Let $x, y \in X_0 \cup X_1$ and $(x_n^{x,y})$ be an admissible linking sequence from x to y in $X_0 \cup X_1$. We have that $h_t(x_k,x_{k+1};(X_0,X_1)) = t\, h_{t^{-1}}(x_k,x_{k+1};(X_1,X_0))$ for all $k = 0,1,\ldots,n-1$, where the labels (X_0,X_1) and (X_1,X_0) are used to differentiate the order in which the spaces compatible metrics is being considered. Then,

$$t \sum_{k=0}^{n-1} h_{t^{-1}}(x_k,x_{k+1};(X_0,X_1)) = \sum_{k=0}^{n-1} h_t(x_k,x_{k+1};(X_1,X_0)).$$

Taking the infimum over all admissible linking sequences from x to y, we have

$$t\,K_M(t^{-1},x,y;(X_0,X_1)) = K_M(t,x,y;(X_1,X_0)).$$

In applying the functional $\Phi_{\theta,q}$,

$$\beta_{1-\theta,q}(x,y;(X_0,X_1)) = \beta_{\theta,q}(x,y;(X_1,X_0)).$$

Thus, we obtain the same relative completion and both have the same metric, i.e.,

$$\mathcal{D}_{1-\theta,q}(x,y;(X_0,X_1)) = \mathcal{D}_{\theta,q}(x,y;(X_1,X_0)).$$

(b) Let $x,y \in X_0 \cap X_1$. We will use the equivalent discrete version of $\beta_{\theta,q}$. Indeed, if $q \le r$, we have that $\ell_q(\mathbb{Z}) \subseteq \ell_r(\mathbb{Z})$. For fixed $\theta \in (0,1)$, consider the sequence with its general term given by

$$c_k = 2^{-k\theta}\,K_M(2^k,x,y), \quad k \in \mathbb{Z}.$$

We have $\| (c_k)_{k\in\mathbb{Z}} \|_{\ell_p} = \| (K_M(2^k,x,y))_{k\in\mathbb{Z}} \|_{\lambda_{\theta,p}}$ for all $p \in [1,\infty]$. Since $\beta_{\theta,q}(x,y)$ and $\| (K_M(2^k,x,y))_{k\in\mathbb{Z}} \|_{\lambda_{\theta,q}}$ are equivalent for all $\theta \in (0,1)$, all $q \in [1,\infty]$, and also $\| (K_M(2^k,x,y))_{k\in\mathbb{Z}} \|_{\lambda_{\theta,q}} \le \| (K_M(2^k,x,y))_{k\in\mathbb{Z}} \|_{\lambda_{\theta,r}}$, for all $\theta \in (0,1)$, the result follows from putting together the inequalities. Furthermore, by Lemma 4, we have

$$\mathcal{D}_{\theta,q}(x,y) \le C'\,\mathcal{D}_{\theta,r}(x,y),$$

for all $x,y \in \overrightarrow{X}_{\theta,r}$.

(c) From the inclusion in the hypothesis, there is $\lambda > 0$ such that $d_0(x,y) \le \lambda\, d_1(x,y)$ for all $x,y \in X_1$. Let $(x_n^{x,y})_{adm}$ be an admissible linking sequence from x to y in $X_0 \cup X_1$. For all $t \ge \lambda$, we have

$$\begin{aligned} d_0(x,y) &\le \sum_{k=0}^{n-1} d_0(x_k,x_{k+1}) \\ &\le \sum_{s\in S} d_0(x_s,x_{s+1}) + (t/\lambda)\sum_{r\in R} d_0(x_r,x_{r+1}) \\ &\le \sum_{s\in S} d_0(x_s,x_{s+1}) + t\sum_{r\in R} d_1(x_r,x_{r+1}) \\ &= \sum_{k=0}^{n-1} h_t(x_k,x_{k+1}), \end{aligned}$$

where S is the set of indices s for which $h_t(x_s,x_{s+1}) = d_0(x_s,x_{s+1})$, and R is the set of indices for which $h_t(x_r,x_{r+1}) = t\,d_1(x_r,x_{r+1})$. This implies that $d_0(x,y) \le K_M(t,x,y)$ for all $t \ge \lambda$. Since the opposite inequality is true, it follows that $d_0(x,y) = K_M(t,x,y)$ for all $t \ge \lambda$. Now, we have that

$$\mathcal{D}^q_{\theta,q}(x,y) = \int_0^1 [t^{-\theta}K_M(t,x,y)]^q \frac{dt}{t} + \int_1^\lambda [t^{-\theta}K_M(t,x,y)]^q \frac{dt}{t} + \int_\lambda^\infty [t^{-\theta}d_0(x,y)]^q \frac{dt}{t}.$$

Since $K_M(t,x,y) \le d_0(x,y)$ for all $t > 0$, it follows from the previous equality that

$$\begin{aligned} \mathcal{D}^q_{\theta,q}(x,y) &= \int_0^1 [t^{-\theta}K_M(t,x,y)]^q \frac{dt}{t} + \int_1^\lambda [t^{-\theta}K_M(t,x,y)]^q \frac{dt}{t} + \\ &\quad + \int_\lambda^\infty [t^{-\theta}d_0(x,y)]^q \frac{dt}{t} \\ &\le \int_0^1 [t^{-\theta}K_M(t,x,y)]^q \frac{dt}{t} + d_0(x,y)\int_1^\infty t^{-\theta q-1}dt. \end{aligned}$$

The first part of the second member of the last inequality satisfies

$$\int_0^1 [t^{-\theta_0}K_M(t,x,y)]^q \frac{dt}{t} \le \int_0^1 [t^{-\theta_1}K_M(t,x,y)]^q \frac{dt}{t},$$

for $0 < \theta_0 < \theta_1 < 1$. Thus,

$$\mathcal{D}^q_{\theta_0,q}(x,y) \leq \mathcal{D}^q_{\theta_1,q}(x,y) + \mu_{\theta_0,q} d_0(x,y),$$

where $\mu_{\theta_0,q} = \int_1^\infty t^{-\theta_0 q - 1} dt$. This inequality shows that $\overrightarrow{X}_{\theta_1,q} \subseteq \overrightarrow{X}_{\theta_0,q}$. □

5. Interpolation of Lipschitz Operators

In this section, we study Lipschitz operators acting on the interpolation spaces of metric spaces. In general, we will show that the Lipschitzian property of an operator is invariant under metric interpolation.

This result is important because through it, we will show that under some conditions, the compactness of these operators can also be preserved. Many problems in analysis rely on properties of compact operators, as in the particular case of compact linear operators, which play an important role in approximation theory.

In this section, let us consider compatible metric space couples $(X_0, X_1)_X$ and $(Y_0, Y_1)_Y$.

Definition 15. *Let (M, d_M) and (N, d_N) be metric spaces, and let $T : M \to N$. Then, T is a Lipschitiz operator if there is a constant $\gamma > 0$ such that*

$$d_N(T(x), T(y)) \leq \gamma\, d_M(x,y),$$

for all $x, y \in M$. Furthermore if T is a Lipschitz operator,

$$\kappa = \inf_{\substack{x,y \in M \\ x \neq y}} \frac{d_N(T(x), T(y))}{d_M(x,y)}$$

is called the Lipschitz constant of T.

To avoid confusion, when we are dealing with two pairs of compatible metric spaces $(X_0, X_1)_X$ and $(Y_0, Y_1)_Y$, we will use the symbols $\overrightarrow{X}$ to refer to the pair $(X_0, X_1)_X$ and $\overrightarrow{Y}$ to refer to the pair $(Y_0, Y_1)_Y$ in each functional or notation that makes this distinction necessary.

Theorem 6. *Let $(X_0, X_1)_X$ and $(Y_0, Y_1)_Y$ be two pairs of compatible metric spaces, and (X, d_X) and (Y, d_Y) be metrics spaces, such that Y is a complete metric space. For fixed $\theta \in (0,1)$ and $q \in [1, \infty]$, $T : X \to Y$ is an operator that satisfies the following:*

(a) $\lim_{n\to\infty} d_X(x_n, x) = 0$ *and* $\lim_{n\to\infty} d_Y(T(x_n), y) = 0$ *imply that* $T(x) = y$.

(b)

$$T_i = T\Big|_{X_i} : X_i \to Y_i, \quad (i = 0, 1).$$

(c) $d_{Y_i}(T_i(x), T_i(z)) \leq \omega_i d_{X_i}(x,z)$, *where* $\omega_i > 0$ *and* $x, z \in X_i$, $i = 0, 1$.

Then,

$$T_{\theta,q} = T\Big|_{\overrightarrow{X}^K_{\theta,q}} : \overrightarrow{X}^K_{\theta,q} \to \overrightarrow{Y}^K_{\theta,q} \tag{28}$$

and

$$\mathcal{D}_{\theta,q,\overrightarrow{Y}}(T_{\theta,q}(x), T_{\theta,q}(z)) \leq \omega_0^{1-\theta}\omega_1^{\theta}\mathcal{D}_{\theta,q,\overrightarrow{X}}(x,z), \quad x, z \in \overrightarrow{X}^K_{\theta,q}. \tag{29}$$

Proof. From properties (b) and (c), we claim that

$$K_M(t, T(x), T(z), \overrightarrow{Y}) \leq \omega_0 K_M\left(\frac{t\omega_1}{\omega_0}, x, z, \overrightarrow{X}\right), \quad x, z \in X_0 \cup X_1.$$

Indeed, let $(x_n^{x,z})_{adm}$ be an admissible linking sequence from x to z in $X_0 \cup X_1$. For every $k = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}\omega_0\, h_{\frac{t\,\omega_1}{\omega_0}}(x_k, x_{k+1}; \overrightarrow{X}) &= \omega_0\, d_{X_0}(x_k, x_{k+1}) \\ &\geq \omega_0\, \omega_0^{-1} d_{Y_0}(T_0(x_k), T_0(x_{k+1})) \\ &\geq h_t(T_0(x_k), T_0(x_{k+1}); \overrightarrow{Y})\end{aligned}$$

and always that $\{x_k, x_{k+1}\} \subseteq X_0$ e $\{x_k, x_{k+1}\} \not\subseteq X_1$. In the same way,

$$\begin{aligned}\omega_0\, h_{\frac{t\,\omega_1}{\omega_0}}(x_k, x_{k+1}; \overrightarrow{X}) &= \frac{\omega_1\, t}{\omega_0}\, \omega_0\, d_{X_1}(x_k, x_{k+1}) \\ &= \omega_1\, t\, d_{X_1}(x_k, x_{k+1}) \\ &\geq \omega_1\, t\, \omega_1^{-1}\, d_{Y_1}(T_1(x_k), T_1(x_{k+1})) \\ &\geq h_t(T_1(x_k), T_1(x_{k+1}); \overrightarrow{Y})\end{aligned}$$

and and always that $\{x_k, x_{k+1}\} \subseteq X_1$ e $\{x_k, x_{k+1}\} \not\subseteq X_0$. Finally, if $x_k, x_{k+1} \in X_1 \cap X_0$,

$$\begin{aligned}\omega_0\, h_{\frac{t\,\omega_1}{\omega_0}}(x_k, x_{k+1}; \overrightarrow{X}) &= \omega_0\, \min\{d_{X_0}(x_k, x_{k+1}), \frac{\omega_1\, t}{\omega_0}\, d_{X_1}(x_k, x_{k+1})\} \\ &\geq \min\{d_{Y_0}(T_0(x_k), T_0(x_{k+1})), t\, d_{Y_1}(T_1(x_k), T_1(x_{k+1}))\} \\ &= h_t(T(x_k), T(x_{k+1}); \overrightarrow{Y}).\end{aligned}$$

Therefore, for every admissible linking sequence $(x_n^{x,z})_{adm}$, we have

$$\sum_{k=0}^{n-1} h_t(T(x_k), T(x_{k+1}); \overrightarrow{Y}) \leq \omega_0 \sum_{k=0}^{n-1} h_{\frac{\omega_1\, t}{\omega_0}}(x_k, x_{k+1}; \overrightarrow{X}).$$

In the above inequality, the indices in the operator T indicating its restrictions were removed, since the operator T is defined in the entire space (X, d_X).

