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Qualitative Analysis for the Solutions of Fractional Stochastic Differential Equations

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Abstract: Fractional pantograph stochastic differential equations (FPSDEs) combine elements of fractional calculus, pantograph equations, and stochastic processes to model complex systems with memory effects, time delays, and random fluctuations. Ensuring the well-posedness of these equations is crucial as it guarantees meaningful, reliable, and applicable solutions across various disciplines. In differential equations, regularity refers to the smoothness of solution behavior. The averaging principle offers an approximation that balances complexity and simplicity. Our research contributes to establishing the well-posedness, regularity, and averaging principle of FPSDE solutions in $\mathscr{L}^{\mathrm{p}}$ spaces with $\mathrm{p} \geq 2$ under Caputo derivatives. The main ingredients in the proof include the use of Hölder, Burkholder–Davis–Gundy, Jensen, and Grönwall–Bellman inequalities, along with the interval translation approach. To understand the theoretical results, we provide numerical examples at the end.

Keywords: pantograph problem; existence and uniqueness; continuous dependency; regularity; averaging principle; Caputo fractional derivative

MSC: 26A33; 69H07; 37C25

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Citation: Mohammed Djaouti, A.; Imran Liaqat, M. Qualitative Analysis for the Solutions of Fractional Stochastic Differential Equations. *Axioms* **2024**, *13*, 438. https://doi.org/10.3390/axioms13070438

Academic Editor: Chris Goodrich

Received: 16 April 2024
Revised: 23 June 2024
Accepted: 25 June 2024
Published: 28 June 2024

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1. Introduction

Traditional calculus provides a foundational framework for understanding and modeling changes and accumulations using integer-order derivatives and integrals. It is characterized by local operations and relatively simple mathematical formulations. Fractional calculus (Fr-Cal) generalizes these concepts to noninteger orders, allowing for a broader range of mathematical modeling. It incorporates nonlocal operations that account for the entire history of a function, making it particularly powerful for describing complex systems with memory effects. This added complexity and generality make Fr-Cal a valuable tool for advanced applications in various scientific and engineering fields.

Fr-Cal is needed for several reasons, particularly in situations where traditional integer-order calculus falls short. Here is why we need Fr-Cal [1–3]:

1. Modeling complex systems: Fr-Cal provides a more accurate and versatile framework for modeling complex systems with memory effects, nonlocal interactions, and fractal characteristics. Many real-world phenomena, such as viscoelastic materials, biological systems, and financial markets, exhibit behaviors that cannot be adequately described by traditional calculus.
2. Anomalous diffusion: Anomalous diffusion processes, where the mean squared displacement of particles does not grow linearly with time, are prevalent in various natural and man-made systems. Fr-Cal offers a natural way to model and analyze such

processes, providing insights into transport phenomena in porous media, crowded environments, and disordered materials.

3. Long-range dependence: Many systems exhibit long-range dependence, where events at distant points in space or time are correlated. Fr-Cal allows for the inclusion of nonlocal interactions, enabling the modeling of processes with long-range dependencies, such as turbulent flows, climate dynamics, and complex networks.
4. Memory effects: Fr-Cal naturally accounts for memory effects, where the current state of a system depends not only on its current state but also on its past states. This is crucial for modeling systems with memory, including viscoelasticity, fractional Brownian motion, and fractional differential equations.
5. Fractal and multifractal systems: Fr-Cal is well-suited for modeling systems with fractal or multifractal characteristics, where patterns or structures exhibit self-similarity across different scales. This includes phenomena in physics, biology, finance, and engineering, such as fractal surfaces, multifractal time series, and self-similar networks.
6. Optimization and control: Fr-Cal plays a role in optimization problems and control theory, offering tools for analyzing and designing systems with fractional-order dynamics. Fractional-order controllers, for example, have been shown to offer advantages in certain applications, such as robotics, aerospace, and biomedical engineering.
7. Interdisciplinary applications: Fr-Cal finds applications across a wide range of disciplines, including physics, engineering, biology, finance, and materials science. Its versatility and applicability make it a valuable tool for addressing complex real-world problems and developing innovative solutions.

Fractional-order derivatives are known for their ability to capture memory effects and nonlocal properties, making them a powerful tool in modeling complex systems where traditional integer-order derivatives fall short. The memory effect refers to the ability of fractional derivatives to account for the past states of a system in determining its current state. Unlike integer-order derivatives, which depend only on the present state and an infinitesimally small neighborhood around it, fractional derivatives incorporate the entire history of the function. The nonlocal property of fractional derivatives means that the derivative at any point depends on the values of the function over an entire interval, rather than just at that point or its immediate neighborhood.

There are several types of fractional derivatives, each with its own unique definition and properties. The choice of fractional derivative depends on the specific application and the nature of the problem. Here are some of the main types of fractional derivatives:

The natural derivative is fundamentally extended into a fractional operator by the Grünwald–Letnikov derivative. Both Anton Karl Grünwald and Aleksey Vasilievich Letnikov introduced it in 1867 and 1868, respectively [4]. It is written as

$$\mathcal{T}_{\tau}^{\lambda}\Omega(\tau) = \lim_{\Theta\to 0}\frac{1}{\Theta^{\lambda}}\sum_{\kappa=0}^{\infty}(-1)^{\kappa}\binom{\lambda}{\kappa}\Omega(\tau-\kappa\Theta),$$

where $\kappa \in \mathbb{N}$, and the gamma function is used to determine the binomial coefficient,

$$\binom{\lambda}{\kappa} = \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-\kappa+1)}{\kappa!} = \frac{\Gamma(\lambda+1)}{\kappa!\Gamma(\lambda-\kappa+1)}.$$

In 1847, R-L defined a new fractional derivative that is called the Riemann–Liouville fractional derivative (RLFD) [5]. It is defined as follows:

$$\mathcal{T}_{\tau}^{\lambda}\Omega(\tau) = \begin{cases} \frac{1}{\Gamma(\mathfrak{n}-\lambda)}\frac{d^{\mathfrak{n}}}{d\tau^{\mathfrak{n}}}\int_0^{\tau}(\tau-\Upsilon)^{\mathfrak{n}-\lambda-1}\Omega(\Upsilon)d\Upsilon & \text{if } \mathfrak{n}-1 \le \lambda < \mathfrak{n}, \\ \frac{d^{\mathfrak{n}}}{d\tau^{\mathfrak{n}}}\Omega(\tau) & \text{if } \lambda = \mathfrak{n}. \end{cases}$$

The Caputo fractional derivative (Cap-FrD) was established in 1967 [6] because the RLFD was ineffective in the description and modeling of some complicated events. It is defined as follows [7,8]:

$$\mathcal{T}_{\tau}^{\lambda}\Omega(\tau) = \begin{cases} \frac{1}{\Gamma(\mathfrak{n}-\lambda)} \int_{\tau_0}^{\tau} (\tau - \mathrm{Y})^{\mathfrak{n}-\lambda-1} \Omega^{\mathfrak{n}}(\mathrm{Y}) d\mathrm{Y}, \ \tau > \tau_0, & \text{if } \mathfrak{n}-1 \leq \lambda < \mathfrak{n}, \\ \frac{d^{\mathfrak{n}}}{d\tau^{\mathfrak{n}}} \Omega(\tau) & \text{if } \lambda = \mathfrak{n}, \end{cases}$$

where $\mathfrak{n} - 1 \leq \lambda < \mathfrak{n} \in \mathbb{Z}^+$. The RLFD of a constant $\mathfrak{C}$ is given by $\frac{\mathfrak{C}\tau^{-\mathfrak{n}}}{\Gamma(1-\lambda)}$. As a result, the ability of Cap-FrD to provide the derivative of a constant zero, as in an ordinary derivative, is one of its strengths. There are more fractional derivatives; the interested reader is referred to [9–11] for further details.

In fractional stochastic differential equations (FSDEs), fractional derivatives are incorporated into stochastic differential equations (SDEs). This allows for the modeling of systems that exhibit both stochastic behavior and nonlocal or memory-like effects. FSDEs find application in various real-life scenarios where systems exhibit memory effects, long-range dependencies, and stochastic behavior. Here are some specific areas where FSDEs are utilized in modeling [12–18]:

1. Finance and economics:
    - Asset price modeling: FSDEs are employed to model asset prices with long memory, capturing phenomena such as volatility clustering and heavy tails observed in financial markets.
    - Risk management: FSDEs help in assessing and managing financial risk by incorporating memory effects into risk models.
    - Option Pricing: Derivative pricing models based on FSDEs account for the memory effects of asset price dynamics, leading to more accurate pricing of financial derivatives.
2. Biological systems:
    - Population dynamics: FSDEs are used to model populations with hereditary traits, where the current population size depends on historical data.
    - Epidemiology: Modeling the spread of diseases with memory effects, considering factors such as past infection rates and immune responses.
    - Ecological interactions: FSDEs capture the dynamics of ecological systems, including predator–prey interactions and species coexistence, where historical interactions influence current dynamics.
3. Climate science and environmental modeling:
    - Climate modeling: FSDEs help in modeling long-term climate trends and variability, considering memory effects in climate processes.
    - Hydrology: Modeling groundwater flow, rainfall–runoff processes, and river discharge, where historical data and long-term dependencies play a crucial role.
    - Environmental pollution: FSDEs are utilized to model the dispersion of pollutants in air and water, considering the memory effects of pollutant transport and degradation processes.
4. Control systems and engineering:
    - Robust control: FSDEs are used to design control systems that account for memory effects and uncertainties, leading to more robust and adaptive control mechanisms.
    - Signal processing: FSDEs are applied in signal processing for filtering and analysis of signals with memory, nonstationarity, and long-range dependencies.
    - Mechanical Systems: Modeling mechanical systems with memory effects, such as viscoelastic materials and structures, for better design and analysis.
5. Telecommunications and networking:
    - Network traffic modeling: FSDEs help in modeling network traffic patterns, including long-range dependence and memory effects, for optimizing network performance and resource allocation.
    - Wireless communications: Modeling wireless channels with memory effects, fading, and interference, for designing efficient communication systems.

6. Physics and Material Science:
    - Anomalous diffusion: FSDEs are used to model diffusion processes in complex materials, porous media, and biological tissues, where particles exhibit anomalous diffusion behavior.
    - Transport phenomena: Modeling transport processes in disordered media, where memory effects and long-range dependencies influence particle movement and dispersion.

Fractional pantograph stochastic differential equations (FP-SDEs) provide a rich and versatile framework for modeling complex systems with memory effects, time delays, and stochastic behavior. They extend traditional stochastic differential equations (SDEs) by incorporating Fr-Cal and delay terms, offering more accurate and comprehensive models for a wide range of applications in science and engineering [19–21]. The development and application of FP-SDEs continue to be an active area of research, with ongoing advancements in both theory and numerical methods.

The existence and uniqueness (EU) of solutions to FSDEs concern whether there exists a solution to the equation, and if it does, whether that solution is unique. This topic is fundamental to the study of FSDEs, as it establishes the mathematical foundation for their analysis and application. For a given FSDE, existence refers to whether there is at least one function that satisfies the equation under consideration. In the context of FSDEs, which involve both fractional derivatives and stochastic processes, establishing the existence of solutions can be challenging due to the combined effects of randomness and nonlocality. Uniqueness, on the other hand, refers to whether there is only one solution that satisfies the FSDE. In many cases, proving uniqueness is as crucial as proving existence, especially in applications where a unique solution is desired for predictive or control purposes.

Continuous dependence is a crucial aspect of the well-posedness of solutions to FSDEs. It refers to the property that small changes in the initial conditions or parameters of the FSDEs result in small changes in the solutions. In other words, if the initial conditions or parameters are perturbed slightly, then the corresponding solutions also change only slightly. This property underpins the stability and reliability of FSDEs in various scientific and engineering applications.

The regularity of a solution to an FSDE refers to the smoothness of the solution function. In mathematical terms, it describes how well-behaved the solution is with respect to its derivatives and how it behaves across the domain of the equation. In the context of FSDEs, the regularity of a solution is particularly important because these equations involve both fractional derivatives and stochastic processes. Since fractional derivatives can capture memory effects and long-range dependencies, the regularity of the solution can vary depending on the order of the fractional derivative and the characteristics of the stochastic process involved. The regularity of a solution to an FSDE is often analyzed using mathematical techniques such as Hölder continuity, Sobolev spaces, or other function space properties. Understanding the regularity of solutions helps in assessing the stability, convergence, and behavior of the system described by the FSDE in various applications, such as in physics, biology, finance, and engineering.

Several writers have recently been actively working on the FSDEs; for instance, Li and Xu [22] developed exponential stability for delay FSDEs. The authors also introduced novel criteria for evaluating stability within the $\mathscr{L}^2$ space. Kexue and Jigen [23] established controllability results for FSDEs using Sadovskii's fixed-point theorem (FPT). Similarly, Cui and Yan [24] applied the same FPT in Hilbert spaces. In another study [25], the authors demonstrated the asymptotic stability results in the $\mathscr{L}^2$ space of FSDEs. In their study [26], the authors presented results related to the EU and stability within the framework of Hyers–Ulam for FSDEs of Cap-FrD, utilizing the Banach FPT. In another work [27], the authors explored stability in terms of exponential and EU results for fuzzy FSDEs under the Lipschitzian condition. Similar results are established in [28,29]. The researchers [30,31] established criteria for the EU of solutions for FSDEs under various assumptions. Karczewska and Lizama [32], as well as Schnaubelt and Veraar [33], explored distinct aspects

of stochastic Volterra equations and FSDEs. Karczewska and Lizama elaborated on various findings concerning perturbations in stochastic Volterra equations while also discussing the EU results for FSDEs. Schnaubelt and Veraar focused on demonstrating the path-wise continuity properties of solutions for the same model. Xiao and Wang [34] utilized the stopping time technique to investigate the stability of FSDEs.

The averaging principle (Ave-Pr) is a fundamental concept used to simplify complex systems described by differential equations. By employing this principle, intricate differential equations can be reduced to more manageable forms, streamlining analysis and enhancing understanding. This approach is particularly valuable in fields such as physics, engineering, biology, and finance, where the complexity of mathematical models can be overwhelming. Through the application of the Ave-Pr, researchers are able to distill essential dynamics from complicated equations, uncovering underlying patterns and facilitating insightful interpretations. In essence, the Ave-Pr serves as a powerful tool for simplifying the analysis of complex systems, making them more accessible for study and application. Khasminskii's pioneering work on the Ave-Pr for stochastic differential equations (SDEs) extended the concept to systems influenced by random fluctuations. His book "Stochastic Stability of Differential Equations" introduced methods for averaging SDEs driven by Brownian motion. Recently, some scholars have shown interest in the concept of averaging in FSDEs. For example, in [35–46], the authors presented findings regarding the principle of averaging.

