

Article

Harmonic Series with Multinomial Coefficient $\binom{4n}{n,n,n,n}$ and Central Binomial Coefficient $\binom{2n}{n}$

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Abstract: Classical hypergeometric series are reformulated as analytic functions of their parameters (in both the numerator and the denominator). Then, the coefficient extraction method is applied to examine hypergeometric series transformations. Several new closed form evaluations are established for harmonic series containing multinomial coefficient $\binom{4n}{n,n,n,n}$ and central binomial coefficient $\binom{2n}{n}$. These results exclusively concern the alternating series of convergence rate “ $-1/4$ ”.

Keywords: hypergeometric series; harmonic number; Riemann zeta function; multinomial coefficient

MSC: 11B65; 33C20; 65B10

1. Introduction and Outline

Harmonic numbers have a very long history that can be traced back to Euler, who initiated his investigation in the course of a correspondence with Goldbach in 1742. Nowadays, these numbers appear frequently in the literature of mathematics (particularly number theory [1–3] and combinatorial analysis [4,5]), physics (standing waves in strings [6]), and computer science (analysis of algorithms [7]), just for examples. They are defined, for $m, n \in \mathbb{N}_0$, by partial sums

$$\begin{aligned} H_n^{(m)} &= \sum_{k=0}^{n-1} \frac{1}{(k+1)^m}, & O_n^{(m)} &= \sum_{k=0}^{n-1} \frac{1}{(2k+1)^m}; \\ \tilde{H}_n^{(m)} &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^m}, & \bar{O}_n^{(m)} &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^m}. \end{aligned}$$

When $m = 1$, it will be suppressed by the above notation. These numbers satisfy the following two simple but useful relations:

$$H_{2n}^{(m)} = O_n^{(m)} + 2^{-m} H_n^{(m)} \quad \text{and} \quad \tilde{H}_{2n}^{(m)} = \bar{O}_n^{(m)} - 2^{-m} H_n^{(m)}.$$

For an indeterminate x and $n \in \mathbb{N}_0$, the rising factorial is defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{for } x \in \mathbb{N}.$$

Denote by $[x^m]\phi(x)$ the coefficient of x^m in the formal power series $\phi(x)$. When $n \in \mathbb{N}$, it is almost routine to check that



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$$\begin{aligned} \mathbf{H}_n &= [x] \frac{n!}{(1-x)_n} = [x] \frac{(1+x)_n}{n!}, \\ 2\mathbf{O}_n &= [x] \frac{(\frac{1}{2})_n}{(\frac{1}{2}-x)_n} = [x] \frac{(\frac{1}{2}+x)_n}{(\frac{1}{2})_n}. \end{aligned}$$

By means of the generating function approach (see Wilf [8]), it is not difficult to establish the following general formulae (cf. Chu [9]):

$$\begin{aligned} [x^m] \frac{n!}{(1-x)_n} &= \Omega_m(\mathbf{H}_n^{(k)}) \quad \text{and} \quad [x^m] \frac{(1-x)_n}{n!} = \Omega_m(-\mathbf{H}_n^{(k)}), \\ [x^m] \frac{(\frac{1}{2})_n}{(\frac{1}{2}-x)_n} &= \Omega_m(2^k \mathbf{O}_n^{(k)}) \quad \text{and} \quad [x^m] \frac{(\frac{1}{2}+x)_n}{(\frac{1}{2})_n} = \Omega_m(2^k \mathbf{O}_n^{(k)}); \end{aligned}$$

where the Bell polynomials (cf. [10] §3.3) are expressed by the multiple sum

$$\Omega_m(T_k) = [x^m] \exp \left\{ \sum_{k \geq 1} \frac{x^k}{k} T_k \right\} = \sum_{\Lambda(m)} \prod_{k=1}^m \frac{T_k^{j_k}}{j_k! k^{j_k}},$$

with $\Lambda(m)$ denoting the set of m -partitions represented by m -tuples $(j_1, j_2, \dots, j_m) \in \mathbb{N}_0^m$ subject to the linear condition $\sum_{k=1}^m k j_k = m$. The first five polynomials are recorded as follows:

$$\begin{aligned} \Omega_1(T_k) &= T_1, \\ \Omega_2(T_k) &= \frac{1}{2} (T_1^2 + T_2), \\ \Omega_3(T_k) &= \frac{1}{6} (T_1^3 + 3T_2T_1 + 2T_3), \\ \Omega_4(T_k) &= \frac{1}{24} (T_1^4 + 6T_2T_1^2 + 8T_3T_1 + 3T_2^2 + 6T_4), \\ \Omega_5(T_k) &= \frac{1}{120} (T_1^5 + 10T_2T_1^3 + 20T_3T_1^2 + 15T_2^2T_1 + 30T_4T_1 + 20T_2T_3 + 24T_5). \end{aligned}$$

There exist different techniques to deal with infinite series with harmonic numbers, for example, Abel's lemma on summation by parts (cf. [11–13]), integral computations (cf. [14–18]), and the Legendre–Fourier series (cf. [19–21]). During the course of our investigation, we find that the “coefficient extraction” method (cf. [9,22]) based on the hypergeometric series is more powerful. For a given hypergeometric series equality, this approach consists of the three main steps below:

- Reformulate the equality by identifying a variable “ x ” and eventual parameters $\{a, b, c\}$ so that both sides of the resulting equality are analytic in x at $x = 0$.
- Determine the formal equality by extracting across the equality and then equating the coefficients $W_m(a, b, c)$ for a fixed monomial x^m .
- Find infinite series identities by computing the coefficients $W_m(a, b, c)$ for particularly specified values of parameters $\{a, b, c\}$.

As sample series, we highlight in advance the following eight representative ones:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{5+12n}{1+2n} - (3+20n) \mathbf{H}_n \right\} = \frac{24 \ln 2}{\pi}, \quad (\text{Equation (2)})$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{1+12n}{1+2n} - 2(3+20n) \mathbf{O}_n \right\} = \frac{8 \ln 2}{\pi}, \quad (\text{Equation (3)})$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n} (4n+1)} \left\{ \frac{1+4n}{n^3} - \frac{2(3+10n)}{n} \mathbf{O}_n^{(2)} \right\} = \zeta(3), \quad (\text{Equation (8)})$$

$$\sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{(1-4n)(3-4n)}{n^4} - \frac{(1-2n)(3-10n)}{n^2} \mathbf{H}_n^{(2)} \right\} = 0, \quad (\text{Equation (4)})$$

$$\sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{1+6n-32n^2}{n^3(2n-1)} + \frac{4(1-2n)(3-10n)}{n^2} \mathbf{O}_n^{(2)} \right\} = 14\zeta(3), \quad (\text{Equation (5)})$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n} (4n+1)} \left\{ \frac{1+3n}{n^3} + \frac{1+6n+10n^2}{n^2(1+2n)} \mathbf{O}_{2n+1} \right\} = \zeta(3) - 2, \quad (\text{Equation (6)})$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n} (4n+1)} \left\{ \frac{1+4n}{4n^4} - \frac{1+6n+10n^2}{n^2(1+2n)} \mathbf{O}_n^{(2)} \right\} = \frac{\pi^4}{360}, \quad (\text{Equation (7)})$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{n^5} - \frac{4\mathbf{O}_n^{(2)}}{n^3} + \frac{4(3+10n)}{n(1+4n)} ((\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}) \right\} = \zeta(5), \quad (\text{Equation (9)})$$

where we have employed the notations for the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{for } \Re(z) > 1$$

and the multinomial coefficient

$$\binom{n_1 + n_2 + \dots + n_m}{n_1, n_2, \dots, n_m} = \frac{(\sum_{k=1}^m n_k)!}{\prod_{k=1}^m n_k!}, \quad \text{in particular, } \binom{m+n}{n} = \binom{m+n}{m, n}.$$