Taking the infimum over all admissible linking sequences from x to z in $X_0 \cup X_1$, we have

$$K_M(t, T(x), T(z); \overrightarrow{Y}) \leq \omega_0\, K_M(\frac{\omega_1\, t}{\omega_0}, x, z; \overrightarrow{X}).$$

Applying the operator $\Phi_{\theta,q}$ on the last inequality and using Lemma 10, we have

$$\beta_{\theta,q,\overrightarrow{Y}}(T(x), T(z)) \leq \omega_0^{1-\theta}\omega_1^{\theta}\beta_{\theta,q,\overrightarrow{X}}(x, z), \quad x, z \in X_0 \cap X_1. \tag{30}$$

Now, let $x \in \overrightarrow{X}^K_{\theta,q}$. We must prove that $T(x) \in \overrightarrow{Y}^K_{\theta,q}$. In fact, let (x_n) be an approximating sequence of x. Since (x_n) is a Cauchy sequence in $(X_0 \cap X_1, \beta_{\theta,q,\overrightarrow{X}})$, it follows from inequality (30) that $(T(x_n))$ is a Cauchy sequence in $(Y_0 \cap Y_1, \beta_{\theta,q,\overrightarrow{Y}})$. From the inclusion of $(Y_0 \cap Y_1, \beta_{\theta,q,\overrightarrow{Y}})$ in (Y, d_Y), it follows that $(T(x_n))$ is a Cauchy sequence in (Y, d_Y). As this last space is complete, there exists $y \in Y$ such that

$$\lim_{n\to\infty} d_Y(T(x_n), y) = 0.$$

From property (a), $T(x) = y$. Thus, $T(x) \in \overrightarrow{Y}^K_{\theta,q}$. Now, let $x, y \in \overrightarrow{X}^K_{\theta,q}$. We have, for any approximating sequences (x_n) of x and (y_n) of y, that

$$\begin{aligned}\mathcal{D}_{\theta,q,\overrightarrow{Y}}(T(x), T(y)) &\leq \overline{\beta_{\theta,q,Y}}(T(x), T(y)) \\ &\leq \lim_{n\to\infty} \beta_{\theta,q,\overrightarrow{Y}}(T(x_n), T(y_n)) \\ &\leq \lim_{n\to\infty} \omega_0^{1-\theta}\omega_1^{\theta}\beta_{\theta,q,\overrightarrow{X}}(x_n, y_n),\end{aligned}$$

where $\overline{\beta_{\theta,q,Y}}$ is the distance associated with the metric $\beta_{\theta,q,Y}$. Taking the infimum over all approximating sequences x and y in $(X_0 \cap X_1, \beta_{\theta,q,X})$, we have

$$\mathcal{D}_{\theta,q,Y}(T(x), T(y)) \leq \omega_0^{1-\theta}\, \omega_1^{\theta}\, \overline{\beta_{\theta,q,X}}(x, y),$$

for all $x, y \in \overrightarrow{X}^K_{\theta,q}$. Now, consider a linking sequence $x_0 = x, x_1, x_2, \ldots, x_n = y \in \overrightarrow{X}^K_{\theta,q}$ from x to y. We have

$$\begin{aligned}\mathcal{D}_{\theta,q,Y}(T(x), T(y)) &\leq \sum_{k=0}^{n-1} \mathcal{D}_{\theta,q,Y}(T(x_k), T(x_{k+1})) \\ &\sum_{k=0}^{n-1} \overline{\beta_{\theta,q,Y}}(T(x_k), T(x_{k+1})) \\ &\omega_0^{1-\theta}\, \omega_1^{\theta} \sum_{k=0}^{n-1} \overline{\beta_{\theta,q,X}}(x_k, x_{k+1}).\end{aligned}$$

Taking the infimum over all linking sequences from x to y in $\overrightarrow{X}^K_{\theta,q}$, we have

$$\mathcal{D}_{\theta,q,Y}(T(x), T(y)) \leq \omega_0^{1-\theta}\, \omega_1^{\theta} \mathcal{D}_{\theta,q,X}(x, y).$$

□

Remark 11. *In Theorem 6, we do not require continuity of T from (X, d_X) to (Y, d_Y), but we require the operator T to satisfy (a), and we also require that (Y, d_Y) be complete. In the next theorem, we obtain the same result as the previous theorem, but we exchange the completeness of (Y, d_Y) and the property (a) for the continuity of T of (X, d_X) in (Y, d_Y).*

Theorem 7. *If the operator T is continuous from (X, d_X) to (Y, d_Y) and satisfies conditions (b) and (c) of Theorem 6, then*

$$T_{\theta,q} = T\Big|_{\overrightarrow{X}^K_{\theta,q}} : \overrightarrow{X}^K_{\theta,q} \to \overrightarrow{Y}^K_{\theta,q}$$

and

$$\mathcal{D}_{\theta,q,\overrightarrow{Y}}(T(x), T(z)) \leq \omega_0^{1-\theta} \omega_1^{\theta} \mathcal{D}_{\theta,q,\overrightarrow{X}}(x, z), \quad x, z \in \overrightarrow{X}^K_{\theta,q}.$$

Proof. We just need to prove that $T(\overrightarrow{X}^K_{\theta,q}) \subseteq \overrightarrow{Y}^K_{\theta,q}$ because the conclusion of the statement is proved in the same way as for the previous theorem.

In fact, let $x \in \overrightarrow{X}^K_{\theta,q}$. Then, there exists a Cauchy sequence $(x_k)_{k\in\mathbb{N}}$ in $(X_0 \cap X_1, \beta_{\theta,q,\overrightarrow{X}})$ such that $\lim_{k\to\infty} d_X(x_k, x) = 0$. Let us show that $w = T(x) \in \overrightarrow{Y}^K_{\theta,q}$. For each $k \in \mathbb{N}$, define $w_k = T(x_k)$. We have, for all $k \in \mathbb{N}$,

$$x_k \in X_0 \cap X_1 \Rightarrow w_k = T(x_k) \in T(X_0 \cap X_1) \subseteq T(X_0) \cap T(X_1).$$

By condition (b), we have that $T(X_i) \subseteq Y_i$, $i = 0, 1$, and it follows that $w_k = T(x_k) \in Y_0 \cap Y_1$, for all $k \in \mathbb{N}$. By inequality (30), we have that (w_k) is a Cauchy sequence in $(Y_0 \cap Y_1, \beta_{\theta,q,\overrightarrow{Y}})$. Furthermore, from the continuity of $T : X \to Y$, it follows that

$$\lim_{k\to\infty} d_X(x_k, x) = 0 \Rightarrow \lim_{k\to\infty} d_Y(T(x_k), T(x)) = \lim_{k\to\infty} d_Y(w_k, w) = 0.$$

Therefore, $(w_k = T(x_k))$ is an approximate sequence of $w = T(x)$, whenever (x_k) is an approximate sequence of x. Consequently, $w = T(x) \in \overrightarrow{Y}^K_{\theta,q}$. □

Corollary 2. *Under the conditions of Theorem 7, consider $X_i = Y_i$, with $i = 0, 1$, $X = Y$, with equal metrics and (X, d_X) complete. If $\omega_0^{1-\theta}\, \omega_1^{\theta} < 1$, then T has a single fixed point at $\overrightarrow{X}^K_{\theta,q}$.*

Proof. Since (X, d_X) is complete, it follows that $(\overrightarrow{X}^K_{\theta,q}, \mathcal{D}_{\theta,q})$ is complete by Proposition 3. Then, we can apply the Banach contraction principle. □

Next, we have a technical lemma that we will need for the following results.