Based on the understanding of the authors, it is important to highlight that the majority of the existing literature focuses on investigating the EU as well as the Ave-Pr results for FSDEs within the context of $\mathscr{L}^2$. Drawing inspiration from these findings, our approach initially establishes the EU of the solutions of the FP-SDEs through the utilization of the Banach FPT in $\mathscr{L}^{\mathrm{p}}$ space. Additionally, we demonstrate the continuous dependency on both initial values and the fractional exponent λ, as well as the regularity of solutions to FP-SDEs in the $\mathscr{L}^{\mathrm{p}}$ space. The objective of the second section is to establish the Ave-Pr result of FP-SDEs in terms of the pth moment. This is achieved by employing various mathematical inequalities such as Grönwall–Bellman's inequality (GB-I), Jensen's inequality (J-I), Burkholder–Davis–Gundy's inequality (BDG-I), Hölder's inequality (H-I), and the interval translation method. To illustrate the theoretical results we established, numerical examples are also presented.

We examined the following FP-SDEs of order $\frac{1}{2} < \lambda < 1$:

$$\begin{cases} \mathcal{T}_{\tau}^{\lambda}\mathfrak{M}(\tau) = \mathbb{U}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) + \mathbb{M}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi))\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(\tau_0) = \chi. \end{cases} \tag{1}$$

where $\phi \in (0, \frac{\xi - \tau_0}{\xi})$ and λ represent the Cap-FrD, $\mathbb{U} : [\tau_0,\ \xi] \times \mathbb{R}^{\theta} \times \mathbb{R}^{\theta} \to \mathbb{R}^{\theta}$, $\mathbb{M} : [\tau_0,\ \xi] \times \mathbb{R}^{\theta} \times \mathbb{R}^{\theta} \to \mathbb{R}^{\theta \times \varsigma}$ are mesurable continuous functions on an underlying complete filtered probability space $(\Omega, \mathbb{F}, \mathscr{F} = (\mathbb{F}_{\tau})_{\tau \in [\tau_0, \infty)}, \mathbb{P})$ with $\theta \in \mathbb{N}$ and $(\mathscr{W}(\tau))_{\tau \in [\tau_0,\ \infty)}$ is a ς-dimensional scalar Brownian motion.

The structure of the research work is as follows: In the subsequent part, we provide definitions, significant findings, and assumptions that form the framework for the FP-SDE results. The well-posedness of the solutions to FP-SDEs is established in the first subsection of Section 3, and regularity is demonstrated in the second subsection. We establish the Ave-Pr theorem in Section 4 and provide numerical examples in Section 5. Finally, the conclusion is presented in Section 6.

2. Preliminaries

In this section, we present some basic concepts, definitions, lemmas, assumptions, and a theorem that are necessary and useful to establish results regarding the FP-SDEs.

Definition 1. *When* $\mathrm{p} \geq 2$, $\tau \in [\tau_0, \infty)$, *suppose* $\mathscr{Z}_\tau^{\mathrm{p}} = \mathscr{L}^{\mathrm{p}}(\Omega, \mathscr{F}_\tau, \mathbb{P})$ *represents all* $\mathscr{F}_\tau$*-measurable,* p^{th} *functions that are integrable* $\mathfrak{M} = (\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_\theta)^T : \Omega \to \mathbb{R}^\theta$ *with*

$$\|\mathfrak{M}\|_{\mathrm{p}} = \left(\sum_{\iota=1}^{\theta} \Xi(|\mathfrak{M}_\iota|^{\mathrm{p}}) \right)^{\frac{1}{\mathrm{p}}}. \tag{2}$$

A measurable procedure $\mathfrak{M}(\tau) : [\tau_0, \xi] \to \mathscr{L}^{\mathrm{p}}(\Omega, \mathscr{F}_\tau, \mathbb{P})$ becomes $\mathbb{F}-$adapted process if $\mathfrak{M}(\tau) \in \mathscr{Z}_\tau^{\mathrm{p}}$ for each $\tau \geq \tau_0$. For $\forall \chi \in \mathscr{Z}_0^{\mathrm{p}}$, a $\mathbb{F}-$adapted process $\mathfrak{M}(\tau)$ is a solution of Equation (1) with an initial condition $\mathfrak{M}(\tau_0) = \chi$ if $\mathfrak{M}(\tau_0) = \chi$ and the subsequent equality satisfy on $\mathscr{Z}_\tau^{\mathrm{p}}$ for $\tau \in [\tau_0, \xi]$. By applying the fractional integral in the Caputo sense and then utilizing the result $\mathrm{I}_\tau^\lambda \mathcal{T}_\tau^\lambda \Omega(\tau) = \Omega(\tau) - \Omega(0)$, we obtained the following integral form of Equation (1).

$$\begin{aligned} \mathfrak{M}(\tau) =& \chi + \frac{1}{\Gamma(\lambda)} \int_{\tau_0}^{\tau} (\tau - \vartheta)^{\lambda-1} \mathbb{U}\big(\vartheta, \mathfrak{M}(\vartheta), \mathfrak{M}(\tau_0 + \phi\vartheta)\big) d\vartheta \\ &+ \frac{1}{\Gamma(\lambda)} \int_{\tau_0}^{\tau} (\tau - \vartheta)^{\lambda-1} \mathbb{M}\big(\vartheta, \mathfrak{M}(\vartheta), \mathfrak{M}(\tau_0 + \phi\vartheta)\big) d\mathscr{W}(\vartheta). \end{aligned} \tag{3}$$

Definition 2. *To establish the results, we suggest the following important assumptions for* $\mathbb{U}$ *and* $\mathbb{M}$.

1. $(\mathbb{A}_1)$ *When* $\forall \omega_1, \omega_2, \Lambda_1, \Lambda_2 \in \mathbb{R}^\theta$ *there are* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *such as*

$$\|\mathbb{U}(\tau, \omega_1, \omega_2) - \mathbb{U}(\tau, \Lambda_1, \Lambda_2)\|_{\mathrm{p}} \leq \mathscr{L}_1\big(\|\omega_1 - \Lambda_1\|_{\mathrm{p}} + \|\omega_2 - \Lambda_2\|_{\mathrm{p}}\big).$$

$$\|\mathbb{M}(\tau, \omega_1, \omega_2) - \mathbb{M}(\tau, \Lambda_1, \Lambda_2)\|_{\mathrm{p}} \leq \mathscr{L}_2\big(\|\omega_1 - \Lambda_1\|_{\mathrm{p}} + \|\omega_2 - \Lambda_2\|_{\mathrm{p}}\big).$$

2. $(\mathbb{A}_2)$ *The drift term* $\mathbb{U}(\tau, \tau_0, \tau_0)$ *and the diffusion* $\mathbb{M}(\tau, \tau_0, \tau_0)$ *are essential bounded in time, i.e.,*

$$\underset{\tau \in [\tau_0, \xi]}{esssup} \|\mathbb{U}(\tau, \tau_0, \tau_0)\|_{\mathrm{p}} < \mathcal{U}, \ \underset{\tau \in [\tau_0, \xi]}{esssup} \|\mathbb{M}(\tau, \tau_0, \tau_0)\|_{\mathrm{p}} < \mathcal{U}. \tag{4}$$

In the context of differential equations, including stochastic and FSDEs, an $\mathscr{L}$-Lipschitz term typically refers to a condition that bounds how much a function can change relative to changes in its input. This is crucial for proving the EU of solutions. While the $\mathscr{L}$-Lipschitz term is commonly assumed to be a constant in differential equations, extending this concept to a linear functional offers significant advantages. Some of them are as follows:

1. Greater flexibility: Extending the Lipschitz condition to a linear functional allows for a more flexible analysis. It can accommodate functions whose rate of change is not uniformly bounded by a constant but still exhibits linear behavior.
2. Enhanced modeling capability: This extension is particularly useful in modeling real-world systems where changes in the state of the system do not follow a uniform rate. For instance, systems with state-dependent dynamics can be better captured using a linear functional.
3. Broader applicability: Many complex systems, especially those described by stochastic and FSDEs, may not satisfy a constant Lipschitz condition. A linear functional provides a more general framework, extending the applicability of the Lipschitz condition to a wider range of problems.

Theorem 1. *Suppose that* $(\mathbb{A}_1)$ *and* $(\mathbb{A}_2)$ *are true. Then, a constant* $\mathcal{S} > 0$ *that depends on* $\lambda, \mathscr{L}, \mathcal{U}, \mathrm{p}, \xi$ *exists, so we have the following [47]:*

$$\|\mathcal{B}_\lambda(\chi, \tau) - \mathcal{B}_\lambda(\chi, \mathfrak{r})\|_{\mathrm{p}} \leq \mathcal{S}|\tau - \mathfrak{r}|^{\lambda - \frac{1}{2}}, \ \forall \tau, \mathfrak{r} \in [\tau_0, \xi], \tag{5}$$

where $\mathcal{B}_\lambda(\chi, \tau)$ *represents the solution of Equation* (1).

Corollary 1. *For every* $\mathrm{Y} \in (0, \lambda - \frac{1}{2})$, *there occurred a modification* $\mathfrak{M}_2(\tau)$ *of* $\mathfrak{M}_1(\tau)$ *with* Φ*-Hölder continuous paths, i.e.,*

$$\mathbb{P}(\mathfrak{M}_1(\tau) = \mathfrak{M}_2(\tau)) = 1, \ \forall \tau \in [\tau_0, \xi]. \tag{6}$$

Proof. By utilizing Equation (5) and Kolmogorov test [48], $\mathfrak{M}(\tau)$ has Y-Hölder continuous modification for all $\mathrm{Y} \in (0, \lambda - \frac{1}{2})$. □

Now, we propose some conditions that are foundational to the results of the Ave-Pr.

1. $(\mathbb{C}_3)$: We make the condition that coefficient $\mathbb{U}$ in Equation (1) when $\forall \omega_1, \omega_2, \Lambda_1, \Lambda_2, \omega, \Lambda \in \mathbb{R}^\theta$, $\tau \in [\tau_0, \xi]$ there is $\mathscr{U}_1 > 0$ such as meet the following:

$$\begin{aligned} \|\mathbb{U}(\tau, \omega_1, \omega_2) - \mathbb{U}(\tau, \Lambda_1, \Lambda_2)\| \vee \|\mathbb{M}(\tau, \omega_1, \omega_2) - \mathbb{M}(\tau, \Lambda_1, \Lambda_2)\| \\ \leq \mathscr{U}_1 \big(\|\omega_1 - \Lambda_1\| + \|\omega_2 - \Lambda_2\|\big). \end{aligned}$$

2. $(\mathbb{C}_4)$: Now we make the condition that coefficient $\mathbb{M}$ in Equation (1) when $\forall \omega_1, \omega_2, \Lambda_1, \Lambda_2, \omega,$
$\Lambda \in \mathbb{R}^\theta$, $\tau \in [\tau_0, \xi]$ there is $\mathscr{U}_2 > 0$ such as satisfy the following:

$$\|\mathbb{U}(\tau, \omega, \Lambda)\| \vee \|\mathbb{M}(\tau, \omega, \Lambda)\| \leq \mathscr{U}_2 \big(1 + \|\omega\| + \|\Lambda\|\big).$$

3. $(\mathbb{C}_5)$: Functions $\widetilde{\mathbb{U}}$ and $\widetilde{\mathbb{M}}$ exist and for $\xi_1 \in [\tau_0, \xi]$, $\tau \in [\tau_0, \xi]$, and $\mathrm{p} \geq 2$, we are able to identify positively bound functions $\mathbb{D}_1(\xi_1)$ and $\mathbb{D}_2(\xi_1)$ that fulfill

$$\frac{1}{\xi_1} \int_{\tau_0}^{\xi_1} \|\mathbb{U}(\tau, \omega, \Lambda) - \widetilde{\mathbb{U}}(\tau, \omega, \Lambda)\|^{\mathrm{p}} d\tau \leq \mathbb{D}_1(\xi_1) \big(1 + \|\omega\|^{\mathrm{p}} + \|\Lambda\|^{\mathrm{p}}\big),$$

$$\frac{1}{\xi_1} \int_{\tau_0}^{\xi_1} \|\mathbb{M}(\tau, \omega, \Lambda) - \widetilde{\mathbb{M}}(\tau, \omega, \Lambda)\|^{\mathrm{p}} d\tau \leq \mathbb{D}_2(\xi_1) \big(1 + \|\omega\|^{\mathrm{p}} + \|\Lambda\|^{\mathrm{p}}\big),$$

where $\lim_{\xi_1 \to \infty} \mathbb{D}_1(\xi_1) = 0$ and $\lim_{\xi_1 \to \infty} \mathbb{D}_2(\xi_1) = 0$.

Lemma 1. *Assume that there are real numbers* $\mathbb{J}_1, \mathbb{J}_2, \ldots, \mathbb{J}_v (v \in \mathbb{N})$ *and meet* $\mathbb{J}_\iota \geq 0$, $(\iota = 1, 2, \ldots, v)$. *Then [49],*

$$\left(\sum_{\iota=1}^{v} \mathbb{J}_\iota \right)^{\mathrm{p}} \leq v^{\mathrm{p}-1} \sum_{\iota=1}^{v} \mathbb{J}_\iota^{\mathrm{p}}, \ \forall \mathrm{p} > 1.$$

Lemma 2. *For any* $\lambda > \frac{1}{2}$ *and* $\tau > 0$, *the following inequality holds [50]:*

$$\frac{\hbar}{\Gamma(2\lambda - 1)} \int_{\tau_0}^{\tau} (\vartheta - \tau_0)^{2\lambda - 2} \mathbb{E}_{2\lambda - 1}(\hbar(\vartheta - \tau_0)^{2\lambda - 1}) d\vartheta \leq \mathbb{E}_{2\lambda - 1}(\hbar(\tau - \tau_0)^{2\lambda - 1}); \tag{7}$$

here, the Mittag–Leffler function $\mathbb{E}_{2\lambda - 1}(.)$ *is defined as*

$$\mathbb{E}_{(2\lambda - 1)}(\tau) = \sum_{\iota=0}^{\infty} \frac{\tau^\iota}{\Gamma((2\xi - 1)\iota + 1)}.$$

In this part, we demonstrate the well-posedness and regularity of the solutions to FP-SDEs.

