

Monogeneity and Power Integral Bases: Recent Developments

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Abstract: Monogeneity is a classical area of algebraic number theory that continues to be actively researched. This paper collects the results obtained over the past few years in this area. Several of the listed results were presented at a series of online conferences titled “Monogeneity and Power Integral Bases”. We also give a collection of the most important methods used in several of these papers. A list of open problems for further research is also given.

Keywords: algebraic number fields; index of an element; generators of power integral bases; monogenic number fields; monogenic polynomials; pure fields; trinomials; relative extensions; algorithms

MSC: 11R04; 11D59; 11D57; 11Y50

1. Introduction

Let $K = \mathbb{Q}(\gamma)$ be an algebraic number field of degree n , generated by the algebraic integer γ , with ring of integers $\mathbb{Z}_K$ and discriminant D_K . It is a classical problem of algebraic number theory, going back to R. Dedekind [1], K. Hensel [2] and H. Hasse [3] to decide if the ring $\mathbb{Z}_K$ can be generated by a single element $\alpha \in \mathbb{Z}_K$, that is, if it is mono-generated, $\mathbb{Z}_K = \mathbb{Z}[\alpha]$. In this case, we say that the ring $\mathbb{Z}_K$, or the field K , is *monogenic*, and the integral basis $\{1, \alpha, \dots, \alpha^{n-1}\}$ is called a *power integral basis*.

Recently, this area has been developing very rapidly. In order to create a suitable forum to present recent results on monogeneity, the author started a series of online meetings “Monogeneity and Power Integral Bases” (<https://sway.cloud.microsoft/F2kZzeZ3bmD4dFfy?ref=Link> accessed on 15 January 2021) in 2021. The purpose was to make contacts, circulate preprints and results, and support collaboration between researchers all over the world working in this area. During the time of pandemy this was the only way to contact, but later on this proved to be an easy and fast way of contacting. Therefore, up to March 2024 we already had nine meetings and we hope to continue.

The main purpose of this paper is to give an overview of the latest developments in monogeneity theory, about the results that were presented at the online meetings and the results that appeared parallel. The paper is also a kind of extension of the book [4] that appeared in 2019. Most of these results are not yet contained there.

In Section 2, we collect the most important tools that were used in several works. These may be useful for further applications. Section 3 collects the most important results, and finally in Section 4 we try to indicate some possible perspectives of further research.

In favour of the reader, we collect some further concepts on monogeneity.

For any primitive element $\alpha \in \mathbb{Z}_K$ (that is $K = \mathbb{Q}(\alpha)$), the *index* of α is defined as the module index

$$I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha]).$$

We obviously have

$$D(\alpha) = I(\alpha)^2 D_K,$$

where $D(\alpha)$ is the discriminant of α ,



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$$D(\alpha) = \prod_{1 \leq i < j \leq n} (\alpha^{(i)} - \alpha^{(j)})^2,$$

$\alpha^{(i)}$ denoting the conjugates of α corresponding to $\gamma^{(i)}$ ($i = 1, \dots, n$) (in the following we shall denote similarly the conjugates of any element of K). Obviously, $I(\alpha) = 1$, if and only if $\mathbb{Z}_K = \mathbb{Z}[\alpha]$, that is, if $\{1, \alpha, \dots, \alpha^{n-1}\}$ is an integral basis, or in other words α generates a power integral basis in K .

The *field index*, $m(K)$, of K is defined as

$$m(K) = \gcd\{I(\alpha) \mid \alpha \in \mathbb{Z}_K, K = \mathbb{Q}(\alpha)\}.$$

If K is monogenic, there are elements of index 1, and the field index is also equal to 1. The converse is not true: the field index may happen to be 1 without the field being monogenic.

If α, β are primitive elements in $\mathbb{Z}_K$ and $\alpha + \beta \in \mathbb{Z}$ or $\alpha - \beta \in \mathbb{Z}$ then obviously their indices are equal. Such elements are called *equivalent*. It was proved by B. J. Birch and J. R. Merriman [5] and then in an effective form by K. Györy [6] that up to equivalence there are only finitely many generators of power integral bases in any number field K .

For any integral basis $(1, \omega_2, \dots, \omega_n)$ of K set

$$L^{(i)}(\underline{X}) = X_1 + \omega_2^{(i)} X_2 + \dots + \omega_n^{(i)} X_n$$

($i = 1, \dots, n$). Then (see [4])

$$D(L(\underline{X})) = \prod_{1 \leq i < j \leq n} (L^{(i)}(\underline{X}) - L^{(j)}(\underline{X})) = I(X_2, \dots, X_n) D_K$$

where $I(X_2, \dots, X_n)$ is a homogeneous polynomial of degree $n(n-1)/2$ with integer coefficients, with the property that for any primitive element $\alpha = x_1 + \omega_2 x_2 + \dots + \omega_n x_n \in \mathbb{Z}_K$ we have

$$I(\alpha) = |I(x_2, \dots, x_n)|.$$

The polynomial $I(X_2, \dots, X_n)$ is called the *index form* corresponding to the integral basis $(1, \omega_2, \dots, \omega_n)$. Since equivalent algebraic integers have the same index, it is independent of X_1 . Therefore, determining elements $\alpha \in \mathbb{Z}_K$ of index m is equivalent to solving the *index form equation*

$$I(x_2, \dots, x_n) = m \text{ in } x_2, \dots, x_n \in \mathbb{Z}.$$

A non-zero irreducible polynomial $f(x) \in \mathbb{Z}[x]$ is called *monogenic* if a root α of $f(x)$ generates a power integral basis in the field $K = \mathbb{Q}(\alpha)$. Obviously, if the polynomial $f(x)$ is monogenic, then K is also monogenic, but the converse is not true. The field K may happen to be monogenic without $f(x)$ being monogenic. The *index* of $f(x)$ is defined as $\text{ind}(f) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$.

2. Tools

2.1. Dedekind's Criterion

Let $\overline{f(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{\ell_i}$ modulo p be the factorization of $f(x)$ modulo p into powers of monic irreducible coprime polynomials of $\mathbb{F}_p[x]$.

For completeness, we recall here a well-known theorem of Dedekind:

Theorem 1 (Chapter I, Proposition 8.3 of [7]).

If p does not divide the index $I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$, then

$$p\mathbb{Z}_K = \prod_{i=1}^r \mathfrak{p}_i^{\ell_i},$$

where $\mathfrak{p}_i = p\mathbb{Z}_K + \phi_i(\alpha)\mathbb{Z}_K$ and the residue degree of $\mathfrak{p}_i$ is $f(\mathfrak{p}_i) = \deg(\phi_i)$.

As indicated above, it is very important to have a tool to determine prime divisors of the indices of algebraic integers. Therefore, the following well-known criterion of Dedekind is very frequently used:

Theorem 2 (Dedekind's criterion [1], see also [8] Theorem 6.1.4, [9] p. 295).

