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The Existence and Representation of the Solutions to the System of Operator Equations $A_iXB_i + C_iYD_i + E_iZF_i = G_i(i = 1, 2)$

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Abstract: In this paper, we give the necessary and sufficient conditions for the existence of general solutions, self-adjoint solutions, and positive solutions to the system of $A_iXB_i + C_iYD_i + E_iZF_i = G_i(i = 1, 2)$ under additional conditions. In addition, we derive the representation of general solutions to the system of $A_iXB_i + C_iYD_i + E_iZF_i = G_i(i = 1, 2)$, and provide the matrix representation of the self-adjoint solutions and the positive solutions in the sense of the star order.

Keywords: operator equation; Moore–Penrose inverse; general solution; self-adjoint solution; positive solution

MSC: 47A05, 47A08, 47A62

1. Introduction

Linear matrix equations are one of the main research topics in matrix theory and its applications. Many articles on matrix equations give different methods for the existence and the representation of solutions [1–16]. Various issues in engineering, math, and other fields may be converted into certain linear matrix equations. For example, Roth [17] gave a necessary and sufficient condition for the solvability of the generalized Sylvester matrix equation

$$AX - YB = C.$$

Lancaster [18] systematically investigated the equation

$$AX + XB = C.$$

After this, many researches have studied some properties of the generalized Sylvester matrix equation [19–22]. In reference [23], the author considered the solvability of the equation

$$AXB + CYD = E,$$

and this equation has many applications in the growth curve model in statistics.

Matrices are operators on finite-dimensional spaces, so can the relevant conclusions of matrix equations be generalized to operator equations on infinite-dimensional spaces?

Douglas [24] formulated the renowned "Douglas Range Inclusion Theorem" and provided some corresponding equivalent conditions for the existence of solutions to the operator equation AX = B. Subsequently, numerous academics studied the solvability of operator equations evolving from AX = B [25–30].

Recently, Vodough and Moslehian [31] considered $AXB = B = BXA$ under the condition of star order and obtained representations of the solutions to the operator equations. Zhang and Ji [32] extended these results and gave the necessary and sufficient conditions for the existence of solutions to the operator equations $AXB = B = BXA$. Cvetković

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Ilić [33] provided the sufficient and necessary conditions for the solvability of the operator equations $AXC = C = CXA$, which is a generalization of [31,32]. Hranislav [28] considered the solvability of the operator equation $A_iXB_i = C_i, i = \overline{1,2}$, and further generalized the problem in [32].

Inspired by the results of [25,28,31–33], in this paper, we study the existence of solutions and the representation of general, self-adjoint, and positive solutions of the operator equation

$$A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2) \tag{1}$$

Moreover, as a corollary, we derive the corresponding conclusion of the operator equation

$$A_iXB_i + C_iYD_i = G_i (i = 1, 2). \tag{2}$$

The structure of this article is as follows:

In Section 2, we provide a review of the definitions of generalized inverses, self-adjoint operator, positive operator, star order, and Lemmas 1–6.

In Section 3, we investigate the operator Equation (1) and give the necessary and sufficient conditions for the existence of solutions under some assumptions. As a corollary, equivalent conditions for the solutions of the operator Equation (2) are provided under certain conditions. Furthermore, we give the necessary and sufficient conditions for the existence of self-adjoint and positive solutions to the operator systems (1) and (2) under a star-order condition.

In Section 4, we study the general solutions, self-adjoint solutions, and positive solutions of the operator Equations (1) and (2) and obtain the explicit representations of these solutions. The general solutions of Equations (1) and (2) are not unique. Indeed, an inner inverse (for $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, elements $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying $TXT = T$, then X is called an inner inverse of T; refer to Preliminaries) is not unique, in general, and we express the solutions through the inner inverse. On the other hand, the representation of solutions has a free parameter S, which is arbitrary. Consequently, the general solutions derived in Theorem 4 and Corollary 9 are not unique. Additionally, if $E_i = 0$ or $F_i = 0$ in Theorem 4, the solutions (16) of Equation (1) coincide with the solutions (24) of Equation (2). Furthermore, it should be noted that the self-adjoint and positive solutions presented in Theorems 5 and 6, respectively, are also non-unique. In the operator matrix representation of the self-adjoint and positive solutions, the skew diagonal elements and the lower right corner of the main diagonal elements are arbitrary.

2. Preliminaries

Considering $\mathcal{H}$ and $\mathcal{K}$ as Hilbert spaces, we represent by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For brevity, we denote $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the notations $\mathcal{R}(T)$, $\mathcal{N}(T)$, and T^* are the range space, the null space, and the adjoint of T, respectively. The operator $T \in \mathcal{B}(\mathcal{H})$ is called self-adjoint if $T = T^*$ and positive if $\langle Tx, x\rangle \geq 0$. $\overline{T}$ represents a bounded extension of T.

For any $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, if there exists an arbitrary operator $X : \mathcal{D}(X) \subseteq \mathcal{K} \to \mathcal{H}$ such that $\mathcal{R}(T) \subseteq \mathcal{D}(X)$ and

$$TXT = T,$$

then X is called an inner inverse of T, which we denote by T^-, and T is called regular. Thus, $X \notin \mathcal{B}(\mathcal{K}, \mathcal{H})$, in general. $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has an inner inverse $X \subseteq \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $\mathcal{R}(T)$ is closed [34]. If X satisfies the equation $XTX = X$, then X is called a reflexive inverse of T. Furthermore, there exists a unique bounded operator that also verifies

$$ST = P_{\overline{\mathcal{R}(T^*)}} \text{ and } TS = P_{\overline{\mathcal{R}(T)}}|_{\mathcal{R}(T)\oplus\mathcal{R}(T)^\perp}.$$

Such an operator S is called the Moore–Penrose generalized inverse of T and will be expressed as $S = T^\dagger$.

In 1978, Drazin introduced a partial order on $\mathcal{B}(\mathcal{H})$, which is called the star order. For $A, B \in \mathcal{B}(\mathcal{H})$, the star order $A \overset{*}{\leq} B$ means $AA^* = BA^*, A^*A = A^*B$; the left star order $A* \leq B$ means $A^*A = A^*B$ and $R(A) \subseteq \mathcal{R}(B)$; the right star order $A \leq *B$ means $AA^* = BA^*$ and $R(A^*) \subseteq \mathcal{R}(B^*)$ [35–41]. In reference [38], the star order, the minus order, and the diamond order were mainly studied. In particular, the star order is a generalization of a positive definite order. For orthogonal projection operators, the star order is a positive definite order; that is, if A and B are orthogonal projection operators, then $A \overset{*}{\leq} B$ is equivalent to $A \leq B$.

Lemma 1 ([35,41], Lemmas 1.1 and 2.1)**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$. The subsequent assertions hold.*

(i) $A \overset{*}{\leq} B \Rightarrow\ R(A) \subseteq R(B),\ \ R(A^*) \subseteq R(B^*)$.

(ii) $A \overset{*}{\leq} B \Leftrightarrow\ A = P_{\overline{\mathcal{R}(A)}}B = BP_{\overline{\mathcal{R}(A^*)}}$.

(iii) $A* \leq B \Rightarrow\ A = P_{\overline{\mathcal{R}(A)}}B$.
$A \leq *B \Rightarrow\ A = BP_{\overline{\mathcal{R}(A^*)}}$.

(iv) $A \overset{*}{\leq} B \Leftrightarrow\ A* \leq B$ *and* $A \leq *B$.

Lemma 2. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{F}, \mathcal{G})$ be closed-range operators and $C \in \mathcal{B}(\mathcal{F}, \mathcal{K})$; then the subsequent assertions are equivalent, as follows:*

(i) Operator equation

$$AXB = C$$

has a solution.

(ii) For some inner inverses A^-, B^-, we have $AA^-CB^-B = C$.

(iii) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.
A representation of the general solution is

$$X = A^-CB^- + U - A^-AUBB^-,$$

where $U \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is an arbitrary operator.

In Theorem 2 [42], when A, B, and C are matrices, the equation $AXB = C$ is solvable, and the form of the general solution is obtained. When A and B are closed-range operators, the solvability and general solution of the operator equation $AXB = C$ is similar to that of the matrix equation.

Lemma 3 ([24], Douglas Lemma)**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$. The subsequent assertions are equivalent, as follows:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) $AA^* \leq \lambda^2 BB^*$ *for some constant* $\lambda > 0$;

(iii) $A = BC$, *for some* $C \in \mathcal{B}(\mathcal{H})$.

If the equivalent conditions $(i) - (iii)$ hold, then there is a unique operator C such that

(a) $\|C\|^2 = inf\{\lambda^2 | AA^* \leq \lambda^2 BB^*\}$;

(b) $\mathcal{N}(A) = \mathcal{N}(C)$;

(c) $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$.

This solution will be called the ***Douglas reduced solution*** *and can be expressed as $B^\dagger A$.*

Lemma 4 ([28], Theorem 2.1)**.** *Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i \in \mathcal{B}(\mathcal{F}, \mathcal{G})$ and $E_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), i = \overline{1,2}$. Assume that $\mathcal{D}(B_1^\dagger) = \mathcal{D}(B_2^\dagger)$ and $\mathcal{R}(E_iB_i^\dagger) \subseteq \mathcal{D}(A_i^\dagger), i = \overline{1,2}$. If $A_1^\dagger E_1 B_1^\dagger = A_2^\dagger E_2 B_2^\dagger$; then the subsequent assertions are equivalent:*

(i) The operator equation system

$$A_1XB_1 = E_1$$

$$A_2XB_2 = E_2$$

is solvable;

(ii) $\mathcal{R}(E_i) \subseteq \mathcal{R}(A_i)$ *and* $\mathcal{R}((A_i^\dagger E_i)^*) \subseteq \mathcal{R}(B_i^*), i = \overline{1,2}$.

Lemma 5 ([28], Theorem 2.2). *Let* $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K}), B_i \in \mathcal{B}(\mathcal{F},\mathcal{G})$, *and* $E_i \in \mathcal{B}(\mathcal{F},\mathcal{K}), i = \overline{1,2}$. *If* $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ *and* $\mathcal{R}(A_2^*) \subseteq \overline{\mathcal{R}(A_1^*)}$, *then the subsequent assertions are equivalent, as follows:*

(i) The system of operator equations

$$A_1XB_1 = E_1$$

$$A_2XB_2 = E_2$$

is solvable;

(ii) $\mathcal{R}(E_1) \subseteq \mathcal{R}(A_1), \mathcal{R}(E_2^*) \subseteq \mathcal{R}(B_2^*)$ *and the system*

$$XB_1 = A_1^\dagger E_1$$

$$A_2X = \overline{E_2B_2^\dagger}$$

is solvable.

Given a closed subspace S of $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, the operator matrix representation of T induced by S is

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \begin{pmatrix} S \\ S^\perp \end{pmatrix} \to \begin{pmatrix} S \\ S^\perp \end{pmatrix} \tag{3}$$

where $T_{11} = P_STP_S|_S \in \mathcal{B}(S)$, $T_{12} = P_ST(I-P_S)|_{S^\perp} \in \mathcal{B}(S^\perp,S)$, $T_{21} = (I-P_S)TP_S|_S \in \mathcal{B}(\mathcal{S},\mathcal{S}^\perp)$, $T_{22} = (I-P_S)T(I-P_S)|_{S^\perp} \in \mathcal{B}(S^\perp)$. The next well-known result characterizes positive operators in terms of their matrix decomposition.

Lemma 6 ([43], Theorem 4.2). *Let* S *be a closed subspace of* $\mathcal{H}$ *and* $T \in \mathcal{B}(\mathcal{H})$ *having an operator matrix representation induced by* S *provided in* (3). *Then* $T \in \mathcal{B}(\mathcal{H})$ *is a positive operator if and only if*

(i) $T_{12} = T_{21}^*$;

(ii) $T_{11} \geq 0$;

(iii) $\mathcal{R}(T_{12}) \subseteq \mathcal{R}(T_{11}^{\frac{1}{2}})$;

(iv) $T_{22} = ((T_{11}^{\frac{1}{2}})^\dagger T_{12})^*(T_{11}^{\frac{1}{2}})^\dagger T_{12} + F$, *where* $F \geq 0$.

3. Existence of Solutions

3.1. The Existence of General Solutions

In the subsequent theorem, we present several equivalent conditions for the existence of the solution to the operator equations $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1,2)$.

For convenience, we define the notation as follows $(i = 1,2)$:

$$\hat{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{K}\oplus\mathcal{K}, \qquad \hat{B} = \begin{pmatrix} B_2 & 0 \\ 0 & B_1 \end{pmatrix} : \mathcal{F}\oplus\mathcal{F} \to \mathcal{G}\oplus\mathcal{G},$$

$$\hat{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{K}\oplus\mathcal{K}, \qquad \hat{D} = \begin{pmatrix} D_2 & 0 \\ 0 & D_1 \end{pmatrix} : \mathcal{F}\oplus\mathcal{F} \to \mathcal{G}\oplus\mathcal{G},$$

$$\hat{E} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{K}\oplus\mathcal{K}, \qquad \hat{F} = \begin{pmatrix} F_2 & 0 \\ 0 & F_1 \end{pmatrix} : \mathcal{F}\oplus\mathcal{F} \to \mathcal{G}\oplus\mathcal{G},$$

$$\hat{G} = \begin{pmatrix} 0 & G_1 \\ G_2 & 0 \end{pmatrix} : \mathcal{F}\oplus\mathcal{F} \to \mathcal{K}\oplus\mathcal{K}, \qquad \hat{X} = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} : \mathcal{G}\oplus\mathcal{G} \to \mathcal{H}\oplus\mathcal{H},$$

$$\hat{Y} = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix} : \mathcal{G}\oplus\mathcal{G} \to \mathcal{H}\oplus\mathcal{H}, \qquad \hat{Z} = \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} : \mathcal{G}\oplus\mathcal{G} \to \mathcal{H}\oplus\mathcal{H},$$

$$A_{S_i} = (I_{\mathcal{K}} - A_i A_i^{\dagger})C_i, \quad B_{S_i} = D_i(I_{\mathcal{F}} - B_i^{\dagger}B_i), \quad M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{C},$$
$$A_{T_i} = (I_{\mathcal{K}} - A_i A_i^{\dagger})G_i, \quad B_{T_i} = G_i(I_{\mathcal{F}} - B_i^{\dagger}B_i), \quad N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{E},$$
$$A_{K_i} = (I_{\mathcal{K}} - A_i A_i^{\dagger})E_i, \quad B_{K_i} = F_i(I_{\mathcal{F}} - B_i^{\dagger}B_i), \quad W_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{G},$$
$$C_{L_i} = I_{\mathcal{K}} - C_i C_i^{\dagger}, \quad D_{L_i} = I_{\mathcal{F}} - D_i^{\dagger}D_i, \quad M_2 = \hat{D}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$
$$C_{S_i} = (I_{\mathcal{K}} - C_i C_i^{\dagger})E_i, \quad D_{S_i} = F_i(I_{\mathcal{F}} - D_i^{\dagger}D_i), \quad N_2 = \hat{F}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$
$$B_{SK_i} = B_{K_i}(I_{\mathcal{F}} - B_{S_i}^{\dagger}B_{S_i}), \quad A_{SK_i} = (I_{\mathcal{K}} - A_{S_i}A_{S_i}^{\dagger})A_{K_i}, \quad W_2 = \hat{G}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$
$$B_{ST_i} = B_{T_i}(I_{\mathcal{F}} - B_{S_i}^{\dagger}B_{S_i}), \quad A_{ST_i} = (I_{\mathcal{K}} - A_{S_i}A_{S_i}^{\dagger})A_{T_i},$$

$$\hat{W} = \begin{pmatrix} 0 & W_1 \\ W_2 & 0 \end{pmatrix} : (\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F}) \to (\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K}),$$
$$P_1 = \begin{pmatrix} M_1 & 0 \\ 0 & \hat{C} \end{pmatrix} : (\mathcal{H}\oplus\mathcal{H})\oplus(\mathcal{H}\oplus\mathcal{H}) \to (\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K}),$$
$$P_2 = \begin{pmatrix} N_1 & 0 \\ 0 & \hat{E} \end{pmatrix} : (\mathcal{H}\oplus\mathcal{H})\oplus(\mathcal{H}\oplus\mathcal{H}) \to (\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K}),$$
$$Q_1 = \begin{pmatrix} M_2 & 0 \\ 0 & \hat{D} \end{pmatrix} : (\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F}) \to (\mathcal{G}\oplus\mathcal{G})\oplus(\mathcal{G}\oplus\mathcal{G}),$$
$$Q_2 = \begin{pmatrix} N_2 & 0 \\ 0 & \hat{F} \end{pmatrix} : (\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F}) \to (\mathcal{G}\oplus\mathcal{G})\oplus(\mathcal{G}\oplus\mathcal{G}),$$
$$\Upsilon' = \begin{pmatrix} 0 & \hat{Y} \\ \hat{Y} & 0 \end{pmatrix} : (\mathcal{G}\oplus\mathcal{G})\oplus(\mathcal{G}\oplus\mathcal{G}) \to (\mathcal{H}\oplus\mathcal{H})\oplus(\mathcal{H}\oplus\mathcal{H}),$$
$$Z' = \begin{pmatrix} 0 & \hat{Z} \\ \hat{Z} & 0 \end{pmatrix} : (\mathcal{G}\oplus\mathcal{G})\oplus(\mathcal{G}\oplus\mathcal{G}) \to (\mathcal{H}\oplus\mathcal{H})\oplus(\mathcal{H}\oplus\mathcal{H}),$$
$$P_{11} = (I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_1P_1^{\dagger})P_2, \qquad W_{11} = (I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_1P_1^{\dagger})\hat{W},$$
$$P_{22} = Q_2(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_1^{\dagger}Q_1), \qquad W_{22} = \hat{W}(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_1^{\dagger}Q_1).$$

Theorem 1. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H},\mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F},\mathcal{G}), (i = 1,2)$ be closed-range operators, and $G_i \in \mathcal{B}(\mathcal{F},\mathcal{K}), (i = 1,2)$. Suppose that*

$$\mathcal{D}(B_{K_i}^{\dagger}) = \mathcal{D}(B_{SK_i}^{\dagger}), \quad \mathcal{D}(F_i^{\dagger}) = \mathcal{D}(D_{S_i}^{\dagger}),$$

$$\mathcal{R}(A_{ST_i}F_i^{\dagger}) \subseteq \mathcal{D}(A_{SK_i}^{\dagger}), \quad \mathcal{R}(C_{L_i}B_{T_i}B_{K_i}^{\dagger}) \subseteq \mathcal{D}(C_{S_i}^{\dagger}), (i = 1,2).$$

If $\mathcal{R}(B_{K_i}) \subseteq \mathcal{R}(B_{SK_i}), \mathcal{R}(F_i) \subseteq \mathcal{R}(D_{S_i}), \mathcal{R}((A_{K_i})^*) \subseteq \overline{\mathcal{R}((A_{SK_i})^*)}, \mathcal{R}((E_i)^*) \subseteq \overline{\mathcal{R}((C_{S_i})^*)}$ and

$$A_{SK_i}^{\dagger}A_{ST_i}F_i^{\dagger} = A_{K_i}^{\dagger}A_{T_i}D_{L_i}D_{S_i}^{\dagger},$$
$$C_{S_i}^{\dagger}C_{L_i}B_{T_i}B_{K_i}^{\dagger} = E_i^{\dagger}B_{ST_i}B_{SK_i}^{\dagger}, (i = 1,2);$$

then the subsequent assertions are equivalent, as follows:

(i) The operator equation system

$$A_1XB_1 + C_1YD_1 + E_1ZF_1 = G_1$$
$$A_2XB_2 + C_2YD_2 + E_2ZF_2 = G_2$$

has a solution.