3. Well-Posedness of Solutions of FP-SDEs under the Standard Lipschitz Condition of Coefficients

To prove the well-posedness of the solutions to FP-SDEs, we need to demonstrate the EU and continuous dependency of the solution on λ and the initial data in order to verify the well-posedness of the solutions.

Suppose $\mathcal{H}^{\mathrm{p}}(\tau_0,\xi)$ is the space of all processes $\mathfrak{M}(\tau)$ that are measurable $\mathscr{F}_\xi$-adapted, with $\mathscr{F}_\xi=(\mathbb{F}_\tau)_{\tau\in[\tau_0,\xi]}$ and satisfy the following:

$$\|\mathfrak{M}(\tau)\|_{\mathcal{H}^{\mathrm{p}}}=\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathfrak{M}(\tau)\|_{\mathrm{p}}<\infty. \tag{8}$$

$\left(\mathcal{H}^{\mathrm{p}}(\tau_0,\xi),\|\cdot\|_{\mathcal{H}^{\mathrm{p}}}\right)$ is surely a Banach space. We construct an operator $\Im_\chi:\mathcal{H}^{\mathrm{p}}(\tau_0,\xi)\to\mathcal{H}^{\mathrm{p}}(\tau_0,\xi)$ by $\Im_\chi(\mathfrak{M}(\tau_0))=\chi$ for any $\chi\in\mathscr{Z}_0^{\mathrm{p}}$ and for $\tau\in[\tau_0,\xi]$, the subsequent equality is valid. Moreover, we follow the procedure [51].

$$\begin{aligned}\Im_\chi(\mathfrak{M}(\tau))=&\chi+\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)d\vartheta\\&+\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)d\mathscr{W}(\vartheta).\end{aligned} \tag{9}$$

The well-defined property of this operator is being demonstrated in the ensuing lemma. The well-defined property of an operator is essential because it ensures that the operator behaves predictably and consistently within its defined domain. When an operator is well-defined, it means that for every input in its domain, there exists a unique output. This property is crucial in mathematical analysis as it guarantees the EU of solutions to equations or problems involving the operator. Without the well-defined property, the behavior of the operator may become ambiguous or unreliable, leading to inconsistencies in mathematical reasoning and potentially invalid results. Therefore, establishing a well-defined property is fundamental for ensuring the validity and reliability of mathematical theories and calculations. The elementary inequality below is being employed in the proof of this result and multiple others that follow.

$$\|\mathfrak{M}_1+\mathfrak{M}_2\|_{\mathrm{p}}^{\mathrm{p}}\le 2^{\mathrm{p}-1}\bigg(\|\mathfrak{M}_1\|_{\mathrm{p}}^{\mathrm{p}}+\big(\|\mathfrak{M}_2\|_{\mathrm{p}}^{\mathrm{p}}\big)\bigg),\ \forall\mathfrak{M}_1,\mathfrak{M}_2\in\mathbb{R}^{\theta}. \tag{10}$$

Lemma 3. *Assume that $\mathbb{A}_1$ and $\mathbb{A}_2$ are valid. The operator $\Im_\chi$ is then well defined for any $\chi\in\mathscr{Z}_0^{\mathrm{p}}$.*

Proof. Suppose for any $\mathfrak{M}(\tau)\in\mathcal{H}^{\mathrm{p}}[\tau_0,\xi]$ with $\forall\tau\in[\tau_0,\xi]$. We obtain the following by utilizing Equations (9) and (10).

$$\begin{aligned}\|\Im_\chi(\mathfrak{M}(\tau))\|_{\mathrm{p}}^{\mathrm{p}}\le&2^{\mathrm{p}-1}\|\chi\|_{\mathrm{p}}^{\mathrm{p}}+\frac{2^{2\mathrm{p}-2}}{\Gamma^{\mathrm{p}}(\lambda)}\bigg\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)d\vartheta\bigg\|_{\mathrm{p}}^{\mathrm{p}}\\&+\frac{2^{2\mathrm{p}-2}}{\Gamma^{\mathrm{p}}(\lambda)}\bigg\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)d\mathscr{W}(\vartheta)\bigg\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned} \tag{11}$$

The H-I gives us the result that

$$\begin{aligned}\bigg\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)d\vartheta\bigg\|_{\mathrm{p}}^{\mathrm{p}}&\le\sum_{\iota=1}^{m}\Xi\bigg(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\big|\mathbb{U}_\iota\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)\big|d\vartheta\bigg)^{\mathrm{p}}\\&\le\sum_{\iota=1}^{m}\Xi\bigg(\bigg(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\frac{(\lambda-1)\mathrm{p}}{(\mathrm{p}-1)}}d\vartheta\bigg)^{\mathrm{p}-1}\\&\quad\int_{\tau_0}^{\tau}\big|\mathbb{U}_\iota\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)\big|^{\mathrm{p}}d\vartheta\bigg)\\&\le\frac{(\xi-\tau_0)^{(\mathrm{p}\lambda-1)}(\mathrm{p}-1)^{(\mathrm{p}-1)}}{(\mathrm{p}\lambda-1)^{(\mathrm{p}-1)}}\\&\quad\int_{\tau_0}^{\tau}\big\|\mathbb{U}\big(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta)\big)\big\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta.\end{aligned} \tag{12}$$

According to $(\mathbb{A}_1)$, we acquire

$$\begin{aligned}\left\|\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & 2^{\mathrm{p}-1}\left(\left\|\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}\right)\\ \leq & 2^{\mathrm{p}-1}\left(\mathscr{L}_1^{\mathrm{p}}\left(\left\|\mathfrak{M}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathfrak{M}(\tau_0+\phi\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}\right)+\left\|\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}\right).\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}\int_{\tau_0}^{\tau}\left\|\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta \leq & 2^{\mathrm{p}-1}\mathscr{L}_1^{\mathrm{p}}\left(\left(esssup_{\vartheta\in[\tau_0,\xi]}\|\mathfrak{M}(\vartheta)\|_{\mathrm{p}}\right)^{\mathrm{p}}+\left(esssup_{\vartheta\in[\tau_0,\xi]}\left\|\mathfrak{M}(\tau_0+\phi\vartheta)\right\|_{\mathrm{p}}\right)^{\mathrm{p}}\right)\\ & \int_{\tau_0}^{\tau}1d\vartheta+2^{\mathrm{p}-1}\int_{\tau_0}^{\tau}\left\|\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\\ & \leq 2^{\mathrm{p}-1}\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)\left(\left\|\mathfrak{M}(\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}+\left\|\mathfrak{M}(\tau_0+\phi\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}\right)+\\ & 2^{\mathrm{p}-1}\int_{\tau_0}^{\tau}\left\|\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta.\end{aligned} \tag{14}$$

By Equations (12) and (14), we obtain

$$\begin{aligned}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & \frac{(\xi-\tau_0)^{(\mathrm{p}\lambda-1)}(2\mathrm{p}-2)^{(\mathrm{p}-2)}}{(\mathrm{p}\lambda-1)^{(\mathrm{p}-1)}}\left(\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)\left(\left\|\mathfrak{M}(\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}\right.\right.\\ & \left.\left.+\left\|\mathfrak{M}(\tau_0+\phi\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}\right)+\int_{\tau_0}^{\tau}\left\|\mathbb{U}(\vartheta,\tau_0,\tau_0)\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\right).\end{aligned} \tag{15}$$

By utilizing $(\mathbb{A}_2)$, we obtain the following from Equation (15).

$$\begin{aligned}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & \frac{(\xi-\tau_0)^{(\mathrm{p}\lambda-1)}(2\mathrm{p}-2)^{(\mathrm{p}-2)}}{(\mathrm{p}\lambda-1)^{(\mathrm{p}-1)}}\left(\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)\left(\left\|\mathfrak{M}(\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}\right.\right.\\ & \left.\left.+\left\|\mathfrak{M}(\tau_0+\phi\vartheta)\right\|_{\mathcal{H}^{\mathrm{p}}}^{\mathrm{p}}\right)+(\xi-\tau_0)\mathcal{U}^{\mathrm{p}}\right).\end{aligned} \tag{16}$$

Now, applying BDG-I and H-I, we obtain

$$\begin{aligned}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & \sum_{\iota=1}^{m}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\left(\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))d\mathscr{W}(\vartheta)\right|^{\mathrm{p}}\right.\\ \leq & \sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left|\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right|^2d\vartheta\right|^{\frac{\mathrm{p}}{2}}\\ \leq & \sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left|\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right|^{\mathrm{p}}d\vartheta\\ & \left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}\\ \leq & \mathcal{C}_{\mathrm{p}}\left(\frac{(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\right)^{\frac{\mathrm{p}-2}{2}}\\ & \int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left\|\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta,\mathfrak{M}(\tau_0+\phi\vartheta))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta.\end{aligned} \tag{17}$$

where $\mathcal{C}_{\mathrm{p}}=\left(\frac{\mathrm{p}^{\mathrm{p}+1}}{2(\mathrm{p}-1)^{\mathrm{p}-1}}\right)^{\frac{\mathrm{p}}{2}}$.

By utilizing $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$, we have the following:

$$\begin{aligned}\left\|\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & 2^{\mathrm{p}-1}\mathscr{L}_2^{\mathrm{p}}\left(\|\mathfrak{M}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\right)+2^{\mathrm{p}-1}\|\mathbb{M}(\vartheta,0,0)\|_{\mathrm{p}}^{\mathrm{p}}\\ \leq & 2^{\mathrm{p}-1}\mathscr{L}_2^{\mathrm{p}}\left(\|\mathfrak{M}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\right)+2^{\mathrm{p}-1}\mathcal{U}^{\mathrm{p}}. \end{aligned} \tag{18}$$

Thus, $\forall\tau\in[\tau_0,\xi]$, we obtain the following:

$$\begin{aligned}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left\|\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta \leq & 2^{\mathrm{p}-1}\mathscr{L}_2^{\mathrm{p}}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left(\left(\underset{\vartheta\in[\tau_0,\xi]}{esssup}\|\mathfrak{M}(\vartheta)\|_{\mathrm{p}}\right)^{\mathrm{p}}\right.\\ & \left.+\left(\underset{\vartheta\in[\tau_0,\xi]}{esssup}\|\mathfrak{M}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}\right)^{\mathrm{p}}\right)d\vartheta\\ & +2^{\mathrm{p}-1}\mathcal{U}^{\mathrm{p}}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\\ \leq & \frac{2^{\mathrm{p}-1}(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\\ & \left(\mathscr{L}_2^{\mathrm{p}}\left(\|\mathfrak{M}(\vartheta)\|_{\mathcal{H}_{\mathrm{p}}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)\|_{\mathcal{H}_{\mathrm{p}}}^{\mathrm{p}}\right)+\mathcal{U}^{\mathrm{p}}\right). \end{aligned} \tag{19}$$

By using Equation (19) in Equation (17), we have

$$\begin{aligned}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & \mathcal{C}_{\mathrm{p}}\left(\frac{(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\right)^{\frac{\mathrm{p}-2}{2}}\frac{2^{\mathrm{p}-1}(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\\ & \left(\mathscr{L}_2^{\mathrm{p}}\left(\|\mathfrak{M}(\vartheta)\|_{\mathcal{H}_{\mathrm{p}}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)\|_{\mathcal{H}_{\mathrm{p}}}^{\mathrm{p}}\right)+\mathcal{U}^{\mathrm{p}}\right). \end{aligned} \tag{20}$$

By employing Equations (16) and (20) in (11), we obtain that $\|\Im(\mathfrak{M}(\tau))\|_{\mathcal{H}_{\mathrm{p}}}<\infty$. As a consequence, the mapping $\Im_\chi$ is well-defined. □

The EU of solutions will be established by showing that the operator $\Im_\chi$ is contractive under a suitable weighted norm ([52], [Remark 2.1]). In this instance, the Mittag–Leffler function $\mathbb{E}_{(2\lambda-1)}(\tau-\tau_0)$ serves as the weight function.

Theorem 2. *If $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$ are valid, then the Equation (1) with $\mathfrak{M}(\tau_0)=\chi$ has unique solution on $[\tau_0,\xi]$ for any $\chi\in\mathscr{Z}_0^{\mathrm{p}}$.*

Proof. First of all, choose a fixed positive constant $\hbar$ as follows:

$$\hbar>\Psi 2^{\mathrm{p}-1}\Gamma(2\lambda-1), \tag{21}$$

where

$$\Psi=\frac{2^{\mathrm{p}-1}}{\Gamma^{\mathrm{p}}(\lambda)}\left(\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)^{(\mathrm{p}-2)\lambda+1}\frac{1}{\left(\frac{(\mathrm{p}-2)\lambda+1}{\mathrm{p}-1}\right)^{\mathrm{p}-1}}+\left(\frac{(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\right)^{\frac{\mathrm{p}-2}{2}}\left(\frac{\mathrm{p}^{\mathrm{p}+1}}{2(\mathrm{p}-1)^{\mathrm{p}-1}}\right)^{\frac{\mathrm{p}}{2}}\mathscr{L}_2^{\mathrm{p}}\right). \tag{22}$$

We establish a weighted norm $\|\cdot\|_\hbar$ over the space $\mathcal{H}^{\mathrm{p}}([\tau_0,\xi])$ as:

$$\|\mathfrak{M}(\tau)\|_\hbar=\underset{\tau\in[\tau_0,\xi]}{esssup}\left(\frac{\|\mathfrak{M}(\tau)\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}\right)^{\frac{1}{\mathrm{p}}},\ \forall\mathfrak{M}(\tau)\in\mathcal{H}^{\mathrm{p}}([\tau_0,\xi]). \tag{23}$$

Two norms, $\|\cdot\|_{\mathcal{H}^{\mathrm{p}}}$ and $\|\cdot\|_{\hbar}$, are equivalent. $\left(\mathcal{H}^{\mathrm{p}}([\tau_0,\xi]),\|\cdot\|_{\hbar}\right)$ is a Banach space as a result. Choose and fix $\chi\in\mathscr{Z}_0^{\mathrm{p}}$ by virtue of Lemma 3, the operator $\Im_\chi$ is well-defined. Now, we prove that the mapping $\Im_\chi$ is contractive with respect to the norm $\|\cdot\|_{\hbar}$. For this purpose, let $\mathfrak{M},\widetilde{\mathfrak{M}}$ be arbitrary. We obtain the following from Equations (9) and (10):

$$\begin{aligned}\|\Im_\chi(\mathfrak{M}(\tau))-\Im_\chi(\widetilde{\mathfrak{M}}(\tau))\|_{\mathrm{p}}^{\mathrm{p}}\leq\\ &\frac{2^{\mathrm{p}-1}}{\Gamma^{\mathrm{p}}(\lambda)}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{U}(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}+\\ &\frac{2^{\mathrm{p}-1}}{\Gamma^{\mathrm{p}}(\lambda)}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{M}(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned}\tag{24}$$