Lemma 12. *If $a > 0$, $x, y \in X_0 \cap X_1$, $(x_n^{x,y})$ is a linking sequence from x to y in $X_0 \cap X_1$, and $(k_0, k_1, \ldots, k_{n-1})$ is an n-tuple of integers, then*

$$[a^{-\theta} \min\{1, a\}]^q \sum_{i\in\mathbb{Z}} \left[2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1})\right]^q$$
$$\sum_{i\in\mathbb{Z}} \left[(2^i a)^{-\theta} \sum_{k_j=i} J_M(2^i a, x_j, x_{j+1})\right]^q$$
$$\leq [a^{-\theta} \max\{1, a\}]^q \sum_{i\in\mathbb{Z}} \left[2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1})\right]^q.$$

Proof. Since $J_M(A, x, y) \leq \max\{1, A/B\}\, J_M(B, x, y)$, for all $A, B \in (0, \infty)$ and

$$\min\{1, B/A\} = \frac{1}{\max\{1, A/B\}},$$

we have

$$\min\{1, B/A\} \leq \frac{J_M(B, x, y)}{J_M(A, x, y)} \leq \max\{1, B/A\},$$

or equivalently,

$$\min\{1, B/A\}\, J_M(A, x, y) \leq J_M(B, x, y) \leq \max\{1, B/A\}\, J_M(A, x, y).$$

Then, for $A = 2^i$ and $B = 2^i a$,

$$\min\{1, a\}\, J_M(2^i, x, y) \leq J_M(2^i a, x, y) \leq \max\{1, a\}\, J_M(2^i, x, y),$$

for any $i \in \mathbb{Z}$. Thus, if $(x_n^{x,y})$ is a linking sequence from x to y in $X_0 \cap X_1$ and $(k_0, k_1, \ldots, k_{n-1})$ is an n-tuple of integers,

$$\min\{1, a\} \sum_{k_j=i} J(2^i, x_j, x_{j+1}) \leq \sum_{k_j=i} J_M(2^i a, x_j, x_{j+1})$$
$$\leq \max\{1, a\} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1}).$$

or

$$a^{-\theta} \min\{1, a\}\, 2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1}) \leq (2^i a)^{-\theta} \sum_{k_j=i} J_M(2^i a, x_j, x_{j+1})$$
$$\leq a^{-\theta} \max\{1, a\}\, 2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1}).$$

Thus,

$$[a^{-\theta} \min\{1, a\}]^q \sum_{i\in\mathbb{Z}} \left[2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1})\right]^q \leq \sum_{i\in\mathbb{Z}} \left[(2^i a)^{-\theta} \sum_{k_j=i} J_M(2^i a, x_j, x_{j+1})\right]^q$$
$$\leq [a^{-\theta} \max\{1, a\}]^q \sum_{i\in\mathbb{Z}} \left[2^{-i\theta} \sum_{k_j=i} J_M(2^i, x_j, x_{j+1})\right]^q.$$

□

Theorem 8. *If $T : X \to Y$ is an operator with properties (a), (b), and (c), then*

$$T : \overrightarrow{X}^J_{\theta,q} \to \overrightarrow{Y}^J_{\theta,q} \tag{31}$$

and

$$d_{\theta,q,\overrightarrow{Y}}(T(x), T(y)) \leq \max\{\omega_0, \omega_1\}\, d_{\theta,q,\overrightarrow{X}}(x, y), \quad x, y \in \overrightarrow{X}^J_{\theta,q}. \tag{32}$$

Proof. Let $x, y \in X_0 \cap X_1$, $(x_n^{x,y})$ be a linking sequence from x to y in $X_0 \cap X_1$ and $(k_0, k_1, \ldots, k_{n-1})$ be a sequence of integers. So,

$$\begin{aligned}
\sum_{k_j=i} J_M(2^i, T(x_j), T(x_{j+1}); \overrightarrow{Y}) &= \sum_{k_j=i} \max\{d_{Y_0}(T_0(x_j), T_0(x_{j+1})), 2^i d_{Y_1}(T_1(x_j), T_1(x_{j+1}))\} \\
&\leq \sum_{k_j=i} \max\{\omega_0\, d_{X_0}(x_j, x_{j+1}), 2^i\, \omega_1\, d_{X_1}(x_j, x_{j+1})\} \\
&\leq \omega_0 \sum_{k_j=i} \max\{d_{X_0}(x_j, x_{j+1}), 2^i\, \omega_1/\omega_0\, d_{X_1}(x_j, x_{j+1})\} \\
&= \omega_0 \sum_{k_j=1} J_M(2^i \omega_1/\omega_0, x_j, x_{j+1}; \overrightarrow{X})\,.
\end{aligned}$$

In applying $\Gamma_{\theta,q}$ to the derived sequences obtained above,

$$\begin{aligned}
\alpha^q_{\theta,q,\overrightarrow{Y}}(T(x), T(y)) &\leq \omega_0^q \sum_{i\in\mathbb{Z}} \left[2^{-\theta i} \sum_{k_j=i} J_M(2^i \omega_1/\omega_0, x_j, x_{j+1}; \overrightarrow{X})\right]^q \\
&= \omega_0^q (\omega_1/\omega_0)^{\theta q} \sum_{i\in\mathbb{Z}} \left[(2^i \omega_1/\omega_0)^{-\theta} \sum_{k_j=i} J_M(2^i \omega_1/\omega_0, x_j, x_{j+1}; \overrightarrow{X})\right]^q \\
&= \omega_0^{(1-\theta)q}\, \omega_1^{\theta q} \sum_{i\in\mathbb{Z}} \left[(2^i \omega_1/\omega_0)^{-\theta} \sum_{k_j=i} J_M(2^i \omega_1/\omega_0, x_j, x_{j+1}; \overrightarrow{X})\right]^q.
\end{aligned}$$

From Lemma 12, with $a = \omega_1/\omega_0$, we have

$$\alpha_{\theta,q,\overrightarrow{Y}}(T(x), T(y)) \leq \max\{\omega_0, \omega_1\} \alpha_{\theta,q,\overrightarrow{X}}(x, y),$$

for all $x, y \in X_0 \cap X_1$. Now, let $x \in \overrightarrow{X}^J_{\theta,q}$. We want to prove that $T(x) \in \overrightarrow{Y}^J_{\theta,q}$.

Indeed, as $x \in \overrightarrow{X}^J_{\theta,q}$, there is an approximating sequence (x_n) of x in $X_0 \cap X_1$. Since

$$\alpha_{\theta,q,\overrightarrow{Y}}(T(x), T(y)) \leq \max\{\omega_0, \omega_1\} \alpha_{\theta,q,\overrightarrow{X}}(x, y),$$

for all $x, y \in X_0 \cap X_1$, it follows that $(T(x_n))$ is a Cauchy sequence in $Y_0 \cap Y_1$. Since $(y_n = T(x_n))$ is a Cauchy sequence in $(Y_0 \cap Y_1, \alpha_{\theta,q,\overrightarrow{Y}})$, which is continuously embedded in (Y, d_Y), which is complete, there is $y \in Y$ such that $\lim_{n\to\infty} d_Y(y_n, y) = 0$. From (a), it follows that $T(x) = y$, yielding $T(x) \in \overrightarrow{Y}^J_{\theta,q}$.

The rest of the proof is analogous to the one of Theorem 6. □

6. Interpolation of Compact Lipschitz Operators in Metric Spaces

The study of the interpolation of compact operators is one of the most important topics within interpolation theory. In the classical article by Lions-Peetre [5], we have the two most famous results for the case of Banach spaces. These were followed by several generalizations over the years, such as the theorems of Persson [12]; Hayakawa [13]; Cobos and Fernandez [14]; Cobos, Edmunds, and Potter [15]; and Cwikel [16,17]. All these results are for linear operators, except for one of Cwikel's papers, which deals with Lipschitz

operators in normed spaces. Here, we present results for Lipschtiz operators, along the lines of the Lions-Peetre theorems.

Definition 16. *Let (A, d_A) and (B, d_B) be metric spaces and $T : A \to B$ be an operator. Then, T is a compact operator if T is continuous and if for every bounded set $L \subseteq A$, we have that $\overline{T(L)} \subseteq B$ is a compact set.*

The above definition is equivalent to saying that T is compact when the range with the T of a bounded sequence in (A, d_A) has a convergent subsequence in (B, d_B).

Lemma 13. *If $X_0 = X_1 = X$ and $d_0 = d_1 = d_X$, given $\theta \in (0,1)$ and $q \in [1, \infty]$, then $\overrightarrow{X}^K_{\theta,q} = X_0$ and $M_{\theta,q}\, d_0 = \mathcal{D}_{\theta,q}$.*

Proof. We have

$$d_0(x,y) \leq \sum_{k=0}^{n-1} d_0(x_k, x_{k+1}) = \sum_{k=0}^{n-1} h_1(x_k, x_{k+1}).$$

Taking the infimum over all admissible linking sequences from x to y in $X_0 \cup X_1$, we have $d_0(x,y) \leq K(1,x,y)$ for all $x, y \in X_0$. Since $K(1,x,y) \leq \max\{1, 1/t\}\, K(t,x,y)$, one has

$$\min\{1,t\}\, d_0(x,y) \leq K(t,x,y).$$

On the other hand, $K(t,x,y) \leq \min\{1,t\}\, J(1,x,y) = d_0(x,y)$. Thus,

$$K(t,x,y) = \min\{1,t\}\, d_0(x,y).$$

In applying the functional $\Phi_{\theta,q}$,

$$\beta_{\theta,q} = M_{\theta,q}\, d_0.$$

Now, since $X_0 = X_0 \cap X_1 \subseteq \overrightarrow{X}^K_{\theta,q} \subseteq X = X_0$, we have $\overrightarrow{X}^K_{\theta,q} = X_0$. □

Proposition 13. *Let $(X_0, X_1)_X$ be a compatible metric space couple, with $\theta \in (0,1)$ and $q \in [1, \infty]$, and let (F, d_F) be a metric space. If T is an operator that satisfies the following three conditions, then $T : F \to \overrightarrow{X}^K_{\theta,q}$ is a compact Lipschitzian operator:*

(i) $T : F \to X$;
(ii) $T : F \to X_0$ *is a compact Lipschitzian operator;*
(iii) $T : F \to X_1$ *is Lipschitzian.*

Proof. From Lemma 13, we have $F = \overrightarrow{F}^K_{\theta,q}$ and $d_F = (M_{\theta,q})^{-1}\, \mathcal{D}_{\theta,q,\overrightarrow{F}}$. By (29) of Theorem 6, we have

$$\mathcal{D}_{\theta,q,\overrightarrow{X}}(T(x), T(y)) \leq \omega_0^{1-\theta} \omega_1^{\theta} \mathcal{D}_{\theta,q,\overrightarrow{F}}(x,y), \quad x, y \in F,$$

where ω_i is the Lipschitz constant of T in each space X_i, $i = 0, 1$. So, $T : F \to \overrightarrow{X}^K_{\theta,q}$ is well defined.

Let (x_n) be a bounded sequence in (F, d_F). As $X_0 \cap X_1$ is dense in $\overrightarrow{X}^K_{\theta,q}$, without loss of generality, we can consider that $T(x_n) = y_n \in X_0 \cap X_1$ for all $n \in \mathbb{N}$. From the compactness of $T : F \to X_0$, there is a subsequence (y_{n_k}) that converges, according to the metric d_0, to a point $y \in X_0$.