Let $f(x) \in \mathbb{Z}[x]$ be a monic non-zero irreducible polynomial with a root α , let $K = \mathbb{Q}(\alpha)$, and let p be a prime number. Let $\overline{f(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{\ell_i} \pmod{p}$ be the factorization of $\overline{f(x)}$ in $\mathbb{F}_p[x]$, with monic $\phi_i \in \mathbb{Z}[x]$, such that their reductions $\overline{\phi_i(x)}$ are irreducible and pairwise coprime over $\mathbb{F}_p$. Set

$$M(x) = \frac{f(x) - \prod_{i=1}^r \phi_i^{\ell_i}(x)}{p}.$$

Then $M(x) \in \mathbb{Z}[x]$ and the following statements are equivalent:

1. p does not divide the index $I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$.
2. For every $i = 1, \dots, r$, either $\ell_i = 1$ or $\ell_i \geq 2$ and $\overline{\phi_i(x)}$ does not divide $\overline{M(x)}$ in $\mathbb{F}_p[x]$.

2.2. The Field Index

We also recall a simple but very important statement of Hensel:

Theorem 3 (K. Hensel [2] p. 280).

The prime factors of the field index are smaller than the degree of the field.

Denote by $v_p(k)$ the highest power of the prime p dividing the integer k .

Theorem 4 (H. T. Engström [10]).

For number fields of degree $n \leq 7$, $v_p(m(K))$ is explicitly determined by the factorization of p into powers of prime ideals of $p\mathbb{Z}_K$.

The corresponding tables of [10] are too long to include here, but they present the explicit exponents.

2.3. Newton Polygon Method

If p divides the index $I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$ then Dedekind's Theorem 1 cannot be applied.

Using Newton polygons, an alternative method was given by Ore [11] to calculate $I(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$, D_K and the prime ideal factorization of primes in $\mathbb{Z}_K$. This was further developed among others by J. Montes and E. Nart [12], Fadil, L.E. J. Montes and E. Nart [13] and L. El Fadil [14]. This theory was extended to so-called higher-order Newton polygons by J. Guardia, J. Montes and E. Nart [15]. The method is also called Montes algorithm.

Here we only give a short introduction to some basic notions and statement of this very technical method, based on the explanation used in [16]. During recent years, a huge amount of papers have applied this method.

For any prime p , let v_p be the p -adic valuation of $\mathbb{Q}$. Denote by $\mathbb{Q}_p$ its p -adic completion and by $\mathbb{Z}_p$ the ring of p -adic integers. Let v_p be the Gauss's extension of v_p to $\mathbb{Q}_p(x)$, $v_p(P) = \min(v_p(a_i), (i = 0, \dots, n))$ for any polynomial $P(x) = \sum_{i=0}^n a_i x^i \in \mathbb{Q}_p[x]$, and extended by $v_p(P/Q) = v_p(P) - v_p(Q)$ for $0 \neq P, Q \in \mathbb{Q}_p[x]$. Let $\phi \in \mathbb{Z}_p[x]$ be a monic polynomial whose reduction is irreducible in $\mathbb{F}_p[x]$, and let $\mathbb{F}_\phi$ be the field $\mathbb{F}_p[x]/(\phi)$. For any monic polynomial $f(x) \in \mathbb{Z}_p[x]$, upon the Euclidean division by successive powers of ϕ , we expand $f(x)$ as follows:

$$f(x) = \sum_{i=0}^{\ell} a_i(x) \phi(x)^i.$$

This is called the ϕ -expansion of $f(x)$ ($\deg(a_i(x)) < \deg(\phi), i = 1, \dots, \ell$). The ϕ -Newton polygon of $f(x)$ with respect to p is the lower boundary convex envelope of the set of points

$\{(i, v_p(a_i(x))), a_i(x) \neq 0\}$ in the Euclidean plane, which we denote by $N_\phi f$. The ϕ -Newton polygon of f is the process of joining the edges $S_1, \dots, S_r$ ordered by increasing slopes, which can be expressed as

$$N_\phi f = S_1 + \dots + S_r.$$

For every side, S_i , of $N_\phi f$, the *length* of S_i , denoted $\ell(S_i)$, is the length of its projection to the x -axis. Its *height*, denoted by $h(S_i)$, is the length of its projection to the y -axis. Let $d(S_i) = \gcd(\ell(S_i), h(S_i))$ be the ramification degree of S . The *principal ϕ -Newton polygon* of f , denoted $N_\phi^+ f$, is the part of the polygon $N_\phi f$, which is determined by joining all sides of negative slopes. For every side, S , of $N_\phi^+ f$, with initial point (s, u_s) and length ℓ , and for every $0 \leq i \leq \ell$, we attach the *residue coefficient* $c_i \in \mathbb{F}_\phi$ as follows:

$$c_i = \begin{cases} 0, & \text{if } (s+i, u_{s+i}) \text{ lies strictly above } S, \\ \left(\frac{a_{s+i}(x)}{p^{u_{s+i}}} \right) \bmod (p, \phi(x)), & \text{if } (s+i, u_{s+i}) \text{ lies on } S, \end{cases}$$

where $(p, \phi(x))$ is the maximal ideal of $\mathbb{Z}_p[x]$ generated by p and ϕ . Let $\lambda = -h/e$ be the slope of S , where h and e are two positive coprime integers. Then $d = \ell/e$ is the *degree* of S . The points with integer coordinates lying on S are exactly

$$(s, u_s), (s+e, u_s-h), \dots, (s+de, u_s-dh).$$

Thus, if i is not a multiple of e , then $(s+i, u_{s+i})$ does not lie in S , and so $c_i = 0$. The polynomial

$$f_S(y) = t_d y^d + t_{d-1} y^{d-1} + \dots + t_1 y + t_0 \in \mathbb{F}_\phi[y],$$

is called the *residual polynomial* of $f(x)$ associated to the side S , where, for every $i = 0, \dots, d$, $t_i = c_{ie}$.

Let $N_\phi^+ f = S_1 + \dots + S_r$ be the principal ϕ -Newton polygon of f with respect to p . We say that f is a ϕ -regular polynomial with respect to p , if $f_{S_i}(y)$ is square free in $\mathbb{F}_\phi[y]$ for every $i = 1, \dots, r$. The polynomial f is said to be p -regular if $\overline{f(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{\ell_i}$ for some monic polynomials $\phi_1, \dots, \phi_t$ of $\mathbb{Z}[x]$, such that $\overline{\phi_1}, \dots, \overline{\phi_t}$ are irreducible coprime polynomials over $\mathbb{F}_p$ and f is a ϕ_i -regular polynomial with respect to p for every $i = 1, \dots, t$.

Let $\phi \in \mathbb{Z}_p[x]$ be a monic polynomial, such that $\overline{\phi(x)}$ is irreducible in $\mathbb{F}_p[x]$. The ϕ -index of $f(x)$ (cf. [13]), denoted by $\text{ind}_\phi(f)$, is $\deg(\phi)$ times the number of points with natural integer coordinates that lie below or on the polygon $N_\phi^+ f$, strictly above the horizontal axis and strictly beyond the vertical axis (see Figure 1).

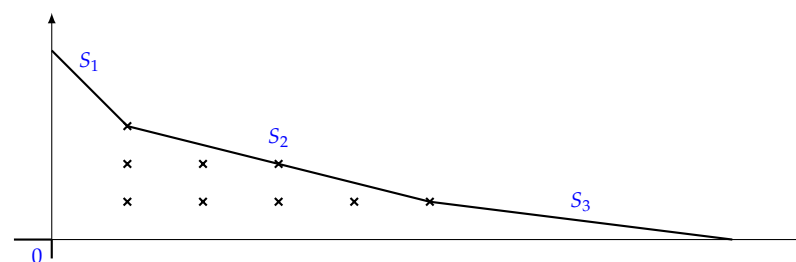


Figure 1. $N_\phi^+ f$.