Using the H-I and $(\mathbb{A}_1)$, we obtain

$$\begin{aligned}&\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{U}(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}\\ &\quad\leq\sum_{\iota=1}^{m}\Xi\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{U}_\iota(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\vartheta\right)^{\mathrm{p}}\\ &\quad\leq\sum_{\iota=1}^{m}\Xi\Bigg(\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\frac{(\lambda-1)(\mathrm{p}-2)}{\mathrm{p}-1}}d\vartheta\right)^{\mathrm{p}-1}\\ &\qquad\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left|\mathbb{U}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{U}_\iota(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\right|\right)\Bigg)\\ &\quad\leq\frac{\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)^{\mathrm{p}\lambda-2\lambda+1}(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\lambda-2\lambda+1)^{\mathrm{p}-1}}\\ &\qquad\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\Big(\|\mathfrak{M}(\vartheta)-\widetilde{\mathfrak{M}}(\vartheta))\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)-\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\|_{\mathrm{p}}^{\mathrm{p}}\Big)d\vartheta.\end{aligned}\tag{25}$$

However, using $(\mathbb{A}_1)$ and the BHDK-Inq, we have

$$\begin{aligned}&\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{M}(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}\\ &=\sum_{\iota=1}^{m}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{M}_\iota(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\Big)d\mathscr{W}(\vartheta)\right|^{\mathrm{p}}\\ &\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left|\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{M}_\iota(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\right|^2d\vartheta\right|^{\frac{\mathrm{p}}{2}}\\ &\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\left|\mathbb{M}_\iota(\vartheta,\mathfrak{M}(\vartheta),\mathfrak{M}(\tau_0+\phi\vartheta))-\mathbb{M}_\iota(\vartheta,\widetilde{\mathfrak{M}}(\vartheta),\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta))\right|^{\mathrm{p}}d\vartheta\\ &\quad\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}\\ &\leq\left(\frac{(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\right)^{\frac{\mathrm{p}-2}{2}}\mathscr{L}_2^{\mathrm{p}}\mathcal{C}_{\mathrm{p}}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\Big(\|\mathfrak{M}(\vartheta)-\widetilde{\mathfrak{M}}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)-\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\Big)d\vartheta.\end{aligned}\tag{26}$$

Thus, $\forall\tau\in[\tau_0,\xi]$, we have

$$\|\Im_\chi(\mathfrak{M}(\tau))-\Im_\chi(\widetilde{\mathfrak{M}}(\tau))\|_{\mathrm{p}}^{\mathrm{p}}\leq\Psi\int_{\tau_0}^{\tau}\Big(\|\mathfrak{M}(\vartheta)-\widetilde{\mathfrak{M}}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathfrak{M}(\tau_0+\phi\vartheta)-\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\Big)(\tau-\vartheta)^{2\lambda-2}d\vartheta,\tag{27}$$

here, Ψ is specified in Equation (22). The result suggests that using the definition of $\|\cdot\|_{\hbar}$ from Equation (23)

$$\frac{\|\Im_\chi \mathfrak{M}(\tau) - \Im_\chi \widetilde{\mathfrak{M}}(\tau)\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})} \leq \frac{\Psi \int_{\tau_0}^{\tau} (\tau-\vartheta)^{2\lambda-2} \frac{\left(\|\mathfrak{M}(\vartheta)-\widetilde{\mathfrak{M}}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}} + \|\mathfrak{M}(\tau_0+\phi\vartheta)-\widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\right)}{\mathbb{E}_{2\lambda-1}(\hbar(\vartheta-\tau_0)^{2\lambda-1})} \mathbb{E}_{2\lambda-1}(\hbar(\vartheta-\tau_0)^{2\lambda-1}) d\vartheta}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}$$

$$\leq \Psi \left(\underset{\vartheta\in[\tau_0,\xi]}{esssup} \left(\frac{\left(\|\mathfrak{M}(\vartheta) - \widetilde{\mathfrak{M}}(\vartheta)\|_{\mathrm{p}}^{\mathrm{p}} + \|\mathfrak{M}(\tau_0+\phi\vartheta) - \widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_{\mathrm{p}}^{\mathrm{p}}\right)}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})} \right)^{\frac{1}{\mathrm{p}}} \right)^{\mathrm{p}}$$

$$\frac{\int_{\tau_0}^{\tau} (\tau-\vartheta)^{2\lambda-2} \mathbb{E}_{2\lambda-1}(\hbar(\vartheta-\tau_0)^{2\lambda-1}) d\vartheta}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}$$

$$\leq \frac{\Psi\Gamma(2\lambda-1)}{\hbar} \left(\|\mathfrak{M}(\vartheta) - \widetilde{\mathfrak{M}}(\vartheta)\|_{\hbar}^{\mathrm{p}} + \|\mathfrak{M}(\tau_0+\phi\vartheta) - \widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_{\hbar}^{\mathrm{p}} \right). \tag{28}$$

By utilizing Lemma 2, we obtain the required result.

$$\|\Im_\chi(\mathfrak{M}(\tau)) - \Im_\chi(\widetilde{\mathfrak{M}}(\tau))\|_\hbar \leq \left(\frac{\Psi\Gamma(2\lambda-1)}{\hbar} \right)^{\frac{1}{\mathrm{p}}} \left(\|\mathfrak{M}(\vartheta) - \widetilde{\mathfrak{M}}(\vartheta)\|_\hbar + \|\mathfrak{M}(\tau_0+\phi\vartheta) - \widetilde{\mathfrak{M}}(\tau_0+\phi\vartheta)\|_\hbar \right). \tag{29}$$

From Equation (21), we obtain $\frac{\Psi\Gamma(2\lambda-1)}{\hbar} < 1$, the operator $\Im_\chi$ on $\left(\mathcal{H}^{\mathrm{p}}([\tau_0,\xi]), \|\cdot\|_\hbar\right)$ is a contractive map. There is a single fixed point of this map in $\mathcal{H}^{\mathrm{p}}([\tau_0,\xi])$, according to the Banach FPT. The unique solution of Equation (1) with the $\mathfrak{M}(\tau_0) = \chi$ is also this fixed point. This theorem is proved. □

In the following theorem, we demonstrate that the solution continuously depends on λ.

Theorem 3. *The solution $\mathcal{B}_\lambda(\tau,\chi)$ depends continuously on λ, i.e.,*

$$\lim_{\lambda\to\tilde{\lambda}} \underset{\tau\in[\tau_0,\xi]}{esssup} \|\mathcal{B}_\lambda(\tau,\chi) - \mathcal{B}_{\tilde{\lambda}}(\tau,\chi)\|_{\mathrm{p}} = 0. \tag{30}$$

Proof. Suppose $\lambda, \tilde{\lambda} \in (\frac{1}{2}, 1)$ further take $\chi \in \mathscr{Z}_0^{\mathrm{p}}$. As $\mathcal{B}_\lambda(\chi,\tau)$ and $\mathcal{B}_{\tilde{\lambda}}(\chi,\tau)$ are solutions to Equation (1), we obtain the following:

$$\begin{aligned}
\mathcal{B}_\lambda(\tau,\chi) - \mathcal{B}_{\tilde{\lambda}}(\chi,\tau) =& \frac{1}{\Gamma(\lambda)} \int_{\tau_0}^{\tau} (\tau-\vartheta)^{\lambda-1} \Big(\mathbb{U}(\vartheta, \mathcal{B}_\lambda(\vartheta,\chi)) - \mathbb{U}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)) \Big) d\vartheta \\
&+ \int_{\tau_0}^{\tau} \left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)} - \frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})} \right) \mathbb{U}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)) d\vartheta \\
&+ \frac{1}{\Gamma(\lambda)} \int_{\tau_0}^{\tau} (\tau-\vartheta)^{\lambda-1} \Big(\mathbb{M}(\vartheta, \mathcal{B}_\lambda(\vartheta,\chi)) - \mathbb{M}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)) \Big) d\mathscr{W}(\vartheta) \\
&+ \int_{\tau_0}^{\tau} \left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)} - \frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})} \right) \mathbb{M}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)) d\mathscr{W}(\vartheta).
\end{aligned} \tag{31}$$

Using Equation (10), we obtain the following result from Equation (31).

$$\begin{aligned}
\|\mathcal{B}_\lambda(\tau,\chi) - \mathcal{B}_{\tilde{\lambda}}(\tau,\chi)\|_{\mathrm{p}}^{\mathrm{p}} \leq& 2^{\mathrm{p}-1}\Psi \int_{\tau_0}^{\tau} (\tau-\vartheta)^{2\lambda-2} \|\mathcal{B}_\lambda(\tau,\chi) - \mathcal{B}_{\tilde{\lambda}}(\chi,\tau)\|_{\mathrm{p}}^{\mathrm{p}} d\vartheta \\
&+ 2^{2\mathrm{p}-2} \left\| \int_{\tau_0}^{\tau} \left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)} - \frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})} \right) \mathbb{U}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi) d\vartheta \right\|_{\mathrm{p}}^{\mathrm{p}} \\
&+ \left\| \int_{\tau_0}^{\tau} \left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)} - \frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})} \right) \mathbb{M}(\vartheta, \mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi) d\mathscr{W}(\vartheta) \right\|_{\mathrm{p}}^{\mathrm{p}}.
\end{aligned} \tag{32}$$

Suppose the following:

$$\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})=\left|\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)}-\frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})}\right|. \tag{33}$$

Now, we simplify Equation (32) one by one. First, by using Equation (10), the H-I, $(\mathbb{A}_1)$, and $(\mathbb{A}_2)$, we obtain the following result:

$$\begin{aligned}
&\left\|\int_{\tau_0}^{\tau}\left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)}-\frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})}\right)\mathbb{U}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}\\
&\quad\leq\sum_{\iota=1}^{m}\Xi\left(\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\left|\mathbb{U}_{\iota}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi))\right|d\vartheta\right)^{\mathrm{p}}\\
&\quad\leq\sum_{\iota=1}^{m}\Xi\left(\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{\frac{\mathrm{p}}{\mathrm{p}-1}}\right)^{\mathrm{p}-1}\int_{\tau_0}^{\tau}\left|\mathbb{U}_{\iota}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right|^{\mathrm{p}}d\vartheta\right)\\
&\quad\leq\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}\right)^{\frac{\mathrm{p}}{2}}\left(\int_{\tau_0}^{\tau}1d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}\int_{\tau_0}^{\tau}\left\|\mathbb{U}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\\
&\quad\leq\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}\right)^{\frac{\mathrm{p}}{2}}(\xi-\tau_0)^{\frac{\mathrm{p}-2}{2}}\int_{\tau_0}^{\tau}2^{\mathrm{p}-1}\left(\mathscr{L}_1^{\mathrm{p}}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}})+\|\mathbb{U}(\vartheta,0)\|_{\mathrm{p}}^{\mathrm{p}}\right)\\
&\quad\leq\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}\right)^{\frac{\mathrm{p}}{2}}(\xi-\tau_0)^{\frac{\mathrm{p}}{2}}2^{\mathrm{p}-1}\left(\mathscr{L}_1^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}})+\mathcal{U}^{\mathrm{p}}\right).
\end{aligned}\tag{34}$$

Now, we simplify the second part of Equation (32). For this, we use the BDG-I, Equation (33), $(\mathbb{A}_1)$, $(\mathbb{A}_2)$. So, we have as a result.

$$\begin{aligned}
&\left\|\int_{\tau_0}^{\tau}\left(\frac{(\tau-\vartheta)^{\lambda-1}}{\Gamma(\lambda)}-\frac{(\tau-\vartheta)^{\tilde{\lambda}-1}}{\Gamma(\tilde{\lambda})}\right)\mathbb{M}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}\\
&\quad\leq\sum_{\iota=1}^{m}\Xi\left|\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\mathbb{M}_{\iota}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)d\mathscr{W}(\vartheta)\right|^{\mathrm{p}} \qquad (35)\\
&\quad\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\left|\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})^2\left|\mathbb{M}_{\iota}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right|^2d\mathscr{W}(\vartheta)\right|^{\frac{\mathrm{p}}{2}}\\
&\quad\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\left[\left(\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})^2\left|\mathbb{M}_{\iota}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right|^{\mathrm{p}}d\vartheta\right)^{\frac{2}{\mathrm{p}}}\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^2d\vartheta\right)^{\frac{\mathrm{p}-2}{\mathrm{p}}}\right]^{\frac{\mathrm{p}}{2}}\\
&\quad=\mathcal{C}_{\mathrm{p}}\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})^2\left\|\mathbb{M}(\vartheta,\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^2d\vartheta\right)^{\frac{\mathrm{p}-2}{\mathrm{p}}}\\
&\quad\leq\mathcal{C}_{\mathrm{p}}\left(\int_{\tau_0}^{\tau}\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})^2d\vartheta\right)^{\frac{\mathrm{p}}{2}}2^{\mathrm{p}-1}\left(\mathscr{L}_2^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right). \qquad (36)
\end{aligned}$$

By utilizing the above results and definition $\|\cdot\|_{\hbar}$, we obtain the following:

$$\begin{aligned}
\frac{\|\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\tilde{\lambda}}(\tau,\chi)\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}\leq&\frac{\Psi 2^{\mathrm{p}-1}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\frac{\left\|\mathcal{B}_{\lambda}(\vartheta,\chi)-\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}{\mathbb{E}_{2\lambda-1}(\hbar(\tau-\tau_0)^{2\lambda-1})}\\
&+2^{3\mathrm{p}-3}\left(\mathscr{L}_1^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^2d\vartheta\right)^{\frac{\mathrm{p}}{2}}\xi^{\frac{\mathrm{p}}{2}}\\
&+2^{3\mathrm{p}-3}\left(\mathscr{L}_2^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)\mathcal{C}_{\mathrm{p}}\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^2d\vartheta\right)^{\frac{\mathrm{p}}{2}}
\end{aligned}$$

$$\leq \frac{\Psi 2^{\mathrm{p}-1}\Gamma(2\lambda-1)}{\hbar}\left\|\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\tilde{\lambda}}(\tau,\chi)\right\|_{\hbar}^{\mathrm{p}}$$
$$+2^{3\mathrm{p}-3}\left(\mathscr{L}_{1}^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta\right)^{\frac{\mathrm{p}}{2}}\xi^{\frac{\mathrm{p}}{2}}$$
$$+2^{3\mathrm{p}-3}\left(\mathscr{L}_{2}^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)\mathcal{C}_{\mathrm{p}}\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta\right)^{\frac{\mathrm{p}}{2}}. \tag{37}$$