We must prove that $y \in \overrightarrow{X}^K_{\theta,q}$ and that $\mathcal{D}_{\theta,q,\overrightarrow{X}}(T(x_{n_k}), y) \to 0$ when $k \to \infty$. For all $r, s \in \mathbb{N}$, we have, from Proposition 11 and from Corollary 1, that

$$\begin{aligned} \beta_{\theta,q,\overrightarrow{X}}(T(x_{n_r}), T(x_{n_s})) &\leq \overline{C}\, [d_0(T(x_{n_r}), T(x_{n_s}))]^{1-\theta}\, [d_1(T(x_{n_r}), T(x_{n_s}))]^{\theta} \\ &\leq \overline{C}\, [d_0(T(x_{n_r}), T(x_{n_s}))]^{1-\theta}\, [\omega_1\, d_F(x_{n_r}, x_{n_s})]^{\theta}, \end{aligned}$$

where $\overline{C}$ is a positive constant arising from the combination of the constant C of Corollary 1 with the constant of Lemma 13. Since (x_n) is bounded in (F, d_F) and (y_{n_k}) is a Cauchy sequence in (X_0, d_0), it follows that (y_{n_k}) is a Cauchy sequence in $(X_0 \cap X_1, \beta_{\theta,q,\overrightarrow{X}})$. As $d_X(x,y) \leq C_0\, d_0(x,y)$, for all $x, y \in X_0$, we have $\lim_{k\to\infty} d_X(y_{n_k}, y) = 0$. So, $y \in \overrightarrow{X}^K_{\theta,q}$. Now, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{D}_{\theta,q}(y_{n_k}, y) &\leq \lim_{j\to\infty} \beta_{\theta,q}(y_{n_k}, y_{n_j}) \\
&\leq \lim_{j\to\infty} \overline{C}\,[d_0(y_{n_k}, y_{n_j})]^{1-\theta}\,[d_1(y_{n_k}, y_{n_j})]^{\theta} \\
&\leq \overline{C} \lim_{j\to\infty} [d_0(y_{n_k}, y_{n_j})]^{1-\theta}\,[\omega_1\, d_F(x_{n_k}, x_{n_j})]^{\theta} \\
&\leq M\,\overline{C} \lim_{j\to\infty} [d_0(y_{n_k}, y_{n_j})]^{1-\theta},
\end{aligned}$$

where $M > 0$ is a constant that does not depend on k. Taking the limit with $k \to \infty$ in the above inequality, we have the target result, since (y_{n_k}) is a Cauchy sequence in (X_0, d_0). Therefore, T is a compact operator. □

7. Interpolation of Metric Spaces vs. Interpolation of Normed Spaces

In this section, we draw a parallel between the interpolation of metric spaces and the usual interpolation of normed spaces. We show that the interpolation of metric spaces developed in this work is still a valid interpolation method for normed spaces, with the advantage of not being dependent on the algebraic structure of the space, whereas Peetre's methods K and J are strongly linked to decomposition into algebraic components for method K and convergent series for method J.

Lemma 14. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(X, \|\cdot\|_X)$ be normed vector spaces such that X is a vector subspace of $\mathcal{X}$, and there is a constant $C > 0$ with $\| x \|_{\mathcal{X}} \leq C \| x \|_X$, for all $x \in X$. Then, the relative completion of $(X, \|\cdot\|_X)$ in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, denoted by $(X_{\mathcal{X}}, \|\cdot\|_{X_{\mathcal{X}}})$ is a vector subspace of $\mathcal{X}$ that contains the subspace X, for which its norm $\|\cdot\|_{X_{\mathcal{X}}}$ satisfies*

$$\| x \|_{\mathcal{X}} \leq C \| x \|_{X_{\mathcal{X}}}, \quad \forall x \in X_{\mathcal{X}}$$

and

$$\| x \|_{X_{\mathcal{X}}} \leq \| x \|_X, \quad \forall x \in X.$$

Proof. Let $x, y, z \in X_{\mathcal{X}}$ and $\lambda \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Since $(X, \|\cdot\|_X)$ is a normed vector space, it follows that $(x_n + \lambda\, y_n)$ is an approximation sequence of $x + \lambda\, y$ whenever (x_n) and (y_n) are sequences that approximate x and y in $(X, \|\cdot\|_X)$. Thus, we obtain that $X_{\mathcal{X}}$ is a vector subspace of $\mathcal{X}$.

Now, let us show that the metric d defined for the relative completion of $(X, \|\cdot\|_X)$ in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a norm for this completion, which we will denote by $\|\cdot\|_{X_{\mathcal{X}}}$. To do so, it is only necessary to show that this metric is invariant by translation and satisfies the property $\| \lambda\, x \|_{X_{\mathcal{X}}} = |\lambda|\, \| x \|_{X_{\mathcal{X}}}$. Let $x, y, z \in X_{\mathcal{X}}$. We prove the translation invariance with two statements:

Claim 1: *The distance function associated with $\|\cdot\|_X$ is translation invariant.*

Let us denote this distance by $\overline{d_X}$. We have

$$\overline{d_X}(x+z, z+y) = \inf \lim_{n\to\infty} \| r_n - s_n \|_X,$$

where the infimum is taken over all approximating sequences of $x + z$ and $z + y$ in $(X, \|\cdot\|_X)$. Given the approximating sequences (x_n), (y_n), and (z_n) of x, y, and z, respectively, in $(X, \|\cdot\|_X)$, one has

$$\lim_{n\to\infty} \| x_n - y_n \|_X = \lim_{n\to\infty} \| (x_n + z_n) - (z_n + y_n) \|_X\,. \tag{33}$$

On the other hand, given the approximation sequences (r_n), (s_n), and (z_n) of $x+z$, $z+y$, and z, respectively, one has

$$\lim_{n\to\infty} \| (r_n - z_n) - (s_n - z_n) \|_X = \lim_{n\to\infty} \| r_n - s_n \|_X . \tag{34}$$

Taking the infimum over all approximating sequences of x and y in (33), we have $\overline{d_X}(x,y) \geq \overline{d_X}(x+z, z+y)$. Note that $(r_n - z_n)$ is a Cauchy sequence in $(X, \|\cdot\|_X)$ and

$$\| r_n - z_n - x \|_{\mathcal{X}} \to \| x + z - z - x \|_{\mathcal{X}} = 0$$

and the same reasoning can be used for $(s_n - z_n)$. Therefore, $(r_n - z_n)$ is an approximating sequence of x in $(X, \|\cdot\|_X)$, and likewise, $(s_n - z_n)$ is an approximating sequence of y in $(X, \|\cdot\|_X)$. Taking the infimum over all approximating sequences of $x+z$ and $z+y$ in (34), we have $\overline{d_X}(x,y) \leq \overline{d_X}(x+z, z+y)$. So, $\overline{d_X}(x,y) = \overline{d_X}(x+z, z+y)$ for all $x, y, z \in X_{\mathcal{X}}$, and we have the invariance by translation.

Claim 2: *For all $x \in X_{\mathcal{X}}$ and $\lambda \in \mathbb{K}$, $\overline{d_X}(\lambda x, 0) = |\lambda| \overline{d_X}(x,0)$.*

Indeed, if (x_n) is an approximating sequence x in $(X, \|\cdot\|_X)$, then (λx_n) is an approximating sequence of λx in $(X, \|\cdot\|_X)$, and we also have the reciprocal for $\lambda \neq 0$. The case of $\lambda = 0$ is trivial. Then, let $\lambda \neq 0$. We have

$$\overline{d_X}(\lambda x, 0) = \inf \lim_{n\to\infty} \| \lambda x_n - \lambda z_n \|_X = |\lambda| \inf \lim_{n\to\infty} \| x_n - z_n \|_X = |\lambda| \overline{d_X}(x,0),$$

where the infimum is taken over all approximating sequences of x and 0 in the space $(X, \|\cdot\|_X)$.

Finally, we prove that the metric of the relative completion is a norm. For $x, y, z \in X_{\mathcal{X}}$, given any linking sequence $x_0 = x, x_1, \ldots, x_n = y \in X_{\mathcal{X}}$, we have that $z_0 = x_0 + z$, $z_1 = x_1 + z, \ldots, z_n = x_n + z$ is a linking sequence from $x+z$ to $z+y$ in $X_{\mathcal{X}}$, and

$$\sum_{k=0}^{n-1} \overline{d_X}(x_k, x_{k+1}) = \sum_{k=0}^{n-1} \overline{d_X}(x_k + z, x_{k+1} + z) = \sum_{k=0}^{n-1} \overline{d_X}(z_k, z_{k+1}).$$

Now, if $m_0 = x+z, m_1, \ldots, m_s = z+y \in X_{\mathcal{X}}$ is a linking sequence from $x+z$ to $z+y$, we have that $c_i = m_i - z \in X_{\mathcal{X}}$, for $i = 0, 1, \ldots, s-1$, is a linking sequence from x to y in $X_{\mathcal{X}}$, and

$$\sum_{k=0}^{s-1} \overline{d_X}(m_k, m_{k+1}) = \sum_{k=0}^{s-1} \overline{d_X}(m_k - z, m_{k+1} - z) = \sum_{k=0}^{s-1} \overline{d_X}(c_k, c_{k+1}).$$

Thus, we obtain the translation invariance of $\|\cdot\|_{\overline{X}_{\mathcal{X}}}$, taking the infimum over the linking sequences consecutively. Using similar reasoning, we have that $\| \lambda x \|_{X_{\mathcal{X}}} = |\lambda| \ \| x \|_{X_{\mathcal{X}}}$. □

Definition 17. *Let A_0 and A_1 be normed vector spaces. We say that A_0 and A_1 are a pair of compatible normed spaces if there exists a topological vector space U with a Hausdorff topology, such that A_0 and A_1 are vector subspaces of U, and each inclusion is continuous. We use $(A_0, A_1)_U$ to denote a pair of compatible normed spaces.*

Proposition 14. *If $(X_0, X_1)_X$ is a pair of compatible normed spaces and $K_M(\cdot,\cdot,\cdot)$ is the K-functional defined for metric spaces, then $K_M(t,\cdot,\cdot)$ is a norm in $X_0 \cap X_1$, considering $X = X_0 + X_1$ and $d_X(x,y) = \| x - y \|_{X_0+X_1}$ for $x, y \in X_0 + X_1$.*

Proof. We just need to show that $K_M(t, x+z, y+z) = K_M(t,x,y)$, for all $x, y \in X_0 \cup X_1$ and $z \in X_0 \cap X_1$, and also that $K_M(t, \lambda x, \lambda y) = |\lambda| K_M(t,x,y)$, for all $\lambda \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

Let $(x_n^{x,y})_{adm}$ be any admissible linking sequence from x to y in $X_0 \cup X_1$. Consider the admissible sequence $(z_n^{x+z,y+z})_{adm}$, given by $z_k = x_k + z$, for $0 \leq k \leq n$. We have

$$\sum_{k=0}^{n-1} h_t(x_k, x_{k+1}) = \sum_{k=0}^{n-1} h_t(z_k, z_{k+1}).$$

Hence, we can obtain that $K_M(t, x+z, y+z) = K_M(t, x, y)$ by just taking the infimum over all admissible linking sequences from x to y in $X_0 \cup X_1$, and then over all admissible linking sequences from $x+z$ to $z+y$ in $X_0 \cup X_1$, obtaining the inverse inequality.