Throughout the paper, the classical hypergeometric series (cf. Bailey [23]) is defined by

$${}_1+mF_m \left[\begin{matrix} a_0, a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_m \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} z^k \frac{(a_0)_k (a_1)_k (a_2)_k \dots (a_m)_k}{k! (b_1)_k (b_2)_k \dots (b_m)_k},$$

where we assume that $|z| < 1$ and that none of the parameters $\{a_i, b_j\}$ is a non-positive integer, so that the series is well-defined and convergent. When a hypergeometric series admits a closed formula, it is generally expressed in terms of the Γ -function quotient (cf. [23–25]):

$$\Gamma \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] = \frac{\Gamma(\alpha) \Gamma(\beta) \dots \Gamma(\gamma)}{\Gamma(A) \Gamma(B) \dots \Gamma(C)},$$

where the Γ -function is defined by the Euler integral

$$\Gamma(x) = \int_0^{\infty} \tau^{x-1} e^{-\tau} d\tau \quad \text{for } \Re(x) > 0.$$

Its logarithmic differentiation results in the digamma function (cf. Rainville [26] §9):

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)},$$

with the Euler–Mascheroni constant being given by the limit

$$\gamma = \lim_{n \rightarrow \infty} (\mathbf{H}_n - \ln n) \approx 0.57721566490.$$

By introducing two further sequences of zeta-variants

$$\begin{aligned} \sigma_1 &= \gamma & \text{and} & \quad \sigma_m = \zeta(m) & \text{for} & \quad m \geq 2, \\ \tau_1 &= \gamma + 2 \ln 2 & \text{and} & \quad \tau_m = (2^m - 1)\zeta(m) & \text{for} & \quad m \geq 2, \end{aligned}$$

we then have the power series expansions of the Γ -function [22]

$$\begin{aligned} \Gamma(1-x) &= \exp \left\{ \sum_{k \geq 1} \frac{\sigma_k}{k} x^k \right\}, \\ \Gamma\left(\frac{1}{2} - x\right) &= \sqrt{\pi} \exp \left\{ \sum_{k \geq 1} \frac{\tau_k}{k} x^k \right\}, \end{aligned}$$

which will be crucial for extracting coefficients from the Γ -function quotients.

The rest of the paper is organized as follows. The next section constitutes the central part of the paper, where the transformation formula previously published by the second author (see [27] Theorem 2.13) will be examined by employing the “coefficient extraction” method, which leads us to numerous infinite series identities with the multinomial coefficient $\binom{4n}{n,n,n,n}$. Finally, the paper will end with Section 3, where a brief concluding discussion will be given.

2. Infinite Series of Convergence Rate “−1/4”

Ten years ago, the second author (see [27] Theorem 2.13) found an accelerating formula between a slowly convergent series (with the convergence rate “1”) and a faster convergent one (with the convergence rate “−1/4”), which can be reproduced as in the lemma below.

Lemma 1. For four parameters $\{a, b, c, d\} \subset \mathbb{C}$ satisfying the condition $\Re(c + d - a - b) > 1$, the following transformation holds:

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} = \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (c-b)_n (d-a)_n (d-b)_{2n}}{(c)_n (d)_{2n+1} (c+d-a-b-1)_{2n+2}} W_n(a, b, c, d),$$

where the cubic polynomial $W_n(a, b, c, d)$ is given by

$$\begin{aligned} W_n(a, b, c, d) &= (d + 2n)(c + d - a - b + 2n)(c + d - b - 1 + 3n) \\ &\quad - (a + n)(c - b + n)(d - b + 2n). \end{aligned}$$

By means of the “coefficient extraction” method, this transformation will be utilized to evaluate the infinite series containing both harmonic numbers and the multinomial coefficient $\binom{4n}{n,n,n,n}$. In order to reformulate the series with respect to k on the left to an analytic function with a fixed variable, we shall frequently make use of Thomae’s transformation formula (cf. Bailey [23] §3.2), which is recorded as below:

$${}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = {}_3F_2 \left[\begin{matrix} b-a, d-a, \Delta \\ c+\Delta, e+\Delta \end{matrix} \middle| 1 \right] \frac{\Gamma(\Delta)\Gamma(b)\Gamma(d)}{\Gamma(a)\Gamma(c+\Delta)\Gamma(e+\Delta)}, \quad (1)$$

where $\Delta = b + d - a - c - e$ denotes the parameter excess.

$$2.1. \boxed{a \rightarrow \frac{1}{2} + ax, b \rightarrow \frac{1}{2} + bx, c \rightarrow 1 + cx, d \rightarrow 1 + dx}$$

Hereafter, we suppose that x is always a variable. Under the parameter setting in the title, the transformation in Lemma 1 can be written equivalently as

$$x(c + d - a - b) \sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (\frac{1}{2} + bx)_k}{(1 + cx)_k (1 + dx)_k} = \sum_{n=0}^{\infty} (-1)^n W_n(\frac{1}{2} + ax, \frac{1}{2} + bx, 1 + cx, 1 + dx) \\ \times \frac{(\frac{1}{2} + ax)_n (\frac{1}{2} - bx + cx)_n (\frac{1}{2} - ax + dx)_n (\frac{1}{2} - bx + dx)_{2n}}{(1 + cx)_n (1 + dx)_{2n+1} (1 - ax - bx + cx + dx)_{2n+1}}.$$

According to D'Alembert's ratio test, the series on the left diverges when $x \rightarrow 0$. We have to reformulate it by means of Thomae's transformation (1) as below, so that it can be expanded into Maclaurin series in x :

$$x(c + d - a - b) \sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (\frac{1}{2} + bx)_k}{(1 + cx)_k (1 + dx)_k} = x(c + d - a - b) {}_3F_2 \left[\begin{matrix} 1, \frac{1}{2} + ax, \frac{1}{2} + bx \\ 1 + cx, 1 + dx \end{matrix} \middle| 1 \right] \\ = {}_3F_2 \left[\begin{matrix} cx, dx, x(c + d - a - b) \\ \frac{1}{2} + x(c + d - a), \frac{1}{2} + x(c + d - b) \end{matrix} \middle| 1 \right] \Gamma \left[\begin{matrix} 1 + cx, 1 + dx, 1 + x(c + d - a - b) \\ \frac{1}{2} + x(c + d - a), \frac{1}{2} + x(c + d - b) \end{matrix} \right].$$

By substitution, we have derived the following transformation formula.

Proposition 1 (Functional equality).

$$\sum_{n=0}^{\infty} (-1)^n W_n(\frac{1}{2} + ax, \frac{1}{2} + bx, 1 + cx, 1 + dx) \frac{(\frac{1}{2} + ax)_n (\frac{1}{2} - bx + cx)_n (\frac{1}{2} - ax + dx)_n (\frac{1}{2} - bx + dx)_{2n}}{(1 + cx)_n (1 + dx)_{2n+1} (1 - ax - bx + cx + dx)_{2n+1}} \\ = {}_3F_2 \left[\begin{matrix} cx, dx, x(c + d - a - b) \\ \frac{1}{2} + x(c + d - a), \frac{1}{2} + x(c + d - b) \end{matrix} \middle| 1 \right] \Gamma \left[\begin{matrix} 1 + cx, 1 + dx, 1 + x(c + d - a - b) \\ \frac{1}{2} + x(c + d - a), \frac{1}{2} + x(c + d - b) \end{matrix} \right].$$

Observing that this is an analytic equality around $x = 0$, we can expand both sides into Maclaurin series in x . Let $A_m(a, b, c, d)$ stand for the resulting equality from the coefficients of x^m extracted across the above equality. As a couple of examples, we illustrate how to deduce infinite series identities from the above transformation.