Finally, by utilizing the Lemma 2, we obtain

$$\left(1-\frac{\Psi 2^{\mathrm{p}-1}\Gamma(2\lambda-1)}{\hbar}\right)\left\|\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\tilde{\lambda}}(\tau,\chi)\right\|_{\hbar}^{\mathrm{p}}\leq 2^{3\mathrm{p}-3}\left(\mathscr{L}_{1}^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)$$
$$\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta\right)^{\frac{\mathrm{p}}{2}}\xi^{\frac{\mathrm{p}}{2}}+2^{3\mathrm{p}-3}\left(\mathscr{L}_{2}^{\mathrm{p}}\underset{\tau\in[\tau_0,\xi]}{esssup}\|\mathcal{B}_{\tilde{\lambda}}(\vartheta,\chi)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{U}^{\mathrm{p}}\right)$$
$$\mathcal{C}_{\mathrm{p}}\left(\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta\right)^{\frac{\mathrm{p}}{2}}. \tag{38}$$

Thus, by Equation (21) and $\mathrm{p}\geq 2$, it is required to demonstrate the following in order to complete the proof:

$$\underset{\tilde{\lambda}\to\lambda}{lim}\ \underset{\tau\in[\tau_0,\xi]}{sup}\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta=0. \tag{39}$$

We have the following:

$$\int_{\tau_0}^{\tau}\left(\mathcal{A}(\tau,\vartheta,\lambda,\tilde{\lambda})\right)^{2}d\vartheta=\int_{\tau_0}^{\tau}\frac{(\tau-\vartheta)^{2\lambda-2}}{\Gamma^{2}(\lambda)}d\vartheta+\int_{\tau_0}^{\tau}\frac{(\tau-\vartheta)^{2\tilde{\lambda}-2}}{\Gamma^{2}(\tilde{\lambda})}d\vartheta-\int_{\tau_0}^{\tau}\frac{(\tau-\vartheta)^{\lambda+\tilde{\lambda}-2}}{\Gamma(\lambda)\Gamma(\tilde{\lambda})}d\vartheta$$
$$=\frac{\tau^{2\lambda-1}}{(2\lambda-1)\Gamma^{2}(\lambda)}+\frac{\tau^{2\tilde{\lambda}-1}}{(2\tilde{\lambda}-1)\Gamma^{2}(\tilde{\lambda})}-\frac{2\tau^{\lambda+\tilde{\lambda}-1}}{(\lambda+\tilde{\lambda}-1)\Gamma(\lambda)\Gamma(\tilde{\lambda})}. \tag{40}$$

Hence,it proved the required result. □

In the following theorem, we demonstrate that the solution continuously depends on the initial value.

Theorem 4. *For any $\chi,\ \gamma\in\mathscr{Z}_{0}^{\mathrm{p}}$ the solution $\mathcal{B}_{\lambda}(\tau,\chi)$ depends Lipschitz continuously on χ, i.e., there exists $\mathscr{L}>0$ such that*

$$\|\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\lambda}(\tau,\gamma)|_{\mathrm{p}}\leq\mathscr{L}\|\chi-\gamma\|_{\mathrm{p}},\ forall\ \tau\in[\tau_0,\ \xi]. \tag{41}$$

Proof. Choose and fix $\chi\in\mathscr{Z}_{0}^{\mathrm{p}}$. Let $\chi\in\mathscr{Z}_{0}^{\mathrm{p}}$ arbitrarily. Since $\mathcal{B}_{\lambda}(\tau,\chi)$ and $\mathcal{B}_{\lambda}(\tau,\gamma)$ are solutions of Equation (1), it follows that

$$\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\lambda}(\tau,\gamma)$$
$$=\chi-\gamma+\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}(\vartheta,\mathcal{B}_{\lambda}(\vartheta,\chi))-\mathbb{U}(\vartheta,\mathcal{B}_{\lambda}(\vartheta,\gamma))\Big)d\vartheta$$
$$+\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\big(\mathbb{M}(\vartheta,\mathcal{B}_{\lambda}(\vartheta,\chi))-\mathbb{M}(\vartheta,\mathcal{B}_{\lambda}(\vartheta,\gamma))\big)d\mathscr{W}(\vartheta). \tag{42}$$

Hence, using the Equation (10).

$$\left\|\mathcal{B}_{\lambda}(\tau,\chi)-\mathcal{B}_{\lambda}(\tau,\gamma)\right\|_{\mathrm{p}}^{\mathrm{p}}\leq 2^{\mathrm{p}-1}\left\|\chi-\gamma\right\|_{\mathrm{p}}^{\mathrm{p}}$$

$$+\frac{2^{\mathrm{p}-1}}{\Gamma(\lambda)}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{U}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\Big)d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}$$
$$+\frac{2^{\mathrm{p}-1}}{\Gamma(\lambda)}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)\big)-\mathbb{M}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)\Big)d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}. \tag{43}$$

Now, we simplify Equation (43) one by one. So, by using Equation (10), H-I, and $(\mathbb{A}_1)$, we obtain the following result:

$$\begin{aligned}
&\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{U}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\Big)d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}\\
&\leq\sum_{\iota=1}^{m}\Xi\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{U}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\Big)d\vartheta\right)^{\mathrm{p}}\\
&\leq\sum_{\iota=1}^{m}\Xi\left(\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\frac{(\lambda-1)(\mathrm{p}-2)}{\mathrm{p}-1}}d\vartheta\right)^{\mathrm{p}-1}\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\big|\mathbb{U}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{U}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\big|\right)\right)\\
&\leq\left(\frac{\mathscr{L}_1^{\mathrm{p}}(\xi-\tau_0)^{\mathrm{p}\lambda-2\lambda+1}(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\lambda-2\lambda+1)^{\mathrm{p}-1}}\right)\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\Big(\big\|\mathcal{B}_\lambda(\vartheta,\chi)-\mathcal{B}_\lambda(\vartheta,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}\Big)d\vartheta.
\end{aligned}\tag{44}$$

Now, using the H-I, BDG-I, and $(\mathbb{A}_1)$, we obtain

$$\begin{aligned}
&\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{M}\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\Big)d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}\\
&=\sum_{\iota=1}^{m}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)d\mathscr{W}(\vartheta)\right|^{\mathrm{p}}\\
&\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\left|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\big|\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\big|^2d\vartheta\right|^{\frac{\mathrm{p}}{2}}\\
&\leq\sum_{\iota=1}^{m}\mathcal{C}_{\mathrm{p}}\Xi\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\big|\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi)\big)-\mathbb{M}_\iota\big(\vartheta,\mathcal{B}_\lambda(\vartheta,\gamma)\big)\big|^{\mathrm{p}}d\vartheta\left(\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}\\
&\leq\mathscr{L}_2^{\mathrm{p}}\mathcal{C}_{\mathrm{p}}\left(\frac{(\xi-\tau_0)^{2\lambda-1}}{2\lambda-1}\right)^{\frac{\mathrm{p}-2}{2}}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\Big(\big\|\mathcal{B}_\lambda(\vartheta,\chi)-\mathcal{B}_\lambda(\vartheta,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}\Big)d\vartheta.
\end{aligned}\tag{45}$$

Utilizing Equations (44) and (45), we can therefore extract the following from Equation (43).

$$\begin{aligned}
\big\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\tau,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}&\leq 2^{\mathrm{p}-1}\|\chi-\gamma\|_{\mathrm{p}}^{\mathrm{p}}\\
&+2^{\mathrm{p}-1}\Psi\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}\Big(\big\|\mathcal{B}_\lambda(\vartheta,\chi)-\mathcal{B}_\lambda(\vartheta,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}\Big)d\vartheta.
\end{aligned}\tag{46}$$

Considering the Grönwall inequality, we obtain the following:

$$\big\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\tau,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}\leq 2^{\mathrm{p}-1}\mathbb{E}_{2\lambda-1}\left(2^{\mathrm{p}-1}\Psi\int_{\tau_0}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\right)\|\chi-\gamma\|_{\mathrm{p}}^{\mathrm{p}}.$$

By utilizing ([53], [Lemma 7.1.1]), we have the following:

$$\big\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\tau,\gamma)\big\|_{\mathrm{p}}^{\mathrm{p}}\leq 2^{\mathrm{p}-1}\mathbb{E}_{2\lambda-1}\left(2^{\mathrm{p}-1}\Psi\Gamma(2\lambda-1)(\tau-\tau_0)^{2\lambda-1}\right)\|\chi-\gamma\|_{\mathrm{p}}^{\mathrm{p}}.$$

Hence,

$$\lim_{\chi\to\gamma}\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\tau,\gamma)\right\|_{\mathrm{p}}=0.$$

Therefore, the proof is complete. □

The Regularity of Solutions to FP-SDEs

In this subsection, we prove the regularity of solutions to FP-SDEs.

Proof of Theorem 1. Take τ and $\mathfrak{r}$ in $[\tau_0,\xi]$, such as $\tau>\mathfrak{r}$. We can derive the following through Equation (10):

$$\begin{aligned}\Gamma^{\mathrm{p}}(\lambda)2^{2-2\mathrm{p}}\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\mathfrak{r},\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}&\le\left\|\int_{\mathfrak{r}}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}\\&+\left\|\int_{\mathfrak{r}}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))d\mathscr{W}(\vartheta)\right\|_{\mathrm{p}}^{\mathrm{p}}\\&+\left\|\int_{\tau_0}^{\mathfrak{r}}\left|(\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right|\mathbb{U}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))d\vartheta\right\|_{\mathrm{p}}^{\mathrm{p}}\\&+\left\|\int_{\tau_0}^{\mathfrak{r}}\left|(\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right|\mathbb{M}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))d\mathscr{W}\right\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned}\tag{47}$$

By utilizing H-I and BDG-I, we obtain the following from Equation (47):

$$\begin{aligned}\Gamma^{\mathrm{p}}(\lambda)2^{2-2\mathrm{p}}\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\mathfrak{r},\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}&\le\frac{(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\lambda-1)^{\mathrm{p}-1}(\tau-\mathfrak{r})^{1-\lambda\mathrm{p}}}\int_{\mathfrak{r}}^{\tau}\left\|\mathbb{U}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\\&+\mathcal{C}_{\mathrm{p}}\int_{\mathfrak{r}}^{\tau}\left\|\mathbb{M}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}(\tau-\vartheta)^{2\lambda-2}d\vartheta\left(\int_{\mathfrak{r}}^{\tau}(\tau-\vartheta)^{2\lambda-2}d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}\\&+\frac{1}{\xi^{\frac{2-\mathrm{p}}{2}}}\int_{\tau_0}^{\mathfrak{r}}\left\|\mathbb{U}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\left(\int_{\tau_0}^{\mathfrak{r}}\left|(\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right|^2d\vartheta\right)^{\frac{\mathrm{p}}{2}}\\&+\mathcal{C}_{\mathrm{p}}\int_{\tau_0}^{\mathfrak{r}}\left((\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right)^2\left\|\mathbb{M}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}d\vartheta\\&\times\left(\int_{\tau_0}^{\mathfrak{r}}\left((\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right)^2d\vartheta\right)^{\frac{\mathrm{p}-2}{2}}.\end{aligned}$$

However, $\mathcal{U}_1>0$ also exists, as $\underset{\tau\in[\tau_0,\xi]}{esssup}\left\|\mathcal{B}_\lambda(\tau,\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}\le\mathcal{U}_1$ because $\mathcal{B}_\lambda(\vartheta,\chi)\in\widetilde{\mathcal{H}}^{\mathrm{p}}([\tau_0,\xi])$. Along with $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$, this implies

$$\left\|\mathbb{U}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}\le2^{\mathrm{p}-1}\left(\mathscr{L}_1^{\mathrm{p}}\left\|\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathbb{U}(\vartheta,0)\right\|_{\mathrm{p}}^{\mathrm{p}}\right)\le2^{\mathrm{p}-1}\left(\mathscr{L}_1^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}}\right).$$

$$\left\|\mathbb{M}(\vartheta,\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}\le2^{\mathrm{p}-1}\left(\mathscr{L}_2^{\mathrm{p}}\left\|\mathcal{B}_\lambda(\vartheta,\chi))\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathbb{M}(\vartheta,0)\right\|_{\mathrm{p}}^{\mathrm{p}}\right)\le2^{\mathrm{p}-1}\left(\mathscr{L}_2^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}}\right).$$

Furthermore,

$$\int_{\tau_0}^{\mathfrak{r}}\left((\tau-\vartheta)^{\lambda-1}-(\mathfrak{r}-\vartheta)^{\lambda-1}\right)^2d\vartheta\le\frac{(\tau-\mathfrak{r})^{2\lambda-1}}{2\lambda-1}.\tag{48}$$

The estimate that results from combining the calculations above is as follows:

$$\Gamma^{\mathrm{p}}(\lambda)2^{2-2\mathrm{p}}\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\mathfrak{r},\chi)\right\|_{\mathrm{p}}^{\mathrm{p}}\le\frac{(2\mathrm{p}-2)^{\mathrm{p}-1}}{(\mathrm{p}\lambda-1)^{\mathrm{p}-1}}(\tau-\mathfrak{r})^{\frac{(2\lambda-1)\mathrm{p}}{2}}\left(\mathscr{L}_1^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}}\right)(\xi-\tau_0)^{\frac{\mathrm{p}}{2}}$$

$$+\frac{1}{(2\lambda-1)^{\frac{\mathrm{p}}{2}}}(\tau-\mathfrak{r})^{\frac{(2\lambda-1)\mathrm{p}}{2}}(\mathscr{L}_2^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})2^{\mathrm{p}-1}\mathcal{C}_{\mathrm{p}}$$
$$+\frac{2^{\mathrm{p}-1}}{(2\lambda-1)^{\mathrm{p}-1}}(\tau-\mathfrak{r})^{\frac{(2\lambda-1)\mathrm{p}}{2}}(\mathscr{L}_1^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})(\xi-\tau_0)^{\frac{\mathrm{p}}{2}}$$
$$+\frac{1}{(2\lambda-1)^{\frac{\mathrm{p}}{2}}}(\tau-\mathfrak{r})^{\frac{(2\lambda-1)\mathrm{p}}{2}}(\mathscr{L}_2^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})2^{\mathrm{p}-1}\mathcal{C}_{\mathrm{p}}.$$