(ii)

$$\begin{aligned} \mathcal{R}(B_{ST_i}) &\subseteq \mathcal{R}(E_i), & \mathcal{R}(C_{L_i}B_{T_i}) &\subseteq \mathcal{R}(C_{S_i}),\\ \mathcal{R}(A_{ST_i}) &\subseteq \mathcal{R}(A_{SK_i}), & \mathcal{R}(A_{T_i}D_{L_i}) &\subseteq \mathcal{R}(A_{K_i}),\\ \mathcal{R}((A_{SK_i}^{\dagger}A_{ST_i})^*) &\subseteq \mathcal{R}((F_i)^*), & \mathcal{R}((C_{S_i}^{\dagger}C_{L_i}B_{T_i})^*) &\subseteq \mathcal{R}((B_{K_i})^*),\\ \mathcal{R}((E_i^{\dagger}B_{ST_i})^*) &\subseteq \mathcal{R}((B_{SK_i})^*), & \mathcal{R}((A_{K_i}^{\dagger}A_{T_i}D_{L_i})^*) &\subseteq \mathcal{R}((D_{S_i})^*), \end{aligned}$$

where i = 1,2.

(iii) $\mathcal{R}(A_{ST_i}) \subseteq \mathcal{R}(A_{SK_i}), \mathcal{R}(C_{L_i}B_{T_i}) \subseteq \mathcal{R}(C_{S_i}), \mathcal{R}((B_{ST_i})^*) \subseteq \mathcal{R}((B_{SK_i})^*), \mathcal{R}((A_{T_i}D_{L_i})^*) \subseteq \mathcal{R}((D_{S_i})^*)$, *and the operator equation system*

$$\begin{aligned} ZF_i &= A_{SK_i}^{\dagger}A_{ST_i},\\ ZB_{K_i} &= C_{S_i}^{\dagger}C_{L_i}B_{T_i},\\ A_{K_i}Z &= \overline{A_{T_i}D_{L_i}D_{S_i}^{\dagger}},\\ E_iZ &= \overline{B_{ST_i}B_{SK_i}^{\dagger}}, \end{aligned}$$

$(i = 1, 2)$ *has a solution.*

Proof. Now, we are going to prove it in four parts.

Claim 1 The operator equation system (1) has a solution if and only if the operator equation system

$$\begin{aligned} M_1\hat{Y}\hat{D} + N_1\hat{Z}\hat{F} &= W_1\\ \hat{C}\hat{Y}M_2 + \hat{E}\hat{Z}N_2 &= W_2 \end{aligned} \tag{4}$$

has a solution.

Indeed, Equation (1) is equivalent to

$$\hat{A}\hat{X}\hat{B} + \hat{C}\hat{Y}\hat{D} + \hat{E}\hat{Z}\hat{F} = \hat{G}. \tag{5}$$

According to Lemma 2, Equation (5) has a solution equivalent to

$$\begin{aligned} \mathcal{R}(\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F}) &\subseteq \mathcal{R}(\hat{A}),\\ \mathcal{R}((\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F})^*) &\subseteq \mathcal{R}(\hat{B}^*). \end{aligned}$$

Then, $\mathcal{R}(\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F}) \subseteq \mathcal{R}(\hat{A})$ if and only if

$$\hat{A}\hat{A}^{\dagger}(\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F}) = \hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F}.$$

It can be obtained by a simple calculation; we have

$$(I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{G} = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{C}\hat{Y}\hat{D} + (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{E}\hat{Z}\hat{F}.$$

That is,

$$M_1\hat{Y}\hat{D} + N_1\hat{Z}\hat{F} = W_1.$$

In a similar way, $\mathcal{R}((\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F})^*) \subseteq \mathcal{R}(\hat{B}^*)$ if and only if

$$\hat{B}^*(\hat{B}^*)^{\dagger}((\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F})^*) = (\hat{G} - \hat{C}\hat{Y}\hat{D} - \hat{E}\hat{Z}\hat{F})^*.$$

Then we have

$$\hat{G}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}) = \hat{C}\hat{Y}\hat{D}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}) + \hat{E}\hat{Z}\hat{F}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}).$$

Namely,

$$\hat{C}\hat{Y}M_2 + \hat{E}\hat{Z}N_2 = W_2.$$

Therefore, the existence of a solution for the system (1) is equivalent to the existence of a solution for the system (4).

Claim 2 The system (4) has a solution if and only if

$$\begin{aligned} P_{11}Z'Q_2 &= W_{11} \\ P_2Z'P_{22} &= W_{22} \end{aligned} \tag{6}$$

has a solution.

In fact, the system (4) is equivalent to

$$P_1Y'Q_1 + P_2Z'Q_2 = \hat{W}. \tag{7}$$

Using Lemma 2 again, Equation (7) has a solution equivalent to

$$\begin{aligned} \mathcal{R}(\hat{W} - P_2Z'Q_2) &\subseteq \mathcal{R}(P_1), \\ R((\hat{W} - P_2Z'Q_2)^*) &\subseteq \mathcal{R}(Q_1^*). \end{aligned}$$

Then, $\mathcal{R}(\hat{W} - P_2Z'Q_2) \subseteq \mathcal{R}(P_1)$ if and only if

$$P_1P_1^{\dagger}(\hat{W} - P_2Z'Q_2) = (\hat{W} - P_2Z'Q_2).$$

By a straightforward computation, we have

$$(I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_1P_1^{\dagger})\hat{W} = (I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_1P_1^{\dagger})P_2Z'Q_2.$$

That is,

$$P_{11}Z'Q_2 = W_{11}.$$

In a similar way, $\mathcal{R}((\hat{W} - P_2Z'Q_2)^*) \subseteq \mathcal{R}(Q_1^*)$ if and only if

$$Q_1^*(Q_1^*)^{\dagger}((\hat{W} - P_2Z'Q_2)^*) = (\hat{W} - P_2Z'Q_2)^*.$$

After a further calculation, we obtain

$$\hat{W}(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_1^{\dagger}Q_1) = P_2Z'Q_2(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_1^{\dagger}Q_1).$$

Namely,

$$P_2Z'P_{22} = W_{22}.$$

Therefore, Equation (4) has a solution that is equivalent to the system (6) having a solution.

Claim 3 The system (6) has a solution equivalent to

$$\begin{aligned} \mathcal{R}(W_{11}) &\subseteq \mathcal{R}(P_{11}), \quad & \mathcal{R}(W_{22}) &\subseteq \mathcal{R}(P_2), \\ \mathcal{R}((P_{11}^{-}W_{11})^*) &\subseteq \mathcal{R}(Q_2^*), \quad & \mathcal{R}((P_2^{-}W_{22})^*) &\subseteq \mathcal{R}(P_{22}^*), \end{aligned}$$

if and only if

$$\begin{aligned} \mathcal{R}(B_{ST_i}) &\subseteq \mathcal{R}(E_i), & \mathcal{R}(C_{L_i}B_{T_i}) &\subseteq \mathcal{R}(C_{S_i}), \\ \mathcal{R}(A_{ST_i}) &\subseteq \mathcal{R}(A_{SK_i}), & \mathcal{R}(A_{T_i}D_{L_i}) &\subseteq \mathcal{R}(A_{K_i}), \\ \mathcal{R}((A_{SK_i}^{\dagger}A_{ST_i})^*) &\subseteq \mathcal{R}((F_i)^*), & \mathcal{R}((C_{S_i}^{\dagger}C_{L_i}B_{T_i})^*) &\subseteq \mathcal{R}((B_{K_i})^*), \\ \mathcal{R}((E_i^{\dagger}B_{ST_i})^*) &\subseteq \mathcal{R}((B_{SK_i})^*), & \mathcal{R}((A_{K_i}^{\dagger}A_{T_i}D_{L_i})^*) &\subseteq \mathcal{R}((D_{S_i})^*), \end{aligned}$$

where i = 1, 2. The equivalence of Claim 3 is obtained by Lemma 4.

Claim 4 The system (6) has a solution equivalent to

$$\mathcal{R}(W_{11}) \subseteq \mathcal{R}(P_{11}), \qquad \mathcal{R}((W_{22})^*) \subseteq \mathcal{R}((P_{22})^*)$$

and the operator equation system

$$Z'Q_2 = P_{11}^{\dagger}W_{11},$$
$$P_2Z' = \overline{W_{22}P_{22}^{\dagger}}$$

has a solution.

Actually, the system (6) has a solution if and only if the following conditions hold:

(a) $\mathcal{R}(A_{ST_i}) \subseteq \mathcal{R}(A_{SK_i})$, $\mathcal{R}(C_{L_i}B_{T_i}) \subseteq \mathcal{R}(C_{S_i})$, $\mathcal{R}((B_{ST_i})^*) \subseteq \mathcal{R}((B_{SK_i})^*)$, $\mathcal{R}((A_{T_i}D_{L_i})^*) \subseteq \mathcal{R}((D_{S_i})^*), (i = 1, 2)$.

(b) The system

$$\begin{aligned} ZF_i &= A_{SK_i}^{\dagger}A_{ST_i}, \\ ZB_{K_i} &= C_{S_i}^{\dagger}C_{L_i}B_{T_i}, \\ A_{K_i}Z &= \overline{A_{T_i}D_{L_i}D_{S_i}^{\dagger}}, \\ E_iZ &= \overline{B_{ST_i}B_{SK_i}^{\dagger}}, \end{aligned} \tag{8}$$

$(i = 1, 2)$ has a solution.

The equivalence of Claim 4 is obtained by Lemma 5.

Therefore, the equivalence of (i) and (ii) in Theorem 1 is obtained from Claim 1, Claim 2, and Claim 3; the equivalence of (i) and (iii) in Theorem 1 is obtained from Claim 1, Claim 2, and Claim 4. □

In the following corollary, we give some equivalent conditions for the existence of the solutions to the system $A_iXB_i + C_iYD_i = G_i (i = 1, 2)$. For convenience, we still use the notation defined earlier in Theorem 1. In particular, M_1, M_2, N_1, and N_2 are of the following form:

$$M_1 = (I_{\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{C}, \qquad N_1 = (I_{\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{E},$$
$$M_2 = \hat{D}(I_{\mathcal{F}} - \hat{B}^{\dagger}\hat{B}), \qquad N_2 = \hat{F}(I_{\mathcal{F}} - \hat{B}^{\dagger}\hat{B}).$$

Corollary 1. *Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i \in \mathcal{B}(\mathcal{F}, \mathcal{G})$ be closed-range operators and $C_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $D_i \in \mathcal{B}(\mathcal{F}, \mathcal{G})$,$G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), S_i = I_{\mathcal{K}} - A_iA_i^{\dagger}, T_i = I_{\mathcal{F}} - B_i^{\dagger}B_i, (i = 1, 2)$. Suppose that*

$$\mathcal{D}(D_i^{\dagger}) = \mathcal{D}((D_iT_i)^{\dagger}),$$
$$\mathcal{R}(S_iG_iD_i^{\dagger}) \subseteq \mathcal{D}((S_iC_i)^{\dagger})$$

and

$$\mathcal{R}(G_iT_i(D_iT_i)^{\dagger}) \subseteq \mathcal{D}(C_i^{\dagger}), (i = 1, 2).$$

If $\mathcal{R}(D_iT_i) \subseteq \mathcal{R}(D_i), \mathcal{R}(C_i^*S_i) \subseteq \overline{\mathcal{R}(C_i^*)}$ and

$$(S_iC_i)^{\dagger}S_iG_iD_i^{\dagger} = C_i^{\dagger}G_iT_i(D_iT_i)^{\dagger}, (i = 1, 2),$$

then the subsequent assertions are equivalent as follows:

(i) The operator equation system

$$A_1XB_1 + C_1YD_1 = G_1$$
$$A_2XB_2 + C_2YD_2 = G_2$$

has a solution.

(ii)

$$\begin{aligned}
&\mathcal{R}(G_iT_i) \subseteq \mathcal{R}(C_i),\\
&\mathcal{R}((C_1^*(S_1)^\dagger) \subseteq \mathcal{R}(D_2^*),\\
&\mathcal{R}((C_2^*(S_2)^\dagger) \subseteq \mathcal{R}(D_1^*),\\
&\mathcal{R}(S_iG_i) \subseteq \mathcal{R}(S_iC_i),\\
&\mathcal{R}(T_iG_i^*(C_i^\dagger)^*) \subseteq \mathcal{R}(T_iD_i^*),
\end{aligned}$$

where $i = 1, 2$.

(iii) $\mathcal{R}(G_i^*S_i) \subseteq \mathcal{R}(D_i^*), \mathcal{R}(G_iT_i) \subseteq \mathcal{R}(C_i)$, *and the system*

$$S_i^*C_iYP_{\overline{\mathcal{R}(D_i)}} = \overline{S_i^*G_iD_i^\dagger},$$

$$P_{\overline{\mathcal{R}(C_i^*)}}YD_iT_i = C_i^\dagger G_iT_i$$

$(i = 1, 2)$ *has a solution.*

Proof. Now, we are going to prove it in three parts.

Claim 1 The system (2) has a solution if and only if

$$\begin{aligned}
M_1\hat{Y}\hat{D} &= N_1\\
\hat{C}\hat{Y}M_2 &= N_2
\end{aligned} \tag{9}$$

has a solution.

Indeed, the system (2) is equivalent to

$$\hat{A}\hat{X}\hat{B} + \hat{C}\hat{Y}\hat{D} = \hat{G}. \tag{10}$$

In view of Lemma 2, Equation (10) has a solution equivalent to

$$\begin{aligned}
&\mathcal{R}(\hat{G} - \hat{C}\hat{Y}\hat{D}) \subseteq \mathcal{R}(\hat{A}),\\
&\mathcal{R}((\hat{G} - \hat{C}\hat{Y}\hat{D})^*) \subseteq \mathcal{R}(\hat{B}^*).
\end{aligned}$$

Then, $\mathcal{R}(\hat{G} - \hat{C}\hat{Y}\hat{D}) \subseteq \mathcal{R}(\hat{A})$ if and only if

$$\hat{A}\hat{A}^\dagger(\hat{G} - \hat{C}\hat{Y}\hat{D}) = \hat{G} - \hat{C}\hat{Y}\hat{D}.$$

It can be obtained by a simple calculation; we have

$$(I_\mathcal{K} - \hat{A}\hat{A}^\dagger)\hat{G} = (I_\mathcal{K} - \hat{A}\hat{A}^\dagger)\hat{C}\hat{Y}\hat{D}.$$

That is

$$M_1\hat{Y}\hat{D} = N_1.$$

In a similar way, $\mathcal{R}((\hat{G} - \hat{C}\hat{Y}\hat{D})^*) \subseteq \mathcal{R}(\hat{B}^*)$ if and only if

$$\hat{B}^*(\hat{B}^*)^\dagger((\hat{G} - \hat{C}\hat{Y}\hat{D})^*) = (\hat{G} - \hat{C}\hat{Y}\hat{D})^*.$$

Therefore, $\hat{G}(I_\mathcal{F} - \hat{B}^\dagger\hat{B}) = \hat{C}\hat{Y}\hat{D}(I_\mathcal{F} - \hat{B}^\dagger\hat{B})$. Namely,

$$\hat{C}\hat{Y}M_2 = N_2.$$

Therefore, the system (2) has a solution that is equivalent to the system (9) having a solution.

Claim 2 The system (9) has a solution equivalent to

$$\begin{aligned}\mathcal{R}(N_1) &\subseteq \mathcal{R}(M_1),\\ \mathcal{R}(N_2) &\subseteq \mathcal{R}(\hat{C}),\\ \mathcal{R}((M_1^{\dagger}N_1)^*) &\subseteq \mathcal{R}(\hat{D}^*),\\ \mathcal{R}((\hat{C}^{\dagger}N_2)^*) &\subseteq \mathcal{R}(M_2^*),\end{aligned}$$

if and only if

$$\begin{aligned}\mathcal{R}(S_iE_i) &\subseteq \mathcal{R}(S_iC_i),\\ \mathcal{R}(E_iT_i) &\subseteq \mathcal{R}(C_i),\\ \mathcal{R}((C_1^*(S_1)^{\dagger}) &\subseteq \mathcal{R}(D_2^*),\\ \mathcal{R}((C_2^*(S_2)^{\dagger}) &\subseteq \mathcal{R}(D_1^*),\\ \mathcal{R}(T_iE_i^*(C_i^{\dagger})^*) &\subseteq \mathcal{R}(T_iD_i^*),\end{aligned}$$

where i = 1, 2.