In the example of Figure 1, $\text{ind}_\phi(f) = 9 \times \deg(\phi)$.

Now assume that $\overline{f(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{\ell_i}$ is the factorization of $\overline{f(x)}$ in $\mathbb{F}_p[x]$ into monic polynomials $\phi_i \in \mathbb{Z}[x]$, which are irreducible and pairwise coprime in $\mathbb{F}_p[x]$ ($i = 1, \dots, r$).

For every $i = 1, \dots, r$, let $N_{\phi_i}^+(f) = S_{i1} + \dots + S_{ir_i}$ be the principal ϕ_i -Newton polygon of f with respect to p . For every $j = 1, \dots, r_i$, let

$$f_{S_{ij}}(y) = \prod_{k=1}^{s_{ij}} \psi_{ijk}^{a_{ijk}}(y)$$

be the factorization of $f_{S_{ij}}(y)$ in $\mathbb{F}_{\phi_i}[y]$. Then we have the following index theorem of Ore.

Theorem 5 (Theorem of Ore, see Theorem 1.7 and Theorem 1.9 in [13], Theorem 3.9 in [14], pp. 323–325 in [11,12]).

1. We have

$$v_p(\text{ind}(f)) \geq \sum_{i=1}^r \text{ind}_{\phi_i}(f).$$

The equality holds if $f(x)$ is p -regular.

2. If $f(x)$ is p -regular, then

$$p\mathbb{Z}_K = \prod_{i=1}^r \prod_{j=1}^{r_i} \prod_{k=1}^{s_{ij}} \mathfrak{p}_{ijk}^{e_{ij}},$$

is the factorization of $p\mathbb{Z}_K$ into powers of prime ideals of $\mathbb{Z}_K$ lying above p , where $e_{ij} = \ell_{ij}/d_{ij}$, ℓ_{ij} is the length of S_{ij} , d_{ij} is the ramification degree of S_{ij} , and $f_{ijk} = \deg(\phi_i) \times \deg(\psi_{ijk})$ is the residue degree of the prime ideal $\mathfrak{p}_{ijk}$ over p .

2.4. Algorithmic Methods

Several of known efficient methods for the resolutions of Diophantine equations are related to Thue equations, cf. [4]. These methods are implemented, e.g., in Magma [17]. Therefore, the most efficient methods for solving index form equations also reduce the index form equation to Thue equations.

In cubic fields, the index form equation is a cubic Thue equation, see [4].

The below method of I. Gaál, A. Pethő and M. Pohst [18,19] reduces the index form equations in quartic fields to a cubic and some corresponding quartic Thue equations. This method is quite often used even nowadays, and therefore we briefly present it.

Let $K = \mathbb{Q}(\xi)$ be a quartic number field and $f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \in \mathbb{Z}[x]$ the minimal polynomial of ξ . We represent any $\alpha \in \mathbb{Z}_K$ in the form

$$\alpha = \frac{a_\alpha + x\xi + y\xi^2 + z\xi^3}{d}, \quad (1)$$

with $a_\alpha, x, y, z \in \mathbb{Z}$, and with a common denominator $d \in \mathbb{Z}$. Consider the solutions of the equation

$$I(\alpha) = m \quad (\alpha \in \mathbb{Z}_K) \quad (2)$$

for $0 < m \in \mathbb{Z}$. We have

Theorem 6 ([18]).

Let $i_m = d^6m/n$, where $n = I(\xi)$. The element α of (1) is a solution of (2), if and only if there is a solution $(u, v) \in \mathbb{Z}^2$ of the cubic equation

$$F(u, v) = u^3 - a_2u^2v + (a_1a_3 - 4a_4)uv^2 + (4a_2a_4 - a_3^2 - a_1^2a_4)v^3 = \pm i_m \quad (3)$$

such that (x, y, z) satisfies

$$\begin{aligned} Q_1(x, y, z) &= x^2 - xya_1 + y^2a_2 + xz(a_1^2 - 2a_2) + yz(a_3 - a_1a_2) \\ &\quad + z^2(-a_1a_3 + a_2^2 + a_4) = u, \\ Q_2(x, y, z) &= y^2 - xz - a_1yz + z^2a_2 = v. \end{aligned} \quad (4)$$

Equation (3) is either trivial to solve (when F is reducible), or it is a cubic Thue equation.

For a solution (u, v) of (3), we set $Q_0(x, y, z) = uQ_2(x, y, z) - vQ_1(x, y, z)$. If α in (1) is a solution of (2), then

$$Q_0(x, y, z) = 0. \quad (5)$$

If $(x_0, y_0, z_0) \in \mathbb{Z}^3$ is a non-trivial solution of (5), with, say, $z_0 \neq 0$ (such a solution can be easily found, see L. J. Mordell [20]), then we can parametrize the solutions x, y, z in the form

$$x = rx_0 + p, y = ry_0 + q, z = rz_0, \quad (6)$$

with rational parameters r, p, q . Substituting these x, y, z into (5), we obtain an equation of the form

$$r(c_1p + c_2q) = c_3p^2 + c_4pq + c_5q^2,$$

with integer coefficients $c_1, \dots, c_5$. Multiply the equations in (6) by $c_1p + c_2q$ and replace $r(c_1p + c_2q)$ by $c_3p^2 + c_4pq + c_5q^2$. Further multiply the equations in (6) by the square of the common denominator of p, q to obtain all integer relations (cf. [19]). We divide those by $\gcd(p, q)^2$ and obtain

$$kx = c_{11}p^2 + c_{12}pq + c_{13}q^2, ky = c_{21}p^2 + c_{22}pq + c_{23}q^2, kz = c_{31}p^2 + c_{32}pq + c_{33}q^2, \quad (7)$$

with integer c_{ij} and integer parameters p, q . Here, k is an integer parameter with the property that k divides the $\det(C)/d_0^2$, where C is the 3×3 matrix with entries c_{ij} and d_0 is the gcd of its entries (cf. [19]). Finally, substituting the x, y, z in (7) into (4) we obtain

$$F_1(p, q) = k^2u, F_2(p, q) = k^2v. \quad (8)$$

According to [19], at least one of the equations in Equation (8) is a quartic Thue equation over the original number field K .

3. Results

3.1. Pure Fields, Trinomials, Quadrinomials, etc.

There is no doubt that the Newton polygon method has been the most powerful tool during the last couple of years. It is frequently combined with the application of Dedekind's criterion. While, in 2014, S. Ahmad, T. Nakahara and M. Syed [21] investigated monogeneity properties of pure sextic fields using their subfield structure and relative monogeneity, in 2017 T. A. Gassert [22] already used Montes algorithm to describe monogeneity of pure fields. Note that this is only about the monogeneity of the polynomials and not the monogeneity of number fields generated by a root of the polynomial (for some corrections, see L. El Fadil [23]).

Together with Newton polygons (or instead of them), Dedekind's criterion and Engström's theorem are also often used. The following results often deal with polynomials of similar shape. It is important to add that, especially using Newton polygons, the whole calculation must be performed separately, even for polynomials of similar shape.