Now, let $\lambda \in \mathbb{K}$. If $(p_m^{x,y})_{adm}$ is any admissible linking sequence from x to y in $X_0 \cup X_1$, we have that $(\lambda\, p_m^{\lambda\, x, \lambda\, y})_{adm}$ is an admissible linking sequence from $\lambda\, x$ to $\lambda\, y$ in $X_0 \cup X_1$. Then,

$$\sum_{k=0}^{m-1} h_t(\lambda\, p_k, \lambda\, p_{k+1}) = |\lambda| \sum_{k=0}^{m-1} h_t(p_k, p_{k+1}).$$

Thus, $K_M(t, \lambda x, \lambda y) = |\lambda|\, K_M(t, x, y)$. □

From Proposition 14, in the next theorem, we obtain another interpolation method for normed spaces.

Theorem 9. *If $(X_0, X_1)_X$ is a pair of compatible normed spaces and $K_M(\cdot,\cdot,\cdot)$ is the K-functional defined for metric spaces, then for fixed $\theta \in (0,1)$ and $q \in [1, \infty]$, the space $\overrightarrow{X}_{\theta,q}^{K}$, defined for the metric spaces, is an exact interpolation space for the pair $(X_0, X_1)_X$, considering $X = X_0 + X_1$, where the metric in X is given by $\|\cdot\|_{X_0+X_1}$.*

Proof. Let $\overrightarrow{X}_{\theta,q}^{K_M}$ be the relative completion of $X_0 \cap X_1$ with respect to the norm

$$\|\cdot\|_{\theta,q,K_M} = \left[\int_0^\infty [t^{-\theta}\, K_M(t,\cdot,\cdot)]^q dt/t\right]^{1/q},$$

in the space $X = X_0 + X_1$ with norm $\|\cdot\|_{X_0+X_1}$. By Proposition 14 and by Lemma 14, the result follows, completing the normed space $(X_0 \cap X_1, \beta_{\theta,q})$ in the normed space $(X_0 + X_1, \|\cdot\|_{X_0+X_1})$. Finally, it is enough to observe that continuous linear operators in normed spaces are Lipschitzian operators. □

Lemma 15. *For every $t > 0$ and every $x \in X_0 \cap X_1$, we have*

$$K_N(t,x) \le K_M(t,x) \le 3K_N(t,x),$$

where $K_M(t,x) := K_M(t,x,0)$, and K_N is a functional K defined for the usual normed space interpolation.

Proof. Let $x \in X_0 \cap X_1$ and $x = x_0 + x_1 \in X_0 + X_1$. Consider the admissible linking sequence from x to 0 in $X_0 \cup X_1$:

$$y_0 = x, y_1 = x_0, y_2 = 0, y_3 = x_1, y_4 = 0$$

and in X_1, consider the norm $t\ \|\cdot\|_1$. We have

$$\begin{aligned} K_M(t,x) &\le \sum_{k=0}^{3} h_t(y_k, y_{k+1}) \\ &= h_t(y_0,y_1) + h_t(y_1,y_2) + h_t(y_2,y_3) + h_t(y_3,y_4) \\ &= h_t(x_1,0) + h_t(x_0,0) + h_t(x_1,0) + h_t(x_1,0) \\ &\le 3(\|\, x_0\,\|_0 + t\ \|\, x_1\,\|_1)\,. \end{aligned}$$

Therefore, taking the infimum over all the decompositions $x_0 + x_1 = x$ in the space $X_0 + X_1$, we obtain

$$K_M(t,x) \le 3\, K_N(t,x), \tag{35}$$

for all $x \in X_0 \cap X_1$ and for all $t > 0$. Now, let $(x_n^{x,0})$ be an admissible linking sequence from x to 0 in $X_0 \cup X_1$. We have

$$\sum_{k=0}^{n-1} (x_k - x_{k+1}) = x, \quad \text{with sum in } X_0 + X_1.$$

For each pair x_k, x_{k+1} that belongs to the intersection, one has

$$K_N(t, x_k - x_{k+1}) \leq \| x_k - x_{k+1} \|_0 + t \| 0 \|_1, \text{ if } x_k, x_{k+1} \in X_0$$

and

$$K_N(t, x_k - x_{k+1}) \leq \| 0 \|_0 + t \| x_k - x_{k+1} \|_1, \text{ if } x_k, x_{k+1} \in X_1.$$

Thus, we can write

$$K_N(t, x_k - x_{k+1}) \leq h_t(x_k, x_{k+1}) = \min\{\| x_k - x_{k+1} \|_0, t \| x_k - x_{k+1} \|_1\}.$$

If $x_k, x_{k+1} \in X_i \setminus X_{1-i}$, with $i \in \{0,1\}$, we have

$$K_N(t, x_k - x_{k+1}) \leq t^i \| x_k - x_{k+1} \|_i = h_t(x_k, x_{k+1}).$$

Then,

$$K_N(t,x) = K_N\left(t, \sum_{k=0}^{n-1} x_k - x_{k+1}\right) \leq \sum_{k=0}^{n-1} K_N(t, x_k - x_{k+1}) \leq \sum_{k=0}^{n-1} h_t(x_k, x_{k+1}).$$

Taking the infimum over all admissible linking sequences from x to 0 in $X_0 \cup X_1$, we obtain $K_N(t,x) \leq K_M(t,x)$. □

Proposition 15. *Given a pair of compatible normed spaces $(X_0, X_1)_X$, $\theta \in (0,1)$ and $1 \leq q < \infty$, let $Z = X_0 \cap X_1$ be equipped with the usual norm of the K-normed interpolated space:*

$$\| z \|_{\theta,q,K_N} = \left(\int_0^\infty [t^{-\theta} K_N(t,z)]^q \frac{dt}{t} \right)^{1/q}.$$

If in $X = X_0 + X_1$, we consider the norm $K_N(1,\cdot)$, and M is the relative completion of Z in $X = X_0 + X_1$, then $M = \overline{X}_{\theta,q}^{K_N}$, where $\overline{X}_{\theta,q}^{K_N}$ is the usual normed interpolated space.

Proof. Let $x \in M$. By the definition of M, there is a sequence (x_n) of points in $Z = X_0 \cap X_1$ such that

$$(1) \qquad \| x_n - x \|_{X_0+X_1} \to 0,$$

$$(2) \qquad \lim_{m,n\to\infty} \| x_m - x_n \|_{\theta,q,K_N} = 0$$

Since

$$\lim_{n\to\infty} K_N(1, x_n - x) = \lim_{n\to\infty} \| x_n - x \|_{X_0+X_1} = 0$$

and $K_N(1,\cdot)$ is equivalent to $K_N(t,\cdot)$, for all $t > 0$, it follows that $\lim_{n\to\infty} K_N(t, x_n - x) = 0$ for all $t > 0$, or $\lim_{n\to\infty} K_N(t, x_n) = K_N(t,x)$, for all $t > 0$. For each $t > 0$ and each $n \in \mathbb{N}$ let $f_n(t) = (t^{-\theta} K_N(t, x_n))^q / t$. Since (x_n) is a Cauchy sequence in $(X_0 \cap X_1, \| \cdot \|_{\theta,q,K_N})$, there is $\gamma > 0$ so that for all $n \in \mathbb{N}$,

$$\int_0^\infty f_n(t)dt = \| x_n \|_{\theta,q,K_N}^q \leq \gamma,$$

since, in a metric space, every Cauchy sequence is bounded. Therefore, each f_n is measurable, and

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} (t^{-\theta} K_N(t, x_n))^q / t = (t^{-\theta} K_N(t,x))^q / t := f(t).$$

From Fatou's Lemma,

$$\int_0^\infty \liminf_{n\to\infty} f_n(t)dt \leq \liminf_{n\to\infty} \int_0^\infty f_n(t)dt,$$

that is,

$$\| x \|^q_{\theta,q,K_N} = \int_0^\infty [t^{-\theta} K_N(t,x)]^q \frac{dt}{t} \leq \liminf_{n\to\infty} \| x_n \|^q_{\theta,q,K_N} \leq \gamma < \infty.$$

Thus, $x \in \overline{X}^{K_N}_{\theta,q}$.

Consider now $x \in \overline{X}^{K_N}_{\theta,q}$. As $q < \infty$, then for $x \in \overline{X}^{K_N}_{\theta,q}$, and by the density of $X_0 \cap X_1$ in $\overline{X}^{K_N}_{\theta,q}$, we conclude that there is a sequence $(x_n) \in X_0 \cap X_1$ such that

$$(*) \qquad \| x_n - x \|_{\theta,q,K_N} \to 0.$$

Then, by $(*)$ we have $\| x_n - x \|_{X_0+X_1} \to 0$ because the interpolated space is continuously included in $X_0 + X_1$. Since every convergent sequence is a Cauchy sequence, (x_n) is a Cauchy sequence on X such that

$$\lim_{n\to\infty} d_X(x_n, x) = \lim_{n\to\infty} \| x_n - x \|_{X_0+X_1} = 0.$$

This shows that $x \in M$. So, we obtain $M = \overline{X}^{K_N}_{\theta,q}$. □

Lemma 16. *Given a pair of compatible normed spaces* $(X_0, X_1)_X$, $\theta \in (0,1)$, *and* $1 \leq q < \infty$, *consider* $X_0 \cap X_1$ *equipped with the usual norm*

$$\beta_{\theta,q,\overline{X}} = \| z \|_{\theta,q,K_N} = \left(\int_0^\infty [t^{-\theta} K_N(t,z)]^q \frac{dt}{t} \right)^{1/q}.$$

Consider in $X = X_0 + X_1$ *the norm* $d_X(x,y) = K_N(1, x-y)$. *If* $\| \cdot \|_{\theta,q,\overline{X}}$ *is the norm of the relative completion of* $(X_0 \cap X_1, \beta_{\theta,q,\overline{X}})$ *in* (X, d_X), *then for all* $x \in \overline{X}^{K_N}_{\theta,q}$,

$$\| x \|_{\theta,q,\overline{X}} = \| x \|_{\theta,q,K_N}.$$

Proof. Let $x \in \overline{X}^{K_N}_{\theta,q}$. By Proposition 15, there is a sequence (x_n) of points in $X = X_0 \cap X_1$ such that

$$(1) \qquad \| x_n - x \|_{X_0+X_1} \to 0,$$

$$(2) \qquad \lim_{m,n\to\infty} \| x_m - x_n \|_{\theta,q,K_N} = 0.$$

Since

$$\lim_{n\to\infty} K_N(1, x_n - x) = \lim_{n\to\infty} \| x_n - x \|_{X_0+X_1} = 0$$

and $K_N(1,\cdot)$ is equivalent to $K_N(t,\cdot)$, for all $t > 0$, it follows that $\lim_{n\to\infty} K_N(t, x_n - x) = 0$, or $\lim_{n\to\infty} K_N(t, x_n) = K_N(t,x)$, for all $t > 0$. For each $t > 0$ and each $n \in \mathbb{N}$, let $f_n(t) = (t^{-\theta} K_N(t, x_n))^q / t$. Since (x_n) is a Cauchy sequence in $(X_0 \cap X_1, \| \cdot \|_{\theta,q,K_N})$, it follows that there is $\gamma > 0$ such that for all $n \in \mathbb{N}$,

$$\int_0^\infty f_n(t)dt = \| x_n \|^q_{\theta,q,K_N} \leq \gamma.$$

Therefore, each f_n is measurable, and

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} (t^{-\theta} K_N(t, x_n))^q / t = (t^{-\theta} K_N(t,x))^q / t := f(t).$$