First, letting $b = d = 0$ in Proposition 1, we can reformulate the resulting expression as

$$\Gamma \left[\begin{matrix} 1 + cx, 1 + cx - ax \\ \frac{1}{2} + cx, \frac{1}{2} + cx - ax \end{matrix} \right] = \sum_{n=0}^{\infty} \binom{4n}{n, n, n, n} \frac{W_n(\frac{1}{2} + ax, \frac{1}{2}, 1 + cx, 1)}{(-1024)^n (2n + 1)^2} \\ \times \frac{(\frac{1}{2} + ax)_n (\frac{1}{2} - ax)_n (\frac{1}{2} + cx)_n n! (2n + 1)!}{(\frac{1}{2})_n^3 (1 + cx)_n (1 - ax + cx)_{2n+1}}.$$

By extracting the coefficient of x across the above equation and then making some simplifications, we obtain the equality

$$\frac{2(a - 2c) \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{3a - 7c + 12n(a - c)}{8(2n + 1)} - \frac{\mathbf{H}_n}{16} (3 + 20n)(a - 3c) - \frac{\mathbf{O}_n}{8} (3 + 20n)(a + c) \right\}.$$

By assigning " $a = 1, c = -1$ " and " $a = 3, c = 1$ ", we find the two summation formulae:

$$\boxed{A_1(1, 0, -1, 0)} \quad \frac{24 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{5 + 12n}{1 + 2n} - (3 + 20n) \mathbf{H}_n \right\}, \quad (2)$$

$$\boxed{A_1(3, 0, 1, 0)} \quad \frac{8 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{1 + 12n}{1 + 2n} - 2(3 + 20n) \mathbf{O}_n \right\}. \quad (3)$$

We point out that (2) and (3) refine the following conjecture by Sun [28] (Conjecture 3.23):

$$\frac{56 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ 12 + (3 + 20n)(\mathbf{H}_{2n} - 3\mathbf{H}_n) \right\}.$$

By carrying out the same process, we can further establish the infinite series identities:

- $A_0(a, b, c, d)$ Ramanujan [29]

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \binom{4n}{n, n, n, n} \frac{3 + 20n}{(-1024)^n}.$$

- $A_1(1, 0, 0, 0)$

$$\frac{16 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{3(1 + 4n)}{1 + 2n} - (3 + 20n)\mathbf{H}_{2n} \right\}.$$

- $A_1(1, 0, 1, 0)$

$$\frac{8 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{2}{1 + 2n} + (3 + 20n)\bar{\mathbf{H}}_{2n} \right\}.$$

- $A_1(1, 0, -1, 2)$

$$\frac{8 \ln 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{3 + 8n}{1 + 2n} + 2(3 + 20n)\mathbf{O}_{2n} \right\}.$$

- $A_2(1, 1, 0, 1)$

$$\frac{4\pi}{3} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{3(1 + 4n)}{(1 + 2n)^2} + \frac{3 + 20n}{4}(\mathbf{H}_n^{(2)} - 12\mathbf{O}_n^{(2)}) \right\}.$$

- $A_2(1 - \sqrt{-3}, 4, 3 + \sqrt{-3}, 4)$

$$\frac{38\pi}{3} - \frac{216 \ln^2 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{6(1 + 4n)}{(1 + 2n)^2} + \frac{6(5 + 12n)\mathbf{H}_n}{1 + 2n} - (3 + 20n)(\mathbf{H}_n^{(2)} + 3\mathbf{H}_n^2) \right\}.$$

- $A_2(1 - 3\sqrt{-3}, 4, -1 - \sqrt{-3}, 4)$

$$3\pi - \frac{4 \ln^2 2}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{1024} \right)^n \binom{4n}{n, n, n, n} \left\{ \frac{11(1 + 4n)}{(1 + 2n)^2} + \frac{2(1 + 12n)\mathbf{O}_n}{1 + 2n} + 2(3 + 20n)(3\mathbf{O}_n^{(2)} - \mathbf{O}_n^2) \right\}.$$

$$2.2. \left[a \rightarrow ax - \frac{1}{2}, b \rightarrow bx - \frac{1}{2}, c \rightarrow cx - \frac{1}{2}, d \rightarrow dx - \frac{1}{2} \right]$$

Under this parameter setting, the transformation in Lemma 1 can be restated as follows:

$$\sum_{k=0}^{\infty} \frac{(ax - \frac{1}{2})_k (bx - \frac{1}{2})_k}{(cx - \frac{1}{2})_k (dx - \frac{1}{2})_k} = \sum_{n=0}^{\infty} (-1)^n W_n(ax - \frac{1}{2}, bx - \frac{1}{2}, cx - \frac{1}{2}, dx - \frac{1}{2}) \times \frac{(ax - \frac{1}{2})_n (cx - bx)_n (dx - ax)_n (dx - bx)_{2n}}{(cx - \frac{1}{2})_n (dx - \frac{1}{2})_{2n+1} (cx + dx - ax - bx - 1)_{2n+2}}.$$

Since the series on the left diverges when $x \rightarrow 0$, we have to reformulate it so that it can be expanded into Maclaurin series in x . Denoting by $L_\alpha(k)$ and $R_\alpha(n)$ the summands on the left and the right

$$L_\alpha(k) = \frac{(ax - \frac{1}{2})_k (bx - \frac{1}{2})_k}{(cx - \frac{1}{2})_k (dx - \frac{1}{2})_k},$$

$$R_\alpha(n) = (-1)^n W_n(ax - \frac{1}{2}, bx - \frac{1}{2}, cx - \frac{1}{2}, dx - \frac{1}{2}) \times \frac{(ax - \frac{1}{2})_n (cx - bx)_n (dx - ax)_n (dx - bx)_{2n}}{(cx - \frac{1}{2})_n (dx - \frac{1}{2})_{2n+1} (cx + dx - ax - bx - 1)_{2n+2}},$$

we can restate the above equality as

$$L_\alpha(0) - R_\alpha(0) + L_\alpha(1) \sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (\frac{1}{2} + bx)_k}{(\frac{1}{2} + cx)_k (\frac{1}{2} + dx)_k} = \sum_{n=1}^{\infty} (-1)^n W_n(ax - \frac{1}{2}, bx - \frac{1}{2}, cx - \frac{1}{2}, dx - \frac{1}{2}) \times \frac{(ax - \frac{1}{2})_n (cx - bx)_n (dx - ax)_n (dx - bx)_{2n}}{(cx - \frac{1}{2})_n (dx - \frac{1}{2})_{2n+1} (cx + dx - ax - bx - 1)_{2n+2}},$$

where the difference of the initial terms can be factorized into

$$L_\alpha(0) - R_\alpha(0) = \frac{(1 - 2ax)(c + d - a - b + 2adx + 2b^2x - 2bcx - 2d^2x)}{2(1 - 2dx)(a + b - c - d)(1 + ax + bx - cx - dx)}.$$