Hence, we obtain

$$\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\mathfrak{r},\chi)\right\|_{\mathrm{p}}\leq\mathcal{S}(\tau-\mathfrak{r})^{\lambda-\frac{1}{2}},\tag{49}$$

where

$$\mathcal{S}^{\mathrm{p}}=2^{2\mathrm{p}-2}\left(\frac{(2\mathrm{p}-2)^{\mathrm{p}-1}}{(\mathrm{p}\lambda-1)^{\mathrm{p}-1}}(\mathscr{L}_1^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})(\xi-\tau_0)^{\frac{\mathrm{p}}{2}}+\frac{1}{(2\lambda-1)^{\frac{\mathrm{p}}{2}}}(\mathscr{L}_2^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})2^{\mathrm{p}-1}\mathcal{C}_{\mathrm{p}}\right)\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}$$
$$+2^{2\mathrm{p}-2}\left(\frac{2^{\mathrm{p}-1}}{(2\lambda-1)^{\mathrm{p}-1}}(\mathscr{L}_1^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})(\xi-\tau_0)^{\frac{\mathrm{p}}{2}}+\frac{1}{(2\lambda-1)^{\frac{\mathrm{p}}{2}}}(\mathscr{L}_2^{\mathrm{p}}\mathcal{U}_1+\mathcal{U}^{\mathrm{p}})2^{\mathrm{p}-1}\mathcal{C}_{\mathrm{p}}\right)\Gamma^{\mathrm{p}}(\lambda).$$

It means that, in addition to the understanding that $\lambda\in(\frac{1}{2},1)$ and $\mathrm{p}\geq 2$,

$$\lim_{\mathfrak{r}\to\tau}\left\|\mathcal{B}_\lambda(\tau,\chi)-\mathcal{B}_\lambda(\mathfrak{r},\chi)\right\|_{\mathrm{p}}=0.\tag{50}$$

□

4. Averaging Principle Result

In this part, we outline the conditions for the growth of $\widetilde{\mathbb{M}}$ and elucidate the key outcome regarding the Ave-Pr for FP-SDEs, as it is established within the framework of $\mathscr{L}^{\mathrm{p}}$.

Lemma 4. *From any ξ_1 within the interval $[\tau_0,\xi]$, we can establish the subsequent growth criteria for $\widetilde{\mathbb{M}}$ by employing conditions $(\mathbb{C}_4)$ and $(\mathbb{C}_5)$.*

$$\|\widetilde{\mathbb{M}}(\omega,\Lambda)\|^{\mathrm{p}}\leq\mathscr{U}_3(1+\|\omega\|^{\mathrm{p}}+\|\Lambda\|^{\mathrm{p}}),$$

where $\mathscr{U}_3=\left(2^{\mathrm{p}-1}\mathbb{D}_2(\xi_1)+6^{\mathrm{p}-1}\mathscr{U}_2^{\mathrm{p}}\right)$.

Proof. Considering conditions $(\mathbb{C}_4)$, $(\mathbb{C}_5)$ and Lemma 1, we obtain the following outcome:

$$\begin{aligned}\|\widetilde{\mathbb{M}}(\omega,\Lambda)\|^{\mathrm{p}}&\leq 2^{\mathrm{p}-1}\|\mathbb{M}(\tau,\omega,\Lambda)-\widetilde{\mathbb{M}}(\omega,\Lambda)\|^{\mathrm{p}}+2^{\mathrm{p}-1}\|\mathbb{M}(\tau,\omega,\Lambda)\|^{\mathrm{p}}\\&\leq 2^{\mathrm{p}-1}\mathbb{D}_2(\xi_1)(1+\|\omega\|^{\mathrm{p}}+\|\Lambda\|^{\mathrm{p}})+2^{\mathrm{p}-1}\mathscr{U}_2^{\mathrm{p}}(1+\|\omega\|+\|\Lambda\|)^{\mathrm{p}}\\&\leq\left(2^{\mathrm{p}-1}\mathbb{D}_2(\xi_1)+6^{\mathrm{p}-1}\mathscr{U}_2^{\mathrm{p}}\right)(1+\|\omega\|^{\mathrm{p}}+\|\Lambda\|^{\mathrm{p}}).\end{aligned}$$

□

Next, we present the time scale change property for the Cap-FrD.

Lemma 5 (time scale change property)**.** *If we take the time scale $\tau=\delta\nu$, we obtain the following [54]:*

$$\mathcal{T}_\nu^\lambda\mathfrak{M}(\delta\nu)=\delta^\lambda\mathcal{T}_\tau^\lambda\mathfrak{M}(\tau).$$

Now, examine the Ave-Pr of FP-SDEs in the sense of $\mathscr{L}^{\mathrm{p}}$ space. First, consider the following:

$$\begin{cases} \mathcal{T}_{\tau}^{\lambda}\mathfrak{M}(\tau) = \mathbb{U}(\frac{\tau}{\varepsilon}, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) + \mathbb{M}(\frac{\tau}{\varepsilon}, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi))\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(\tau_0) = \chi. \end{cases} \tag{51}$$

When we suppose $\frac{\tau}{\varepsilon} = \nu$ and then employ Lemma 5, Equation (51) can be rearranged in the following manner:

$$\begin{cases} \varepsilon^{-\lambda}\mathcal{T}_{\nu}^{\lambda}\mathfrak{M}(\varepsilon\nu) = \mathbb{U}(\nu, \mathfrak{M}(\varepsilon\nu), \mathfrak{M}(\varepsilon\nu_0 + \varepsilon\nu\phi)) + \mathbb{M}(\nu, \mathfrak{M}(\varepsilon\nu), \mathfrak{M}(\varepsilon\nu_0 + \varepsilon\nu\phi))\frac{d\mathscr{W}(\varepsilon\nu)}{\varepsilon d\nu}, \\ \mathfrak{M}(\varepsilon\nu_0) = \chi, \end{cases}$$

where $\tau = \varepsilon\nu$ and $\tau_0 = \varepsilon\nu_0$.

By considering $d\mathscr{W}(\varepsilon\nu) = \sqrt{\varepsilon}d\mathscr{W}(\nu)$ and representing $\mathfrak{M}(\varepsilon\nu) = \mathfrak{M}_{\varepsilon}(\nu)$ and $\mathfrak{M}(\varepsilon\nu_0 + \varepsilon\nu\phi) = \mathfrak{M}_{\varepsilon}(\nu_0 + \nu\phi)$, we obtain the following outcomes from the above system:

$$\begin{cases} \mathcal{T}_{\nu}^{\lambda}\mathfrak{M}_{\varepsilon}(\nu) = \varepsilon^{\lambda}\mathbb{U}(\nu, \mathfrak{M}_{\varepsilon}(\nu), \mathfrak{M}_{\varepsilon}(\nu_0 + \nu\phi)) + \varepsilon^{\lambda-\frac{1}{2}}\mathbb{M}(\nu, \mathfrak{M}_{\varepsilon}(\nu), \mathfrak{M}_{\varepsilon}(\nu_0 + \nu\phi))\frac{d\mathscr{W}(\nu)}{d\nu}, \\ \mathfrak{M}_{\varepsilon}(\nu_0) = \chi. \end{cases}$$

We can indicate $\nu := \tau$ without losing generality. The standard form of Equation (1) is provided as follows by utilizing the natural time scaling $\frac{\tau}{\varepsilon} \mapsto \tau$ and Lemma 5.

$$\begin{cases} \mathcal{T}_{\tau}^{\lambda}\mathfrak{M}_{\varepsilon}(\tau) = \varepsilon^{\lambda}\mathbb{U}(\tau, \mathfrak{M}_{\varepsilon}(\tau), \mathfrak{M}_{\varepsilon}(\tau_0 + \tau\phi)) + \varepsilon^{\lambda-\frac{1}{2}}\mathbb{M}(\tau, \mathfrak{M}_{\varepsilon}(\tau), \mathfrak{M}_{\varepsilon}(\tau_0 + \tau\phi))\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}_{\varepsilon}(\tau_0) = \chi. \end{cases} \tag{52}$$

Equation (52) can thus be represented in its integral form as

$$\begin{aligned} \mathfrak{M}_{\varepsilon}(\tau) =& \chi + \varepsilon^{\lambda}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{U}\big(\vartheta, \mathfrak{M}_{\varepsilon}(\vartheta), \mathfrak{M}_{\varepsilon}(\tau_0 + \phi\vartheta)\big)d\vartheta \\ &+ \varepsilon^{\lambda-\frac{1}{2}}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\mathbb{M}\big(\vartheta, \mathfrak{M}_{\varepsilon}(\vartheta), \mathfrak{M}_{\varepsilon}(\tau_0 + \phi\vartheta)\big)d\mathscr{W}(\vartheta), \end{aligned} \tag{53}$$

here, $\varepsilon \in (0, \varepsilon_0]$ represents a small positive parameter, with ε_0 being a predetermined constant. Additionally, $\mathbb{U}$ and $\mathbb{M}$ fulfill the criteria outlined in conditions $(\mathbb{C}_3)$ and $(\mathbb{C}_4)$. The averaged form of Equation (53) is consequently presented below.

$$\begin{aligned} \mathfrak{M}_{\varepsilon}^{*}(\tau) =& \chi + \varepsilon^{\lambda}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\widetilde{\mathbb{U}}\big(\mathfrak{M}_{\varepsilon}^{*}(\vartheta), \mathfrak{M}_{\varepsilon}^{*}(\tau_0 + \phi\vartheta)\big)d\vartheta \\ &+ \varepsilon^{\lambda-\frac{1}{2}}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\widetilde{\mathbb{M}}\big(\mathfrak{M}_{\varepsilon}^{*}(\vartheta), \mathfrak{M}_{\varepsilon}^{*}(\tau_0 + \phi\vartheta)\big)d\mathscr{W}(\vartheta), \end{aligned} \tag{54}$$

where $\widetilde{\mathbb{U}} : \mathbb{R}^{\theta} \times \mathbb{R}^{\theta} \to \mathbb{R}^{\theta}, \widetilde{\mathbb{M}} : \mathbb{R}^{\theta} \times \mathbb{R}^{\theta} \to \mathbb{R}^{\theta\times\varsigma}$.

Theorem 5. *Assume that from* $(\mathbb{C}_3)$ *to* $(\mathbb{C}_5)$ *are satisfied. We can determine the corresponding* $\varepsilon_1 \in (0, \varepsilon_0], \varrho > 0, \beth \in (0,1)$ *satisfies* $\forall\varepsilon \in (0, \varepsilon_1]$ *when* $\mathrm{p} \in [2, (1-\lambda)^{-1})$ *and for* $\mho > 0$*, which is an arbitrarily small number. The result is derived in the following manner:*

$$\Xi\Big[\sup_{\tau\in[\tau_0,\ \varrho\varepsilon^{-\beth}]}\big\|\mathfrak{M}_{\varepsilon}(\tau) - \mathfrak{M}_{\varepsilon}^{*}(\tau)\big\|^{\mathrm{p}}\Big] \leq \mho. \tag{55}$$

Proof. We obtain the subsequent result for $\tau \in [\tau_0, \mathfrak{a}] \subset [\tau_0, \xi]$ via Equations (53) and (54).

$$\begin{aligned} &\mathfrak{M}_{\varepsilon}(\tau) - \mathfrak{M}_{\varepsilon}^{*}(\tau) \\ &= (\chi - \chi) + \varepsilon^{\lambda}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\big(\mathbb{U}\big(\vartheta, \mathfrak{M}_{\varepsilon}(\vartheta), \mathfrak{M}_{\varepsilon}(\tau_0 + \phi\vartheta)\big) - \widetilde{\mathbb{U}}\big(\mathfrak{M}_{\varepsilon}^{*}(\vartheta), \mathfrak{M}_{\varepsilon}^{*}(\tau_0 + \phi\vartheta)\big)\big)d\vartheta \end{aligned}$$

$$+\varepsilon^{\lambda-\frac{1}{2}}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\big(\mathbb{M}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{M}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\big)d\mathscr{W}(\vartheta). \tag{56}$$

By using J-I, we obtain the following from Equation (56) as a result

$$\begin{aligned}\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}} &\le 2^{\mathrm{p}-1}\\ &\left\|\varepsilon^{\lambda}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{U}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\vartheta\right\|^{\mathrm{p}}\\ &+2^{\mathrm{p}-1}\\ &\left\|\varepsilon^{\lambda-\frac{1}{2}}\frac{1}{\Gamma(\lambda)}\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{M}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\mathscr{W}(\vartheta)\right\|^{\mathrm{p}}\\ &\le\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{\mathrm{p}-1}\varepsilon^{\mathrm{p}\lambda}\\ &\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{U}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\vartheta\right\|^{\mathrm{p}}\\ &+\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{\mathrm{p}-1}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})}\\ &\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{M}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\mathscr{W}(\vartheta)\right\|^{\mathrm{p}}.\end{aligned} \tag{57}$$

Utilizing Equation (57) in Equation (55).

$$\begin{aligned}&\Xi\left[\sup_{\tau_0\le\tau\le\mathfrak{a}}\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}}\right]\\ &\quad\le\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{\mathrm{p}-1}\varepsilon^{\mathrm{p}\lambda}\\ &\quad\Xi\left[\sup_{\tau_0\le\tau\le\mathfrak{a}}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{U}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\vartheta\right\|^{\mathrm{p}}\right]\\ &\quad+\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{\mathrm{p}-1}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})}\\ &\quad\Xi\left[\sup_{\tau_0\le\tau\le\mathfrak{a}}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{M}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{M}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\phi\vartheta)\big)\Big)d\mathscr{W}(\vartheta)\right\|^{\mathrm{p}}\right]\\ &\quad=\mathbb{V}_1+\mathbb{V}_2.\end{aligned} \tag{58}$$