From Fatou's Lemma,

$$\int_0^\infty \liminf_{n\to\infty} f_n(t)dt \leq \liminf_{n\to\infty} \int_0^\infty f_n(t)dt,$$

that is,

$$\| x \|^q_{\theta,q,K_N} = \int_0^\infty [t^{-\theta} K_N(t,x)]^q \frac{dt}{t} \leq \liminf_{n\to\infty} \| x_n \|^q_{\theta,q,K_N} = \lim_{n\to\infty} \| x_n \|^q_{\theta,q,K_N}.$$

Taking the infimum over all approximating sequences x in $(X_0 \cap X_1, \beta_{\theta,q,\overline{X}})$, we have

$$\| x \|^q_{\theta,q,K_N} \leq \overline{\beta_{\theta,q,\overline{X}}}(x,0),$$

where $\overline{\beta_{\theta,\overline{X}}}$ is the distance associated with the metric $\beta_{\theta,q,\overline{X}}$. As this inequality is valid for all $x \in \overline{X}^{K_N}_{\theta,q}$, it follows that

$$\| x \|_{\theta,q,K_N} \leq \| x \|_{\theta,q,\overline{X}}, \tag{36}$$

for all $x \in \overline{X}_{\mathcal{X}}$. Since $X_0 \cap X_1$ is dense in the completion of $(X_0 \cap X_1, \beta_{\theta,q,\overline{X}})$ in (X, d_X), for all x, there is a sequence (x_n) of points in $X_0 \cap X_1$ such that

$$\| x_n - x \|_{\theta,q,\overline{X}} \to 0.$$

Hence, from (36), (x_n) is a sequence convergent to x in the norm $\| \cdot \|_{\theta,q,K_N}$. Since $(X_0 \cap X_1, \| \cdot \|_{\theta,q,K_N})$ is continuously embedded in the relative completion $(\overline{X}^{K_N}_{\theta,q}, \| \cdot \|_{\theta,q,\overline{X}})$, for all $n \in \mathbb{N}$, we have

$$\| x_n \|_{\theta,q,\overline{X}} \leq \| x_n \|_{\theta,q,K_N}.$$

Taking $n \to \infty$ in the above inequality, we have

$$\| x \|_{\theta,q,\overline{X}} \leq \| x \|_{\theta,q,K_N},$$

for any $x \in \overline{X}_{\mathcal{X}}$. So, we obtain $\| x \|_{\theta,q,\overline{X}} = \| x \|_{\theta,q,K_N}$ as we wanted. □

Corollary 3. *Let (X_0, X_1) be a pair of compatible normed spaces, $\theta \in (0,1)$ and $q \in [1,\infty)$, and let $\overrightarrow{X}^{K_M}_{\theta,q}$ be the interpolated space obtained with the functional K_M, with norm $\| \cdot \|_{\theta,q,K_M}$. If $\overline{X}^{K_N}_{\theta,q}$ is the usual interpolated space of normed spaces, with norm $\| \cdot \|_{\theta,q,K_N}$, then, considering $\mathcal{X} = X_0 + X_1$ with metric $\| \cdot \|_{X_0+X_1}$, we have*

$$\overline{X}^{K_N}_{\theta,q} = \overrightarrow{X}^{K_M}_{\theta,q},$$

with continuous inclusion and norms satisfying

$$\| x \|_{\theta,q,K_N} \leq \| x \|_{\theta,q,K_M} \leq 3 \| x \|_{\theta,q,K_N}.$$

Proof. The proof follows from Lemmas 4, 15, and 16. □

The J-Metric Method in the Context of Normed Spaces

Another question that arises is whether the J-space of the normed case coincides with the J-space of the metric case when $\overrightarrow{X} = (X_0, X_1)$ is a pair of compatible normed spaces. In what follows, J_N denotes the usual J functional.

Lemma 17. *Let $\overrightarrow{X} = (X_0, X_1)$ be a compatible pair of normed spaces. So, $\alpha_{\theta,q,\overrightarrow{X}}$ is a norm over $X_0 \cap X_1$.*

Proof. Since $\alpha_{\theta,q,\overrightarrow{X}}$ is a metric over $X_0 \cap X_1$, we only need to show that it is translation invariant, and that given $\lambda \in \mathbb{R}$, we have $\alpha_{\theta,q,\overrightarrow{X}}(\lambda x, \lambda y) = |\lambda| \, \alpha_{\theta,q,\overrightarrow{X}}(x,y)$.

Given $x, y, z \in X_0 \cap X_1$, let $(x^{x,y}_n)$ be a linking sequence from x to y in $X_0 \cap X_1$ and let $(k_0, k_1, \ldots, k_{n-1})$ be a sequence of integers. Then,

$$\sum_{k_j=i} J_M(2^i, x_j, x_{j+1}) = \sum_{k_j=i} J_M(2^i, x_j + z, x_{j+1} + z).$$

Therefore, the sequence derived from x to y is equal to the sequence derived from $x+z$ to $y+z$. This shows that $\alpha_{\theta,q,\overrightarrow{X}}$ is translation invariant on $X_0 \cap X_1$. Now, if $\lambda \in \mathbb{R}$,

$$\sum_{k_j=i} J_M(2^i, \lambda\, x_j, \lambda\, x_{j+1}) = |\lambda| \sum_{k_j=i} J_M(2^i, x_j, x_{j+1}).$$

Consequently, $\alpha_{\theta,q,\overrightarrow{X}}(\lambda\, x, y) = |\lambda|\, \alpha_{\theta,q,\overrightarrow{X}}(x, y)$. □

Proposition 16. *Let $\overrightarrow{X} = (X_0, X_1)$ be a compatible pair of normed spaces. Then,*

$$\alpha_{\theta,q}(a, 0) \geq C \parallel a \parallel_{\theta,q,J_N}$$

and

$$\alpha_{\theta,q}(a, 0) \leq \left(1 + 2^{(1-\theta)q}\right)^{1/q} \parallel a \parallel_{\theta,q,J_N} + J_N(1, a),$$

for all $a \in X_0 \cap X_1$.

Proof. We have that $\alpha_{\theta,q}(a, 0) \geq C \parallel a \parallel_{\theta,q,J_N}$ follows from

$$\alpha_{\theta,q}(a, 0) \geq \gamma\, \beta_{\theta,q}(a, 0) \sim \parallel a \parallel_{\theta,q,K} \sim \parallel a \parallel_{\theta,q,J_N}.$$

Now, let $(a_j)_{j\in\mathbb{Z}}$ be such that the following hold:

(i) $a_j \in X_0 \cap X_1$ for all $j \in \mathbb{Z}$,

(ii) $\sum_{j\in\mathbb{Z}} a_j = a$, (convergence in $X_0 + X_1$),

(iii) $\parallel (J_N(2^j, a_j)) \parallel_{\lambda_{\theta,q}} < \infty$.

Consider the following linking sequence from 0 to a,

$$x_0 = 0, \;\; x_1 = a_1, \;\; x_2 = 0 \quad \text{e} \quad x_3 = a,$$

and the triple of integers given by $\{(1, 2, 0)\}$. For each $i \in \mathbb{Z}$, we have

$$c_i = \begin{cases} J_M(2^1, x_0, x_1), & \text{if } i = 1 \\ J_M(2^2, x_1, x_2), & \text{if } i = 2 \\ J_M(1, x_2, x_3), & \text{if } i = 0 \\ 0 & \text{a.o.} \end{cases}$$

Note that

$$\begin{aligned}\Phi_{\theta,q}((c)) &= \left(\sum_{i\in\mathbb{Z}}[2^{-i\theta}c_i]^q\right)^{1/q} = \left([2^{-\theta}c_1]^q + [2^{-2\theta}c_2]^q + [c_0]^q\right)^{1/q}\\ &= \left([2^{-\theta}J_M(2^1,0,x_1)]^q + [2^{-2\theta}J_M(2^2,x_1,0)]^q + [J_M(1,0,a)]^q\right)^{1/q}\\ &\le \left([2^{-\theta}J_N(2^1,a_1)]^q + [2^{-2\theta}J_N(2^2,a_1)]^q + [J_N(1,a)]^q\right)^{1/q}\\ &\le \left[(1+2^{(1-\theta)q})[2^{-\theta}J_N(2^1,a_1)]^q + [J_N(1,a)]^q\right]^{1/q}\\ &\le \left[(1+2^{(1-\theta)q})\sum_{i\in\mathbb{Z}}[2^{-i\theta}J_N(2^i,a_i)]^q\right]^{1/q} + J_N(1,a)\\ &= \left(1+2^{(1-\theta)q}\right)^{1/q} \| (J_N(2^i,a_i))_{i\in\mathbb{Z}} \|_{\lambda_{\theta,q}} + J_N(1,a), \quad \forall n\in\mathbb{Z}.\end{aligned}$$

The result follows by taking the infimum over all discrete canonical representations of a. □

8. Examples of Metric Interpolation

This section provides examples of the method of the interpolation of metric spaces that cannot be applied with the interpolation of usual normed spaces. We also present normed vector spaces, but with scalar fields different from $\mathbb{R}$ and $\mathbb{C}$, which appear in the literature and which lack a specific treatment in interpolation theory.

8.1. The Spaces $\mathbb{Q}[\sqrt{r}]$ and $\mathbb{Q}[\sqrt{s}]$

Given a prime number p, we define the set

$$\mathbb{Q}[\sqrt{p}] := \{a + b\sqrt{p};\ \ a, b \in \mathbb{Q}\}.$$

This set appears frequently in algebra, especially in field extension theory. It is a field obtained by isomorphism with the quotient of the ring of polynomials with rational coefficients $\mathbb{Q}[x]$, by the maximal ideal generated by the irreducible polynomial $x^2 - p$. This set is a normed vector space, of dimension 2, over the field of rational numbers.

We will show that although this set cannot be considered under the interpolation of normed spaces, because it is not a real vector space, we can consider it in the theory of interpolation in metric spaces.