The equivalence of Claim 2 is obtained by Lemma 4.

Claim 3 The system (9) has a solution equivalent to the system $M_1Y = \overline{N_1\hat{D}^{\dagger}}$, $YM_2 = \hat{C}^{\dagger}N_2$, where $Y = \overline{\hat{C}^{\dagger}\hat{C}\hat{Y}\hat{D}\hat{D}^{\dagger}}$ and $R(N_1^*) \subseteq R(\hat{D}^*)$, $R(N_2) \subseteq R(\hat{C})$.

In fact, the system (9) has a solution if and only if the following conditions hold:

(a) $\mathcal{R}(E_i^*S_i) \subseteq \mathcal{R}(D_i^*)$, $\mathcal{R}(E_iT_i) \subseteq \mathcal{R}(C_i)$, $(i = 1, 2)$.

(b) The system

$$\begin{aligned}S_i^*C_iYP_{\overline{\mathcal{R}(D_i)}} &= \overline{S_i^*E_iD_i^{\dagger}},\\ P_{\overline{\mathcal{R}(C_i^*)}}YD_iT_i &= C_i^{\dagger}E_iT_i\end{aligned}$$

(i = 1, 2) has a solution.

The equivalence of Claim 3 is obtained by Lemma 5.

Therefore, the equivalence of (i) and (ii) in Corollary 1 is obtained from Claim 1 and Claim 2; the equivalence of (ii) and (iii) in Corollary 1 is obtained from Claim 1 and Claim 3. □

In Theorem 1, take $A_1 = B_2 = A, B_1 = A_2 = B, C_i = D_i = E_i = F_i = 0, G_i = G$, $(i = 1, 2)$. Then the following results are obtained:

Corollary 2 ([32], Proposition 2.1)**.** *Let $A, B, G \in \mathcal{B}(\mathcal{H})$. If A and B have a closed range, $A^{\dagger}GB^{\dagger} = B^{\dagger}GA^{\dagger}$, then the subsequent assertions are equivalent, as follows:*

(1) The system $AXB = G = BXA$ is solvable;

(2) $AA^{\dagger}GA^{\dagger}A = BB^{\dagger}GB^{\dagger}B = G$;

(3) $\mathcal{R}(G) \subseteq \mathcal{R}(A), \mathcal{R}(G) \subseteq \mathcal{R}(B), \mathcal{R}(G^*) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(G^*) \subseteq \mathcal{R}(B^*)$.

Remark 1. *In Theorem 1, the system (6) can be expressed as follows:*

$$\begin{aligned}A_{SK_i}ZF_i &= A_{ST_i},\\ C_{S_i}ZB_{K_i} &= C_{L_i}B_{T_i},\\ A_{K_i}ZD_{S_i} &= A_{T_i}D_{L_i},\\ E_iZB_{SK_i} &= B_{ST_i},\end{aligned} \tag{11}$$

where i = 1,2. Then the system (11) has a solution if and only if there exist some $U_1, V_1 \in \mathcal{B}(\mathcal{F}, \mathcal{K})$ and $U_2, V_2 \in \mathcal{B}(\mathcal{F}, \mathcal{K})$ such that the system

$$\begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix} Z \begin{pmatrix} F_i & B_{K_i} \end{pmatrix} = \begin{pmatrix} A_{ST_i} & U_1 \\ V_1 & C_{L_i}B_{T_i} \end{pmatrix},$$
$$\begin{pmatrix} A_{K_i} \\ E_i \end{pmatrix} Z \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix} = \begin{pmatrix} A_{T_i}D_{L_i} & U_2 \\ V_2 & B_{ST_i} \end{pmatrix} \tag{12}$$

$(i = 1, 2)$ *has a solution. Let*

$$A'_{1i} = \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}, \quad B'_{1i} = \begin{pmatrix} F_i & B_{K_i} \end{pmatrix}, \quad C'_{1i} = \begin{pmatrix} A_{ST_i} & U_1 \\ V_1 & C_{L_i}B_{T_i} \end{pmatrix},$$
$$A'_{2i} = \begin{pmatrix} A_{K_i} \\ E_i \end{pmatrix}, \quad B'_{2i} = \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix}, \quad C'_{2i} = \begin{pmatrix} A_{T_i}D_{L_i} & U_2 \\ V_2 & B_{ST_i} \end{pmatrix}.$$

Hence, Equation (12) can be expressed as

$$A'_{1i}ZB'_{1i} = C'_{1i},$$
$$A'_{2i}ZB'_{2i} = C'_{2i},$$

where i = 1, 2.

Remark 2. *Under the assumptions of Theorem 1, it can be further concluded that a solution of*

$$ZB'_{1i} = A'^{\dagger}_{1i}C'_{1i},$$
$$A'_{2i}Z = \overline{C'_{2i}B'^{\dagger}_{2i}}, \tag{13}$$

$(i = 1, 2)$ *is also a solution of the system (12). Additionally, if* $Z \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ *satisfies the system (12), we obtain the bounded linear extension*

$$Z_0 = \overline{P_{\mathcal{R}\left((A'_{1i})^*\right)} Z P_{\mathcal{R}(B'_{2i})}}, \quad i = 1, 2$$

which are solutions of the system (13).

In fact, suppose that Z' *satisfies the system (12). Then, clearly,*

$$\mathcal{R}\begin{pmatrix} A_{ST_i} & U_1 \\ V_1 & C_{L_i}B_{T_i} \end{pmatrix} \subseteq \mathcal{R}\begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix},$$

$$\mathcal{R}\left(\begin{pmatrix} A_{T_i}D_{L_i} & U_2 \\ V_2 & B_{ST_i} \end{pmatrix}^*\right) \subseteq \mathcal{R}\left(\begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix}^*\right),$$

i.e.,

$$\mathcal{R}(A_{ST_i}) \subseteq \mathcal{R}(A_{SK_i}),$$
$$\mathcal{R}(C_{L_i}B_{T_i}) \subseteq \mathcal{R}(C_{S_i}),$$
$$\mathcal{R}((B_{ST_i})^*) \subseteq \mathcal{R}((B_{SK_i})^*),$$
$$\mathcal{R}((A_{T_i}D_{L_i})^*) \subseteq \mathcal{R}((D_{S_i})^*).$$

Additionally, from

$$\mathcal{R}(B_{K_i}) \subseteq \mathcal{R}(B_{SK_i}),$$
$$\mathcal{R}(F_i) \subseteq \mathcal{R}(D_{S_i}),$$
$$\mathcal{R}((A_{K_i})^*) \subseteq \overline{\mathcal{R}((A_{SK_i})^*)},$$
$$\mathcal{R}((E_i)^*) \subseteq \overline{\mathcal{R}((C_{S_i})^*)},$$

we obtain that

$$\begin{pmatrix} F_i & B_{K_i} \end{pmatrix} = \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix} \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix}^{\dagger} \begin{pmatrix} F_i & B_{K_i} \end{pmatrix},$$
$$\begin{pmatrix} A_{K_i} \\ E_i \end{pmatrix} = \begin{pmatrix} A_{K_i} \\ E_i \end{pmatrix} \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}^{\dagger} \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}, (i = 1, 2),$$

i.e.,

$$B'_{1i} = B'_{2i} B'^{\dagger}_{2i} B'_{1i},$$
$$A'_{2i} = A'_{2i} A'^{\dagger}_{1i} A'_{1i}.$$

Let

$$\widetilde{Z} := \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}^{\dagger} \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix} Z' \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix} \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix}^{\dagger}$$
$$= A'^{\dagger}_{1i} A'_{1i} Z' B'_{2i} B'^{\dagger}_{2i} \in \mathcal{B}(\mathcal{D}(B'^{\dagger}_{2i}), \mathcal{H}).$$

Then

$$\widetilde{Z} B'_{1i} = A'^{\dagger}_{1i} A'_{1i} Z' B'_{2i} B'^{\dagger}_{2i} B'_{1i} = A'^{\dagger}_{1i} A'_{1i} Z' B'_{1i} = A'^{\dagger}_{1i} C'_{1i},$$
$$A'_{2i} \widetilde{Z} = A'_{2i} A'^{\dagger}_{1i} A'_{1i} Z' B'_{2i} B'^{\dagger}_{2i} = A'_{2i} Z' B'_{2i} B'^{\dagger}_{2i} = C'_{2i} B'^{\dagger}_{2i}.$$

We denote the continuous bounded linear extension of $\widetilde{Z}$ *as* $Z_0 \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. *By* $\mathcal{R}(B'_{1i}) \subseteq \mathcal{R}(B'_{2i}) \subseteq \mathcal{D}(B'_{2i}) = \mathcal{D}(\widetilde{Z})$, *we obtain* $Z_0 B'_{1i} = \widetilde{Z} B'_{1i} = A'^{\dagger}_{1i} C'_{1i}$. *From* $\mathcal{R}(C'^{*}_{2i}) \subseteq \mathcal{R}(B'^{*}_{2i})$, *combined with Douglas' Lemma 3, there exists* $\mathring{Z} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ *such that*

$$C'_{2i} = \mathring{Z} B'_{2i}, (i = 1, 2).$$

It follows that

$$C'_{2i} B'^{\dagger}_{2i} = \mathring{Z} B'_{2i} B'^{\dagger}_{2i} \in \mathcal{B}(\mathcal{D}(B'^{\dagger}_{2i}), \mathcal{H}).$$

For any $z \in \mathcal{G}, z_n \in \mathcal{D}(B'^{\dagger}_{2i})$ *such that* $z_n \to z \ \ (n \to \infty)$. *We have*

$$A'_{2i} Z_0 z = A'_{2i} (\lim_{n\to\infty} \widetilde{Z} z_n) = \lim_{n\to\infty} A'_{2i} \widetilde{Z} z_n$$
$$= \lim_{n\to\infty} C'_{2i} B'^{\dagger}_{2i} z_n$$
$$= \overline{C'_{2i} B'^{\dagger}_{2i}} z.$$

Then Z_0 *is a solution of the system (13).*

On the other hand, in view of

$$\mathcal{R}(A_{ST_i}) \subseteq \mathcal{R}(A_{SK_i}),$$
$$\mathcal{R}(C_{L_i} B_{T_i}) \subseteq \mathcal{R}(C_{S_i}),$$
$$\mathcal{R}((B_{ST_i})^*) \subseteq \mathcal{R}((B_{SK_i})^*),$$
$$\mathcal{R}((A_{T_i} D_{L_i})^*) \subseteq \mathcal{R}((D_{S_i})^*),$$

we obtain

$$C'_{1i} = A'_{1i} A'^{\dagger}_{1i} C'_{1i},$$
$$C'_{2i} = C'_{2i} B'^{\dagger}_{2i} B'_{2i}, (i = 1, 2).$$

Now suppose that $\widetilde{Z} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ *satisfies the system (13). Due to*

$$\mathcal{R}(B'_{2i}) \subseteq \mathcal{D}(B'^{\dagger}_{2i}) = \mathcal{D}(C'_{2i} B'^{\dagger}_{2i}),$$

we have that

$$A'_{1i} \widetilde{Z} B'_{1i} = A'_{1i} A'^{\dagger}_{1i} C'_{1i} = C'_{1i},$$
$$A'_{2i} \widetilde{Z} B'_{2i} = \overline{C'_{2i} B'^{\dagger}_{2i}} B'_{2i} = C'_{2i} B'^{\dagger}_{2i} B'_{2i} = C'_{2i},$$

where i = 1, 2. Therefore, the system (12) has a solution.

3.2. The Existence of Self-Adjoint Solutions

In this section, we present several equivalent conditions for the existence of self-adjoint solutions to the system $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$ in sense of star order.

For convenience, we define the notation as follows (i = 1, 2):

$$\begin{aligned}
&A_{S_i} = (I_{\mathcal{K}} - A_iA_i^{\dagger})C_i, && A_{T_i} = (I_{\mathcal{K}} - A_iA_i^{\dagger})G_i, && A_{K_i} = (I_{\mathcal{K}} - A_iA_i^{\dagger})E_i,\\
&B_{S_i} = D_i(I_{\mathcal{F}} - B_i^{\dagger}B_i), && B_{T_i} = G_i(I_{\mathcal{F}} - B_i^{\dagger}B_i), && B_{K_i} = F_i(I_{\mathcal{F}} - B_i^{\dagger}B_i),\\
&C_{L_i} = I_{\mathcal{K}} - C_iC_i^{\dagger}, && C_{S_i} = (I_{\mathcal{K}} - C_iC_i^{\dagger})E_i, && D_{L_i} = I_{\mathcal{F}} - D_i^{\dagger}D_i,\\
&D_{S_i} = F_i(I_{\mathcal{F}} - D_i^{\dagger}D_i), && B_{SK_i} = B_{K_i}(I_{\mathcal{F}} - B_{S_i}^{\dagger}B_{S_i}), && A_{SK_i} = (I_{\mathcal{K}} - A_{S_i}A_{S_i}^{\dagger})A_{K_i},\\
&B_{ST_i} = B_{T_i}(I_{\mathcal{F}} - B_{S_i}^{\dagger}B_{S_i}), && A_{ST_i} = (I_{\mathcal{K}} - A_{S_i}A_{S_i}^{\dagger})A_{T_i},
\end{aligned}$$

$$A'_{1i} = \begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}, \quad B'_{1i} = \begin{pmatrix} F_i & B_{K_i} \end{pmatrix}, \quad C'_{1i} = \begin{pmatrix} A_{ST_i} & U_1 \\ V_1 & C_{L_i}B_{T_i} \end{pmatrix},$$

$$A'_{2i} = \begin{pmatrix} A_{K_i} \\ E_i \end{pmatrix}, \quad B'_{2i} = \begin{pmatrix} D_{S_i} & B_{SK_i} \end{pmatrix}, \quad C'_{2i} = \begin{pmatrix} A_{T_i}D_{L_i} & U_2 \\ V_2 & B_{ST_i} \end{pmatrix},$$

for some $U_1, V_1, U_2, V_2 \in \mathcal{B}(\mathcal{F}, \mathcal{K})$.

$$\hat{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \quad \hat{B} = \begin{pmatrix} B_2 & 0 \\ 0 & B_1 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G} \oplus \mathcal{G},$$

$$\hat{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \quad \hat{D} = \begin{pmatrix} D_2 & 0 \\ 0 & D_1 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G} \oplus \mathcal{G},$$

$$\hat{E} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \quad \hat{F} = \begin{pmatrix} F_2 & 0 \\ 0 & F_1 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G} \oplus \mathcal{G},$$

$$\hat{G} = \begin{pmatrix} 0 & G_1 \\ G_2 & 0 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K}, \quad N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{E}, \quad N_2 = \hat{F}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$

$$P_2 = \begin{pmatrix} N_1 & 0 \\ 0 & \hat{E} \end{pmatrix} : (\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$Q_2 = \begin{pmatrix} N_2 & 0 \\ 0 & \hat{F} \end{pmatrix} : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{G} \oplus \mathcal{G}) \oplus (\mathcal{G} \oplus \mathcal{G}),$$

$$W_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{G}, \quad W_2 = \hat{G}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$

$$\hat{W} = \begin{pmatrix} 0 & W_1 \\ W_2 & 0 \end{pmatrix} : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$P'_{11} = (I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_2P_2^{-})P_1, \quad P'_{22} = Q_2(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_2^{-}Q_2),$$

$$W'_{11} = (I_{(\mathcal{K}\oplus\mathcal{K})\oplus(\mathcal{K}\oplus\mathcal{K})} - P_2P_2^{\dagger})\hat{W}, \quad W'_{22} = \hat{W}(I_{(\mathcal{F}\oplus\mathcal{F})\oplus(\mathcal{F}\oplus\mathcal{F})} - Q_2^{\dagger}Q_2),$$

$$M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - \hat{A}\hat{A}^{\dagger})\hat{C}, \quad M_2 = \hat{D}(I_{\mathcal{F}\oplus\mathcal{F}} - \hat{B}^{\dagger}\hat{B}),$$

$$P_1 = \begin{pmatrix} M_1 & 0 \\ 0 & \hat{C} \end{pmatrix} : (\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$Q_1 = \begin{pmatrix} M_2 & 0 \\ 0 & \hat{D} \end{pmatrix} : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{G} \oplus \mathcal{G}) \oplus (\mathcal{G} \oplus \mathcal{G}).$$

Theorem 2. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), (i = 1, 2)$ be closed-range operators, and $G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$. Suppose that*

$$C'_{1i} \overset{*}{\leq} A'_{2i} \overset{*}{\leq} A'_{1i},$$
$$C'_{2i} \overset{*}{\leq} B'_{1i} \overset{*}{\leq} B'_{2i},$$
$$W'_{11} \overset{*}{\leq} P_2 \overset{*}{\leq} P'_{11},$$
$$W'_{22} \overset{*}{\leq} Q_1 \overset{*}{\leq} P'_{22},$$

and

$$A'^{\dagger}_{1i}A'_{1i}|_{\mathcal{D}(B'^{\dagger}_{2i})} = B'_{2i}B'^{\dagger}_{2i}, (i = 1, 2),$$
$$P'^{\dagger}_{11}P'_{11}|_{\mathcal{D}(P'^{\dagger}_{22})} = P'_{22}P'^{\dagger}_{22}.$$

If $X = A_i^{\dagger}(G_i - C_iYD_i - E_iZF_i)B_i^{\dagger}$ *is a self-adjoint operator, and the system of operator equations* $A_iXB_i + C_iYD_i + E_iZF_i = G_i, (i = 1, 2)$ *has a solution, then the subsequent assertions are equivalent, as follows:*

(i) The system

$$A_1XB_1 + C_1YD_1 + E_1ZF_1 = G_1$$
$$A_2XB_2 + C_2YD_2 + E_2ZF_2 = G_2$$

has a self-adjoint solution.