According to Thomae's transformation (1), the series on the left can be reformulated as follows:

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (\frac{1}{2} + bx)_k}{(\frac{1}{2} + cx)_k (\frac{1}{2} + dx)_k} = {}_3F_2 \left[\begin{matrix} 1, \frac{1}{2} + ax, \frac{1}{2} + bx \\ \frac{1}{2} + cx, \frac{1}{2} + dx \end{matrix} \middle| 1 \right]$$

$$= \Gamma \left[\begin{matrix} \frac{1}{2} + cx, \frac{1}{2} + dx, cx + dx - ax - bx - 1 \\ x(c + d - a) - \frac{1}{2}, x(c + d - b) - \frac{1}{2} \end{matrix} \right]$$

$$\times {}_3F_2 \left[\begin{matrix} cx - \frac{1}{2}, dx - \frac{1}{2}, x(c + d - a - b) - 1 \\ x(c + d - a) - \frac{1}{2}, x(c + d - b) - \frac{1}{2} \end{matrix} \middle| 1 \right].$$

By making a substitution, we derive, after dividing by $\frac{(ax - \frac{1}{2})(cx - bx)(dx - ax)(dx - bx)}{(\frac{1}{2} - cx)(\frac{1}{2} - dx)(ax + bx - cx - dx)_2}$ across the resulting expression, the transformation formula as in the following proposition.

Proposition 2 (Functional equality).

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^n \frac{W_n(ax - \frac{1}{2}, bx - \frac{1}{2}, cx - \frac{1}{2}, dx - \frac{1}{2}) (\frac{1}{2} + ax)_{n-1} (1 - ax + dx)_{n-1} (1 - bx + cx)_{n-1} (1 + dx - bx)_{2n}}{(dx - bx + 2n) (\frac{1}{2} + cx)_{n-1} (\frac{1}{2} + dx)_{2n} (1 - ax - bx + cx + dx)_{2n}} \\ &= \frac{(1 - 2cx)(c + d - a - b + 2adx + 2b^2x - 2bcx - 2d^2x)}{4x^2(a - d)(b - c)(b - d)} + \Gamma \left[\begin{matrix} \frac{1}{2} + cx, \frac{1}{2} + dx, 1 + x(c + d - a - b) \\ \frac{1}{2} + cx + dx - ax, \frac{1}{2} + cx + dx - bx \end{matrix} \right] \\ & \times {}_3F_2 \left[\begin{matrix} cx - \frac{1}{2}, dx - \frac{1}{2}, x(c + d - a - b) - 1 \\ x(c + d - a) - \frac{1}{2}, x(c + d - b) - \frac{1}{2} \end{matrix} \middle| 1 \right] \frac{(\frac{1}{2} - bx)(\frac{1}{2} + ax - cx - dx)(\frac{1}{2} + bx - cx - dx)}{x^3(a - d)(b - c)(b - d)}. \end{aligned}$$

Denoting by $B_m(a, b, c, d)$ the resulting equality from the coefficients of x^m extracted across the above equation, we derive the following infinite series identities.

- $B_0(a, b, c, d)$

$$4 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \frac{(2n-1)(3-10n)}{n^2}.$$

- $B_1(0, 1, 0, 1)$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{4(2n-1)}{n^2} + \frac{(2n-1)(3-10n)}{n^2} \mathbf{O}_{2n} \right\}.$$

- $B_1(0, 1, 1, 0)$

$$2\pi^2 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{30n-3-64n^2}{n^2} - \frac{(2n-1)(3-10n)}{n^2} (\mathbf{H}_n + 6\mathbf{O}_n) \right\}.$$

- $B_2(1, 0, 1, 0)$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{(1-4n)(3-4n)}{n^4} - \frac{(1-2n)(3-10n)}{n^2} \mathbf{H}_n^{(2)} \right\}. \quad (4)$$

- $\Im\{B_2(1, 2\sqrt{-2}, 1 + 2\sqrt{-2}, 0)\}$

$$14\zeta(3) = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{1+6n-32n^2}{n^3(2n-1)} + \frac{4(1-2n)(3-10n)}{n^2} \mathbf{O}_n^{(2)} \right\}. \quad (5)$$

- $B_2(1, 1, 1, 1)$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{2}{n^2} - \frac{8(2n-1)}{n^2} \mathbf{O}_{2n} + \frac{(1-2n)(3-10n)}{n^2} (\mathbf{O}_{2n}^{(2)} + \mathbf{O}_{2n}^2) \right\}.$$

- $B_3(1, 1, 1, 1)$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{(1-2n)(3-10n)}{n^2} (2\mathbf{O}_{2n}^{(3)} + \mathbf{O}_{2n}^3 + 3\mathbf{O}_{2n} \mathbf{O}_{2n}^{(2)}) + \frac{6}{n^2} \mathbf{O}_{2n} - \frac{12(2n-1)}{n^2} (\mathbf{O}_{2n}^{(2)} + \mathbf{O}_{2n}^2) \right\}.$$

$$\bullet \quad \lim_{b \rightarrow 0} B_3(1, b, 1, b) / b$$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{4(1-2n)}{n^4} - \frac{(1-2n)(3-10n)}{n^2} \mathbf{O}_{2n} \mathbf{H}_n^{(2)} \right\} + \frac{(1-4n)(3-4n)}{n^4} \mathbf{O}_{2n} - \frac{4(1-2n)}{n^2} \mathbf{H}_n^{(2)} \Bigg\}.$$

$$\bullet \quad B_4(1, 0, 1, 0)$$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{2(1-4n)(3-4n)}{n^4} \mathbf{H}_{n-1}^{(2)} - \frac{(1-2n)(3-10n)}{n^2} [(\mathbf{H}_n^{(2)})^2 - \mathbf{H}_n^{(4)}] \right\}.$$

$$\bullet \quad B_6(1, 0, 1, 0)$$

$$0 = \sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{4n}{n, n, 2n}} \left\{ \frac{(1-2n)(3-10n)}{6n^2} [(\mathbf{H}_n^{(2)})^3 - 3\mathbf{H}_n^{(2)} \mathbf{H}_n^{(4)} + 2\mathbf{H}_n^{(6)}] \right. \\ \left. + \frac{(3-4n)(1-4n)}{n^4} \left(\frac{\mathbf{H}_{n-1}^{(2)}}{n^2} - \frac{(\mathbf{H}_n^{(2)})^2 - \mathbf{H}_n^{(4)}}{2} \right) \right\}.$$

$$2.3. \quad a \rightarrow ax, b \rightarrow bx - \frac{1}{2}, c \rightarrow cx - \frac{1}{2}, d \rightarrow dx$$

Under this parameter setting, the transformation in Lemma 1 becomes

$$\sum_{k=0}^{\infty} \frac{(ax)_k (bx - \frac{1}{2})_k}{(cx - \frac{1}{2})_k (dx)_k} = \sum_{n=0}^{\infty} (-1)^n W_n(ax, bx - \frac{1}{2}, cx - \frac{1}{2}, dx) \\ \times \frac{(ax)_n (-bx + cx)_n (-ax + dx)_n (\frac{1}{2} - bx + dx)_{2n}}{(cx - \frac{1}{2})_n (dx)_{2n+1} (-1 - ax - bx + cx + dx)_{2n+2}}.$$