The first results investigated monogeneity in pure fields (or radical extensions) generated by a root of an irreducible binomial of type $x^n - m$. Assuming that m is squarefree, conditions were given for the monogeneity (or non-monogeneity) of such pure fields, for $n = 6, 8, 12, \dots$, etc. A following step was to consider general exponents like $n = 2^k, 2^k \cdot 3^\ell, \dots$, etc., and later on $n = p^k$ with a prime p . For some exponents, the more complicated case of a composite m was also investigated. Here is a list of such results, for brevity indicating only the exponents considered:

- Z. S. Aygin and K. D. Nguyen [24]: $n = 3$;
- L. El Fadil [25]: $n = 12$;
- L. El Fadil [26]: $n = 18$;
- L. El Fadil [27]: $n = 20$;
- L. El Fadil [28]: $n = 24$;

- L. El Fadil [29]: $n = 36$;
- Fadil, L.E. H. Ben Yakkou and J. Didi [30]: $n = 42$;
- Fadil, L.E. H. Choulli and O. Kchit [31]: $n = 60$;
- L. El Fadil and M. Faris [32]: $n = 84$;
- H. Ben Yakkou and O. Kchit [33]: $n = 3^k$;
- L. El Fadil [34]: $n = 2 \cdot 3^k$;
- L. El Fadil [35]: $n = 6, 2^k \cdot 3^\ell$;
- Yakkou, H.B. A. Chillali and L. El Fadil [36]: $n = 2^k \cdot 5^\ell$;
- L. El Fadil [37]: $n = 3^k \cdot 7^\ell$;
- L. El Fadil and A. Najim [38]: $n = 2^k \cdot 3^\ell$;
- L. El Fadil and O. Kchit [39]: $n = 2^k \cdot 7^\ell$;
- L. El Fadil [40]: $n = 2^k \cdot 3^\ell \cdot 5^t$;
- H. Ben Yakkou and L. El Fadil [41]: $n = p^k$;
- L. El Fadil [42]: $n = 6, m$ composite;
- L. El Fadil and I. Gaál [16]: $n = 8, m$ composite.

The exponents $n \leq 9$ with a squarefree m were investigated by I. Gaál and L. Remete [43], which was extended to arbitrary m by L. El Fadil and I. Gaál [16,44].

A typical statement from this list is the following:

Theorem 7 (L. El Fadil and A. Najim [38]).

Let α be a root of the irreducible polynomial $x^{2^k \cdot 3^\ell} - m$ with a squarefree m . If $m \not\equiv 1 \pmod{4}$ and $m \not\equiv \pm 1 \pmod{9}$ then α generates a power integral basis in $K = \mathbb{Q}(\alpha)$. If $m \equiv 1 \pmod{4}$, or $m \equiv 1 \pmod{9}$, or $k = 2$ and $m \equiv -1 \pmod{9}$, then K is not monogenic.

A next step was to consider monogeneity properties of number fields generated by a root of an irreducible trinomial of type $x^n + ax^m + b$. The field index is also often determined by using Engström's theorem. In the following list, again we only indicate the type of trinomials considered:

- L. El Fadil [45]: $x^4 + ax + b$;
- L. El Fadil and I. Gaál [46]: $x^4 + ax^2 + b$;
- H. Smith [47]: $x^4 + ax + b, x^4 + cx^3 + d$;
- L. Jones [48] showed that there exist exactly three distinct monogenic trinomials of the form $x^4 + bx^2 + d$ with Galois C_4 ;
- Jakhar, A. S. Kaur and S. Kumar [49]: $x^5 + ax + b$;
- L. El Fadil [50]: $x^5 + ax^2 + b$;
- L. El Fadil [51]: $x^5 + ax^3 + b$;
- L. El Fadil [52]: $x^6 + ax + b$;
- A. Jakhar and S. Kumar [53]: $x^6 + ax + b$;
- L. El Fadil [54]: $x^6 + ax^3 + b$;
- L. El Fadil and O. Kchit [55]: $x^6 + ax^5 + b$;
- A. Jakhar and S. Kaur [56]: $x^6 + ax^m + b$;
- R. Ibarra, H. Lembeck, M. Ozaslan, H. Smith and K. E. Stange [57]: $x^n + ax + b, x^n + cx^{n-1} + d$ for $n = 5, 6$;
- L. El Fadil and O. Kchit [58]: $x^7 + ax^3 + b$;
- H. Ben Yakkou [59]: $x^7 + ax^5 + b$;
- Jakhar, A. S. Kaur and S. Kumar [60]: $x^7 + ax + b$;
- H. Ben Yakkou [61]: $x^8 + ax + b$;
- H. Ben Yakkou and B. Boudine [62]: $x^8 + ax + b$;
- Jakhar, A. S. Kaur and S. Kumar [63]: $x^8 + ax^m + b$;
- L. Jones [64] considered monogenic trinomials of type $x^8 + ax^4 + b$ with prescribed Galois group;

- O. Kchit [65]: $x^9 + ax + b$;
- H. Ben Yakkou and P. Tiebekabe [66]: $x^9 + ax + b$;
- L. El Fadil and O. Kchit [67]: $x^9 + ax^2 + b$;
- L. El Fadil and O. Kchit [68]: $x^{12} + ax^m + b$;
- H. Ben Yakkou [69]: $x^{2^r} + ax^m + b$;
- H. Ben Yakkou and L. El Fadil [70]: $x^n + ax + b$, $n = 5, 6, 3^k, 2^k \cdot 3^\ell, 2^k \cdot 3^\ell + 1$;
- A. Jakhar and S. Kumar [71] gave explicit conditions for the non-monogeneity of $x^{q^s} - ax - b$;
- A. Jakhar [72]: $x^{p^s} - ax^m - b$;
- B. Jhorar and S. K. Khanduja [73]: $x^n + ax + b$, showed also that $f(x) = x^n - x - 1$ is monogenic, if and only if $|D(f)| = n^n - (n-1)^{n-1}$ is squarefree;
- H. Ben Yakkou [74]: $x^n + ax^m + b$, $n = p^k, s \cdot p^k, 2^k \cdot 3^\ell$;
- L. El Fadil [75]: $x^n + ax^m + b$, $n = 2^k \cdot 3^\ell$;
- A. Jakhar [76]: $x^n - ax^m - b$;
- Jakhar, A. S. Khanduja and N. Sangwan [77]: $x^n + ax^m + b$;
- Jakhar, A. S. Khanduja and N. Sangwan [78] gave necessary and sufficient conditions in terms of a, b, m, n for a given prime, p , to divide $I(\theta)$, where θ is a root of $x^n + ax^m + b$;
- L. Jones [79] considered monogenic reciprocal trinomials of type $x^{2m} + Ax^m + 1$;
- L. Jones [80] showed that there are infinitely many primes p , such that $x^6 + px^3 + 1$ is monogenic with Galois group D_6 ;
- L. Jones [81] showed that $x^n + x + 1$ is monogenic, if and only if its discriminant is squarefree;
- L. Jones and T. Phillips [82] showed that $x^n + ax + b$ is monogenic infinitely often;
- L. Jones and D. White [83] found new infinite families of monogenic trinomials of type $x^n + Ax^m + B$.

A typical statement from this list is the following:

Theorem 8 (L. Jones and D. White [83]).

Let $n \geq 2$ be an integer, with $m \geq 1$ a proper divisor of n . Let $t = n/m$ and let κ denote the squarefree kernel of m . Let A and B be positive integers with $\gcd(A, B) > 1$, and define

$$D := \frac{t^t B^{t-1} + (1-t)^{t-1} A^t}{\gcd(A, B)^{t-1}}.$$

If B and D are squarefree, and $\gcd(A, B) = 0 \pmod{\kappa}$, then $f(x) = x^n + Ax^m + B$ is monogenic. Moreover, $D(f)$ is not squarefree if $m \geq 2$.