(ii) The system

$$ZB'_{1i} = A'^{\dagger}_{1i}C'_{1i},$$
$$A'_{2i}Z = \overline{C'_{2i}B'^{\dagger}_{2i}},$$

(i = 1,2) has a self-adjoint solution.

Proof. $(i) \Rightarrow (ii)$ Suppose that (i) holds. Analogous to the demonstration of Theorem 1, the system (1) has a solution equivalent to the system (6) having a solution. Additionally, according to Remark 1, the system (6) is equivalent to the system (12) having a solution.

Suppose that, in the system (12), there exists a self-adjoint solution Z. Then, from Remark 2,

$$Z_0 = \overline{P_{\mathcal{R}\left(\begin{pmatrix} A_{SK_i} \\ C_{S_i} \end{pmatrix}^*\right)} Z P_{\mathcal{R}\left(D_{S_i}\ B_{SK_i}\right)}} = \overline{P_{\mathcal{R}\left((A'_{1i})^*\right)} Z P_{\mathcal{R}\left(B'_{2i}\right)}} \in \mathcal{B}(\mathcal{H}), i = 1, 2.$$

is a solution of the system (13).

In addition, for an arbitrary $z \in \mathcal{H}$ and $z_{n_i} \in \mathcal{D}(B'_{2i})$, such that $z_{n_i} \to z, (i = 1, 2)$. Since

$$A'^{\dagger}_{1i}A'_{1i} = B'_{2i}B'^{\dagger}_{2i}, (i = 1, 2),$$

on $\mathcal{D}(B'^{\dagger}_{2i})$, we obtain

$$\begin{aligned}
\langle Z_0 z, z\rangle &= \lim_{n\to\infty}\langle Z_0 z_{n_i}, z_{n_i}\rangle = \lim_{n\to\infty}\langle A'^{\dagger}_{1i}A'_{1i}ZB'_{2i}B'^{\dagger}_{2i}z_{n_i}, z_{n_i}\rangle \\
&= \lim_{n\to\infty}\langle A'^{\dagger}_{1i}A'_{1i}ZA'^{\dagger}_{1i}A'_{1i}z_{n_i}, z_{n_i}\rangle = \lim_{n\to\infty}\langle z_{n_i}, A'^{\dagger}_{1i}A'_{1i}Z^*A'^{\dagger}_{1i}A'_{1i}z_{n_i}\rangle \\
&= \lim_{n\to\infty}\langle z_{n_i}, A'^{\dagger}_{1i}A'_{1i}ZB'_{2i}B'^{\dagger}_{2i}z_{n_i}\rangle = \lim_{n\to\infty}\langle z_{n_i}, Z_0 z_{n_i}\rangle \\
&= \langle z, Z_0 z\rangle, (i = 1, 2).
\end{aligned}$$

Therefore, Z_0 is a self-adjoint solution of the system (13).

$(ii) \Rightarrow (i)$ If Z is a self-adjoint solution of the system (13), from condition

$$C'_{1i} \overset{*}{\leq} A'_{2i} \overset{*}{\leq} A'_{1i},$$
$$C'_{2i} \overset{*}{\leq} B'_{1i} \overset{*}{\leq} B'_{2i},$$

we can deduce

$$\mathcal{R}((C'_{2i})^*) \subseteq \mathcal{R}((B'_{2i})^*),$$
$$\mathcal{R}(C'_{1i}) \subseteq \mathcal{R}(A'_{1i}).$$

Further, we obtain

$$C'_{1i} = A'_{1i}A'^{\dagger}_{1i}C'_{1i},$$
$$C'_{2i} = C'_{2i}B'^{\dagger}_{2i}B'_{2i}.$$

Therefore,

$$A'_{1i}ZB'_{1i} = A'_{1i}A'^{\dagger}_{1i}C'_{1i} = C'_{1i},$$
$$A'_{2i}ZB'_{2i} = \overline{C'_{2i}B'^{\dagger}_{2i}}B'_{2i} = C'_{2i}B'^{\dagger}_{2i}B'_{2i} = C'_{2i}, (i = 1, 2).$$

It follows that Z is a self-adjoint solution of the system (12).

We know that a self-adjoint solution to the system (12) implies a self-adjoint solution to the system (6). This inference is based on the developed equivalence relation. Consequently, the existence of solutions for the system (6) indicates that the system (1) possesses a self-adjoint solution Z.

Next, we show that Y is also a self-adjoint solution of the system (1).

In Claim 2 of Theorem 1, we also know that the system (4) has a solution equivalent to the system (1) having a solution. As a matter of fact, the system (4) has a solution if and only if the system

$$P_1Y'Q_1 + P_2Z'Q_2 = \hat{W}$$

has a solution.

From Lemma 2, Equation (7) is solvable equivalent to the following two inclusion relations holding simultaneously:

$$\mathcal{R}(\hat{W} - P_1Y'Q_1) \subseteq \mathcal{R}(P_2),$$
$$\mathcal{R}((\hat{W} - P_1Y'Q_1)^*) \subseteq \mathcal{R}(Q_2^*).$$

Similar to the previous proof, we can obtain that the system (4) has a solution that is equivalent to the system

$$\begin{aligned} P'_{11}Y'Q_1 &= W'_{11}, \\ P_2Y'P'_{22} &= W'_{22}, \end{aligned} \tag{14}$$

having a solution. Combined with

$$W'_{11} \overset{*}{\leq} P_2 \overset{*}{\leq} P'_{11},$$
$$W'_{22} \overset{*}{\leq} Q_1 \overset{*}{\leq} P'_{22}$$
$$P'^{\dagger}_{11}P'_{11} = P'_{22}P'^{\dagger}_{22},$$

we obtain that a self-adjoint solution to the system (14) is equivalent to a self-adjoint solution to (1).

Additionally, since $X = A_i^{\dagger}(G_i - C_iYD_i - E_iZF_i)B_i^{\dagger}$ is a self-adjoint operator, the system (1) has three self-adjoint solutions. □

The following results can be obtained by taking $E_i = F_i = 0, (i = 1, 2)$ from Theorem 2.

Corollary 3. *Let $A_i, B_i \in \mathcal{B}(\mathcal{H})$ be closed-range operators and $C_i, D_i, G_i \in \mathcal{B}(\mathcal{H}), S_i = I_{\mathcal{H}} - A_iA_i^{\dagger}, T_i = I_{\mathcal{H}} - B_i^{\dagger}B_i, (i = 1, 2)$ such that*

$$G_iT_i \overset{*}{\leq} S_i^*C_i \overset{*}{\leq} C_i,$$
$$S_i^*G_i \overset{*}{\leq} D_iT_i \overset{*}{\leq} D_i$$

and

$$C_i^\dagger C_i|_{\mathcal{D}(D_i^\dagger)} = D_i D_i^\dagger, (i = 1, 2).$$

If $A_i^\dagger(G_i - C_i Y D_i)B_i^\dagger$ *is a self-adjoint operator, and the system* $A_i X B_i + C_i Y D_i = G_i (i = 1, 2)$ *has a solution, then the subsequent assertions are equivalent, as follows:*

(i) The system

$$A_1 X B_1 + C_1 Y D_1 = G_1$$
$$A_2 X B_2 + C_2 Y D_2 = G_2$$

has self-adjoint solution.

(ii) The system

$$S_i C_i Y P_{\overline{\mathcal{R}(D_i)}} = \overline{S_i G_i D_i^\dagger},$$
$$P_{\overline{\mathcal{R}(C_i^*)}} Y D_i T_i = C_i^\dagger G_i T_i$$

$(i = 1, 2)$ *has self-adjoint solution.*

Proof. $(i) \Rightarrow (ii)$ Suppose that (i) holds. Similar to the proof of Corollary 1, the system (2) has a solution equivalent to the system (9) having a solution. Additionally, the system (9) is equivalent to

$$\begin{aligned} S_i^* C_i Y D_i &= S_i^* G_i P_{\overline{\mathcal{R}(D_i^*)}}, \\ C_i Y D_i T_i &= P_{\overline{\mathcal{R}(C_i)}} G_i T_i, (i = 1, 2). \end{aligned} \tag{15}$$

Suppose that, in the system (15), there exists a self-adjoint solution Y. Then, from the proof of Corollary 1,

$$Y_i = \overline{C_i^\dagger C_i Y D_i D_i^\dagger} \in \mathcal{B}(\mathcal{H}), (i = 1, 2)$$

is a solution of the operator system

$$S_i^* C_i Y P_{\overline{\mathcal{R}(D_i)}} = \overline{S_i^* G_i D_i^\dagger},$$
$$P_{\overline{\mathcal{R}(C_i^*)}} Y D_i T_i = C_i^\dagger G_i T_i.$$

In addition, for an arbitrary $y \in \mathcal{H}$ and $y_{n_i} \in \mathcal{D}(D_i^\dagger)$ such that $y_{n_i} \to y, (i = 1, 2)$. Since $C_i^\dagger C_i|_{\mathcal{D}(D_i^\dagger)} = D_i D_i^\dagger, (i = 1, 2)$, we obtain

$$\begin{aligned} \langle Y_i y, y\rangle &= \lim_{n\to\infty} \langle Y_i y_{n_i}, y_{n_i}\rangle = \lim_{n\to\infty} \langle C_i^\dagger C_i Y D_i D_i^\dagger y_{n_i}, y_{n_i}\rangle \\ &= \lim_{n\to\infty} \langle C_i^\dagger C_i Y C_i^\dagger C_i y_{n_i}, y_{n_i}\rangle = \lim_{n\to\infty} \langle y_{n_i}, C_i^\dagger C_i Y^* C_i^\dagger C_i y_{n_i}\rangle \\ &= \lim_{n\to\infty} \langle y_{n_i}, C_i^\dagger C_i Y C_i^\dagger C_i y_{n_i}\rangle = \lim_{n\to\infty} \langle y_{n_i}, C_i^\dagger C_i Y D_i D_i^\dagger y_{n_i}\rangle \\ &= \langle y, Y_i y\rangle, (i = 1, 2). \end{aligned}$$

Therefore, $Y_i = \overline{C_i^\dagger C_i Y D_i D_i^\dagger} \in \mathcal{B}(\mathcal{H}), (i = 1, 2)$ is a self-adjoint solution of

$$S_i^* C_i Y P_{\overline{\mathcal{R}(D_i)}} = \overline{S_i^* G_i D_i^\dagger}, P_{\overline{\mathcal{R}(C_i^*)}} Y D_i T_i = C_i^\dagger G_i T_i.$$

$(ii) \Rightarrow (i)$ Assume that Y is a self-adjoint solution of $S_i^* C_i Y P_{\overline{\mathcal{R}(D_i)}} = \overline{S_i^* G_i D_i^\dagger}$, $P_{\overline{\mathcal{R}(C_i^*)}} Y D_i T_i = C_i^\dagger G_i T_i, (i = 1, 2)$. From the condition $G_i T_i \overset{*}{\leq} S_i^* C_i \overset{*}{\leq} C_i$, $S_i^* G_i \overset{*}{\leq} D_i T_i \overset{*}{\leq} D_i$, we can deduce $\mathcal{R}(G_i^* S_i) \subseteq \mathcal{R}(D_i^*), \mathcal{R}(G_i T_i) \subseteq \mathcal{R}(C_i)$; further, we obtain $G_i T_i = C_i C_i^\dagger G_i T_i$, $S_i^* G_i = S_i^* E_i D_i^\dagger D_i$. Therefore,

$$S_i^* C_i Y D_i = \overline{S_i^* G_i D_i^\dagger} D_i = S_i^* G_i D_i^\dagger D_i = S_i^* G_i P_{\overline{\mathcal{R}(D_i^*)}},$$
$$C_i Y D_i T_i = C_i C_i^\dagger G_i T_i = P_{\overline{\mathcal{R}(C_i)}} G_i T_i, (i = 1, 2).$$

We know that a self-adjoint solution to the system (15) means a self-adjoint solution to the system (9). Therefore, it follows from the equivalence relation that the system (9) and the system (2) have solutions that the system (2) has a self-adjoint solution Y. Additionally, since $X = A_i^\dagger (G_i - C_i Y D_i) B_i^\dagger$ is a self-adjoint operator, the system (2) has a pair of self-adjoint solutions. □

The following results can be obtained by taking $C_i = D_i = E_i = F_i = 0, (i = 1, 2)$ from Theorem 2:

Corollary 4 ([28], Theorem 2.3). *Let $A_i, B_i, G_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,2}$ such that $G_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $G_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ and $A_1^\dagger A_1|_{\mathcal{D}(B_2^\dagger)} = B_2 B_2^\dagger$. If the system $A_i X B_i = G_i, i = \overline{1,2}$ is solvable, then the subsequent assertions are equivalent, as follows:*

(i) The operator equation system

$$A_1 X B_1 = G_1$$
$$A_2 X B_2 = G_2$$

has a self-adjoint solution;

(ii) The operator equation system

$$X B_1 = A_1^\dagger G_1,$$
$$A_2 X = \overline{G_2 B_2^\dagger}$$

has a self-adjoint solution.

The following results can be obtained by taking $A_1 = B_2 = A, B_1 = A_2 = B$, $C_i = D_i = E_i = F_i = 0, G_i = G, (i = 1, 2)$ from Theorem 2.

Corollary 5 ([32], Corollary 2.7). *Let $A, B, G \in \mathcal{B}(\mathcal{H})$. If A has a closed range with $A^\dagger A = A A^\dagger$ and $G \overset{*}{\leq} B \overset{*}{\leq} A$, then the subsequent assertions are equivalent as follows:*

(i) There exists a self-adjoint operator $\widetilde{X} \in \mathcal{B}(\mathcal{H})$ such that $A\widetilde{X}B = G = B\widetilde{X}A$;

(ii) There exists a self-adjoint operator $\widetilde{Y} \in \mathcal{B}(\mathcal{H})$ such that $B\widetilde{Y} = GA^\dagger$ and $\widetilde{Y}B = A^\dagger G$.

3.3. The Existence of Positive Solutions

In the subsequent theorem, we present several equivalent conditions for the existence of positive solutions to the system $A_i X B_i + C_i Y D_i + E_i Z F_i = G_i (i = 1, 2)$.

For convenience, we can refer to the symbols defined in Theorem 2.

Theorem 3. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), (i = 1, 2)$ be closed-range operators, and $G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$. Suppose that*

$$C_{1i}' \overset{*}{\leq} A_{2i}' \overset{*}{\leq} A_{1i}',$$
$$C_{2i}' \overset{*}{\leq} B_{1i}' \overset{*}{\leq} B_{2i}',$$
$$W_{11}' \overset{*}{\leq} P_2 \overset{*}{\leq} P_{11}',$$
$$W_{22}' \overset{*}{\leq} Q_1 \overset{*}{\leq} P_{22}',$$

and

$$A_{1i}'^\dagger A_{1i}'|_{\mathcal{D}(B_{2i}'^\dagger)} = B_{2i}' B_{2i}'^\dagger, (i = 1, 2),$$
$$P_{11}'^\dagger P_{11}'|_{\mathcal{D}(P_{22}'^\dagger)} = P_{22}' P_{22}'^\dagger.$$

If $X = A_i^{\dagger}(G_i - C_iYD_i - E_iZF_i)B_i^{\dagger}$ *is a positive operator, and the system* $A_iXB_i + C_iYD_i + E_iZF_i = G_i, (i = 1,2)$ *has a solution, then the subsequent assertions are equivalent, as follows:*

(i) The operator equation system

$$A_1XB_1 + C_1YD_1 + E_1ZF_1 = G_1$$
$$A_2XB_2 + C_2YD_2 + E_2ZF_2 = G_2$$

has positive solution.

(ii) The operator equation system

$$ZB'_{1i} = A'^{\dagger}_{1i}C'_{1i},$$
$$A'_{2i}Z = \overline{C'_{2i}B'^{\dagger}_{2i}},$$

(i=1, 2) has positive solution.

Proof. The demonstration technique parallels the validation of Theorem 2, and is therefore omitted. □

The following results can be obtained by taking $E_i = F_i = 0, (i = 1,2)$ from Theorem 3.

Corollary 6. *Let* $A_i, B_i \in \mathcal{B}(\mathcal{H})$ *be closed-range operators and* $C_i, D_i, G_i \in \mathcal{B}(\mathcal{H}), S_i = I_{\mathcal{H}} - A_iA_i^{\dagger}, T_i = I_{\mathcal{H}} - B_i^{\dagger}B_i, (i = 1,2)$ *such that*

$$G_iT_i \overset{*}{\leq} S_i^*C_i \overset{*}{\leq} C_i,$$
$$S_i^*G_i \overset{*}{\leq} D_iT_i \overset{*}{\leq} D_i$$

and

$$C_i^{\dagger}C_i|_{\mathcal{D}(D_i^{\dagger})} = D_iD_i^{\dagger}, (i = 1,2).$$

If $A_i^{\dagger}(G_i - C_iYD_i)B_i^{\dagger}|_{\mathcal{D}(B_i^{\dagger})}$ *is a positive operator, and the system* $A_iXB_i + C_iYD_i = G_i (i = 1,2)$ *has a solution, then the subsequent assertions hold equivalently, as follows:*

(i) The operator equation system

$$A_1XB_1 + C_1YD_1 = G_1$$

$$A_2XB_2 + C_2YD_2 = G_2$$

has positive solution.

(ii) The operator equation system

$$S_iC_iYP_{\overline{\mathcal{R}(D_i)}} = \overline{S_iG_iD_i^{\dagger}},$$

$$P_{\overline{\mathcal{R}(C_i^*)}}YD_iT_i = C_i^{\dagger}G_iT_i$$

$(i = 1,2)$ *has positive solution.*

Proof. The demonstration technique parallels the demonstration in Corollary 3, which is omitted. □

The following results can be obtained by taking $C_i = D_i = E_i = F_i = 0, (i = 1,2)$ from Theorem 3.