Denoting by $L_{\beta}(k)$ and $R_{\beta}(n)$ the summands respectively on the left and the right as

$$L_{\beta}(k) = \frac{(ax)_k (bx - \frac{1}{2})_k}{(cx - \frac{1}{2})_k (dx)_k}, \\ R_{\beta}(n) = (-1)^n W_n(ax, bx - \frac{1}{2}, cx - \frac{1}{2}, dx) \\ \times \frac{(ax)_n (-bx + cx)_n (-ax + dx)_n (\frac{1}{2} - bx + dx)_{2n}}{(cx - \frac{1}{2})_n (dx)_{2n+1} (-1 - ax - bx + cx + dx)_{2n+2}},$$

we can rewrite the above equality as

$$L_{\beta}(0) - R_{\beta}(0) + L_{\beta}(1) \sum_{k=0}^{\infty} \frac{(1+ax)_k (\frac{1}{2} + bx)_k}{(\frac{1}{2} + cx)_k (1+dx)_k} \\ = \sum_{n=1}^{\infty} (-1)^n W_n(ax, bx - \frac{1}{2}, cx - \frac{1}{2}, dx) \frac{(ax)_n (cx - bx)_n (dx - ax)_n (\frac{1}{2} - bx + dx)_{2n}}{(cx - \frac{1}{2})_n (dx)_{2n+1} (cx + dx - ax - bx - 1)_{2n+2}},$$

where the difference of the initial terms can be decomposed into

$$L_{\beta}(0) - R_{\beta}(0) = \frac{a(c - b + 2b^2x - 2bcx + 2adx - 2d^2x)}{2d(a + b - c - d)(1 + ax + bx - cx - dx)}.$$

According to Thomae's transformation (1), the series on the left can be expressed as

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(1+ax)_k (\frac{1}{2}+bx)_k}{(\frac{1}{2}+cx)_k (1+dx)_k} &= {}_3F_2 \left[\begin{matrix} 1, 1+ax, \frac{1}{2}+bx \\ \frac{1}{2}+cx, 1+dx \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} dx, cx+dx-ax-bx-1, cx-\frac{1}{2} \\ cx+dx-bx, cx+dx-ax-\frac{1}{2} \end{matrix} \middle| 1 \right] \times \Gamma \left[\begin{matrix} 1+dx, \frac{1}{2}+cx, cx+dx-ax-bx-1 \\ cx+dx-bx, cx+dx-ax-\frac{1}{2} \end{matrix} \right]. \end{aligned}$$

By making substitutions first and then multiplying $\frac{d(cx-\frac{1}{2})(ax+bx-cx-dx)_2}{a(ax-dx)(bx-cx)}$ across the resulting equation, we find, after some simplifications, the transformation as in the proposition below.

Proposition 3 (Functional equality).

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n W_n(ax, bx-\frac{1}{2}, cx-\frac{1}{2}, dx) (1+ax)_n (1+cx-bx)_n (1+dx-ax)_n (\frac{1}{2}-bx+dx)_{2n}}{(n+ax)(n-bx+cx)(n-ax+dx)(\frac{1}{2}+cx)_{n-1} (1+dx)_{2n} (1+cx+dx-ax-bx)_{2n}} \\ = \frac{(cx-\frac{1}{2})(c-b+2b^2x-2bcx+2adx-2d^2x)}{2x(a-d)(b-c)} + \frac{(c+d-b)(bx-\frac{1}{2})(cx+dx-ax-\frac{1}{2})}{x(a-d)(b-c)} \\ \times \Gamma \left[\begin{matrix} 1+dx, 1-ax-bx+cx+dx, \frac{1}{2}+cx \\ 1+cx+dx-bx, \frac{1}{2}+cx+dx-ax \end{matrix} \right] {}_3F_2 \left[\begin{matrix} dx, cx+dx-ax-bx-1, cx-\frac{1}{2} \\ cx+dx-bx, cx+dx-ax-\frac{1}{2} \end{matrix} \middle| 1 \right]. \end{aligned}$$

Let $C_m(a, b, c, d)$ stand for the resulting equality from the coefficients of x^m extracted across the above equality. Then, we derive the following infinite series identities.

$$\bullet \quad \boxed{C_0(a, b, c, d)}$$

$$2 \ln 2 - 2 = \sum_{n=1}^{\infty} \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \frac{(1-2n)(9-20n)}{n(-4)^n}.$$

$$\bullet \quad \boxed{C_1(1, -2, 1, -2)}$$

$$3 \ln^2 2 - 4 \ln 2 - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{13-60n+64n^2}{2n^2} - \frac{(1-2n)(9-20n)}{n} \mathbf{H}_n \right\}.$$

$$\bullet \quad \boxed{C_1(1, 2, 1, 2)}$$

$$\frac{\pi^2}{8} - \frac{\ln^2 2}{2} + 2 \ln 2 = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{1+28n-64n^2}{4n^2} + \frac{(1-2n)(9-20n)}{n} \mathbf{O}_n \right\}.$$

$$\bullet \quad \boxed{C_1(1, 0, -1, 2)}$$

$$2 - 2 \ln 2 - \frac{\pi^2}{8} - \frac{\ln^2 2}{2} = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{1-12n+24n^2}{4n^2} - \frac{(1-2n)(9-20n)}{n} \mathbf{O}_{2n} \right\}.$$

$$\bullet \quad \boxed{C_1(0, 1, 0, 1)}$$

$$2 \ln^2 2 - 4 \ln 2 - \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{(1-2n)(3-16n)}{n^2} - \frac{(1-2n)(9-20n)}{n} \mathbf{H}_{2n} \right\}.$$

$$\bullet \quad \boxed{C_1(1, 0, 1, 0)}$$

$$\frac{\pi^2}{12} + \ln^2 2 = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{7-16n}{2n^2} + \frac{(1-2n)(9-20n)}{n} \bar{\mathbf{H}}_{2n} \right\}.$$

$$\bullet \quad \boxed{C_2(1, 2, 1, 2)}$$

$$\frac{7\zeta(3)}{2} + \frac{\pi^2}{2} + \frac{\ln^3 2}{3} - \frac{\pi^2}{4} \ln 2 - 2 \ln^2 2 = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{9-36n+64n^2}{4n^3} + \frac{1+28n-64n^2}{n^2} \mathbf{O}_n \right\}.$$

$$\bullet \quad \boxed{C_2(1, 0, -1, 2)}$$

$$\frac{\pi^2}{2} + \frac{\ln^3 2}{3} + \frac{\pi^2}{4} \ln 2 + 2 \ln^2 2 = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \left\{ \frac{(1-4n)(9-8n)}{4n^3} - \frac{1-12n+24n^2}{n^2} \mathbf{O}_{2n} \right\}.$$

$$\bullet \quad \boxed{C_2(1, -2, 1, -2)}$$

$$\frac{2\pi^2}{3} + 6\zeta(3) + 12 \ln^3 2 - 24 \ln^2 2 - \pi^2 \ln 2 = \sum_{n=1}^{\infty} \left(\frac{-1}{4} \right)^n \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} \times \left\{ \frac{97-380n+320n^2}{n^3} - \frac{2(13-60n+64n^2)}{n^2} \mathbf{H}_n \right\}.$$

$$2.4. \quad \boxed{a \rightarrow \frac{1}{2} + ax, b \rightarrow bx, c \rightarrow cx, d \rightarrow 1 + dx}$$