Let $r, s \in \mathbb{N}$ be distinct prime numbers. Consider $X_0 = \mathbb{Q}[\sqrt{r}]$, $X_1 = \mathbb{Q}[\sqrt{s}]$, and $X = \mathbb{R}$ provided with the induced metric of the absolute value of $\mathbb{R}$. We have that $X_0 \cap X_1 = \mathbb{Q}$ and, for $x, y \in X_0 \cap X_1$ and $t > 0$,

$$\min\{1,\, t\} d_{\mathcal{X}}(x,y) \le K_M(t,x,y) \le \min\{1,\, t\}\, J_M(1,x,y).$$

Note that $d_{\mathcal{X}}(x,y) = |x-y|$ and $J_M(1,x,y) = |x-y|$. Then,

$$\min\{1,\, t\}|x-y| \le K_M(t,x,y) \le \min\{1,\, t\}\, |x-y|.$$

Consequently,

$$\mathcal{D}_{\theta,q}(x,y) = M_{\theta,q}\, |x-y|,$$

where $M_{\theta,q} = \Phi_{\theta,q}(\min\{1,\cdot\})$.

This gives that the interpolated space is the whole line, because if $z \in \mathbb{R}$, there is a Cauchy sequence (z_n) in $(\mathbb{Q}, |\cdot|)$ such that $|z_n - z| \to 0$. Consequently, (z_n) is a Cauchy sequence in $(\mathbb{Q}, M_{\theta,q}\,|\cdot|)$ and $M_{\theta,q}\,|z_n - z| \to 0$. So, $z \in \overrightarrow{X}^{K}_{\theta,q}$. Since the reverse inclusion is immediate from the definition of Cauchy completion, we have $\mathbb{R} = \overrightarrow{X}^{K_M}_{\theta,q}$.

Let us now determine what is the interpolated space $\overrightarrow{X}_{\theta,q}^{J_M}$. From the inclusion of J_M-space in K_M-space, it follows that

$$\frac{1}{2\,\gamma_{\theta,\infty,1}} D_{\theta,q}(x,y) \leq d_{\theta,q}(x,y),$$

for all $x, y \in \mathbb{Q}$. We know that $\mathcal{D}_{\theta,q}(x,y) = M_{\theta,q}\,|x-y|$ and

$$d_{\theta,q}(x,y) \leq 2^{\theta}\, t^{-\theta}\, J_M(t,x,y),$$

for all $x, y \in \mathbb{Q}$ and for all $t > 0$. From these inequalities, we have, for $x, y \in \mathbb{Q}$,

$$\frac{M_{\theta,q}}{2\,\gamma_{\theta,\infty,1}}\,|x-y| \leq d_{\theta,q}(x,y) \leq 2^{\theta}\, t^{-\theta}\, J_M(t,x,y), \;\; \forall t > 0.$$

For

$$t = 2\left[\frac{4}{2-2^{\theta}} + \frac{2}{2^{\theta}-1}\right]^{1/\theta} [(1-\theta)\theta\, q]^{1/(\theta q)},$$

we have

$$\frac{M_{\theta,q}}{2\,\gamma_{\theta,\infty,1}}\,|x-y| \leq d_{\theta,q}(x,y) \leq \frac{M_{\theta,q}}{2\,\gamma_{\theta,\infty,1}}\, J_M(t,x,y).$$

If for the values of θ and q, we have $t \leq 1$, then $d_{\theta,q}(x,y) = \frac{M_{\theta,q}}{2\,,\gamma_{\theta,\infty,1}}|x-y|$ for all $x, y \in \mathbb{Q}$. Then, $\mathbb{R} = \overrightarrow{X}_{\theta,q}^{K_M} = \overrightarrow{X}_{\theta,q}^{J_M}$ with equivalent metrics.

8.2. Two Examples of Metric Interpolation in Sequence Spaces

Here, we will consider a Fréchet sequence space. For more details on Fréchet spaces, see [18]. Let $\mathbb{R}^{\omega} = \{(x_1, x_2, x_3, \ldots);\;\; x_k \in \mathbb{R},\; \forall\, k \in \mathbb{N}\}$. Over $\mathbb{R}^{\omega}$, consider the pointwise operations of sum and scalar multiplication. Endowed with these operations, $\mathbb{R}^{\omega}$ is a linear space over $\mathbb{R}$.

Definition 18 (The pointwise convergence in $\mathbb{R}^{\omega}$)**.** *Let (x^i) be a sequence in $\mathbb{R}^{\omega}$. We say that (x^i) converges to $x = (x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^{\omega}$, if for every $k \in \mathbb{N}$, the numeric sequence (x_k^i), formed by the k-th coordinates of the sequence points (x^i), converges to the k-th coordinate of x, that is,*

$$\lim_{i\to\infty} x^i = x \Longleftrightarrow \lim_{i\to\infty} x_k^i = x_k\ ,$$

for all $k \in \mathbb{N}$.

Proposition 17. *Consider in $\mathbb{R}^{\omega}$ the pointwise convergence. Then, $\mathbb{R}^{\omega}$ does not admit a norm.*

Proof. Suppose that there is a norm $\|\cdot\|$ on $\mathbb{R}^{\omega}$ that represents its pointwise convergence; that is, if (x^i) converges to x, then $\|x - x^i\| \to 0$ if $i \to \infty$. For every $i \in \mathbb{N}$, let e_i be the sequence with 1 in the i-th coordinate and 0 in all the others. Now, define $x^i = \dfrac{e_i}{\|\, e_i\,\|}$. Then, we have that x^i converges to the null sequence $(0, 0, \ldots)$ since

$$\lim_{i\to\infty} x^i = (0, 0, \ldots)\,.$$

But

$$\lim_{i\to\infty} \|\, x^i\,\| = 1 \neq 0.$$

Therefore, no norm can define pointwise convergence in $\mathbb{R}^{\omega}$. □

Proposition 18. *The space $\mathbb{R}^{\omega}$ endowed with the function*

$$d_\omega((x_k),(y_k)) := \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1+|x_k-y_k|)}, \tag{37}$$

where $(x_k), (y_k) \in \mathbb{R}^\omega$*, is a complete metric space, and then,* $(\mathbb{R}^\omega, d_\omega)$ *is a Fréchet space. Furthermore, this metric is compatible with its pointwise convergence.*

Proof. We only prove the triangular inequality and the completeness of the space, since the other properties follow from the definition. For the triangular inequality, consider the function

$$f(x) = \frac{x}{x+1}.$$

Then, we have

$$\begin{aligned} f(x+y) \le f(x)+f(y) &\iff \frac{x+y}{1+x+y} \le \frac{x}{1+x} + \frac{y}{1+y} \\ &\iff 1 - \frac{1}{1+x+y} \le 2 - \frac{1}{1+x} - \frac{1}{1+y} \\ &\iff \frac{1}{1+x} + \frac{1}{1+y} \le 1 + \frac{1}{1+x+y} \\ &\iff (1+x+y)(1+y) + (1+x+y)(1+x) \\ &\le (2+x+y)(1+x)(1+y) \\ &\iff 1+x+y \le (1+x)(1+y) \\ &\iff xy \ge 0, \end{aligned}$$

and the result comes straight from this.

For the completeness, let (x^k) be a Cauchy sequence in $(\mathbb{R}^\omega, d_\omega)$. Let x_j^k stand for the j-th coordinate of the k-th term of the sequence (x^k). We have that the function $f(x)$ is increasing for $x \ge 0$. Since (x^k) is a Cauchy sequence, given $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that for $m, n \ge k_0$ and $j \in \mathbb{N}$, we have $d_\omega(x^m, x^n) < \frac{2^{-j}\varepsilon}{1+\varepsilon}$. Thus,

$$\frac{2^{-j}|x_j^m - x_j^n|}{1+|x_j^m - x_j^n|} \le \sum_{k=1}^{\infty} \frac{2^{-k}|x_k^m - x_k^n|}{1+|x_k^m - x_k^n|} = d_\omega(x^m, x^n) < \frac{2^{-j}\varepsilon}{1+\varepsilon}.$$

That is,

$$f(|x_j^m - x_j^n|) = \frac{|x_j^m - x_j^n|}{1+|x_j^m - x_j^n|} < \frac{\varepsilon}{1+\varepsilon} = f(\varepsilon),$$

implies that $|x_j^m - x_j^n| < \varepsilon$ for every $j \in \mathbb{N}$ and $m, n \ge k_0$. Thus, we have that (x_j^m) is a Cauchy sequence of real numbers for every $m \ge k_0$ and all $j \in \mathbb{N}$. Since $\mathbb{R}$ is complete, for every $m \ge k_0$, there is $x_m \in \mathbb{R}$ such that $|x_j^m - x_m| \to 0$ as $j \to \infty$. If we set $x = (x_1, x_2, x_3, \ldots)$, we have $d_\omega(x^k, x) \to 0$ as $k \to \infty$. This concludes the proof. □

Proposition 19. *Let*

$$\ell^\infty := \{(x_n) \in \mathbb{R}^\omega;\ (x_n) \textit{ is bounded}\}.$$

Then, $\ell^\infty \subset \mathbb{R}^\omega$*, and this inclusion is Lipschitz-continuous.*

Proof. For a sequence $(z_n) \in \ell^\infty$, we set

$$\| (z_n) \|_\infty := \sup_{n\in\mathbb{N}} |z_n|.$$

This is the usual norm for this space, and $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space. Let (x_i) and (y_i) be two sequences in ℓ^∞. Now, for a fixed $j \in \mathbb{N}$, we have

$$\sum_{k=1}^{j} \frac{|x_k - y_k|}{2^k(1+|x_k - y_k|)} \leq \sum_{k=1}^{j} \frac{\| (x_i) - (y_i) \|_\infty}{2^k(1+|x_k - y_k|)} \leq \sum_{k=1}^{j} \frac{\| (x_i) - (y_i) \|_\infty}{2^k}.$$

Taking the limit as $j \to \infty$, we have

$$d_\omega((x_i),(y_i)) \leq \| (x_i) - (y_i) \|_\infty . \tag{38}$$

□

8.2.1. First Example

Let c_{00} be the space of the sequences with only finitely many non-null terms. It is clear that $c_{00} \subset \ell^\infty$. Now, set $X_0 = c_{00}$ and $d_{X_0} = \| \cdot - \cdot \|_\infty$ and $X = \mathbb{R}^\omega$ with $d_X = d_\omega$. For $\theta \in (0,1)$ and $q \in [1,\infty)$, we have

$$c_{00} \subseteq (X, X_0)_{\theta,q,K} \subseteq \mathbb{R}^\omega, \tag{39}$$

with Lipschitz-continuous inclusions.