The research continued into the direction considering monogeneity properties of quadri-nomials, quintinomials, etc., that is, polynomials with four, five, etc., terms and the number fields generated by a root of these polynomials:

- T. A. Gassert, H. Smith and K. E. Stange [84]: $x^4 - 6x^2 - kx - 3$;
- H. Ben Yakkou [85]: $x^4 + ax^3 + bx + c$;
- J. Harrington and L. Jones [86] constructed new families of quartic polynomials with various Galois groups, which are monogenic infinitely often;
- A. Jakhar and R. Kalwaniya [87]: $x^6 + ax^m + bx + c$;
- L. Jones [64]: $x^8 + ax^6 + bx^4 + ax^2 + 1$;
- L. Jones [88] constructed infinitely many monogenic polynomials of degree p for every odd prime p ;
- L. Jones [89]: $x^p - 2ptx^{p-1} + p^2t^2x^{p-2} + 1$;
- Jakhar, A. S. Kaur and S. Kumar [90]: $x^n + ax^2 + bx + c$;
- Jakhar, A. S. Kaur and S. Kumar [91]: $x^{p^s} - ax^n - bx^m - c$;
- A. Jakhar [92]: $x^n + ax^{n-1} + bx^{n-2} + c$;

- L. Jones [93] constructed infinite families of reciprocal monogenic polynomials with prescribed Galois group;
- L. Jones [94] showed that if $4 \leq n \geq m \geq 0$ and $\gcd(n, m) = \gcd(n, k) = 1$ then $x^{n-m}(x+k)^m + p$ is monogenic for infinitely many primes p ;
- L. Jones [95]: $x^n + A(Bx + 1)^m$;
- L. Jones [96]: $x^n + t \cdot g(x)$ with $n > \deg(g)$, when $g(x)$ is monic and $\deg(g) \in \{2, 3, 4\}$;
- L. Jones [97] constructed reciprocal monogenic quintinomials of type $x^{2^n} + Ax^{3 \cdot 2^{n-2}} + Bx^{2^{n-1}} + Ax^{2^{n-2}} + 1$;
- L. Jones [98] considered infinite families of monogenic quadrinomials, quintinomials and sextinomials.

3.2. The Relative Case

In addition to the absolute case (extension of $\mathbb{Q}$), several authors considered monogeneity problems in the relative case (extensions of an algebraic number field), or even similar problems in Dedekind rings. Mostly, Dedekind's criterion is used.

- M. E. Charkani and A. Deajim [99]; (see also A. Deajim and L. El Fadil [100]): $x^p - m$ over number fields;
- M. Sahmoudi and M. E. Charkani [101] considered relative pure cyclic extensions;
- A. Soullami, M. Sahmoudi and O. Boughaleb [102]: $x^{3^n} + ax^{3^s} - b$ over number fields;
- O. Boughaleb, A. Soullami and M. Sahmoudi [103]: $x^{p^n} + ax^{p^s} - b$ over number fields;
- H. Smith [104] studied relative radical extensions;
- S. K. Khanduja and B. Jhorar [105] gave equivalent versions of Dedekind's criterion in general rings;
- S. Arpin, S. Bozlee, L. Herr and H. Smith [106,107] studied monogeneity of number rings from a modul-theoretic perspective;
- R. Sekigawa [108] constructed an infinite number of cyclic relative extensions of prime degree that are relative monogenic.

3.3. Composite Polynomials

Several authors considered monogeneity of composites of polynomials, composites of binomials, etc. The authors mainly use Dedekind's criterion.

- J. Harrington and L. Jones [109] gave conditions for the monogeneity of $(x^m - b)^n - a$, and the composition of $x^n - a$ and $x^m - b$;
- Jakhar, A. R. Kalwaniya and P. Yadav [110] considered monogeneity of $(x^m - b)^n - a$, and the composition of $x^n - a$ and $x^m - b$ using a refined version of the Dedekind criterion;
- J. Harrington and L. Jones [111] considered monogeneity of $\Phi_{p^a}(\Phi_{2^b}(x))$, where $\Phi_N(x)$ is the cyclotomic polynomial of index N ;
- L. Jones [112] considered monotonically stable polynomials of type $g(f^n(x))$;
- L. Jones [113] constructed infinite collections of monic Eisenstein polynomials $f(x)$, such that $f(x^{d^n})$ are monogenic for all integers $n \geq 0$ and $d > 1$;
- L. Jones [114] considered monogeneity of $S_k(x^p)$, where $S_k(x) = x^3 - kx^2 - (k+3)x - 1$ the Shanks polynomial;
- L. Jones [115] considered monogeneity of $f(x^p)$, where $f(x)$ is the characteristic polynomial of an N th order linear recurrence;
- J. Harrington and L. Jones [116] gave conditions for the monogeneity of $f(x^{p^n})$, where $f(x) = x^m + ax^{m-1} + b$;
- S. Kaur, S. Kumar and L. Remete [117] considered monogeneity of $f(x^k)$, where $f(x) = x^d + A \cdot h(x)$, $\deg h < d$.

Let us recall a typical statement:

Theorem 9 (J. Harrington and L. Jones [111]). *Let a and b be positive integers, and let p be a prime. Then the polynomial $\Phi_{p^a}(\Phi_{2^b}(x))$ is monogenic, where $\Phi_N(x)$ is the cyclotomic polynomial of index N .*

3.4. Connection with primes

L. Jones [118–120], and J. Harrington and L. Jones [121] detected relations of monogeneity of power compositional polynomials with properties of primes. We present here one of these statements.

For a recurrence sequence $U_0 = 0, U_1 = 1$ and $U_n = kU_{n-1} + U_{n-2}$, (U_n) is periodic modulo any integer. Denote by $\pi_k(m)$ its period length modulo m . The prime p is called a k -Wall–Sun–Sun prime, if

$$U_{\pi_k(p)} \equiv 0 \pmod{p^2}.$$

Theorem 10 (L. Jones [118]).

Let $D = k^2 + 4$ if $k \equiv 1 \pmod{2}$, and $D = (k/2)^2 + 1$ if $k \equiv 0 \pmod{2}$. Suppose that $k \not\equiv 0 \pmod{4}$ and that D is squarefree. Let h denote the class number of $\mathbb{Q}(\sqrt{D})$. Let $s \geq 1$ be an integer, such that, for every odd prime divisor p of s , D is not a square modulo p and $\gcd(p, hD) = 1$. Then

$$x^{2s^n} - kx^{s^n} - 1$$

is monogenic for all integers $n \geq 1$, if and only if no prime divisor of s is a k -Wall–Sun–Sun prime.

3.5. Number of Generators of Power Integral Bases

Some further results considered the number of non-equivalent generators of power integral bases:

- M. Kang and D. Kim [122] considered the number of monogenic orders in pure cubic fields;
- J. H. Evertse [123] considered “rationally monogenic” orders of number fields;
- S. Akhtari [124] showed that a positive proportion of cubic number fields, when ordered by their discriminant, are not monogenic;
- L. Alpöge, M. Bhargava, A. Shnidman [125] showed that, if isomorphism classes of cubic fields are ordered by absolute discriminant, then a positive proportion are not monogenic and yet have no local obstruction to being monogenic (that is, the index form equations represent $+1$ or $-1 \pmod{p}$ for all primes p);
- M. Bhargava [126] proved that an order O in a quartic number field can have at most 2760 inequivalent generators of power integral bases (and at most 182 if $|D(O)|$ is sufficiently large). The problem is reduced to counting the number solutions of cubic and quartic Thue equations, somewhat analogously like described in Section 2.4, using a refined enumeration;
- S. Akhtari [127] gave another proof of Bhargava’s result [126]: she used the more direct approach of Section 2.4 and applied sharp bounds for the numbers of solutions of cubic and quartic Thue equations.