Under this parameter setting, we can rewrite the transformation in Lemma 1 as

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (bx)_k}{(cx)_k (1 + dx)_k} = \sum_{n=0}^{\infty} (-1)^n \mathbf{W}_n \left(\frac{1}{2} + ax, bx, cx, 1 + dx \right) \\ \times \frac{(\frac{1}{2} + ax)_n (cx - bx)_n (\frac{1}{2} + dx - ax)_n (1 + dx - bx)_{2n}}{(cx)_n (1 + dx)_{2n+1} (cx + dx - ax - bx - \frac{1}{2})_{2n+2}}.$$

In view of Thomae's transformation (1), the above series on the left can be reformulated as

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2} + ax)_k (bx)_k}{(cx)_k (1 + dx)_k} = {}_3F_2 \left[\begin{matrix} 1, \frac{1}{2} + ax, bx \\ cx, 1 + dx \end{matrix} \middle| 1 \right] \\ = \Gamma \left[\begin{matrix} cx, 1 + dx, cx + dx - ax - bx - \frac{1}{2} \\ cx + dx - ax - \frac{1}{2}, cx + dx - bx \end{matrix} \right] \\ \times {}_3F_2 \left[\begin{matrix} cx - 1, dx, cx + dx - ax - bx - \frac{1}{2} \\ cx + dx - ax - \frac{1}{2}, cx + dx - bx \end{matrix} \middle| 1 \right].$$

By making substitutions, we derive, after multiplying by $\frac{(cx)(cx + dx - ax - bx - \frac{1}{2})}{(cx - bx)}$ across the resulting expression, the transformation formula as in the following proposition.

Proposition 4 (Functional equality).

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^n W_n \left(\frac{1}{2} + ax, bx, cx, 1 + dx \right) \frac{\left(\frac{1}{2} + ax \right)_n (1 + cx - bx)_{n-1} \left(\frac{1}{2} + dx - ax \right)_n (1 + dx - bx)_{2n}}{(1 + cx)_{n-1} (1 + dx)_{2n+1} \left(\frac{1}{2} + cx + dx - ax - bx \right)_{2n+1}} \\ &= \left\{ \frac{cx(c+d-b)}{b-c} + \frac{cx(1+2ax)(1-bx+dx)}{(1+dx)(1-2ax-2bx+2cx+2dx)} \right\} - \frac{b-c-d}{b-c} \left(\frac{1}{2} + ax - cx - dx \right) \\ & \times \Gamma \left[\begin{matrix} 1 + cx, 1 + dx, \frac{1}{2} + cx + dx - ax - bx \\ 1 + cx + dx - bx, \frac{1}{2} + cx + dx - ax \end{matrix} \right] {}_3F_2 \left[\begin{matrix} cx - 1, dx, cx + dx - ax - bx - \frac{1}{2} \\ cx + dx - ax - \frac{1}{2}, cx + dx - bx \end{matrix} \middle| 1 \right]. \end{aligned}$$

Denote by $D_m(a, b, c, d)$ the resulting equality from the coefficients of x^m extracted across the above equation. Particular care should be paid when determining the related coefficients from the above ${}_3F_2$ -series on the right. This is quite different from previous subsections, since the following k -sums are indispensable:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mathbf{H}_k}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j=1}^k \frac{1}{j} = \sum_{j=1}^{\infty} \frac{1}{j^2} = \zeta(2), \\ \sum_{k=1}^{\infty} \frac{\mathbf{O}_k}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j=1}^k \frac{1}{2j-1} = \sum_{j=1}^{\infty} \frac{1}{j(2j-1)} = 2 \ln 2; \\ \sum_{k=1}^{\infty} \frac{\mathbf{H}_k}{k^2(k+1)} &= \sum_{k=1}^{\infty} \left\{ \frac{\mathbf{H}_k}{k^2} - \frac{\mathbf{H}_k}{k(k+1)} \right\} = 2\zeta(3) - \zeta(2), \\ \sum_{k=1}^{\infty} \frac{\mathbf{O}_k}{k^2(k+1)} &= \sum_{k=1}^{\infty} \left\{ \frac{\mathbf{O}_k}{k^2} - \frac{\mathbf{O}_k}{k(k+1)} \right\} = \frac{7}{4}\zeta(3) - 2 \ln 2; \\ \sum_{k=1}^{\infty} \frac{\mathbf{H}_k^{(2)}}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j=1}^k \frac{1}{j^2} = \sum_{j=1}^{\infty} \frac{1}{j^3} = \zeta(3), \\ \sum_{k=1}^{\infty} \frac{\mathbf{O}_k^{(2)}}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j=1}^k \frac{1}{(2j-1)^2} = \sum_{j=1}^{\infty} \frac{1}{j(2j-1)^2} = \frac{\pi^2}{4} - 2 \ln 2. \end{aligned}$$

They can be shown easily by means of series rearrangements.

Another series that we need is

$$\sum_{k=1}^{\infty} \frac{\bar{\mathbf{H}}_{2k}^2}{k(k+1)} = 2 \ln 2 + 2 \ln^2 2 - \zeta(3) - \frac{\pi^2}{12}.$$

This can be confirmed by writing it first as a double integral and then computing it manually (or by *Mathematica* 11):

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\bar{\mathbf{H}}_{2k}^2}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \int_0^1 \int_0^1 \frac{(1-x^{2k})(1-y^{2k})}{(1+x)(1+y)} dx dy \\ &= \int_0^1 \int_0^1 \frac{(1-x^2y^2) \ln(1-x^2y^2) - 2(1-x^2)y^2 \ln(1-x^2)}{(1+x)(1+y)x^2y^2} dx dy. \end{aligned}$$

We remark that this double integral can be refined by the following two integrals:

$$\begin{aligned} \int_0^1 \int_0^1 \frac{(1-x^2y^2) \ln(1-x^2y^2)}{(1+x)(1+y)x^2y^2} dx dy &= 2 \ln 2 - 2 \ln^2 2 - \zeta(3) + \frac{\pi^2}{6} \ln 2 - \frac{\pi^2}{12}, \\ \int_0^1 \int_0^1 \frac{(1-x) \ln(1-x^2)}{(1+y)x^2} dx dy &= \ln 2 \int_0^1 \frac{(1-x) \ln(1-x^2)}{x^2} dx = \frac{\pi^2}{12} \ln 2 - 2 \ln^2 2. \end{aligned}$$

The infinite series identities derived from Proposition 4 are recorded as follows.

$$\bullet \quad \boxed{D_0(a, b, c, d)}$$

$$-\frac{1}{4} = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{n(1+5n)}{(1+4n)} \right\}.$$

$$\bullet \quad \boxed{D_1(0, 2, 1, 1)}$$

$$1 - 2 \ln 2 = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{5+24n}{1+4n} - \frac{6n(1+5n)}{1+4n} \mathbf{H}_n \right\}.$$

$$\bullet \quad \boxed{D_1(2, 0, 1, 1)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1+2n}{1+4n} + \frac{4n(1+5n)}{1+4n} \mathbf{O}_n \right\}.$$

$$\bullet \quad \boxed{D_1(1, 0, 2, 0)}$$

$$\frac{1}{4} = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{2n}{1+4n} - \frac{n(1+5n)}{1+4n} \mathbf{O}_{2n+1} \right\}.$$

$$\bullet \quad \boxed{D_2(1, 0, 1, 0)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{1+4n} + \frac{4n(1+5n)}{1+4n} \mathbf{O}_n^{(2)} \right\}.$$