Corollary 7 ([28], Theorem 2.4)**.** *Let* $A_i, B_i, G_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,2}$ *such that* $G_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1$, $G_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$ *and* $A_1^{\dagger}A_1|_{\mathcal{D}(B_2^{\dagger})} = B_2B_2^{\dagger}$. *If the system* $A_iXB_i = G_i, i = \overline{1,2}$ *is solvable, then the subsequent assertions hold equivalently, as follows:*

(i) *The operator equation system*

$$A_1XB_1 = G_1$$

$$A_2XB_2 = G_2$$

has a positive solution;

(ii) *The operator equation system*

$$XB_1 = A_1^{\dagger}G_1,$$

$$A_2X = \overline{G_2B_2^{\dagger}}$$

has a positive solution.

The following results can be obtained by taking $A_1 = B_2 = A, B_1 = A_2 = B$, $C_i = D_i = E_i = F_i = 0, G_i = G, (i = 1, 2)$ from Theorem 3.

Corollary 8 ([32], Corollary 2.8)**.** *Let $A, B, G \in \mathcal{B}(\mathcal{H})$. If A has a closed range with $A^{\dagger}A = AA^{\dagger}$ and $G \overset{*}{\leq} B \overset{*}{\leq} A$, then the subsequent assertions hold equivalently, as follows:*

(1) There exists a positive operator $\widetilde{X} \in \mathcal{B}(\mathcal{H})$ such that $A\widetilde{X}B = G = B\widetilde{X}A$;

(2) There exists a positive operator $\widetilde{Y} \in \mathcal{B}(\mathcal{H})$ such that $B\widetilde{Y} = GA^{\dagger}$ and $\widetilde{Y}B = A^{\dagger}G$.

4. Representation of Solutions

For convenience, we define the notation as follows $(i = 1, 2)$:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad D = \begin{pmatrix} D_1 & D_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad F = \begin{pmatrix} F_1 & F_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$

$$G = \begin{pmatrix} G_1 & U_3 \\ V_3 & G_2 \end{pmatrix}, \ \textit{for some } U_3, V_3 \in \mathcal{B}(\mathcal{F}, \mathcal{K}),$$

$$M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^{-})C : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad M_1^{'} = D(I_{\mathcal{F}\oplus\mathcal{F}} - B^{-}B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$

$$M_2 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^{-})E : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad M_2^{'} = F(I_{\mathcal{F}\oplus\mathcal{F}} - B^{-}B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$

$$N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^{-})G : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K}, \qquad N_2 = G(I_{\mathcal{F}\oplus\mathcal{F}} - B^{-}B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K},$$

$$M_E = \begin{pmatrix} M_2 \\ E \end{pmatrix} : \mathcal{H} \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$M_C = \begin{pmatrix} M_1 \\ C \end{pmatrix} : \mathcal{H} \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$M_F = \begin{pmatrix} F & M_2^{'} \end{pmatrix} : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to \mathcal{G},$$

$$M_D = \begin{pmatrix} D & M_1^{'} \end{pmatrix} : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to \mathcal{G},$$

$$N_N = \begin{pmatrix} N_1 & U_4 \\ V_4 & N_2 \end{pmatrix}, \ \textit{for some } U_4, V_4 \in \mathcal{B}(\mathcal{F} \oplus \mathcal{F}, \mathcal{K} \oplus \mathcal{K}),$$

$$M_{M_E} = (I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_E M_E^{-})M_C : \mathcal{H} \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$M_{M_F} = M_D(I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_F^{-}M_F) : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to \mathcal{G},$$

$$N_{M_E} = (I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_E M_E^{-})N_N : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$N_{M_F} = N_N(I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_F^{-}M_F) : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$M_E^{'} = (I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_C M_C^{-})M_E : \mathcal{H} \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$N_N^{'} = (I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_C M_C^{-})N_N : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}),$$

$$M_F^{'} = M_F(I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_D^{-}M_D) : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to \mathcal{G},$$

$$N_N^{''} = N_N(I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_D^{-}M_D) : (\mathcal{F} \oplus \mathcal{F}) \oplus (\mathcal{F} \oplus \mathcal{F}) \to (\mathcal{K} \oplus \mathcal{K}) \oplus (\mathcal{K} \oplus \mathcal{K}).$$

4.1. The Representation of General Solutions

In the subsequent theorem, we derive the representation of the solutions of the equation system $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$.

Theorem 4. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F}, \mathcal{G})$,$G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be closed-range operators. If*

$$N_{M_E} \overset{*}{\leq} M_{M_F} \overset{*}{\leq} M_D,$$
$$N_{M_F} \overset{*}{\leq} M_{M_E} \overset{*}{\leq} M_C,$$

then the general solutions of the operator system $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$ are

$$\begin{aligned}
Y &= M_{M_E}^- N_{M_E} M_D^- + L - M_{M_E}^- M_{M_E} L M_D M_D^-, \\
Z &= M_E^- N_N M_F^- - M_E^- M_C M_{M_E}^- N_{M_E} M_D^- M_D M_F^- - M_E^- M_C L M_D M_F^- \\
&\quad + M_E^- M_{M_E} L M_D M_F^-, \\
X &= A^- G B^- - A^- C M_{M_E}^- N_{M_E} M_D^- D B^- + A^- C M_{M_E}^- M_{M_E} L M_D M_D^- D B^- \\
&\quad - A^- C L D B^- + A^- E M_E^- M_C M_{M_E}^- N_{M_E} M_D^- M_D M_F^- F B^- \\
&\quad - A^- E M_E^- N_N M_F^- F B^- + A^- E M_E^- M_C L M_D M_F^- F B^- \\
&\quad - A^- E M_{M_E} L M_D M_F^- F B^-,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
L &= (M_C - M_{M_E})^- (N_{M_F} - M_C M_{M_E}^- N_{M_E} M_D^- M_{M_F}) M_{M_F}^- + S \\
&\quad - (M_C - M_{M_E})^- (M_C - M_{M_E}) S M_{M_F} M_{M_F}^-.
\end{aligned}$$

In the above equation, $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator.

Proof. The system (1) has a solution if and only if there exists $U_3, V_3 \in \mathcal{B}(\mathcal{F}, \mathcal{K})$ such that the equation

$$\begin{aligned}
&\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Y \begin{pmatrix} D_1 & D_2 \end{pmatrix} \\
&\qquad + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} Z \begin{pmatrix} F_1 & F_2 \end{pmatrix} = \begin{pmatrix} G_1 & U_3 \\ V_3 & G_2 \end{pmatrix}
\end{aligned} \tag{17}$$

has a solution; i.e.,

$$AXB = G - CYD - EZF \tag{18}$$

has a solution. By Lemma 2, Equation (18) has a solution if and only if

$$\mathcal{R}(G - CYD - EZF) \subseteq \mathcal{R}(A),$$
$$\mathcal{R}((G - CYD - EZF)^*) \subseteq \mathcal{R}(B^*).$$

Then, $\mathcal{R}(G - CYD - EZF) \subseteq \mathcal{R}(A)$ if and only if

$$(I_{\mathcal{K}\oplus\mathcal{K}} - AA^-)G = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^-)CYD + (I_{\mathcal{K}\oplus\mathcal{K}} - AA^-)EZF.$$

That is,

$$M_1 YD + M_2 ZF = N_1. \tag{19}$$

In a similar way, $\mathcal{R}((G - CYD - EZF)^*) \subseteq \mathcal{R}(B^*)$ if and only if

$$G(I_{\mathcal{F}\oplus\mathcal{F}} - B^- B) = CYD(I_{\mathcal{F}\oplus\mathcal{F}} - B^- B) + EZF(I_{\mathcal{F}\oplus\mathcal{F}} - B^- B).$$

Namely,

$$CYM_1' + EZM_2' = N_2. \tag{20}$$

Equation (1) has a solution if and only if the systems (19) and (20) have a solution. The systems (19) and (20) have a solution if and only if there exists $U_4, V_4 \in \mathcal{B}(\mathcal{F} \oplus \mathcal{F}, \mathcal{K} \oplus \mathcal{K})$ such that the equation

$$\begin{pmatrix} M_1 \\ C \end{pmatrix} Y \begin{pmatrix} D & M_1' \end{pmatrix} + \begin{pmatrix} M_2 \\ E \end{pmatrix} Z \begin{pmatrix} F & M_2' \end{pmatrix} = \begin{pmatrix} N_1 & U_4 \\ V_4 & N_2 \end{pmatrix}$$

has a solution; i.e.,

$$M_C Y M_D + M_E Z M_F = N_N \tag{21}$$

has a solution. According to Lemma 2, Equation (21) has a solution if and only if

$$\mathcal{R}(N_N - M_C Y M_D) \subseteq \mathcal{R}(M_E),$$
$$\mathcal{R}((N_N - M_C Y M_D)^*) \subseteq \mathcal{R}(M_F^*).$$

Then, $\mathcal{R}(N_N - M_C Y M_D) \subseteq \mathcal{R}(M_E)$ if and only if

$$(I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_E M_E^-) N_N = (I_{\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}\oplus\mathcal{K}} - M_E M_E^-) M_C Y M_D.$$

That is,

$$M_{M_E} Y M_D = N_{M_E}. \tag{22}$$

In a similar way, $\mathcal{R}((N_N - M_C Y M_D)^*) \subseteq \mathcal{R}(M_F^*)$ if and only if

$$N_N (I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_F^- M_F) = M_C Y M_D (I_{\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}\oplus\mathcal{F}} - M_F^- M_F).$$

Namely,

$$M_C Y M_{M_F} = N_{M_F}. \tag{23}$$

Equation (21) has a solution if and only if the systems (22) and (23) have a solution.

In view of Lemma 2, the general solutions of Equation (22) are

$$Y = M_{M_E}^- N_{M_E} M_D^- + L - M_{M_E}^- M_{M_E} L M_D M_D^-,$$

where $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is an arbitrary operator. If Y satisfies Equation (23), then

$$M_C (M_{M_E}^- N_{M_E} M_D^- + L - M_{M_E}^- M_{M_E} L M_D M_D^-) M_{M_F} = N_{M_F},$$

and then

$$M_C M_{M_E}^- N_{M_E} M_D^- M_{M_F} + M_C L M_{M_F} - M_C M_{M_E}^- M_{M_E} L M_D M_D^- M_{M_F} = N_{M_F}.$$

By Lemma 1 and from

$$M_{M_E} \overset{*}{\leq} M_C \Rightarrow M_{M_E} = M_C P_{\overline{\mathcal{R}((M_{M_E})^*)}} = M_C M_{M_E}^- M_{M_E},$$

$$M_{M_F} \overset{*}{\leq} M_D \Rightarrow \mathcal{R}(M_{M_F}) \subseteq \mathcal{R}(M_D) \Rightarrow M_{M_F} = M_D M_D^- M_{M_F},$$

it implies that

$$M_C L M_{M_F} - M_{M_E} L M_{M_F} = N_{M_F} - M_C M_{M_E}^- N_{M_E} M_D^- M_{M_F}.$$

It is evident that L is a solution of the equation

$$(M_C - M_{M_E}) L M_{M_F} = N_{M_F} - M_C M_{M_E}^- N_{M_E} M_D^- M_{M_F}.$$

It follows from Lemma 2 that

$$L = (M_C - M_{M_E})^-(N_{M_F} - M_C M_{M_E}^- N_{M_E} M_D^- M_{M_F})M_{M_F}^- + S - (M_C - M_{M_E})^-(M_C - M_{M_E})SM_{M_F}M_{M_F}^-,$$

where $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator. Substituting the representation for Y into (21) and by $M_{M_E} = M_C M_{M_E}^- M_{M_E}$ yields (16) for Z. Using the representation of Y and Z, we obtain (16) for X. □

For convenience, we define the notation as follows $(i = 1, 2)$:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad D = \begin{pmatrix} D_1 & D_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$G = \begin{pmatrix} G_1 & U \\ V & G_2 \end{pmatrix}, \text{for some } U, V \in \mathcal{B}(\mathcal{F}, \mathcal{K}).$$
$$M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^-)C : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K},$$
$$N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^-)G : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K},$$
$$M_2 = D(I_{\mathcal{F}\oplus\mathcal{F}} - B^-B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$N_2 = G(I_{\mathcal{F}\oplus\mathcal{F}} - B^-B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K}.$$

The following results can be obtained by taking $E_i = F_i = 0, (i = 1, 2)$ from Theorem 4.

Corollary 9. *Let $A_i, C_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be closed-range operators. If*

$$N_1 \overset{*}{\leq} M_2 \overset{*}{\leq} D,$$
$$N_2 \overset{*}{\leq} M_1 \overset{*}{\leq} C,$$

then the general solutions of $A_i X B_i + C_i Y D_i = G_i (i = 1, 2)$ are

$$\begin{aligned} Y &= M_1^- N_1 D^- + L - M_1^- M_1 L D D^-, \\ X &= A^- G B^- - A^- C M_1^- N_1 D^- D B^- - A^- C L D B^- + A^- M_1 L D B^-, \end{aligned} \tag{24}$$

where

$$L = (C - M_1)^-(N_2 - C M_1^- N_1 D^- M_2)M_2^- + S - (C - M_1)^-(C - M_1)SM_2M_2^-.$$

In the above equation, $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator.

Proof. The system (2) has a solution if and only if $U, V \in \mathcal{B}(\mathcal{F}, \mathcal{K})$ exists such that the equation

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Y \begin{pmatrix} D_1 & D_2 \end{pmatrix} = \begin{pmatrix} G_1 & U \\ V & G_2 \end{pmatrix}$$

has a solution; i.e., the equation

$$AXB = G - CYD \tag{25}$$

has a solution. According to Lemma 2, Equation (25) has a solution if and only if

$$\mathcal{R}(G - CYD) \subseteq \mathcal{R}(A),$$
$$\mathcal{R}((G - CYD)^*) \subseteq \mathcal{R}(B^*).$$

Then, $\mathcal{R}(G - CYD) \subseteq \mathcal{R}(A)$ if and only if $(I - AA^-)G = (I - AA^-)CYD$. That is,

$$M_1YD = N_1. \tag{26}$$

In a similar way, $\mathcal{R}((G - CYD)^*) \subseteq \mathcal{R}(B^*)$ if and only if $G(I - B^-B) = CYD(I - B^-B)$. Namely,

$$CYM_2 = N_2. \tag{27}$$

Equation (25) has a solution if and only if the systems (26) and (27) have a solution; the solution of the systems (26) and (27) is the solution of system (2). According to Lemma 2, the general solutions of the equation $M_1YD = N_1$ are

$$Y = M_1^- N_1 D^- + L - M_1^- M_1 LDD^-,$$

where $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is an arbitrary operator. If Y satisfies the equation $CYM_2 = N_2$, then

$$C(M_1^- N_1 D^- + L - M_1^- M_1 LDD^-)M_2 = N_2,$$

and hence,

$$CM_1^- N_1 D^- M_2 + CLM_2 - CM_1^- M_1 LDD^- M_2 = N_2.$$

By Lemma 1 and from

$$M_1 \overset{*}{\leq} C \Rightarrow M_1 = CP_{\overline{\mathcal{R}(M_1^*)}} = CM_1^- M_1,$$

$$M_2 \overset{*}{\leq} D \Rightarrow \mathcal{R}(M_2) \subseteq \mathcal{R}(D) \Rightarrow M_2 = DD^- M_2,$$

it implies that

$$CLM_2 - M_1 LM_2 = N_2 - CM_1^- N_1 D^- M_2.$$

It is evident that L is a solution of the equation

$$(C - M_1)LM_2 = N_2 - CM_1^- N_1 D^- M_2.$$

Using Lemma 2 again, we obtain

$$L = (C - M_1)^-(N_2 - CM_1^- N_1 D^- M_2)M_2^- + S - (C - M_1)^-(C - M_1)SM_2M_2^-,$$

where $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator. Substituting the representation for Y into (25) yields (24) for X. □

In [28], the author studies the existence of the solution of the system $A_1XB_1 = G_1$, $A_2XB_2 = G_2$, and does not give the representation of the solution. Next, we give the representation of the solution of $A_1XB_1 = G_1, A_2XB_2 = G_2$.

The general solutions of $A_1XB_1 = G_1, A_2XB_2 = G_2$ can be obtained by taking $C_i = D_i = E_i = F_i = 0, (i = 1, 2)$ from Theorem 4.

Corollary 10. *Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be closed-range operators. If*

$$G_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1,$$
$$G_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2,$$

then the general solutions of the system $A_iXB_i = G_i (i = 1, 2)$ are

$$\begin{aligned} X = {} & A_2^- G_2 B_2^- + (A_1 - A_2)^-(G_1 - A_1 A_2^- G_2 B_2^- B_1)B_1^- + S \\ & - A_2^- A_2(A_1 - A_2)^-(G_1 - A_1 A_2^- G_2 B_2^- B_1)B_1^- B_2 B_2^- \\ & - (A_1 - A_2)^-(A_1 - A_2)SB_1 B_1^- - A_2^- A_2 SB_2 B_2^- \\ & + A_2^- A_2(A_1 - A_2)^-(A_1 - A_2)SB_1 B_1^- B_2 B_2^-, \end{aligned}$$

where $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator.

The following results can be obtained by taking $A_2 = B_1 = I$ from Corollary 10.

Corollary 11. *Let $A_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_2 \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be closed-range operators. If*

$$G_1 \overset{*}{\leq} A_1,$$
$$G_2 \overset{*}{\leq} B_2,$$

then the general solutions of the system $A_1X = G_1, XB_2 = G_2$ are

$$\begin{aligned} X = {} & G_2B_2^- + (A_1 - I)^-(G_1 - A_1G_2B_2^-) + S - (A_1 - I)^-(A_1 - I)S \\ & - (A_1 - I)^-(G_1 - A_1G_2B_2^-)B_2B_2^- - SB_2B_2^- \\ & + (A_1 - I)^-(A_1 - I)SB_2B_2^-, \end{aligned}$$

where $S \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ is an arbitrary operator.