3.6. Miscellaneous

In addition to the above lists, there were several further interesting statements achieved for monogeneity. We try to recall them here.

- H. H. Kim [128] showed that the number of monogenic dihedral quartic extensions with absolute discriminant $\leq X$ is of size $O(X^{3/4}(\log X)^3)$;
- N. Khan, S. Katayama, T. Nakahara and T. Uehara [129] proved that the composite of a totally real field with a cyclotomic field of odd conductor ≥ 3 or even ≥ 8 has no power integral basis;
- N. Khan, T. Nakahara and H. Sekiguchi [130] proved that there are exactly three monogenic cyclic sextic fields of prime-power conductor, namely $\mathbb{Q}(\zeta_7)$, $\mathbb{Q}(\zeta_9)$ and the maximal real subfield of $\mathbb{Q}(\zeta_{13})$;

- D. Gil-Muñoz and M. Tinková [131] considered the indices of non-monogenic simplest cubic polynomials;
- L. Jones [132] considered infinite families of monogenic Pisot (anti-Pisot) polynomials;
- A. Jakhar and S. K. Khanduja [133] gave lower bounds for the p -index of a polynomial;
- M. Castillo, [134] showed, e.g., that $\mathbb{Q}(\alpha_n)$, $n \geq 1$ is monogenic, where $\alpha_0 = 1$ and $\alpha_n = \sqrt{2 + \alpha_{n-1}}$ for $n \geq 1$;
- T. Kashio and R. Sekigawa [135] showed that a monogenic normal cubic field is a simplest cubic field for some parameter;
- F. E. Tanoé [136] considered monogeneity of biquadratic fields using a special integer basis;
- K. V. Kouakou and F. E. Tanoé [137,138], and F. E. Tanoé and V. Kouassi [139] considered monogeneity of triquadratic fields;
- Aruna C. and P. Vanchinathan [140] showed that an infinite number of so-called exceptional quartic fields are monogenic.

3.7. Explicit Calculations, Algorithms

The powerful methods of Dedekind's criterion and Newton polygons often decide about the monogeneity of number fields. However, to explicitly determine all inequivalent generators of power integral bases one needs to perform calculations. These algorithms usually involve Baker-type estimates, reduction methods and enumeration algorithms, cf. [4]. There are efficient algorithms for low degree fields and some more complicated methods for higher degree fields. Since these procedures usually require considerable CPU time, if the number field is of high degree, or we need information about a large number of fields, then we turn to the so-called "fast" algorithms for determining "small" solutions. This yields a fast method to determine solutions of the index form equation with absolute values, say $\leq 10^{100}$. These algorithms determine all solutions with a high probability but do not exclude extremely large solutions (which, however, nobody has ever met).

We collect here some recent results involving explicit determination of generators of power integral bases.

- Z. Franušić and B. Jadrijević [141] calculated generators of relative power integral bases in a family of quartic extensions of imaginary quadratic fields;
- I. Gaál [142] showed that index form equations in composites of a totally real cubic field and a complex quadratic field can be reduced to absolute Thue equations;
- I. Gaál [143] showed that the index form equations in composites of a totally real field and a complex quadratic field can be reduced to the absolute index form equations of the totally real field;
- I. Gaál [144] considered generators of power integral bases in fields generated by monogenic trinomials of type $x^6 + 3x^3 + 3a$;
- I. Gaál [145] considered generators of power integral bases in fields generated by monogenic binomial compositions of type $(x^3 - b)^2 + 1$;
- I. Gaál [146] gave an efficient method to determine all generators of power integral bases of pure sextic fields;
- I. Gaál and L. Remete [147] considered monogeneity in octic fields of type $K = \mathbb{Q}(\sqrt[4]{a + bi})$;
- I. Gaál [148] determined "small" solutions of the index form equation in $K = \mathbb{Q}(\sqrt[6]{m})$, for $-5000 < m < 0$, such that $x^6 - m$ is monogenic (1521 fields). Experience: $\sqrt[6]{m}$ is the only generator of power integral bases;
- I. Gaál [149] determined "small" solutions of index form equations in $K = \mathbb{Q}(\sqrt[8]{m})$, $-5000 < m < 0$, such that $x^8 - m$ is monogenic (2024 fields). Experience: $\sqrt[8]{m}$ is the only generator of power integral bases, except for $m = -1$;
- I. Gaál [150] extended [46] on monogeneity properties of trinomials of type $x^4 + ax^2 + b$;
- I. Gaál [151] calculated generators of power integral bases in families of number fields generated by a root of monogenic quartic polynomials considered in [86].

In [150,151], the method described in Section 2.4 was used, in [141,147,149], its relative analogue, see [4,152].

Also, here we recall some typical statements:

Theorem 11 (I. Gaál [143]).

Let L be a totally real number field, $M = \mathbb{Q}(\sqrt{d})$, $d < 0$ squarefree, assume $\gcd(D_L, D_M) = 1$. If α generates a power integral basis in $K = LM$, then $\alpha = a + \beta \pm \omega$, where $a \in \mathbb{Z}$, β generates a power integral basis in L and $(1, \omega)$ is integral basis in M .

Theorem 12 (L. El Fadil and I. Gaál [46]).

Assume $a > 1, b > 1$ and $f(x) = x^4 + ax^2 + b$ is irreducible and monogenic. If a, b are not of type

$$a = \frac{u \pm 1}{v}, b = \frac{u^2 - 1}{4v^2}$$

for some $u, v \in \mathbb{Z}, v \neq 0, u \neq 1$, then up to equivalence the root α of $f(x)$ is the only generator of power integral bases in $K = \mathbb{Q}(\alpha)$.

4. Further Research

The above lists of results indicate what has already been already done and what is still missing. It would be very interesting to somehow describe monogeneity properties of quartic fields and maybe quintic fields. This would require study of quintinomials and sextinomials.

What general exponents of binomials and trinomials can still be considered? Is it possible to describe in general monogeneity properties of arbitrary trinomials of degree ≤ 9 ?

How can one extend the available algorithms to be able to calculate solutions of index form equations in higher degree fields?

All these and several other questions are to be answered. As it is seen from the above lists, in addition to some new ideas, often the application of old, forgotten methods may also help.