$$\bullet \quad \boxed{D_2(1 - \sqrt{-1}, 0, -1 - \sqrt{-1}, 2)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{1+4n} + \frac{2(1+2n)}{1+4n} \mathbf{O}_n + \frac{4n(1+5n)}{1+4n} \mathbf{O}_n^2 \right\}.$$

$$\bullet \quad \boxed{\Im \{D_2(1 - \sqrt{-1}, 4, 3 - \sqrt{-1}, 2)\}}$$

$$\frac{\pi^2}{24} - \frac{1}{2} = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{2+13n+16n^2}{2n(1+2n)(1+4n)} - \frac{n(1+5n)}{1+4n} \mathbf{H}_n^{(2)} \right\}.$$

$$\bullet \quad \boxed{2D_2(0, 2, 1, 1) - D_2(1, 0, 1, 0)}$$

$$\frac{\pi^2}{3} + 4 \ln 2 - 4 \ln^2 2 - 3 = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{6+39n+52n^2}{n(1+2n)(1+4n)} - \frac{3(5+24n)}{1+4n} \mathbf{H}_n + \frac{9n(1+5n)}{1+4n} \mathbf{H}_n^2 \right\}.$$

$$\bullet \quad \boxed{D_2(0, 0, 1, 0)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{1+4n} - \frac{8n}{1+4n} \mathbf{O}_{2n+1} + \frac{2n(1+5n)}{1+4n} (\mathbf{O}_{2n+1}^{(2)} + \mathbf{O}_{2n+1}^2) \right\}.$$

$$\bullet \quad \boxed{D_3(0, 0, 1, 0)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{2n(1+5n)}{3(1+4n)} (2\mathbf{O}_{2n+1}^{(3)} + \mathbf{O}_{2n+1}^3 + 3\mathbf{O}_{2n+1} \mathbf{O}_{2n+1}^{(2)}) \right. \\ \left. + \frac{\mathbf{O}_{2n+1}}{1+4n} - \frac{4n(\mathbf{O}_{2n+1}^{(2)} + \mathbf{O}_{2n+1}^2)}{1+4n} \right\}.$$

$$\bullet \quad \boxed{D_3(1 - \sqrt{-1}, 0, -1 - \sqrt{-1}, 2)}$$

$$0 = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{\mathbf{O}_n}{1+4n} + \frac{1+2n}{1+4n} \mathbf{O}_n^2 + \frac{4n(1+5n)}{3(1+4n)} (\mathbf{O}_n^3 - \mathbf{O}_n^{(3)}) \right\}.$$

$$\bullet \quad \boxed{\lim_{a \rightarrow 1} D_3(a, 0, 1, 0)/(a-1)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{\mathbf{O}_{2n+1}}{1+4n} - \frac{8n\mathbf{O}_n^{(2)}}{1+4n} + \frac{4n(1+5n)}{1+4n} \mathbf{O}_{2n+1} \mathbf{O}_n^{(2)} \right\}.$$

$$\bullet \quad \boxed{D_4(1, 0, 1, 0)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{\mathbf{O}_n^{(2)}}{1+4n} + \frac{2n(1+5n)}{1+4n} [(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}] \right\}.$$

$$\bullet \quad \boxed{D_6(1, 0, 1, 0)}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{3(\mathbf{O}_n^{(2)})^2 - 3\mathbf{O}_n^{(4)}}{1+4n} + \frac{4n(1+5n)}{1+4n} [(\mathbf{O}_n^{(2)})^3 - 3\mathbf{O}_n^{(2)} \mathbf{O}_n^{(4)} + 2\mathbf{O}_n^{(6)}] \right\}.$$

$$2.5. \quad \boxed{a \rightarrow ax, b \rightarrow \frac{1}{2} + bx, c \rightarrow 1 + cx, d \rightarrow \frac{1}{2} + dx}$$

Under this parameter setting, the transformation in Lemma 1 can be reformulated as

$$x(c+d-a-b) \sum_{k=0}^{\infty} \frac{(ax)_k (\frac{1}{2} + bx)_k}{(1+cx)_k (\frac{1}{2} + dx)_k} = \sum_{n=0}^{\infty} (-1)^n W_n(ax, \frac{1}{2} + bx, 1 + cx, \frac{1}{2} + dx) \\ \times \frac{(ax)_n (\frac{1}{2} - bx + cx)_n (\frac{1}{2} - ax + dx)_n (dx - bx)_{2n}}{(1+cx)_n (\frac{1}{2} + dx)_{2n+1} (1 - ax - bx + cx + dx)_{2n+1}}.$$

Denote by $L_{\gamma}(k)$ and $R_{\gamma}(n)$ the summands respectively on the left and the right:

$$L_{\gamma}(k) = \frac{(ax)_k (\frac{1}{2} + bx)_k}{(1+cx)_k (\frac{1}{2} + dx)_k}, \\ R_{\gamma}(n) = (-1)^n W_n(ax, \frac{1}{2} + bx, 1 + cx, \frac{1}{2} + dx) \\ \times \frac{(ax)_n (\frac{1}{2} - bx + cx)_n (\frac{1}{2} - ax + dx)_n (dx - bx)_{2n}}{(1+cx)_n (\frac{1}{2} + dx)_{2n+1} (1 - ax - bx + cx + dx)_{2n+1}}.$$

By factorizing the difference of the two initial terms

$$L_{\gamma}(0) - R_{\gamma}(0) = \frac{ax\{ax - cx - 2dx - 1 + 2x^2(ad + b^2 - bc - d^2)\}}{(1+2dx)(1+cx+dx-ax-bx)}$$

and then applying Thomae's transformation (1),

$$\begin{aligned} & x(c+d-a-b) \sum_{k=1}^{\infty} \frac{(ax)_k (\frac{1}{2}+bx)_k}{(1+cx)_k (\frac{1}{2}+dx)_k} \\ &= \frac{ax^2(c+d-a-b)(\frac{1}{2}+bx)}{(1+cx)(\frac{1}{2}+dx)} {}_3F_2 \left[\begin{matrix} 1, 1+ax, \frac{3}{2}+bx \\ 2+cx, \frac{3}{2}+dx \end{matrix} \middle| 1 \right] \\ &= (ax)\Gamma \left[\begin{matrix} 1+cx, \frac{1}{2}+dx \\ 1+x(c+d-b), \frac{1}{2}+bx \end{matrix} \right] {}_3F_2 \left[\begin{matrix} dx-bx, \frac{1}{2}-bx+cx, x(c+d-a-b) \\ 1+x(c+d-b), 1+x(c+d-a-b) \end{matrix} \middle| 1 \right], \end{aligned}$$

we can express, after dividing by " $ax^2(d-b)$ " across, the resulting transformation as follows.

Proposition 5 (Functional equality).