The solution obtained in Corollary 11 is coincide with the result in reference [44].

The following results can be obtained by taking $A_1 = B_2, B_1 = A_2, G_1 = G_2 = G$ from Corollary 10.

Corollary 12 ([32], Theorem 3.1)**.** *Let $A, B, G \in \mathcal{B}(\mathcal{H})$. If A has a closed range and*

$$G \overset{*}{\leq} B \overset{*}{\leq} A,$$

then the general solutions of $AXB = G = BXA$ are

$$\begin{aligned} X = {} & B^-(G - BA^-GB^-A)(A - B)^- - B^-BS(A - B)(A - B)^- \\ & - A^-AB^-(G - BA^-GB^-A)(A - B)^-BB^- + A^-GB^- + S \\ & + A^-AB^-BS(A - B)(A - B)^-BB^- - A^-ASBB^-, \end{aligned}$$

where $S \in \mathcal{H}$ is arbitrary.

4.2. The Representation of Self-Adjoint Solutions

In the following theorem, combined with the notation defined in Theorem 4, we give the operator matrix representation of self-adjoint solutions for the system $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$.

Theorem 5. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be closed-range operators. If $A^\dagger(G - CY_0D - EZ_0F)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is a self-adjoint operator, $N_{M_E} \overset{*}{\leq} M_{M_F} \overset{*}{\leq} M_D, N_{M_F} \overset{*}{\leq} M_{M_E} \overset{*}{\leq} M_C, N'_N \overset{*}{\leq} M'_F \overset{*}{\leq} M_F, N''_N \overset{*}{\leq} M'_E \overset{*}{\leq} M_E, M_C = M'_E$, and*

$$\begin{aligned} M_C^\dagger M_C|_{\mathcal{D}(M_C^\dagger)} &= M_CM_C^\dagger = M_D^\dagger M_D|_{\mathcal{D}(M_D^\dagger)} = M_DM_D^\dagger \\ &= A^\dagger A|_{\mathcal{D}(A^\dagger)} = AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger \\ &= C^\dagger C|_{\mathcal{D}(C^\dagger)} = CC^\dagger = D^\dagger D|_{\mathcal{D}(D^\dagger)} = DD^\dagger \\ &= E^\dagger E|_{\mathcal{D}(E^\dagger)} = EE^\dagger = F^\dagger F|_{\mathcal{D}(F^\dagger)} = FF^\dagger \\ &= G^\dagger G|_{\mathcal{D}(G^\dagger)} = GG^\dagger = (M'_E)^\dagger M'_E|_{\mathcal{D}((M'_E)^\dagger)} = M'_E(M'_E)^\dagger \\ &= (M'_F)^\dagger M'_F|_{\mathcal{D}((M'_F)^\dagger)} = M'_F(M'_F)^\dagger. \end{aligned} \tag{28}$$

Then $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$ has self-adjoint solutions. Under space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, the self-adjoint solutions have a matrix representation

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix} \tag{29}$$

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix} \tag{30}$$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \tag{31}$$

where

$$Y_{11} = P_{\overline{\mathcal{R}(M_C^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} \ ,$$
$$Z_{11} = P_{\overline{\mathcal{R}((M_E')^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}}$$

are self-adjoint operators, satisfying that

$$Y_0 = \overline{M_C^\dagger M_C Y M_D M_D^\dagger},$$
$$Z_0 = \overline{(M_E')^\dagger M_E' Z_0 M_F M_F^\dagger}$$

are self-adjoint solutions of the equation system

$$M_{M_E} Y = \overline{N_{M_E} M_D^\dagger} \ ,$$
$$Y M_{M_F} = M_C^\dagger N_{M_F} \ ,$$

and

$$M_E' Z = \overline{N_N' M_F^\dagger} \ ,$$
$$Z M_F' = M_E^\dagger N_N'' \ ,$$

respectively; Y_{12}, Z_{12}, and X_{12} are arbitrary operators; and Y_{22}, Z_{22}, and X_{22} are arbitrary self-adjoint operators.

Proof. If $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has matrix representation (29). According to Theorem 4, the systems (22) and (23) have a solution equivalent to the system (1) having a solution.

According to Lemma 1, since $M_{M_F} \overset{*}{\leq} M_D$, $M_{M_E} \overset{*}{\leq} M_C$ and (28) holds, then under space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, $M_{M_E}, M_C, M_D, M_{M_F}$, A, B, C, D, E, F, G have the matrix representations

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_C = \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_D = \begin{pmatrix} M_{21} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{M_E} = \begin{pmatrix} M_{31} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{M_F} = \begin{pmatrix} M_{41} & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting them into the systems (22) and (23), we obtained

$$M_{M_E} Y M_D = \begin{pmatrix} M_{31} Y_{11} M_{21} & 0 \\ 0 & 0 \end{pmatrix} = M_{M_E} P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_D,$$

$$M_C Y M_{M_F} = \begin{pmatrix} M_{11} Y_{11} M_{41} & 0 \\ 0 & 0 \end{pmatrix} = M_C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_{M_F},$$

where $P_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}} = M_C^\dagger M_C Y_0 M_D M_D^\dagger \in \mathcal{B}(\mathcal{D}(M_D^\dagger), \mathcal{H})$. Combined with

$$\begin{aligned}\mathcal{R}(N_{M_F}) &\subseteq \mathcal{R}(M_C),\\ \mathcal{R}(M_{M_F}) &\subseteq \mathcal{R}(M_D),\\ \mathcal{R}((N_{M_E})^*) &\subseteq \mathcal{R}((M_D)^*),\\ \mathcal{R}((M_{M_E})^*) &\subseteq \mathcal{R}((M_C)^*),\end{aligned}$$

we obtain

$$\begin{aligned}M_{M_E}YM_D &= M_{M_E}P_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}}M_D = M_{M_E}M_C^\dagger M_C Y_0 M_D\\ &= M_{M_E}Y_0M_D = \overline{N_{M_E}M_D^\dagger}M_D = N_{M_E}M_D^\dagger M_D = N_{M_E},\\ M_CYM_{M_F} &= M_CP_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}}M_{M_F} = M_CY_0P_{\overline{\mathcal{R}(M_D)}}M_{M_F}\\ &= M_CY_0M_DM_D^\dagger M_{M_F} = M_CY_0M_{M_F} = M_CM_C^\dagger N_{M_F} = N_{M_F}.\end{aligned}$$

Therefore, $P_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}}$ is a solution of the systems (22) and (23) and the system (1).

In addition, for an arbitrary $y \in \overline{\mathcal{R}((M_C)^*)} = \overline{\mathcal{R}(M_D)}$,

$$\begin{aligned}\langle Y_{11}y, y\rangle &= \langle P_{\overline{\mathcal{R}((M_C)^*)}}Y_0y, y\rangle = \langle Y_0y, P_{\overline{\mathcal{R}((M_C)^*)}}y\rangle\\ &= \langle y, Y_0^*y\rangle = \langle P_{\overline{\mathcal{R}((M_C)^*)}}y, Y_0y\rangle\\ &= \langle y, P_{\overline{R((M_C)^*)}}Y_0y\rangle = \langle y, Y_{11}y\rangle.\end{aligned}$$

Therefore, Y_0 is a self-adjoint solution of the system of the operator Equation (1).

Assume that $Z \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has matrix representation (30). In the proof of Theorem 4, we also know that the systems (19) and (20) have a solution equivalent to the system (1) having a solution. Additionally, Equations (19) and (20) have a solution if and only if the equation

$$M_CYM_D + M_EZM_F = N_N$$

has a solution.

Using Lemma 2 again, Equation (21) has a solution equivalent to

$$\begin{aligned}\mathcal{R}(N_N - M_EZM_F) &\subseteq \mathcal{R}(M_C),\\ \mathcal{R}((N_N - M_EZM_F)^*) &\subseteq \mathcal{R}((M_D)^*).\end{aligned}$$

According to Lemma 3 and a simple calculation, we get that the systems (19) and (20) have a solution that is equivalent to the system

$$\begin{aligned}M_E'ZM_F &= N_N',\\ M_EZM_F' &= N_N'',\end{aligned} \tag{32}$$

having a solution. Combined with

$$\begin{aligned}N_N' &\overset{*}{\leq} M_F' \overset{*}{\leq} M_F,\\ N_N'' &\overset{*}{\leq} M_E' \overset{*}{\leq} M_E,\\ M_E'^\dagger M_E' &= M_F'M_F'^\dagger,\end{aligned}$$

we get that a self-adjoint solution Z of the system (32) is also a self-adjoint solution of (1).

In fact, similar to the proof above, under space decomposition $\overline{\mathcal{R}(M_F')} \oplus \mathcal{N}(M_F') \to \overline{\mathcal{R}(M_F')} \oplus \mathcal{N}(M_F')$, M_E', M_F, M_E, M_F' have the matrix representations

$$M'_E = \begin{pmatrix} M'_{11} & 0 \\ 0 & 0 \end{pmatrix}, M_F = \begin{pmatrix} M'_{21} & 0 \\ 0 & 0 \end{pmatrix}, M_E = \begin{pmatrix} M'_{31} & 0 \\ 0 & 0 \end{pmatrix}, M'_F = \begin{pmatrix} M'_{41} & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting them into the system (32), we obtained

$$M'_E Z M_F = \begin{pmatrix} M'_{11} Z_{11} M'_{21} & 0 \\ 0 & 0 \end{pmatrix} = M'_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_F,$$

$$M_E Z M'_F = \begin{pmatrix} M'_{31} Z_{11} M'_{41} & 0 \\ 0 & 0 \end{pmatrix} = M_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_{M'_F},$$

where $P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} = (M'_E)^\dagger M'_E Z_0 M_F M_F^\ddagger \in \mathcal{B}(\mathcal{D}(M_F^\ddagger), \mathcal{H})$. Combined with

$$\begin{aligned} \mathcal{R}(N''_N) &\subseteq \mathcal{R}(M_E), \\ \mathcal{R}(M'_F) &\subseteq \mathcal{R}(M_F), \\ \mathcal{R}((N'_N)^*) &\subseteq \mathcal{R}((M_F)^*), \\ \mathcal{R}((M'_E)^*) &\subseteq \mathcal{R}((M_E)^*), \end{aligned}$$

we obtain

$$\begin{aligned} M'_E Z M_F &= M'_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_F = M'_E (M_E)^\dagger M_E Z_0 M_F \\ &= M'_E Z_0 M_F = \overline{N'_N M_F^\ddagger} M_F = N'_N M_F^\ddagger M_F = N'_N, \\ M_E Z M'_F &= M_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M'_F = M_E Z_0 P_{\overline{\mathcal{R}(M_F)}} M'_F \\ &= M_E Z_0 M_F (M_F)^\dagger M'_F = M_E Z_0 M'_F = M_E M_E^\ddagger N''_N = N''_N. \end{aligned}$$

Therefore, $P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}}$ is a solution of the system (32) and the system of the operator Equation (1).

In addition, for an arbitrary $z \in \overline{\mathcal{R}((M'_E)^*)} = \overline{\mathcal{R}(M_F)}$,

$$\begin{aligned} \langle Z_{11} z, z \rangle &= \langle P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 z, z \rangle = \langle Z_0 z, P_{\overline{\mathcal{R}((M'_E)^*)}} z \rangle \\ &= \langle z, Z_0^* z \rangle = \langle P_{\overline{\mathcal{R}((M'_E)^*)}} z, Z_0 z \rangle \\ &= \langle z, P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 z \rangle = \langle z, Z_{11} z \rangle. \end{aligned}$$

We can deduce that Z_0 is a self-adjoint solution of the system (1).

On the same assumption, $X \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (31). Substituting the matrix representations of Y, Z, and X into (18), we obtain

$$A_{11} X_{11} B_{11} + C_{11} Y_{11} D_{11} + E_{11} Z F_{11} = G_{11},$$

which is equivalent to

$$AXB = G - C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} D - E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} F,$$

then

$$\begin{aligned} X &= \overline{A^\dagger (G - C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} D - E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} F) B^\dagger} \\ &= \overline{A^\dagger (G - C Y_0 D - E Z_0 F) B^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H}). \end{aligned}$$

In addition, for an arbitrary $x \in \mathcal{G}$, combined with the self-adjointness of $A^\dagger (G - C Y_0 D - E Z_0 F) B^\dagger|_{\mathcal{D}(B^\dagger)}$,

$$\begin{aligned}\langle X_{11}x, x\rangle &= \langle \overline{A^\dagger(G - CY_0D - EZ_0F)B^\dagger}x, x\rangle \\ &= \langle x, \overline{(B^\dagger)^*(G - CY_0D - EZ_0F)^*(A^\dagger)^*}x\rangle \\ &= \langle x, \overline{A^\dagger(G - CY_0D - EZ_0F)B^\dagger}x\rangle \\ &= \langle x, X_{11}x\rangle.\end{aligned}$$

We have $X_{11}^* = X_{11}$. Therefore, X is also a self-adjoint solution of the system (1).

Conversely, if $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a self-adjoint solution to Equation (18), then Y is also the self-adjoint solution of the systems (22) and (23). Under the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, set

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

We denote the continuous bounded linear extension of $M_C^\dagger M_C Y M_D M_D^\dagger$ as

$$Y_0 = \overline{M_C^\dagger M_C Y M_D M_D^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H}).$$

By $\mathcal{R}(M_{M_F}) \subseteq \mathcal{R}(M_D) \subseteq \mathcal{D}(M_D^\dagger) = \mathcal{D}(Y)$, we obtain $Y_0 M_{M_F} = Y M_{M_F} = M_C^\dagger N_{M_F}$. From $\mathcal{R}((N_{M_E})^*) \subseteq \mathcal{R}((M_D)^*)$, combined with Lemma 3, there exists $\widetilde{Y} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that $N_{M_E} = \widetilde{Y} M_D$. It follows that $N_{M_E} M_D^\dagger = \widetilde{Y} M_D M_D^\dagger \in \mathcal{B}(\mathcal{D}(M_D^\dagger), \mathcal{H})$. In that way, for any $y \in \mathcal{G}, y_n \in \mathcal{D}(M_D^\dagger)$ such that $y_n \to y$. Therefore,

$$M_{M_E} Y_0 y = M_{M_E}(\lim_{n\to\infty} Y_0 y_n) = \lim_{n\to\infty} M_{M_E} Y_0 y_n = \lim_{n\to\infty} N_{M_E} M_D^\dagger y_n = \overline{N_{M_E} M_D^\dagger} y.$$

Then Y_0 is a solution of the system

$$\begin{aligned} M_{M_E} Y_0 &= \overline{N_{M_E} M_D^\dagger}, \\ Y_0 M_{M_F} &= M_C^\dagger N_{M_F}. \end{aligned}$$

By a simple calculation, we obtain

$$\begin{aligned} P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} &= P_{\overline{\mathcal{R}((M_C)^*)}} \overline{P_{\overline{\mathcal{R}((M_C)^*)}} Y P_{\overline{\mathcal{R}(M_D)}}|_{\mathcal{D}(M_D^\dagger)}} P_{\overline{\mathcal{R}(M_D)}} \\ &= P_{\overline{\mathcal{R}((M_C)^*)}} Y P_{\overline{\mathcal{R}(M_D)}} \in \mathcal{B}(\mathcal{G}, \mathcal{H}). \end{aligned}$$

It follows that $Y_{11} = P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}}$. Since Y is a self-adjoint operator, we have that Y_{12} is an arbitrary operator and satisfies $Y_{21} = Y_{12}^*$, and Y_{22} is an arbitrary self-adjoint operator. Therefore, Y has the matrix representation of (29).

Similar to the proof above, we have $Z_{11} = P_{\overline{\mathcal{R}((M_E')^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}}$. Since Z is a self-adjoint operator, we have that Z_{12} is an arbitrary operator and satisfies $Z_{21} = Z_{12}^*$, and Z_{22} is an arbitrary self-adjoint operator. Therefore, Z has the matrix representation of (30).

Suppose that $X \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a self-adjoint solution to the system (18). Under the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, set

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Substituting X into equation $AXB + CYD + EZF = G$, we obtain

$$\begin{aligned} X &= \begin{pmatrix} A_{11}^\dagger (G_{11} - C_{11} Y_{11} D_{11} - E_{11} Z F_{11}) B_{11}^\dagger & 0 \\ 0 & 0 \end{pmatrix} \\ &= \overline{A^\dagger(G - CYD - EZF)B^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H}); \end{aligned}$$

according to the known assumption, $A^\dagger(G - CYD - EZF)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is self-adjoint operator, so we get that

$$X_{11} = A^\dagger(G - CP_{\overline{\mathcal{R}((M_C)^*)}}YP_{\overline{\mathcal{R}(M_D)}}D - EP_{\overline{\mathcal{R}((M_C)^*)}}YP_{\overline{\mathcal{R}(M_D)}}F)B^\dagger$$

is self-adjoint operator. X_{12} is an arbitrary operator and satisfies $X_{21} = X_{12}^*$, and X_{22} is an arbitrary self-adjoint operator. Therefore, the matrix representation of X is (31). □

For convenience, we define the notation as follows $(i = 1, 2)$:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad D = \begin{pmatrix} D_1 & D_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$G = \begin{pmatrix} G_1 & U \\ V & G_2 \end{pmatrix}, \text{for some } U, V \in \mathcal{B}(\mathcal{F}, \mathcal{K}).$$
$$M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^\dagger)C : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K},$$
$$N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^\dagger)G : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K},$$
$$M_2 = D(I_{\mathcal{F}\oplus\mathcal{F}} - B^\dagger B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$N_2 = G(I_{\mathcal{F}\oplus\mathcal{F}} - B^\dagger B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K}.$$

The following results can be obtained by taking $E_i = F_i = 0, (i = 1, 2)$ from Theorem 5.