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References

1. Dedekind, R. Über den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen. *Göttingen Abh.* **1878**, *23*, 1–23.
2. Hensel, K. *Theorie der Algebraischen Zahlen*; Teubner: Leipzig, Germany, 1908.
3. Hasse, H. *Zahlentheorie*; Akademie: Berlin, Germany, 1963.
4. Gaál, I. Diophantine equations and power integral bases. In *Theory and algorithms*, 2nd ed.; Birkhäuser: Boston, MA, USA, 2019.
5. Birch, B.J.; Merriman, J.R. Finiteness theorems for binary forms with given discriminant. *Proc. Lond. Math. Soc.* **1972**, *24*, 385–394. [[CrossRef](#)]
6. Györy, K. Sur les polynômes à coefficients entiers et de discriminant donne, III. *Publ. Math. (Debrecen)* **1976**, *23*, 141–165. [[CrossRef](#)]
7. Neukirch, J. *Algebraic Number Theory*; Springer: Berlin, Germany, 1999.
8. Cohen, H. *A Course in Computational Algebraic Number Theory*, GTM 138; Springer: Berlin/Heidelberg, Germany, 1993.
9. Pohst, M.P.; Zassenhaus, H. *Algorithmic Algebraic Number Theory*; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, UK, 1989.
10. Engstrom, H.T. On the common index divisor of an algebraic number field. *Trans. Am. Math. Soc.* **1930**, *32*, 223–237. [[CrossRef](#)]
11. Ore, O. Newtonsche Polygone in der Theorie der algebraischen Körper. *Math. Ann.* **1928**, *99*, 84–117. [[CrossRef](#)]
12. Montes, J.; Nart, E. On a theorem of Ore. *J. Algebra* **1992**, *146*, 318–334. [[CrossRef](#)]
13. Fadil, L.E.; Montes, J.; Nart, E. Newton polygons and p -integral bases of quartic number fields. *J. Algebra Appl.* **2012**, *11*, 1250073. [[CrossRef](#)]

14. Fadil, L.E. On Newton polygon techniques and factorization of polynomial over Henselian valued fields. *J. Algebra Appl.* **2020**, *19*, 2050188. [\[CrossRef\]](#)
15. Guardia, J.; Montes, J.; Nart, E. Newton polygons of higher order in algebraic number theory. *Trans. Amer. Math. Soc.* **2012**, *364*, 361–416. [\[CrossRef\]](#)
16. Fadil, L.E.; Gaál, I. Integral bases and monogeneity of pure number fields with non-square free parameters up to degree 9. *Tatra Mt. Math. Publ.* **2023**, *83*, 61–86. [\[CrossRef\]](#)
17. Bosma, W.; Cannon, J.; Playoust, C. The Magma Algebra System. I. The User Language. *J. Symb. Comput.* **1997**, *24*, 235–265. [\[CrossRef\]](#)
18. Gaál, I.; Pethő, A.; Pohst, M. On the resolution of index form equations in quartic number fields. *J. Symb. Comput.* **1993**, *16*, 563–584. [\[CrossRef\]](#)
19. Gaál, I.; Pethő, A.; Pohst, M. Simultaneous representation of integers by a pair of ternary quadratic forms—With an application to index form equations in quartic number fields. *J. Number Theory* **1996**, *57*, 90–104. [\[CrossRef\]](#)
20. Mordell, L.J. *Diophantine Equations*; Academic Press: New York, NY, USA; London, UK, 1969.
21. Ahmad, S.; Nakahara, T.; Syed, M. Power integral bases for certain pure sextic fields. *Int. J. Number Theory* **2014**, *10*, 2257–2265. [\[CrossRef\]](#)
22. Gassert, T.A. A note on the monogeneity of power maps. *Albanian J. Math.* **2017**, *11*, 3–12. [\[CrossRef\]](#)
23. Fadil, L.E. A note on monogeneity of pure number fields. *arXiv* **2021**, arXiv:2106.00004.
24. Aygin, Z.S.; Nguyen, K.D. Monogenic pure cubics. *J. Number Theory* **2021**, *219*, 356–367; Erratum in *J. Number Theory* **2023**, *242*, 244. [\[CrossRef\]](#)
25. Fadil, L.E. On power integral bases for certain pure number fields. *Publ. Math. Debr.* **2022**, *100*, 219–231. [\[CrossRef\]](#)
26. Fadil, L.E. On power integral bases for certain pure number fields defined by $x^{18} - m$. *Commentat. Math. Univ. Carol.* **2022**, *63*, 11–19.
27. Fadil, L.E. On monogeneity of certain pure number fields defined by $x^{20} - m$. *Sao Paulo J. Math. Sci.* **2022**, *16*, 1063–1071. [\[CrossRef\]](#)
28. Fadil, L.E. On power integral bases for certain pure number fields defined by $x^{24} - m$. *Stud. Sci. Math. Hung.* **2020**, *57*, 397–407. [\[CrossRef\]](#)
29. Fadil, L.E. On power integral bases for certain pure number fields defined by $x^{36} - m$. *Stud. Sci. Math. Hung.* **2021**, *58*, 371–380.
30. Fadil, L.E.; Yakkou, H.B.; Didi, J. On power integral bases of certain pure number fields defined by $x^{42} - m$. *Bol. Soc. Mat. Mex. III. Ser.* **2021**, *27*, 81. [\[CrossRef\]](#)
31. Fadil, L.E.; Choulli, H.; Kchit, O. On monogeneity of certain pure number fields defined by $x^{60} - m$. *Acta Math. Vietnam* **2023**, *48*, 283–293. [\[CrossRef\]](#)
32. Fadil, L.E.; Faris, M. On power integral bases of certain pure number fields defined by $x^{84} - m$. *Rev. Unión Mat. Argent.* **2023**, *65*, 197–211. [\[CrossRef\]](#)
33. Yakkou, H.B.; Kchit, O. On power integral bases of certain pure number fields defined by $x^{3^r} - m$. *Sao Paulo J. Math. Sci.* **2022**, *16*, 1072–1079. [\[CrossRef\]](#)
34. Fadil, L.E. On power integral bases for certain pure number fields defined by $x^{2 \cdot 3^k} - m$. *Acta Arith.* **2021**, *201*, 269–280. [\[CrossRef\]](#)
35. Fadil, L.E. On power integral bases for certain pure sextic fields. *Bol. Soc. Parana. Mat.* **2022**, *40*, 143. [\[CrossRef\]](#)
36. Yakkou, H.B.; Chillali, A.; Fadil, L.E. On power integral bases for certain pure number fields defined by $x^{2^r \cdot 5^s} - m$. *Commun. Algebra* **2021**, *49*, 2916–2926. [\[CrossRef\]](#)
37. Fadil, L.E. On power integral bases of certain pure number fields defined by $x^{3^r \cdot 7^s} - m$. *Colloq. Math.* **2022**, *169*, 307–317. [\[CrossRef\]](#)
38. Fadil, L.E.; Najim, A. On monogeneity of certain pure number fields defined by $x^{2^u \cdot 3^v} - m$. *Acta Sci. Math.* **2022**, *88*, 581–594. [\[CrossRef\]](#)
39. Fadil, L.E.; Kchit, O. On monogeneity of certain pure number fields defined by $x^{2^r \cdot 7^s} - m$. *Bol. Soc. Parana. Mat.* **2023**, *41*, 138.
40. Fadil, L.E. On monogeneity of certain pure number fields defined by $x^{2^u \cdot 3^v \cdot 5^t} - m$. *arXiv* **2022**, arXiv:2204.02436.
41. Yakkou, H.B.; Fadil, L.E. On monogeneity of certain pure number fields defined by $x^{p^r} - m$. *Int. J. Number Theory* **2021**, *17*, 2235–2242. [\[CrossRef\]](#)
42. Fadil, L.E. On integral bases and monogeneity of pure sextic number fields with non-squarefree coefficients. *J. Number Theory* **2021**, *228*, 375–389. [\[CrossRef\]](#)
43. Gaál, I.; Remete, L. Integral bases and monogeneity of pure fields. *J. Number Theory* **2017**, *173*, 129–146. [\[CrossRef\]](#)
44. Fadil, L.E.; Gaál, I. On integral bases and monogeneity of pure octic number fields with non-square free parameters. *arXiv* **2022**, arXiv:2202.04417.
45. Fadil, L.E. A note on indices of quartic number fields defined by trinomials $x^4 + ax + b$. *Commun. Algebra* **2024**, *52*, 1349–1359. [\[CrossRef\]](#)
46. Fadil, L.E.; Gaál, I. On the monogeneity of quartic number fields defined by $x^4 + ax^2 + b$. *arXiv* **2022**, arXiv:2204.03226.
47. Smith, H. Two families of monogenic S_4 quartic number fields. *Acta Arith.* **2018**, *186*, 257–271. [\[CrossRef\]](#)
48. Jones, L. Monogenic cyclic quartic trinomials. *arXiv* **2024**, arXiv:2404.17869.
49. Jakhar, A.; Kaur, S.; Kumar, S. Common index divisor of the number fields defined by $x^5 + ax + b$. *Proc. Edinb. Math. Soc. II. Ser.* **2022**, *65*, 1147–1161. [\[CrossRef\]](#)