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^n \frac{W_n(ax, \frac{1}{2}+bx, 1+cx, \frac{1}{2}+dx)}{(ax+n)(dx-bx+2n)} \frac{(1+ax)_n (\frac{1}{2}-bx+cx)_n (\frac{1}{2}-ax+dx)_n (1+dx-bx)_{2n}}{(1+cx)_n (\frac{1}{2}+dx)_{2n+1} (1-ax-bx+cx+dx)_{2n+1}} \\ &= \frac{ax-cx-2dx-1+2x^2(ad+b^2-bc-d^2)}{(1+2dx)(dx-bx)(1+cx+dx-ax-bx)} + \frac{1}{dx-bx} \Gamma \left[\begin{matrix} 1+cx, \frac{1}{2}+dx \\ 1+x(c+d-b), \frac{1}{2}+bx \end{matrix} \right] \\ & \quad \times {}_3F_2 \left[\begin{matrix} dx-bx, \frac{1}{2}-bx+cx, x(c+d-a-b) \\ 1+x(c+d-b), 1+x(c+d-a-b) \end{matrix} \middle| 1 \right]. \end{aligned}$$

Denoting by $E_m(a, b, c, d)$ the resulting equality from the coefficients of x^m extracted across the above equation, and then making use of the following easily proved k -sums

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k}{k! \times k^2} &= \frac{\pi^2}{6} - 2 \ln^2 2, \\ \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k}{k! \times k^3} &= 2\zeta(3) - \frac{\pi^2}{3} \ln 2 + \frac{4}{3} \ln^3 2, \\ \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k \mathbf{H}_k}{k! \times k^2} &= \frac{9}{2} \zeta(3) - \frac{2\pi^2}{3} \ln 2, \\ \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k \mathbf{O}_k}{k! \times k^2} &= \frac{7}{2} \zeta(3) - \frac{\pi^2}{2} \ln 2, \end{aligned}$$

we can establish infinite series identities as below.

$$\bullet \quad E_0(a, b, c, d)$$

$$2 - 4 \ln 2 = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n} (3+10n)}{\binom{4n}{2n} n(4n+1)}.$$

$$\bullet \quad E_1(0, 1, 0, 1) - E_1(1, 1, 1, 1)$$

$$\frac{\pi^2}{6} - 2 = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n} (4n+1)} \left\{ \frac{1+6n+10n^2}{n^2(1+2n)} \right\}.$$

$$\bullet \quad E_1(5, 4, 13, 0)$$

$$\frac{\pi^2}{12} + 8 \ln^2 2 - 7 = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n} (4n+1)} \left\{ \frac{7}{1+2n} + \frac{3(3+10n)}{n} \mathbf{H}_n \right\}.$$

- $E_1(1, 4, 1, 0)$

$$\frac{\pi^2}{4} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{1}{1+2n} - \frac{2(3+10n)}{n} \mathbf{O}_n \right\}.$$

- $E_1(0, 1, 0, 1)$

$$\frac{\pi^2}{2} - 4 = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{3+8n}{n(1+2n)} - \frac{3+10n}{n} \mathbf{O}_{2n+1} \right\}.$$

- $\Re\{E_2(1, 1 + \sqrt{-1}, 1, 0)\}$

$$1 - \frac{5}{4}\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(1+4n)} \left\{ \frac{1+8n+20n^2+14n^3}{2n^3(1+2n)^2} + \frac{1+6n+10n^2}{n^2(1+2n)} \mathbf{O}_n \right\}.$$

- $E_2(1, 1, 1, 1) - E_2(0, 1, 0, 1) + E_2(1, 0, 1, 0)$

$$\zeta(3) - 2 = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{1+3n}{n^3} + \frac{1+6n+10n^2}{n^2(1+2n)} \mathbf{O}_{2n+1} \right\}. \quad (6)$$

- $E_3(1, 0, 1, 0)$

$$\frac{\pi^4}{360} = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{1+4n}{4n^4} - \frac{1+6n+10n^2}{n^2(1+2n)} \mathbf{O}_n^{(2)} \right\}. \quad (7)$$

- $E_2(1, 0, 1, 0)$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{1+4n}{n^3} - \frac{2(3+10n)}{n} \mathbf{O}_n^{(2)} \right\}. \quad (8)$$

- $\Im\{E_2(1, 1 + \sqrt{-1}, 1, 0)\}$

$$-\frac{3}{4}\zeta(3) = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{8n^3} + \frac{\mathbf{O}_n}{2n^2} + \frac{3+10n}{n(1+4n)} \mathbf{O}_n^2 \right\}.$$

- $E_2(1, 1, 1, 1)$

$$4 - 6\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(4n+1)} \left\{ \frac{1+2n}{n^3} + \frac{2(1+n)}{n^2} \mathbf{O}_{2n+1} + \frac{3+10n}{n} (\mathbf{O}_{2n+1}^{(2)} + \mathbf{O}_{2n+1}^2) \right\}.$$

- $E_4(1, 0, 1, 0)$

$$\zeta(5) = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{n^5} - \frac{4\mathbf{O}_n^{(2)}}{n^3} + \frac{4(3+10n)}{n(1+4n)} ((\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}) \right\}. \quad (9)$$

- $E_5(1, 0, 1, 0)$

$$\frac{\pi^6}{945} = \sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \frac{1}{n^6} - \frac{4\mathbf{O}_n^{(2)}}{n^4} + \frac{8(1+6n+10n^2)}{n^2(1+2n)(1+4n)} ((\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}) \right\}.$$

$$\begin{aligned}
 & \bullet \quad \Re\{E_2(1, \sqrt{-1} - 1, \sqrt{-4} - 1, 0)\} \\
 & \quad \frac{12 + 4\pi^2 \ln 2 - 27\zeta(3)}{6} = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(1+4n)} \left\{ \frac{2(1+3n)(1+5n+5n^2)}{n^3(1+2n)^2} \right. \\
 & \quad \left. - \frac{3(1+6n+10n^2)}{n^2(1+2n)} \mathbf{H}_n - \frac{3+10n}{n} \mathbf{H}_n^{(2)} \right\}. \\
 & \bullet \quad 2E_2(1, -1, -1, 0) - E_2(1, 0, 1, 0) \\
 & \quad \frac{3 + 4\pi^2 \ln 2 - 16 \ln^3 2 - 21\zeta(3)}{27} = \sum_{n=1}^{\infty} \frac{(-1)^n \binom{2n}{n}}{\binom{4n}{2n}(1+4n)} \left\{ \frac{17+136n+340n^2+264n^3}{72n^3(1+2n)^2} \right. \\
 & \quad \left. - \frac{9+54n+76n^2}{12n^2(1+2n)} \mathbf{H}_n + \frac{3+10n}{4n} \mathbf{H}_n^2 \right\}.
 \end{aligned}$$

3. Concluding Comments

The paper is entitled “Harmonic Series with Multinomial Coefficient $\binom{4n}{n,n,n,n}$ and Central Binomial Coefficient $\binom{2n}{n}$ ” since we have evaluated many series in closed form that have the following four binomial structures in their summands:

$$\left\{ \binom{4n}{n,n,n,n}, \frac{1}{\binom{4n}{n,n,2n}} = \frac{\binom{2n}{n}}{\binom{4n}{n,n,n}}, \frac{\binom{4n}{2n}}{\binom{2n}{n}^2} = \frac{\binom{4n}{n,n,n,n}}{\binom{2n}{n}^4}, \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{\binom{2n}{n}^3}{\binom{4n}{n,n,n,n}} \right\}.$$

Even though it is quite common to encounter sums involving central binomial coefficients (see [30–35]), there exist few infinite series containing the above binomial/multinomial components in the literature (cf. [36–40]). Most of the results presented in this paper are new (except for those explicitly indicated).

According to what we have achieved, the “coefficient extraction” method is successful in dealing with series of harmonic numbers. However, there are serious difficulties to go further when the involved harmonic numbers have orders beyond two. In this case, the n -series to be evaluated have more complicated summands (with harmonic numbers of higher order). Another obstacle comes from the k -sums as in Sections 2.4 and 2.5, which may not admit closed expressions. To tackle these problems, alternative approaches will be necessary. The interested reader is enthusiastically encouraged to make further exploration.

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