From $\mathbb{V}_1$

$$\begin{aligned}\mathbb{V}_1\le&\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{2\mathrm{p}-2}\varepsilon^{\mathrm{p}\lambda}\\ &\Xi\left[\sup_{\tau_0\le\tau\le\mathfrak{a}}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathfrak{M}_\varepsilon(\vartheta),\mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big)-\mathbb{U}\big(\vartheta,\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\vartheta\right\|^{\mathrm{p}}\right]\\ &+\frac{1}{\Gamma^{\mathrm{p}}(\lambda)}2^{2\mathrm{p}-2}\varepsilon^{\mathrm{p}\lambda}\\ &\Xi\left[\sup_{\tau_0\le\tau\le\mathfrak{a}}\left\|\int_{\tau_0}^{\tau}(\tau-\vartheta)^{\lambda-1}\Big(\mathbb{U}\big(\vartheta,\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)-\widetilde{\mathbb{U}}\big(\mathfrak{M}^*_\varepsilon(\vartheta),\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big)\Big)d\vartheta\right\|^{\mathrm{p}}\right]\\ &=\mathbb{V}_{11}+\mathbb{V}_{12}.\end{aligned} \tag{59}$$

Using H-I, J-I, and $(\mathbb{C}_3)$ on $\mathbb{V}_{11}$, we obtain the following result:

$$\begin{aligned}
\mathbb{V}_{11} \leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}\lambda} \left(\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{\frac{(\lambda-1)\mathrm{p}}{\mathrm{p}-1}} d\vartheta \right)^{\mathrm{p}-1} \\
& \Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \int_{\tau_0}^{\tau} \| \mathbb{U}(\vartheta, \mathfrak{M}_\varepsilon(\vartheta), \mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)) - \mathbb{U}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)) \|^{\mathrm{p}} d\vartheta \right] \\
\leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{3\mathrm{p}-3} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-1} \mathscr{U}_2^{\mathrm{p}} \left(\frac{\mathrm{p}-1}{\lambda\mathrm{p}-1} \right)^{\mathrm{p}-1} \\
& \Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \int_{\tau_0}^{\tau} \| \mathfrak{M}_\varepsilon(\vartheta) - \mathfrak{M}^*_\varepsilon(\vartheta) \|^{\mathrm{p}} d\vartheta \right] + \Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \int_{\tau_0}^{\tau} \| \mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta) - \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta) \|^{\mathrm{p}} d\vartheta \right] \\
= & \mathbb{Y}_{11} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-1} \left(\int_{\tau_0}^{\mathfrak{a}} \Xi \left[\sup_{0 \leq \rho \leq \vartheta} \| \mathfrak{M}_\varepsilon(\rho) - \mathfrak{M}^*_\varepsilon(\rho) \|^{\mathrm{p}} \right] d\vartheta \right. \\
& \left. + \int_{\tau_0}^{\mathfrak{a}} \Xi \left[\sup_{0 \leq \rho \leq \vartheta} \| \mathfrak{M}_\varepsilon(\tau_0+\rho\phi) - \mathfrak{M}^*_\varepsilon(\tau_0+\rho\phi) \|^{\mathrm{p}} \right] d\vartheta \right),
\end{aligned} \tag{60}$$

where $\mathbb{Y}_{11} = 2^{3\mathrm{p}-3} \mathscr{U}_2^{\mathrm{p}} \left(\frac{\mathrm{p}-1}{\lambda\mathrm{p}-1} \right)^{\mathrm{p}-1} \frac{1}{\Gamma^{\mathrm{p}}(\lambda)}$.

Using H-I, J-I, and $(\mathbb{C}_5)$ on $\mathbb{V}_{12}$, we obtain the following result:

$$\begin{aligned}
\mathbb{V}_{12} \leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}\lambda} \Big(\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{\frac{(\lambda-1)\mathrm{p}}{\mathrm{p}-1}} d\vartheta \Big)^{\mathrm{p}-1} \\
& \Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \int_{\tau_0}^{\tau} \left\| \mathbb{U}\big(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big) - \widetilde{\mathbb{U}}\big(\mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big) \right\|^{\mathrm{p}} d\vartheta \right] \\
\leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}\lambda} \left(\frac{\mathrm{p}-1}{\lambda\mathrm{p}-1} \right)^{\mathrm{p}-1} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}} \mathbb{D}_1(\mathfrak{a}) \\
& \big(1 + \Xi \|\mathfrak{M}^*_\varepsilon(\vartheta)\|^{\mathrm{p}} + \Xi \|\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\|^{\mathrm{p}}\big) \\
= & \mathbb{Y}_{12} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}},
\end{aligned} \tag{61}$$

where $\mathbb{Y}_{12} = 2^{2\mathrm{p}-2} \mathbb{D}_1(\mathfrak{a}) \Big(1 + \Xi \|\mathfrak{M}^*_\varepsilon(\vartheta)\|^{\mathrm{p}} + \Xi \|\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\|^{\mathrm{p}}\Big) \left(\frac{\mathrm{p}-1}{\lambda\mathrm{p}-1} \right)^{\mathrm{p}-1} \frac{1}{\Gamma^{\mathrm{p}}(\lambda)}$.

Through the use of J-I, $\mathbb{V}_2$ provides the following:

$$\begin{aligned}
\mathbb{V}_2 \leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \\
& \left(\Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \left\| \int_{\tau_0}^{\tau} (\tau-\vartheta)^{\lambda-1} \Big[\mathbb{M}\big(\vartheta, \mathfrak{M}_\varepsilon(\vartheta), \mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)\big) - \mathbb{M}\big(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big) \Big] d\mathscr{W}(\vartheta) \right\|^{\mathrm{p}} \right] \right) \\
& + \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \\
& \left(\Xi \left[\sup_{\tau_0 \leq \tau \leq \mathfrak{a}} \left\| \int_{\tau_0}^{\tau} (\tau-\vartheta)^{\lambda-1} \Big[\mathbb{M}\big(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big) - \widetilde{\mathbb{M}}\big(\mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\big) \Big] d\mathscr{W}(\vartheta) \right\|^{\mathrm{p}} \right] \right) \\
= & \mathbb{V}_{21} + \mathbb{V}_{22}.
\end{aligned} \tag{62}$$

Using $(\mathbb{C}_3)$, H-I, and BDG-I on $\mathbb{V}_{21}$, we achieve the following outcomes:

$$\begin{aligned}
\mathbb{V}_{21} \leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \Big(2(\mathrm{p}-1)^{1-\mathrm{p}} \mathrm{p}^{\mathrm{p}+1} \Big)^{\frac{\mathrm{p}}{2}} \\
& \Xi \left[\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{2\lambda-2} \| \mathbb{M}(\vartheta, \mathfrak{M}_\varepsilon(\vartheta), \mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)) - \mathbb{M}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)) \|^2 d\vartheta \right]^{\frac{\mathrm{p}}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \mathfrak{a}^{\frac{\mathrm{p}}{2}-1} \left(\mathrm{p}^{\mathrm{p}+1} 2(\mathrm{p}-1)^{1-\mathrm{p}}\right)^{\frac{\mathrm{p}}{2}} \\
&\Xi\left[\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}} \|\mathbb{M}(\vartheta, \mathfrak{M}_\varepsilon(\vartheta), \mathfrak{M}_\varepsilon(\tau_0+\phi\vartheta)) - \mathbb{M}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta))\|^{\mathrm{p}} d\vartheta\right] \\
&\leq \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{3\mathrm{p}-3} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \mathfrak{a}^{\frac{\mathrm{p}}{2}-1} \mathscr{U}_2^{\mathrm{p}} \left(\mathrm{p}^{\mathrm{p}+1} 2(1-\mathrm{p})^{\mathrm{p}-1}\right)^{\frac{\mathrm{p}}{2}} \int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}} \\
&\Xi\left[\sup_{\tau_0\leq\rho\leq\vartheta} \left[\|\mathfrak{M}_\varepsilon(\rho) - \mathfrak{M}^*_\varepsilon(\rho)\|^{\mathrm{p}} + \|\mathfrak{M}_\varepsilon(\rho\phi) - \mathfrak{M}^*_\varepsilon(\rho\phi)\|^{\mathrm{p}}\right] d\vartheta\right] \\
&= \mathbb{Y}_{21} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\frac{\mathrm{p}}{2}-1} \left(\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}} \Xi\left[\sup_{\tau_0\leq\rho\leq\vartheta} \|\mathfrak{M}_\varepsilon(\rho) - \mathfrak{M}^*_\varepsilon(\rho)\|^{\mathrm{p}} d\vartheta\right] \right. \\
&\left. + \int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}} \Xi\left[\sup_{0\leq\rho\leq\vartheta} \|\mathfrak{M}_\varepsilon(\tau_0+\rho\phi) - \mathfrak{M}^*_\varepsilon(\tau_0+\rho\phi)\|^{\mathrm{p}} d\vartheta\right] \right),
\end{aligned} \tag{63}
$$

where $\mathbb{Y}_{21} = 2^{3\mathrm{p}-3} \mathscr{U}_2^{\mathrm{p}} \left(\frac{\mathrm{p}^{\mathrm{p}+1}}{2(\mathrm{p}-1)^{\mathrm{p}-1}}\right)^{\frac{\mathrm{p}}{2}} \frac{1}{\Gamma^{\mathrm{p}}(\lambda)}$.

Again using H-I and BDG-I on $\mathbb{V}_{22}$, we achieve the following outcomes:

$$
\begin{aligned}
\mathbb{V}_{22} \leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \left(2(\mathrm{p}-1)^{1-\mathrm{p}} \mathrm{p}^{\mathrm{p}+1}\right)^{\frac{\mathrm{p}}{2}} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} \\
& \Xi\left[\int_{\tau_0}^{\mathfrak{a}} \left\|\mathbb{U}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)) - \widetilde{\mathbb{M}}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta))\right\|^2 (\mathfrak{a}-\vartheta)^{2\lambda-2} d\vartheta\right]^{\frac{\mathrm{p}}{2}} \\
\leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} 2^{2\mathrm{p}-2} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\frac{\mathrm{p}}{2}-1} \left(2(\mathrm{p}-1)^{\mathrm{p}-1} \mathrm{p}^{\mathrm{p}+1}\right)^{\frac{\mathrm{p}}{2}} \Xi\left[\int_{\tau_0}^{\mathfrak{a}} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}} \right. \\
& \left. \left(\|\mathbb{M}(\vartheta, \mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta))\|^{\mathrm{p}} + \left\|\widetilde{\mathbb{M}}(\mathfrak{M}^*_\varepsilon(\vartheta), \mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta))\right\|^{\mathrm{p}}\right) d\vartheta\right] \\
\leq & \frac{1}{\Gamma^{\mathrm{p}}(\lambda)} \frac{2^{3\mathrm{p}-3} 3^{\mathrm{p}-1} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}} \mathscr{U}_2^{\mathrm{p}} \left(\mathscr{U}_2^{\mathrm{p}} + \mathscr{U}_3\right)^{\mathrm{p}}}{((\lambda-1)\mathrm{p}+1)} \\
& \left(2(\mathrm{p}-1)^{1-\mathrm{p}} \mathrm{p}^{\mathrm{p}+1}\right)^{\frac{\mathrm{p}}{2}} \left(1 + \Xi[\|\mathfrak{M}^*_\varepsilon(\vartheta)\|^{\mathrm{p}}] + \Xi[\|\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\|^{\mathrm{p}}]\right) \\
= & \mathbb{Y}_{22} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}},
\end{aligned} \tag{64}
$$

where

$$
\begin{aligned}
\mathbb{Y}_{22} = & 2^{3\mathrm{p}-3} 3^{\mathrm{p}-1} \mathscr{U}_2^{\mathrm{p}} \left(\mathscr{U}_2^{\mathrm{p}} + \mathscr{U}_3\right)^{\mathrm{p}} \frac{1}{((\lambda-1)\mathrm{p}+1)} \\
& \left(2(\mathrm{p}-1)^{1-\mathrm{p}} \mathrm{p}^{\mathrm{p}+1}\right)^{\frac{\mathrm{p}}{2}} \left(1 + \Xi[\|\mathfrak{M}^*_\varepsilon(\vartheta)\|^{\mathrm{p}}] + \Xi[\|\mathfrak{M}^*_\varepsilon(\tau_0+\phi\vartheta)\|^{\mathrm{p}}]\right) \frac{1}{\Gamma^{\mathrm{p}}(\lambda)}.
\end{aligned}
$$

By utilizing Equations (59)–(64) in (58), as a result, we obtain the following outcomes:

$$
\begin{aligned}
&\Xi\left[\sup_{\tau_0\leq\tau\leq\mathfrak{a}} \|\mathfrak{M}_\varepsilon(\tau) - \mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}}\right] \leq \mathbb{Y}_{12} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}} + \mathbb{Y}_{22} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}} \\
&+ \int_{\tau_0}^{\mathfrak{a}} \left[\mathbb{Y}_{11} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-1} + \mathbb{Y}_{21} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\frac{\mathrm{p}}{2}-1} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}}\right] \Xi\left[\sup_{\tau_0\leq\rho\leq\vartheta} \|\mathfrak{M}_\varepsilon(\rho) - \mathfrak{M}^*_\varepsilon(\rho)\|^{\mathrm{p}} d\vartheta\right] \\
&+ \int_{\tau_0}^{\mathfrak{a}} \left[\mathbb{Y}_{11} \varepsilon^{\mathrm{p}\lambda} (\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-1} + \mathbb{Y}_{21} \varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})} (\mathfrak{a}-\tau_0)^{\frac{\mathrm{p}}{2}-1} (\mathfrak{a}-\vartheta)^{(\lambda-1)\mathrm{p}}\right] \\
&\Xi\left[\sup_{\tau_0\leq\rho\leq\vartheta} \|\mathfrak{M}_\varepsilon(\tau_0+\rho\phi) - \mathfrak{M}^*_\varepsilon(\tau_0+\rho\phi)\|^{\mathrm{p}} d\vartheta\right].
\end{aligned} \tag{65}
$$

Consequently, we obtain the subsequent outcome from Equation (65).

$$\Xi\Big[\sup_{\tau_0\le\tau\le\mathfrak{a}}\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}}\Big]\le\Bigg(\mathbb{Y}_{12}\varepsilon^{\mathrm{p}\lambda}(\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}}+\mathbb{Y}_{22}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})}(\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\Bigg)$$
$$\exp\Bigg(2\mathbb{Y}_{11}\varepsilon^{\mathrm{p}\lambda}(\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}}+\frac{2\mathbb{Y}_{21}}{(\lambda-1)\mathrm{p}+1}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})}(\mathfrak{a}-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\Bigg).$$

This implies that for any $\tau\in\big[\tau_0,\varrho\varepsilon^{-\beth}\big]\subseteq[\tau_0,\xi]$, there are $\varrho>0$ and $\beth\in(0,1)$ as well.