Corollary 13. *Let $A_i, C_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$. If $A^\dagger(G - CYD)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is a self-adjoint operator, $N_1 \overset{*}{\leq} M_2 \overset{*}{\leq} D$, $N_2 \overset{*}{\leq} M_1 \overset{*}{\leq} C$ and*

$$\begin{aligned} C^\dagger C|_{\mathcal{D}(C^\dagger)} = CC^\dagger = D^\dagger D|_{\mathcal{D}(D^\dagger)} = DD^\dagger = A^\dagger A|_{\mathcal{D}(A^\dagger)} \\ = AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger = G^\dagger G|_{\mathcal{D}(G^\dagger)} = GG^\dagger \end{aligned} \tag{33}$$

Then $A_iXB_i + C_iYD_i = G_i (i = 1, 2)$ has a pair of self-adjoint solutions. Under the space decomposition $\overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C) \to \overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C)$, the self-adjoint solutions have the matrix representation

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix} \tag{34}$$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \tag{35}$$

where

$$Y_{11} = P_{\overline{\mathcal{R}(C^*)}}Y_0P_{\overline{\mathcal{R}(D)}}$$

is a self-adjoint operator, satisfying that $Y_0 = \overline{C^\dagger CYDD^\dagger}$ is a self-adjoint solution of

$$M_1Y = \overline{N_1D^\dagger},$$
$$YM_2 = C^\dagger N_2 \ ,$$

Y_{12} and X_{12} are arbitrary operators, Y_{22} and X_{22} are arbitrary self-adjoint operators.

Proof. The proof method is similar to the proof of Theorem 5, which is omitted here. □

The self-adjoint solutions of $A_1XB_1 = G_1, A_2XB_2 = G_2$ can be obtained by taking $C_i = D_i = 0, (i = 1, 2)$ from Corollary 13.

Corollary 14 ([28], Theorem 2.5). *Let $A_i, B_i, G_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,2}$ such that*

$$G_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1, \quad G_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$$

and

$$A^\dagger A|_{\mathcal{D}(A^\dagger)} = AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger.$$

If the system $A_iXB_i = G_i, i = \overline{1,2}$ has a self-adjoint solution, then the general self-adjoint solution has the matrix representation

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$, where X_{22} is a self-adjoint operator and $X_{11} = P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} P_{\overline{\mathcal{R}(B_2)}}$ satisfying that $\widetilde{Y}$ is a self-adjoint solution of

$$A_2X = \overline{G_2B_2^\dagger},$$
$$XB_1 = A_1^\dagger G_1.$$

The following results can be obtained by taking $A_1 = B_2, B_1 = A_2, G_1 = G_2 = G$ from Corollary 14.

Corollary 15 ([32], Theorem 3.3). *Let $A, B, G \in \mathcal{B}(\mathcal{H})$ such that A has a closed range with $A^\dagger A = AA^\dagger$ and $G \overset{*}{\leq} B \overset{*}{\leq} A$. If the system $AXB = G = BXA$ has a self-adjoint solution, then the general self-adjoint solutions have the matrix representation*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$$

in terms of $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, where X_{22} is self-adjoint and $X_{11} = P_{\mathcal{R}(A^*)} \widetilde{Y} P_{\mathcal{R}(A^*)}$ satisfying that $\widetilde{Y}$ is a self-adjoint solution of $BX = GA^\dagger, XB = A^\dagger G$.

4.3. The Representation of Positive Solutions

In the following theorem, combined with the notation defined in Theorem 4, we give the operator matrix representation of positive solutions for the system $A_iXB_i + C_iYD_i + E_iZF_i = G_i (i = 1, 2)$.

Theorem 6. *Let $A_i, C_i, E_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i, F_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$ be a closed-range operator. If $A^\dagger(G - CY_0D - EZ_0F)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is a positive operator, $N_{M_E} \overset{*}{\leq} M_{M_F} \overset{*}{\leq} M_D$, $N_{M_F} \overset{*}{\leq} M_{M_E} \overset{*}{\leq} M_C$, $N_N' \overset{*}{\leq} M_F' \overset{*}{\leq} M_F$, $N_N'' \overset{*}{\leq} M_E' \overset{*}{\leq} M_E, M_C = M_E'$, and*

$$\begin{aligned} M_C^\dagger M_C|_{\mathcal{D}(M_C^\dagger)} &= M_C M_C^\dagger = M_D^\dagger M_D|_{\mathcal{D}(M_D^\dagger)} = M_D M_D^\dagger \\ &= A^\dagger A|_{\mathcal{D}(A^\dagger)} = AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger \\ &= C^\dagger C|_{\mathcal{D}(C^\dagger)} = CC^\dagger = D^\dagger D|_{\mathcal{D}(D^\dagger)} = DD^\dagger \\ &= E^\dagger E|_{\mathcal{D}(E^\dagger)} = EE^\dagger = F^\dagger F|_{\mathcal{D}(F^\dagger)} = FF^\dagger \\ &= G^\dagger G|_{\mathcal{D}(G^\dagger)} = GG^\dagger = (M_E')^\dagger M_E'|_{\mathcal{D}((M_E')^\dagger)} = M_E'(M_E')^\dagger \\ &= (M_F')^\dagger M_F'|_{\mathcal{D}((M_F')^\dagger)} = M_F'(M_F')^\dagger. \end{aligned}$$

Then $A_iXB_i + C_iYD_i + E_iZF_i = G_i(i = 1,2)$ *has positive solutions. Under the space decomposition* $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$*, the positive solutions have the matrix representation*

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & ((Y_{11}^{\frac{1}{2}})^\dagger Y_{12})^*(Y_{11}^{\frac{1}{2}})^\dagger Y_{12} + f \end{pmatrix} \tag{36}$$

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & ((Z_{11}^{\frac{1}{2}})^\dagger Z_{12})^*(Z_{11}^{\frac{1}{2}})^\dagger Z_{12} + f \end{pmatrix} \tag{37}$$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f \end{pmatrix} \tag{38}$$

where

$$Y_{11} = P_{\overline{\mathcal{R}(M_C^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} \ ,$$
$$Z_{11} = P_{\overline{\mathcal{R}((M_E')^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}}$$

are positive operators, satisfying that

$$Y_0 = \overline{M_C^\dagger M_C Y M_D M_D^\dagger},$$
$$Z_0 = \overline{(M_E')^\dagger M_E' Z_0 M_F M_F^\dagger}$$

are positive solutions of the equation system

$$M_{M_E} Y = \overline{N_{M_E} M_D^\dagger} \ ,$$
$$Y M_{M_F} = M_C^\dagger N_{M_F} \ ,$$

and

$$M_E' Z = \overline{N_N' M_F^\dagger} \ ,$$
$$Z M_F' = M_E^\dagger N_N'' \ ,$$

respectively, and Y_{12}, Z_{12}, *and* X_{12} *are arbitrary operators and satisfy*

$$\mathcal{R}(Y_{12}) \subseteq \mathcal{R}(Y_{11}^{\frac{1}{2}}), \quad Y_{21} = Y_{12}^*,$$
$$\mathcal{R}(Z_{12}) \subseteq \mathcal{R}(Z_{11}^{\frac{1}{2}}), \quad Z_{21} = Z_{12}^*,$$
$$\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}}), \quad X_{21} = X_{12}^*,$$

$$Y_{22} = ((Y_{11}^{\frac{1}{2}})^\dagger Y_{12})^*(Y_{11}^{\frac{1}{2}})^\dagger Y_{12} + f,$$
$$Z_{22} = ((Z_{11}^{\frac{1}{2}})^\dagger Z_{12})^*(Z_{11}^{\frac{1}{2}})^\dagger Z_{12} + f,$$
$$X_{22} = ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f,$$

and f *is positive.*

Proof. Suppose that $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (36). According to Theorem 4, the systems (22) and (23) have a solution equivalent to the systems (1) having a solution.

According to Lemma 1, since $M_{M_F} \overset{*}{\leq} M_D$, $M_{M_E} \overset{*}{\leq} M_C$ and (28) holds, then under the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, $M_{M_E}, M_C, M_D, M_{M_F}$, A, B, C, D, E, F, G have the matrix representations

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}, G = \begin{pmatrix} G_{11} & 0 \\ 0 & 0 \end{pmatrix}, M_C = \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix},$$
$$M_D = \begin{pmatrix} M_{21} & 0 \\ 0 & 0 \end{pmatrix}, M_{M_E} = \begin{pmatrix} M_{31} & 0 \\ 0 & 0 \end{pmatrix}, M_{M_F} = \begin{pmatrix} M_{41} & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting them into the systems (22) and (23), we obtained

$$M_{M_E} Y M_D = \begin{pmatrix} M_{31} Y_{11} M_{21} & 0 \\ 0 & 0 \end{pmatrix} = M_{M_E} P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_D,$$

$$M_C Y M_{M_F} = \begin{pmatrix} M_{11} Y_{11} M_{41} & 0 \\ 0 & 0 \end{pmatrix} = M_C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_{M_F},$$

where $P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} = M_C^\dagger M_C Y_0 M_D M_D^\dagger \in \mathcal{B}(\mathcal{D}(M_D^\dagger), \mathcal{H})$. Combined with

$$\begin{aligned} \mathcal{R}(N_{M_F}) &\subseteq \mathcal{R}(M_C), \\ \mathcal{R}(M_{M_F}) &\subseteq \mathcal{R}(M_D), \\ \mathcal{R}((N_{M_E})^*) &\subseteq \mathcal{R}((M_D)^*), \\ \mathcal{R}((M_{M_E})^*) &\subseteq \mathcal{R}((M_C)^*), \end{aligned}$$

we obtain

$$\begin{aligned} M_{M_E} Y M_D &= M_{M_E} P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_D = M_{M_E} M_C^\dagger M_C Y_0 M_D \\ &= M_{M_E} Y_0 M_D = \overline{N_{M_E} M_D^\dagger} M_D = N_{M_E} M_D^\dagger M_D = N_{M_E}, \\ M_C Y M_{M_F} &= M_C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} M_{M_F} = M_C Y_0 P_{\overline{\mathcal{R}(M_D)}} M_{M_F} \\ &= M_C Y_0 M_D M_D^\dagger M_{M_F} = M_C Y_0 M_{M_F} = M_C M_C^\dagger N_{M_F} = N_{M_F}. \end{aligned}$$

Therefore, $P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}}$ is a solution of the systems (22) and (23), as well as the system (1).

In addition, for an arbitrary $y \in \overline{\mathcal{R}((M_C)^*)} = \overline{\mathcal{R}(M_D)}$,

$$\begin{aligned} \langle Y_{11} y, y \rangle &= \langle P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 y, y \rangle = \langle Y_0 y, P_{\overline{\mathcal{R}((M_C)^*)}} y \rangle \\ &= \langle Y P_{\overline{\mathcal{R}((M_C)^*)}} y, P_{\overline{\mathcal{R}((M_C)^*)}} y \rangle \geq 0. \end{aligned}$$

Therefore, Y_0 is a positive solution of the system (1).

Assume that $Z \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (30). In the proof of Theorem 4, we also know that the systems (19) and (20) have a solution equivalent to the system (1) having a solution. Additionally, the Equations (19) and (20) have a solution if and only if the system

$$M_C Y M_D + M_E Z M_F = N_N$$

has a solution.

Using Lemma 2 again, Equation (21) has a solution equivalent to

$$\begin{aligned} \mathcal{R}(N_N - M_E Z M_F) &\subseteq \mathcal{R}(M_C), \\ \mathcal{R}((N_N - M_E Z M_F)^*) &\subseteq \mathcal{R}((M_D)^*). \end{aligned}$$

According to Lemma 3 and a simple calculation, we get that the systems (19) and (20) have a solution that is equivalent to the system

$$\begin{aligned} M_E' Z M_F &= N_N', \\ M_E Z M_F' &= N_N'', \end{aligned} \tag{39}$$

having a solution. Combined with

$$N'_N \overset{*}{\leq} M'_F \overset{*}{\leq} M_F,$$
$$N''_N \overset{*}{\leq} M'_E \overset{*}{\leq} M_E,$$
$$M'^{\dagger}_E M'_E = M'_F M'^{\dagger}_F,$$

we get that a positive solution Z of the system (37) is also a positive solution of (1).

In reality, similar to the proof above, under the space decomposition $\overline{\mathcal{R}(M'_F)} \oplus \mathcal{N}(M'_F) \to \overline{\mathcal{R}(M'_F)} \oplus \mathcal{N}(M'_F)$, M'_E, M_F, M_E, M'_F have the matrix representations

$$M'_E = \begin{pmatrix} M'_{11} & 0 \\ 0 & 0 \end{pmatrix}, M_F = \begin{pmatrix} M'_{21} & 0 \\ 0 & 0 \end{pmatrix}, M_E = \begin{pmatrix} M'_{31} & 0 \\ 0 & 0 \end{pmatrix}, M'_F = \begin{pmatrix} M'_{41} & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting them into the system (39), we obtained

$$M'_E Z M_F = \begin{pmatrix} M'_{11} Z_{11} M'_{21} & 0 \\ 0 & 0 \end{pmatrix} = M'_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_F,$$

$$M_E Z M'_F = \begin{pmatrix} M'_{31} Z_{11} M'_{41} & 0 \\ 0 & 0 \end{pmatrix} = M_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_{M'_F},$$

where $P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} = (M'_E)^{\dagger} M'_E Z_0 M_F M_F^{\dagger} \in \mathcal{B}(\mathcal{D}(M_F^{\dagger}), \mathcal{H})$. Combined with

$$\mathcal{R}(N''_N) \subseteq \mathcal{R}(M_E),$$
$$\mathcal{R}(M'_F) \subseteq \mathcal{R}(M_F),$$
$$\mathcal{R}((N'_N)^*) \subseteq \mathcal{R}((M_F)^*),$$
$$\mathcal{R}((M'_E)^*) \subseteq \mathcal{R}((M_E)^*),$$

we obtain

$$\begin{aligned} M'_E Z M_F &= M'_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M_F = M'_E (M_E)^{\dagger} M_E Z_0 M_F \\ &= M'_E Z_0 M_F = \overline{N'_N M_F^{\dagger}} M_F = N'_N M_F^{\dagger} M_F = N'_N, \\ M_E Z M'_F &= M_E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} M'_F = M_E Z_0 P_{\overline{\mathcal{R}(M_F)}} M'_F \\ &= M_E Z_0 M_F (M_F)^{\dagger} M'_F = M_E Z_0 M'_F = M_E M_E^{\dagger} N''_N = N''_N. \end{aligned}$$

Therefore, $P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}}$ is a solution of the system (39), as well as the system (1).

In addition, for an arbitrary $z \in \overline{\mathcal{R}((M'_E)^*)} = \overline{\mathcal{R}(M_F)}$,

$$\begin{aligned} \langle Z_{11} z, z \rangle &= \langle P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 z, z \rangle = \langle Z_0 z, P_{\overline{\mathcal{R}((M'_E)^*)}} z \rangle \\ &= \langle Z_0 P_{\overline{\mathcal{R}((M'_E)^*)}} z, P_{\overline{\mathcal{R}((M'_E)^*)}} z \rangle \geq 0. \end{aligned}$$

we can conclude that Z_0 is a positive solution of the system (1).

On the same assumption, $X \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (38). Substituting the matrix representation of Y, Z, and X into (18), we obtain

$$A_{11} X_{11} B_{11} + C_{11} Y_{11} D_{11} + E_{11} Z F_{11} = G_{11},$$

which is equivalent to

$$AXB = G - C P_{\overline{\mathcal{R}((M_C)^*)}} Y_0 P_{\overline{\mathcal{R}(M_D)}} D - E P_{\overline{\mathcal{R}((M'_E)^*)}} Z_0 P_{\overline{\mathcal{R}(M_F)}} F,$$

then

$$\begin{aligned} X &= \overline{A^\dagger(G - CP_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}}D - EP_{\overline{\mathcal{R}((M_E')^*)}}Z_0P_{\overline{\mathcal{R}(M_F)}}F)B^\dagger} \\ &= \overline{A^\dagger(G - CY_0D - EZ_0F)B^\dagger} \in \mathcal{B}(\mathcal{G},\mathcal{H}). \end{aligned}$$

In addition, for an arbitrary $x \in \mathcal{G}$, combined with the positivity of $A^\dagger(G - CY_0D - EZ_0F)B^\dagger|_{\mathcal{D}(B^\dagger)}$,

$$\langle X_{11}x, x\rangle = \langle \overline{A^\dagger(G - CY_0D - EZ_0F)B^\dagger}x, x\rangle \geq 0.$$

We have $X_{11}^* = X_{11}$. Therefore, X is also a positive solution of the system (1).

Conversely, assume that $Y \in \mathcal{B}(\mathcal{G},\mathcal{H})$ is a positive solution to Equation (18); then Y is also the positive solution of the systems (22) and (23). Under the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, set

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

We denote the continuous bounded linear extension of $M_C^\dagger M_C Y M_D M_D^\dagger$ as $Y_0 = \overline{M_C^\dagger M_C Y M_D M_D^\dagger} \in \mathcal{B}(\mathcal{G},\mathcal{H})$. By $\mathcal{R}(M_{M_F}) \subseteq \mathcal{R}(M_D) \subseteq \mathcal{D}(M_D^\dagger) = \mathcal{D}(Y)$, we obtain $Y_0M_{M_F} = YM_{M_F} = M_C^\dagger N_{M_F}$. From $\mathcal{R}((N_{M_E})^*) \subseteq \mathcal{R}((M_D)^*)$, combined with Lemma 3, there exists $\widetilde{Y} \in \mathcal{B}(\mathcal{G},\mathcal{H})$ such that $N_{M_E} = \widetilde{Y}M_D$. It follows that $N_{M_E}M_D^\dagger = \widetilde{Y}M_DM_D^\dagger \in \mathcal{B}(\mathcal{D}(M_D^\dagger),\mathcal{H})$. In that way, for any $y \in \mathcal{G}, y_n \in \mathcal{D}(M_D^\dagger)$ such that $y_n \to y$. Therefore,

$$M_{M_E}Y_0y = M_{M_E}(\lim_{n\to\infty} Y_0y_n) = \lim_{n\to\infty} M_{M_E}Y_0y_n = \lim_{n\to\infty} N_{M_E}M_D^\dagger y_n = \overline{N_{M_E}M_D^\dagger}y.$$

Then Y_0 is a solution of the system $M_{M_E}Y_0 = \overline{N_{M_E}M_D^\dagger}, Y_0M_{M_F} = M_C^\dagger N_{M_F}$.