Proposition 20. *X_0 is not a dense subspace of (X, d_X).*

Proof. It is sufficient to prove that

$$c_0 := \{(x_n) \in \mathbb{R}^\omega;\ x_n \to 0\}$$

is closed in $(\mathbb{R}^\omega, d_\omega)$, since $c_{00} \subset c_0$. Let (x^k) be a sequence of points in c_0 such that $d_X(x^k, x) \to 0$ for some $x \in X$. If x_j^k stands for the j-th coordinate of the k-th term of (x^k) and $x = (x_j)$, from $d_X(x^k, x) \to 0$, we can write

$$\lim_{j\to\infty} |x_j^k - x_j| = 0,\ \ \forall k \in \mathbb{N}.$$

Then, since

$$|x_j| \leq |x_j^k - x_j| + |x_j^k|,\ \ \forall j,k \in \mathbb{N},$$

it follows that $x \in c_0$. This proves that c_0 is a closed set in the metric topology. □

Corollary 4. $(X, X_0)_{\theta,q,K_M} \neq X$.

Proof. It is easy to see that $c_{00} \subseteq (X, X_0)_{\theta,q,K_M} \subseteq \overline{c_{00}} \subset c_0$, where $\overline{c_{00}}$ stands for the metric closure of c_{00}. The sequence $(1,1,1,\ldots) \in \mathbb{R}^\omega \setminus c_0$. Then, we have $(X, X_0)_{\theta,q,K_M} \neq X$, as desired. □

Proposition 21. $c_{00} \neq (X, X_0)_{\theta,q,K_M}$.

Proof. Consider the inequality

$$C_{\theta,q} d_X(x,y) \leq \beta_{\theta,q}(x,y) \leq B_{\theta,q} \| x - y \|_\infty,\ \forall x,y \in X_0, \tag{40}$$

where $C_{\theta,q}, B_{\theta,q} > 0$ are suitable constants depending only on θ and q.

Since c_{00} is dense in $(\ell^r, \| \cdot \|_r)$ (for $1 \leq r < +\infty$) and $\| \cdot \|_\infty \leq \eta_r \| \cdot \|_r$ in ℓ^r, where $\eta_r > 0$ is a suitable constant, it follows that if $x \in \ell^r$, then there is a sequence (x^k) in c_{00} such that $\| x^k - x \|_\infty \to 0$. Due to inequality (40), we have that $x \in (X, X_0)_{\theta,q,K_M}$ for all $x \in \ell^r$. Then, $\ell^r \subseteq (X, X_0)_{\theta,q,K_M}$. Since $\ell^r \setminus c_{00} \neq \emptyset$, we have the result. □

As the final result, we have $c_{00} \subseteq (X, X_0)_{\theta,q,K_M} \subseteq \mathbb{R}^\omega$ with strict inclusions.

Remark 12. *Actually, we proved more above. We also have that*

$$c_{00} \subset \bigcup_{r\geq 1} \ell^r \subseteq (X, X_0)_{\theta,q,K_M} \subset \mathbb{R}^\omega .$$

Remark 13. *If we want to generate spaces that are strictly intermediary, we have to seek the case where X_0 is neither dense nor closed in (X, d_X). Observe that it does not depend directly on the metric d_{X_0}.*

8.2.2. Second Example: A Trivial Interpolation

Now, we interpolate ℓ^∞ itself to search for what kind of new elements we can gain.

Proposition 22. *ℓ^∞ is closed in $(\mathbb{R}^\omega, d_\omega)$.*

Proof. Let (x^k) be a convergent sequence of points in ℓ^∞ such that $d_\omega(x^k, x) \to 0$ as $k \to \infty$, for some $x \in \mathbb{R}^\omega$. Then, if x_j^k stands for the j-th coordinate of the k-th term of (x^k) and $x = (x_j)$, from $d_\omega(x^k, x) \to 0$, we have

$$\lim_{j\to\infty} |x_j^k - x_j| = 0, \ \ \forall k \in \mathbb{N}.$$

Notice that

$$|x_j| \leq |x_j - x_j^k| + |x_j^k|, \ \forall j, k \in \mathbb{N}.$$

Since (x^k) is convergent, it follows that $d_\omega(x^k, 0) \leq M$ for all $k \in \mathbb{N}$. Then, $|x_j^k| \leq M$ for all $j, k \in \mathbb{N}$. The result is straightforward based on these statements. □

Since ℓ^∞ is closed in $\mathbb{R}^\omega$ with respect to the metric d_ω, if we interpolate ℓ^∞ in $\mathbb{R}^\omega$, we obtain $\ell^\infty = (\mathbb{R}^\omega, \ell^\infty)_{\theta,q,K_M}$, which follows from $\ell^\infty \subseteq (\mathbb{R}^\omega, \ell^\infty)_{\theta,q,K_M} \subseteq \overline{\ell^\infty} = \ell^\infty$.

8.3. Non-Trivial Interpolation: Conditions

Analyzing the examples, we obtain some conditions to guarantee that the interpolation space does not get stuck in X_0 and does not increase to the whole space X. We can call these cases trivial since we do not obtain a new space.

Proposition 23. *If X_0 is closed in (X, d_X), then the interpolation space is equal to X_0.*

Proof. We have the inclusions $X_0 \subseteq (X, X_0)_{\theta,q,K_M} \subseteq \overline{X_0}$ for $\theta \in (0, 1)$ and $q \geq 1$. This statement arises from observing that if (x^k) is a Cauchy sequence in (X_0, d_{X_0}), then (x^k) is a Cauchy sequence in (X, d_X), and it gives that the interpolation space is contained in the closure of X_0 in (X, d_X). □

Proposition 24. *If X_0 is not dense in (X, d_X), then $(X, X_0)_{\theta,q,K_M} \neq X$.*

Proof. If X_0 is not dense, then $\overline{X_0} \neq X$. Since $X_0 \subseteq (X, X_0)_{\theta,q,K_M} \subseteq \overline{X_0}$, we have the result. □

Corollary 5. *If X_0 is not dense and not closed in (X, d_X), then for all $0 < \theta < 1$ and $1 \leq q < \infty$, we have*

$$X_0 \subset (X, X_0)_{\theta,q,K_M} \subset X$$

and these inclusions are strict.

Proof. Immediate. □

Remark 14. *It is worth mentioning that due to Corollary 3, the same conditions are valid for normed spaces in the cases where $\theta \in (0, 1)$ and $q \in [1, \infty)$.*

9. Conclusions

In this work, we developed an interpolation theory for metric spaces inspired by the real method of interpolation. Under certain conditions, these interpolation spaces preserve Lipschitz operators. We also demonstrate that this method, while valid in metric spaces, retains its validity in normed spaces without requiring any algebraic structure. Furthermore, we applied this metric space interpolation method to normed spaces and found that it is equivalent to the K-method widely studied in the literature. As an application, we considered the interpolation of Fréchet sequence spaces. For future research, we suggest obtaining more general results for the interpolation of compact operators, as well as a reiteration-like theorem.

Also, fixed point theory is a very important research topic in metric and Banach spaces. Recent works have obtained fixed-point results for multi-valued mappings in b-metric spaces as well as in some contexts of interpolation theory; see [19,20]. Thus, we suggest studying possible connections between these works and the theory developed in the current work.

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References

1. Bergh, J.; Löfström, J. *Interpolation Spaces: An Introduction*; Springer: Berlin/Heidelberg, Germany, 1976.
2. Calderón, A.P. Intermediate spaces and interpolation. *Studia Math.* **1964**, **1**, 31–34.
3. Lions, J.L.; Peetre, J. Sur une classe d'espaces d'interpolation. *Inst. Hautes Etudes Sci. Publ. Math.* **1964**, *253*, 5–68. [CrossRef]
4. Aronszajn, N.; Gagliardo, E. Interpolation space and interpolation methods. *Ann. Mat. Pura Appl.* **1965**, *68*, 51–117. [CrossRef]
5. Peetre, J. *Theory of Interpolation of Normed Spaces*; Notes de Brasilia; Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas: Rio de Janeiro, Brazil, 1968; Volume 39.
6. Peetre, J.; Sparr, G. Interpolation of normed Abelian groups. *Ann. Mat. Pura Appl.* **1972**, *92*, 217–262. [CrossRef]
7. Brudnyî, Y.; Krugljak, Y.N. *Interpolation Functors and Interpolation Spaces*; Elsevier: Amsterdam, The Netherlands, 1991.
8. Krée, P. Interpolation d'espaces qui ne sont ni norés ni complets. *Ann. Inst. Fourier* **1968**, *17*, 137–174. [CrossRef]
9. Peetre, J. Interpolation of Lipschitz operators and metric spaces. *Mathematica* **1970**, *12*, 325–334.
10. Gustavsson, J. *Interpolation of Metric Spaces*; Technical Report; Lund University: Lund, Sweden, 1971.
11. Chrzaszcz, K.; Jachymski, J.; Turobos, F. Two refinements of Frink's metrization theorem and fixed point results for Lipschitzian mappings on quasimetric spaces. *Aequat. Math.* **2019**, *93*, 277–297. [CrossRef]
12. Persson, A. Compact linear mappings between interpolation spaces. *Ark. Mat.* **1964**, *5*, 215–219. [CrossRef]
13. Hayakawa, K. Interpolation by the real method preserves compactness of operators. *J. Math. Soc. Jpn.* **1969**, *21*, 31–34. [CrossRef]
14. Cobos, F.; Fernandez, D.L. On interpolation of compact operators. *Ark. Mat.* **1989**, *27*, 211–217. [CrossRef]
15. Cobos, F.; Edmunds, D.E.; Potter, A.J.B. Real interpolation and compact linear operators. *J. Func. Anal.* **1990**, *88*, 351–365. [CrossRef]
16. Cwikel, M. Real and complex interpolation and extrapolation of compact operators. *Duke Math. J.* **1992**, **65**, 333–343. [CrossRef]
17. Cwikel, M.; Kalton, N. Interpolation of compact operators by the methods of Calderón and Gustavsson-Peetre. *arXiv* **1992**, arXiv:math/9210206.
18. Meise, R.; Vogt, D. *Introduction to Functional Analysis*; Oxford University Press: Oxford, UK, 1997.
19. Abbas, M.; Anjum, R.; Riasat, S. A new type of fixed point theorem via interpolation of operators with application in homotopy theory. *Arab. J. Math.* **2023**, *12*, 277–288. [CrossRef]
20. Ćirić, L.; Abbas, M.; Rajovic, M.; Ali, B. Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two b-metrics. *App. Math. Comp.* **2012**, *219*, 1712–1723. [CrossRef]

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