$$\Xi\Big[\sup_{\tau_0\le\tau\le\varrho\varepsilon^{-\beth}}\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}}\Big]\le\mathcal{Z}\varepsilon^{1-\beth},\tag{66}$$

where

$$\mathcal{Z}=\Bigg(\mathbb{Y}_{12}(\varrho-\tau_0)^{\lambda\mathrm{p}}\varepsilon^{\mathrm{p}\lambda+\beth-\lambda\beth\mathrm{p}-1}+\mathbb{Y}_{22}(\varrho-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})(1+\beth)+\beth-\lambda\beth\mathrm{p}-1}\Bigg)$$
$$\exp\Bigg[2\mathbb{Y}_{11}(\varrho-\tau_0)^{\lambda\mathrm{p}}\varepsilon^{\mathrm{p}\lambda(1-\lambda\beth)}+\frac{2\mathbb{Y}_{21}}{(\lambda-1)\mathrm{p}+1}(\varrho-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})(1+\beth)-\lambda\beth\mathrm{p}}\Bigg]$$

is a constant. As a result, when $\forall\mho>0$, finding $\varepsilon_1\in(0,\varepsilon_0]$ that satisfies $\forall\varepsilon\in(0,\varepsilon_1]$ and $\tau\in\big[\tau_0,\varrho\varepsilon^{-\beth}\big]$ allows us to deduce

$$\Xi\Big[\sup_{\tau_0\le\tau\le\varrho\varepsilon^{-\beth}}\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\|^{\mathrm{p}}\Big]\le\mho.$$

□

Corollary 2. *Assume that the assumptions* $(\mathbb{C}_3)$ *and* $(\mathbb{C}_4)$ *are valid. For any arbitrary positive number* $\mho_1>0$*, the following criteria are defined:* $\beth$ *is within the interval* $(0,1)$*,* ϱ *is greater than 0, and* ε_1 *lies in the interval* $(0,\varepsilon_0)$*. We have* $\forall\varepsilon\in(0,\varepsilon_1]$*:*

$$\lim_{\varepsilon\to0}\mathfrak{P}\Big[\sup_{\tau\in[\tau_0,\,\varrho\varepsilon^{-\beth}]}\big\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\big\|>\mho_1\Big]=0.\tag{67}$$

Proof. By employing the Chebyshev–Markov inequality and Theorem 5, it is possible to infer the following for any positive number $\mho_1$.

$$\begin{aligned}\mathfrak{P}\Big[\sup_{\tau\in[\tau_0,\,\varrho\varepsilon^{-\beth}]}\big\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\big\|>\mho_1\Big]&\le\frac{1}{\mho_1^2}\Xi\Big[\sup_{\tau\in[\tau_0,\,\varrho\varepsilon^{-\beth}]}\big\|\mathfrak{M}_\varepsilon(\tau)-\mathfrak{M}^*_\varepsilon(\tau)\big\|^2\Big]\\&\le\frac{\mathcal{Z}\varepsilon^{1-\beth}}{\mho_1^2}\\&\le0\text{ as }\varepsilon\to0,\end{aligned}$$

where

$$\mathcal{Z}=\Bigg(\mathbb{Y}_{12}(\varrho-\tau_0)^{\lambda\mathrm{p}}\varepsilon^{\mathrm{p}\lambda+\beth-\lambda\beth\mathrm{p}-1}+\mathbb{Y}_{22}(\varrho-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})(1+\beth)+\beth-\lambda\beth\mathrm{p}-1}\Bigg)$$
$$\exp\Bigg[2\mathbb{Y}_{11}(\varrho-\tau_0)^{\lambda\mathrm{p}}\varepsilon^{\mathrm{p}\lambda(1-\lambda\beth)}+\frac{2\mathbb{Y}_{21}}{(\lambda-1)\mathrm{p}+1}(\varrho-\tau_0)^{\lambda\mathrm{p}-\frac{\mathrm{p}}{2}}\varepsilon^{\mathrm{p}(\lambda-\frac{1}{2})(1+\beth)-\lambda\beth\mathrm{p}}\Bigg].$$

It ends the proof. □

5. Examples

In this section, we provide four examples to demonstrate the practical value of our presented outcome.

Example 1. *Consider the subsequent FP-SDE:*

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}(\tau) = 2\varepsilon^{\lambda}\cos^2(\tau)\mathfrak{M}_{\varepsilon}(\tau) - \varepsilon^{\lambda}\mathfrak{M}_{\varepsilon}\sin^2(\frac{1}{2}\tau) + \varepsilon^{\lambda-\frac{1}{2}}\frac{d\mathscr{W}(\tau)}{d\tau}, \ \tau \in [0,\pi], \\ \mathfrak{M}(0) = \chi, \end{cases} \tag{68}$$

where $\lambda = 0.8$, $\tau_0 = 0$ *and* $\phi \in (0, \frac{\pi-0}{\pi}) = (0,1)$.

$$\begin{aligned} \mathbb{U}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= 2\cos^2(\tau)\mathfrak{M}_{\varepsilon}(\tau) - \mathfrak{M}_{\varepsilon}\sin^2\left(\frac{1}{2}\tau\right), \\ \mathbb{M}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= 1. \end{aligned}$$

Since both drift and diffusion factors in Equation (68) *meet the necessary conditions of Theorem* 2*, the solution to Equation* (68) *exists and is unique.*

The averages of $\mathbb{U}$ *and* $\mathbb{M}$ *can be expressed in the following manner:*

$$\begin{aligned} \widetilde{\mathbb{U}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \frac{1}{\pi}\int_0^{\pi}\left(2\cos^2(\tau)\mathfrak{M}_{\varepsilon}(\tau) - \mathfrak{M}_{\varepsilon}\sin^2\left(\frac{1}{2}\tau\right)\right)d\tau = \frac{1}{2}\mathfrak{M}_{\varepsilon}(\tau), \\ \widetilde{\mathbb{M}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \frac{1}{\pi}\int_0^{\pi} 1 d\tau = 1. \end{aligned}$$

Thus, we have the corresponding averaged FP-SDEs

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}^{*}(\tau) = \frac{1}{2}\varepsilon^{\lambda}\mathfrak{M}_{\varepsilon}(\tau) + \varepsilon^{\lambda-\frac{1}{2}}\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(0) = \chi. \end{cases}$$

Example 2. *Consider the subsequent FP-SDEs:*

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}(\tau) = \varepsilon^{\lambda}\sin^2(\tau)\mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau) + \varepsilon^{\lambda-\frac{1}{2}}\sin\left(\mathfrak{M}_{\varepsilon}(\tau)\right)\frac{d\mathscr{W}(\tau)}{d\tau}, \ \tau \in [0.5,\pi], \\ \mathfrak{M}(0.5) = \chi, \end{cases} \tag{69}$$

where $\lambda = 0.8$, $\tau_0 = 0.5$, $\phi \in (0, \frac{\pi-0.5}{\pi})$ *and*

$$\begin{aligned} \mathbb{U}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \sin^2(\tau)\mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau), \\ \mathbb{M}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \sin(\mathfrak{M}_{\varepsilon}(\tau)). \end{aligned}$$

Since both $\sin^2(\tau)\mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau)$ *and* $\sin(\mathfrak{M}_{\varepsilon}(\tau))$ *factors in Equation* (69) *meet the necessary conditions of Theorem* 2*, the solution to Equation* (69) *exists and is unique.*

We can express the averages of $\mathbb{U}$ *and* $\mathbb{M}$ *as follows:*

$$\begin{aligned} \widetilde{\mathbb{U}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \frac{1}{\pi}\int_{0.5}^{\pi}\sin^2(\tau)\mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau)d\tau = \frac{1}{\pi}\left(\frac{1}{2}((\pi - 0.5) + \frac{1}{4}\sin 1)\right)\mathfrak{M}_{\varepsilon}^{*}(\tau_0 + \tau\phi), \\ \widetilde{\mathbb{M}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) &= \frac{1}{\pi}\int_{0.5}^{\pi}\sin(\mathfrak{M}_{\varepsilon}(\tau))d\tau = \frac{1}{\pi}(\pi - 0.5)\sin(\mathfrak{M}_{\varepsilon}(\tau)). \end{aligned}$$

Thus, we have the corresponding averaged FP-SDEs

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}^{*}(\tau) = \frac{1}{2}\varepsilon^{\lambda}\mathfrak{M}_{\varepsilon}^{*}(\tau\phi) + \varepsilon^{\lambda-\frac{1}{2}}\left(\frac{1}{\pi}(\pi - 0.5)\sin\left(\mathfrak{M}_{\varepsilon}(\tau)\right)\right)\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(0.5) = \chi. \end{cases}$$

Example 3. *Consider the subsequent FP-SDEs:*

$$\begin{cases} \mathcal{T}_{\tau}^{0.9}\mathfrak{M}_{\varepsilon}(\tau) = \frac{1}{2}\varepsilon^{\lambda}\mathfrak{M}_{\varepsilon}(\tau\phi) + \frac{3\pi}{4}\varepsilon^{\lambda-\frac{1}{2}}\sin^{3}\tau.\mathfrak{M}_{\varepsilon}(\tau)\frac{d\mathscr{W}(\tau)}{d\tau}, \ \tau \in [0,\pi], \\ \mathfrak{M}(0) = \chi, \end{cases} \tag{70}$$

where $\lambda = 0.9$, $\tau_0 = 0$ *and* $\phi \in (0, \frac{\pi-0}{\pi}) = (0,1)$.

$$\mathbb{U}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{1}{2}\mathfrak{M}_{\varepsilon}(\tau\phi),$$
$$\mathbb{M}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{3\pi}{4}\sin^{3}\tau.\mathfrak{M}_{\varepsilon}(\tau).$$

Since both drift $\frac{1}{2}\mathfrak{M}_{\varepsilon}(\tau\phi)$ *and diffusion* $\frac{3\pi}{4}\sin^{3}\tau.\mathfrak{M}_{\varepsilon}(\tau)$ *factors in Equation* (70) *meet the necessary conditions of Theorem* 2*, the solution to Equation* (70) *exists and is unique.*

The following forms represent the averages of $\mathbb{U}$ *and* $\mathbb{M}$.

$$\widetilde{\mathbb{U}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{2}\mathfrak{M}_{\varepsilon}(\tau\phi)d\lambda = \frac{1}{2}\mathfrak{M}_{\varepsilon}^{*}(\tau\phi),$$
$$\widetilde{\mathbb{M}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{1}{\pi}\frac{3\pi}{4}\int_{0}^{\pi}\sin^{3}\tau.\mathfrak{M}_{\varepsilon}(\tau)d\lambda = \mathfrak{M}_{\varepsilon}^{*}(\tau).$$

Thus, we have the corresponding averaged FP-SDEs

$$\begin{cases} \mathcal{T}_{\tau}^{0.9}\mathfrak{M}_{\varepsilon}^{*}(\tau) = \frac{1}{2}\varepsilon^{\lambda}\mathfrak{M}_{\varepsilon}^{*}(\tau\phi) + \varepsilon^{\lambda-\frac{1}{2}}\mathfrak{M}_{\varepsilon}^{*}(\tau)\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(0) = \chi. \end{cases}$$

Example 4. *Consider the subsequent FP-SDE:*

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}(\tau) = \frac{3}{5}\varepsilon^{\lambda}\exp^{-\tau} + \varepsilon^{\lambda-\frac{1}{2}}\mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau)\frac{d\mathscr{W}(\tau)}{d\tau}, \ \tau \in [0.3,\pi], \\ \mathfrak{M}(0.3) = \chi, \end{cases} \tag{71}$$

where $\lambda = 0.8$, $\tau_0 = 0.3$, $\phi \in (0, \frac{\pi-0.3}{\pi})$, *and*

$$\mathbb{U}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{3}{5}\exp^{-\tau},$$
$$\mathbb{M}(\tau, \mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \mathfrak{M}_{\varepsilon}(\tau_0 + \phi\tau).$$

Since both drift and diffusion factors in Equation (71) *meet the necessary conditions of Theorem* 2*, the solution to Equation* (71) *exists and is unique.*

The averages of $\mathbb{U}$ *and* $\mathbb{M}$ *can be expressed in the following manner:*

$$\widetilde{\mathbb{U}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{1}{\pi}\int_{0.3}^{\pi}\left(\frac{3}{5}\exp^{-\tau}\right)d\tau = \frac{3}{5}\frac{1}{\pi}(\exp^{-0.3} - \exp^{-\pi}),$$
$$\widetilde{\mathbb{M}}(\mathfrak{M}(\tau), \mathfrak{M}(\tau_0 + \tau\phi)) = \frac{1}{\pi}\int_{0.3}^{\pi}1d\tau = \frac{1}{\pi}(\pi - 0.3).$$

Thus, we have the corresponding averaged FP-SDEs

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\mathfrak{M}_{\varepsilon}^{*}(\tau) = \varepsilon^{\lambda}\left(\frac{3}{5}\frac{1}{\pi}(\exp^{-0.3} - \exp^{-\pi})\right) + \varepsilon^{\lambda-\frac{1}{2}}\left(\frac{1}{\pi}(\pi - 0.3)\right)\frac{d\mathscr{W}(\tau)}{d\tau}, \\ \mathfrak{M}(\tau_0) = \chi. \end{cases}$$

6. Conclusions

In this research work, we generalize the results concerning the existence, uniqueness, continuous dependency, and regularity of solutions to FP-SDEs, along with the Ave-Pr in the $\mathscr{L}^{\mathrm{p}}$ space. The application of the contraction mapping concept is utilized to explore the

EU of the problem under discussion. Moreover, we illustrate the Ave-Pr for FP-SDEs in the $\mathscr{L}^{\mathrm{p}}$ space using GB-I, BDG-I, H-I, J-I, and an interval translation approach. Ultimately, we present three examples to help elucidate the established results and demonstrate the effectiveness of our findings.

In our future research, we plan to employ numerical methods to tackle various real-world problems modeled by FP-SDEs.

Author Contributions: Conceptualization, A.M.D. and M.I.L.; methodology, A.M.D. and M.I.L.; software, A.M.D. and M.I.L.; validation, A.M.D. and M.I.L.; formal analysis, A.M.D. and M.I.L.; investigation, A.M.D. and M.I.L.; resources, A.M.D.; writing—original draft preparation, A.M.D. and M.I.L.; writing—review and editing, A.M.D. and M.I.L.; visualization, A.M.D. and M.I.L.; funding acquisition, A.M.D. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Project No. GrantA225).

Data Availability Statement: No data were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflicts of interest.

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