By a simple calculation, we obtain

$$\begin{aligned} \langle Y_0y, y\rangle &= \lim_{n\to\infty}\langle Y_0y_n, y_n\rangle = \lim_{n\to\infty}\langle M_C^\dagger M_C Y M_D M_D^\dagger y_n, y_n\rangle \\ &= \lim_{n\to\infty}\langle YM_DM_D^\dagger y_n, M_DM_D^\dagger y_n\rangle \geq 0. \end{aligned}$$

It follows that $Y_{11} = P_{\overline{\mathcal{R}((M_C)^*)}}Y_0P_{\overline{\mathcal{R}(M_D)}}$. According to Lemma 6, Y is a positive operator implying that Y_{12} is an arbitrary operator and satisfying $\mathcal{R}(Y_{12}) \subseteq \mathcal{R}(Y_{11}^{\frac{1}{2}})$, $Y_{21} = Y_{12}^*$, $Y_{22} = ((Y_{11}^{\frac{1}{2}})^\dagger Y_{12})^*(Y_{11}^{\frac{1}{2}})^\dagger Y_{12} + f$.

Similar to the proof above, we have $Z_{11} = P_{\overline{\mathcal{R}((M_E')^*)}}Z_0P_{\overline{\mathcal{R}(M_F)}}$. Since Z is positive operator, we have Z_{12} as an arbitrary operator and satisfying $\mathcal{R}(Z_{12}) \subseteq \mathcal{R}(Z_{11}^{\frac{1}{2}})$, $Z_{21} = Z_{12}^*$, $Z_{22} = ((Z_{11}^{\frac{1}{2}})^\dagger Z_{12})^*(Z_{11}^{\frac{1}{2}})^\dagger Z_{12} + f$. Therefore, Z has the matrix representation of (37).

Suppose that $X \in \mathcal{B}(\mathcal{G},\mathcal{H})$ is a positive solution to the system (18); then find the operator matrix representation of X. Under the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$, set

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Substituting X into the equation $AXB + CYD + EZF = G$, we obtain

$$\begin{aligned} X &= \begin{pmatrix} A_{11}^\dagger(G_{11} - C_{11}Y_{11}D_{11} - E_{11}ZF_{11})B_{11}^\dagger & 0 \\ 0 & 0 \end{pmatrix} \\ &= \overline{A^\dagger(G - CYD - EZF)B^\dagger} \in \mathcal{B}(\mathcal{G},\mathcal{H}), \end{aligned}$$

according to the known assumptions; $A^\dagger(G - CYD - EZF)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is positive operator, and we get that

$$X_{11} = A^\dagger(G - CP_{\overline{\mathcal{R}((M_C)^*)}}YP_{\overline{\mathcal{R}(M_D)}}D - EP_{\overline{\mathcal{R}((M_C)^*)}}YP_{\overline{\mathcal{R}(M_D)}}F)B^\dagger$$

is a positive operator. X_{12} is an arbitrary operator and satisfies $\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}})$, $X_{21} = X_{12}^*$, $X_{22} = ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f$. Therefore, the matrix representation of X is (38). □

For convenience, we define the notation as follows ($i = 1, 2$):

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}, \qquad D = \begin{pmatrix} D_1 & D_2 \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$G = \begin{pmatrix} G_1 & U \\ V & G_2 \end{pmatrix}, \text{for some } U, V \in \mathcal{B}(\mathcal{F}, \mathcal{K}).$$
$$M_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^\dagger)C : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K},$$
$$N_1 = (I_{\mathcal{K}\oplus\mathcal{K}} - AA^\dagger)G : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K},$$
$$M_2 = D(I_{\mathcal{F}\oplus\mathcal{F}} - B^\dagger B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{G},$$
$$N_2 = G(I_{\mathcal{F}\oplus\mathcal{F}} - B^\dagger B) : \mathcal{F} \oplus \mathcal{F} \to \mathcal{K} \oplus \mathcal{K}.$$

Corollary 16. *Let $A_i, C_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B_i, D_i \in \mathcal{B}(\mathcal{F}, \mathcal{G}), G_i \in \mathcal{B}(\mathcal{F}, \mathcal{K}), (i = 1, 2)$. If $A^\dagger(G - CYD)B^\dagger|_{\mathcal{D}(B^\dagger)}$ is a positive operator, $N_1 \overset{*}{\leq} M_2 \overset{*}{\leq} D$, $N_2 \overset{*}{\leq} M_1 \overset{*}{\leq} C$ and*

$$C^\dagger C|_{\mathcal{D}(C^\dagger)} = CC^\dagger = D^\dagger D|_{\mathcal{D}(D^\dagger)} = DD^\dagger = A^\dagger A|_{\mathcal{D}(A^\dagger)}$$
$$= AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger = G^\dagger G|_{\mathcal{D}(G^\dagger)} = GG^\dagger.$$

Then $A_iXB_i + C_iYD_i = G_i (i = 1, 2)$ has a pair of positive solutions. Under the space decomposition $\overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C) \to \overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C)$, the positive solutions have the matrix representation

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & ((Y_{11}^{\frac{1}{2}})^\dagger Y_{12})^*(Y_{11}^{\frac{1}{2}})^\dagger Y_{12} + f \end{pmatrix} \tag{40}$$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f \end{pmatrix} \tag{41}$$

where $Y_{11} = P_{\overline{\mathcal{R}(C^*)}} Y P_{\overline{\mathcal{R}(D)}}$ and $X_{11} = A^\dagger(G - CP_{\overline{\mathcal{R}(C^*)}} Y P_{\overline{\mathcal{R}(D)}} D)B^\dagger$ are positive operators, satisfying that $Y_0 = \overline{C^\dagger CYDD^\dagger}$ is a positive solution to the system

$$M_1 Y = \overline{N_1 D^\dagger},$$
$$YM_2 = C^\dagger N_2,$$

Y_{12} and X_{12} are arbitrary operators, Y_{22} and X_{22} are arbitrary positive operators, $R(Y_{12}) \subseteq R(Y_{11}^{\frac{1}{2}})$ and $R(X_{12}) \subseteq R(X_{11}^{\frac{1}{2}})$, f is positive.

Proof. Assume that $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (40). Since $M_2 \overset{*}{\leq} D$, $M_1 \overset{*}{\leq} C$ and (33) hold, then, under the space decomposition $\overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C) \to \overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C)$, M_1, C, D, M_2, A, B, G have the matrix representations

$$M_1 = \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} M_{21} & 0 \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}, G = \begin{pmatrix} G_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting them into the systems (26) and (27), we obtained

$$M_1YD = \begin{pmatrix} M_{11}Y_{11}D_{11} & 0 \\ 0 & 0 \end{pmatrix} = M_1P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}D,$$

$$CYM_2 = \begin{pmatrix} C_{11}Y_{11}M_{21} & 0 \\ 0 & 0 \end{pmatrix} = CP_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}M_2,$$

where $P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}} = C^\dagger CYDD^\dagger \in \mathcal{B}(\mathcal{D}(D^\dagger)\mathcal{H})$. Combined with

$$\begin{aligned} &\mathcal{R}(N_2) \subseteq \mathcal{R}(C), \\ &\mathcal{R}(M_2) \subseteq \mathcal{R}(D), \\ &\mathcal{R}(N_1^*) \subseteq \mathcal{R}(D^*), \\ &\mathcal{R}(M_1^*) \subseteq \mathcal{R}(C^*), \end{aligned}$$

we obtain

$$M_1YD = M_1P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}D = M_1C^\dagger CYD = M_1YD = N_1D^\dagger D = N_1,$$

$$CYM_2 = CP_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}M_2 = CYDD^\dagger M_2 = CYM_2 = CC^\dagger N_2 = N_2.$$

Therefore, Y is a solution of the systems (26) and (27), as well as the system (2).

In addition, for an arbitrary $y \in \overline{\mathcal{R}(C^*)} = \overline{\mathcal{R}(D)}$,

$$\langle Y_{11}y, y\rangle = \langle P_{\overline{\mathcal{R}(C^*)}}Yy, y\rangle = \langle Yy, P_{\overline{\mathcal{R}(C^*)}}y\rangle = \langle YP_{\overline{\mathcal{R}(C^*)}}y, P_{\overline{\mathcal{R}(C^*)}}y\rangle \geq 0.$$

Therefore, Y is a positive solution of the system (2).

On the same assumption, $X \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ has the matrix representation (41). Substituting the matrix form of Y and X into (25), we obtain

$$A_{11}X_{11}B_{11} + C_{11}Y_{11}D_{11} = G_{11},$$

which is equivalent to $AXB = G - CP_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}D$, then

$$X = \overline{A^\dagger(G - CP_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}D)B^\dagger} = \overline{A^\dagger(G - CYD)B^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H}).$$

In addition, for an arbitrary $x \in \mathcal{G}$, combined with the positivity of $A^\dagger(G - CYD)B^\dagger|_{\mathcal{D}(B^\dagger)}$,

$$\langle X_{11}x, x\rangle = \overline{\langle A^\dagger(G - CYD)B^\dagger x, x\rangle} \geq 0.$$

Therefore, X is also one positive solution of the system (2).

Conversely, assuming that $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a positive solution to the systems (25), then Y is also the positive solution of the system (26) and (27). Under the space decomposition $\overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C) \to \overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C)$, set

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

We denote the continuous bounded linear extension of $C^\dagger CYDD^\dagger$ as $Y_0 = \overline{C^\dagger CYDD^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. By $\mathcal{R}(M_2) \subseteq \mathcal{R}(D) \subseteq \mathcal{D}(D^\dagger) = \mathcal{D}(Y)$, we obtain $Y_0M_2 = YM_2 = C^\dagger N_2$. From $\mathcal{R}(N_1^*) \subseteq \mathcal{R}(D^*)$, combined with Lemma 3, there exists $\widetilde{Z} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that

$N_1 = \widetilde{Z}D$. It follows that $N_1D^\dagger = \widetilde{Z}DD^\dagger \in \mathcal{B}(\mathcal{D}(D^\dagger), \mathcal{H})$. In that way, for any $y \in \mathcal{G}$, $y_n \in \mathcal{D}(D^\dagger)$ such that $y_n \to y$. Therefore,

$$M_1Y_0y = M_1(\lim_{n\to\infty} Y_0y_n) = \lim_{n\to\infty} M_1Y_0y_n = \lim_{n\to\infty} N_1D^\dagger y_n = \overline{N_1D^\dagger}y.$$

Then Y_0 is a solution of the system $M_1Y = \overline{N_1D^\dagger}, YM_2 = C^\dagger N_2$. Additionally, we have

$$\begin{aligned}\langle Y_0y, y\rangle &= \lim_{n\to\infty}\langle Y_0y_n, y_n\rangle = \lim_{n\to\infty}\langle C^\dagger CYDD^\dagger y_n, y_n\rangle \\ &= \lim_{n\to\infty}\langle YDD^\dagger y_n, DD^\dagger y_n\rangle \geq 0.\end{aligned}$$

Therefore, $Y_0 \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a positive solution of the system $M_1Y = \overline{N_1D^\dagger}, YM_2 = C^\dagger N_2$. After a simple calculation, we obtain

$$\begin{aligned}P_{\overline{\mathcal{R}(C^*)}}Y_0P_{\overline{\mathcal{R}(D)}} &= P_{\overline{\mathcal{R}(C^*)}}\overline{P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}|_{\mathcal{D}(D^\dagger)}}P_{\overline{\mathcal{R}(D)}} \\ &= P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}} \in \mathcal{B}(\mathcal{G}, \mathcal{H}).\end{aligned}$$

Therefore, it follows that $Y_{11} = P_{\overline{\mathcal{R}(C^*)}}Y_0P_{\overline{\mathcal{R}(D)}} = P_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}$. According to Lemma 6, Y is positive, implying that $\mathcal{R}(Y_{12}) \subseteq \mathcal{R}(Y_{11}^{\frac{1}{2}}), Y_{21} = Y_{12}^*,\ Y_{22} = ((Y_{11}^{\frac{1}{2}})^\dagger Y_{12})^*(Y_{11}^{\frac{1}{2}})^\dagger Y_{12} + f$. Suppose that $X \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ is a positive solution to the system (25); then find the operator matrix representation of X. Under the space decomposition $\overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C) \to \overline{\mathcal{R}(C^*)} \oplus \mathcal{N}(C)$, set

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Substituting X into the equation $AXB + CYD = G$, we obtain

$$X = \begin{pmatrix} A_{11}^\dagger(G_{11} - C_{11}Y_{11}D_{11})B_{11}^\dagger & 0 \\ 0 & 0 \end{pmatrix} = \overline{A^\dagger(G - CYD)B^\dagger} \in \mathcal{B}(\mathcal{G}, \mathcal{H}),$$

using Lemma 6 again, due to X being a positive operator, combined with the known assumptions $A^\dagger(G - CYD)B^\dagger|_{\mathcal{D}(B^\dagger)}$ being a positive operator, we get that $X_{11} = A^\dagger(G - CP_{\overline{\mathcal{R}(C^*)}}YP_{\overline{\mathcal{R}(D)}}D)B^\dagger$ is a positive operator. X_{12} is an arbitrary operator and satisfies $\mathcal{R}(X_{12}) \subseteq \mathcal{R}(X_{11}^{\frac{1}{2}}), X_{21} = X_{12}^*,\ X_{22} = ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f$. Therefore, the matrix representation of X is (41). □

The positive solutions of $A_1XB_1 = G_1, A_2XB_2 = G_2$ can be obtained by taking $C_i = D_i = 0, (i = 1, 2)$ from Corollary 16.

Corollary 17 ([28], Theorem 2.7). *Let $A_i, B_i, G_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,2}$ such that*

$$G_1 \overset{*}{\leq} A_2 \overset{*}{\leq} A_1,$$
$$G_2 \overset{*}{\leq} B_1 \overset{*}{\leq} B_2$$

and

$$A^\dagger A|_{\mathcal{D}(A^\dagger)} = AA^\dagger = B^\dagger B|_{\mathcal{D}(B^\dagger)} = BB^\dagger.$$

If the system $A_iXB_i = G_i, i = \overline{1,2}$ has a positive solution, then the general positive solution has the matrix representation

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^*(X_{11}^{\frac{1}{2}})^\dagger X_{12} + f \end{pmatrix}$$

in terms of $\mathcal{H} = \overline{\mathcal{R}(A_1^*)} \oplus \mathcal{N}(A_1)$, *where f is positive,* $R(X_{12}) \subseteq R(X_{11}^{\frac{1}{2}})$ *and* $X_{11} = P_{\overline{\mathcal{R}(A_1^*)}} \widetilde{Y} P_{\overline{\mathcal{R}(B_2)}}$, *satisfying that* $\widetilde{Y}$ *is a positive solution of*

$$\begin{aligned} A_2 X &= \overline{G_2 B_2^\dagger}, \\ X B_1 &= A_1^\dagger G_1. \end{aligned}$$

The following result can be obtained by taking $A_1 = B_2, B_1 = A_2, G_1 = G_2 = G$ from Corollary 16.

Corollary 18 ([32], Theorem 3.4)**.** *Let* $A, B, G \in \mathcal{B}(\mathcal{H})$ *such that* A *has a closed range with* $A^\dagger A = A A^\dagger$ *and* $G \overset{*}{\leq} B \overset{*}{\leq} A$. *If the system* $AXB = G = BXA$ *has a positive solution, then the general positive solutions have the matrix representation*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & ((X_{11}^{\frac{1}{2}})^\dagger X_{12})^* (X_{11}^{\frac{1}{2}})^\dagger X_{12} + f \end{pmatrix}$$

with respect to $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, *where f is positive,* $R(X_{12}) \subseteq R(X_{11}^{\frac{1}{2}})$ *and* $X_{11} = P_{\mathcal{R}(A^*)} \widetilde{Y} P_{\mathcal{R}(A^*)}$ *satisfying that* $\widetilde{Y}$ *is a positive solution of*

$$\begin{aligned} BX &= GA^\dagger, \\ XB &= A^\dagger G. \end{aligned}$$

5. Conclusions

The purpose of this paper is to study the necessary and sufficient conditions for the existence of general, self-adjoint, and positive solutions of the operator system $A_i X B_i + C_i Y D_i + E_i Z F_i = G_i (i = 1, 2)$ under additional assumptions. Furthermore, we discuss the representation of general solutions of the operator system $A_i X B_i + C_i Y Di + EiZFi = Gi(I = 1, 2)$ using star-order conditions. Additionally, we provide the operator matrix representation for self-adjoint and positive solutions with respect to the space decomposition $\overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C) \to \overline{\mathcal{R}((M_C)^*)} \oplus \mathcal{N}(M_C)$.

Moreover, by means of space decomposition, generalized inverse, and star order, we can establish some equivalent conditions for solvability and the representation of general, self-adjoint, and positive solutions to the system

$$\begin{cases} A_1 X B_1 + C_1 Y D_1 + E_1 Z F_1 = G_1, \\ A_2 X B_2 + C_2 Y D_2 + E_2 Z F_2 = G_2, \\ H X = G_3, \\ X J = G_4. \end{cases}$$

$$\begin{cases} A_1 X B_1 + C_1 Y D_1 + E_1 Z F_1 = G_1, \\ A_2 X B_2 + C_2 Y D_2 + E_2 Z F_2 = G_2, \\ H X = G_3, \\ Y J = G_4, \end{cases}$$

and so on [45–47]. We hope that such equations, like their classical prototypes, will find applications in control theory, information theory, linear system theory, sampling, and other areas.

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