50. Fadil, L.E. On common index divisors and monogeneity of certain number fields defined by $x^5 + ax^2 + b$. *Commun. Algebra* **2022**, *50*, 3102–3112. [\[CrossRef\]](#)
51. Fadil, L.E. On the index divisors and monogeneity of number fields defined by $x^5 + ax^3 + b$. *Quaest. Math.* **2023**, *46*, 2355–2365. [\[CrossRef\]](#)
52. Fadil, L.E. On common index divisor and monogeneity of certain number fields defined by trinomials $X^6 + AX + B$. *Quaest. Math.* **2023**, *46*, 1609–1627. [\[CrossRef\]](#)
53. Jakhar, A.; Kumar, S. On nonmonogenic number fields defined by $x^6 + ax + b$. *Can. Math. Bull.* **2022**, *65*, 788–794. [\[CrossRef\]](#)
54. Fadil, L.E. On non monogeneity of certain number fields defined by trinomials $x^6 + ax^3 + b$. *J. Number Theory* **2022**, *239*, 489–500. [\[CrossRef\]](#)
55. Fadil, L.E.; Kchit, O. On index divisors and monogeneity of certain sextic number fields defined by $x^6 + ax^5 + b$. *arXiv* **2022**, arXiv:2206.05529.
56. Jakhar, A.; Kaur, S. A note on non-monogeneity of number fields arising from sextic trinomials. *Quaest. Math.* **2023**, *46*, 833–840. [\[CrossRef\]](#)
57. Ibarra, R.; Lembeck, H.; Ozaslan, M.; Smith, H.; Stange, K.E. Monogenic fields arising from trinomials. *Involve* **2022**, *15*, 299–317. [\[CrossRef\]](#)
58. Fadil, L.E.; Kchit, O. On index divisors and monogeneity of certain septic number fields defined by $x^7 + ax^3 + b$. *Commun. Algebra* **2023**, *51*, 2349–2363. [\[CrossRef\]](#)
59. Yakkou, H.B. The index of the septic number field defined by $x^7 + ax^5 + b$. *arXiv* **2022**, arXiv:2206.14345.
60. Jakhar, A.; Kaur, S.; Kumar, S. On common index divisor of the number fields defined by $x^7 + ax + b$. *arXiv* **2023**, arXiv:2301.00365.
61. Yakkou, H.B. On monogeneity of certain number fields defined by $x^8 + ax + b$. *Acta Math. Hung.* **2022**, *166*, 614–623. [\[CrossRef\]](#)
62. Yakkou, H.B.; Boudine, B. On the index of the octic number field defined by $x^8 + ax + b$. *Acta Math. Hung.* **2023**, *170*, 585–607. [\[CrossRef\]](#)
63. Jakhar, A.; Kaur, S.; Kumar, S. Non-monogeneity of certain octic number fields defined by trinomials. *Colloq. Math.* **2023**, *171*, 145–152. [\[CrossRef\]](#)
64. Jones, L. Monogenic even octic polynomials and their Galois groups. *arXiv* **2024**, arXiv:2404.17921.
65. Kchit, O. On the index divisors and monogeneity of certain nonic number fields. *arXiv* **2023**, arXiv:2307.03284.
66. Yakkou, H.B.; Tiebekabe, P. On common index divisors and monogeneity of the nonic number field defined by a trinomial $x^9 + ax + b$. *arXiv* **2022**, arXiv:2212.05029.
67. Fadil, L.E.; Kchit, O. The index of certain nonic number fields defined by $x^9 + ax^2 + b$. *arXiv* **2023**, arXiv:2310.13509.
68. Fadil, L.E.; Kchit, O. On index divisors and monogeneity of certain number fields defined by $x^{12} + ax^m + b$. *Ramanujan J.* **2024**, *63*, 451–482. [\[CrossRef\]](#)
69. Yakkou, H.B. On monogeneity of certain number fields defined by $x^{2^r} + ax^m + b$. *arXiv* **2022**, arXiv:2203.10413.
70. Yakkou, H.B.; Fadil, L.E. On monogeneity of certain number fields defined by trinomials. *Funct. Approx. Comment. Math.* **2022**, *67*, 199–221. [\[CrossRef\]](#)
71. Jakhar, A.; Kumar, S. Non-monogeneity of some number fields generated by binomials or trinomials of prime-power degree. *J. Algebra Appl.* **2024**, *23*, 2450095. [\[CrossRef\]](#)
72. Jakhar, A. On nonmonogenic algebraic number fields. *Rocky Mt. J. Math.* **2023**, *53*, 103–110. [\[CrossRef\]](#)
73. Jhorar, B.; Khanduja, S.K. On power basis of a class of algebraic number fields. *Int. J. Number Theory* **2016**, *12*, 2317–2321. [\[CrossRef\]](#)
74. Yakkou, H.B. On nonmonogenic number fields defined by trinomials of type $x^n + ax^m + b$. *Rocky Mt. J. Math.* **2023**, *53*, 685–699.
75. Fadil, L.E. On index and monogeneity of certain number fields defined by trinomials. *Math. Slovaca* **2023**, *73*, 861–870. [\[CrossRef\]](#)
76. Jakhar, A. Nonmonogeneity of number fields defined by trinomials. *N. Y. J. Math.* **2022**, *28*, 650–658.
77. Jakhar, A.; Khanduja, S.K.; Sangwan, N. On prime divisors of the index of an algebraic integer. *J. Number Theory* **2016**, *166*, 47–61. [\[CrossRef\]](#)
78. Jakhar, A.; Khanduja, S.K.; Sangwan, N. Characterization of primes dividing the index of a trinomial. *Int. J. Number Theory* **2017**, *13*, 2505–2514. [\[CrossRef\]](#)
79. Jones, L. Monogenic reciprocal trinomials and their Galois groups. *J. Algebra Appl.* **2022**, *21*, 2250026. [\[CrossRef\]](#)
80. Jones, L. Sextic reciprocal monogenic dihedral polynomials. *Ramanujan J.* **2021**, *56*, 1099–1110. [\[CrossRef\]](#)
81. Jones, L. Infinite families of non-monogenic trinomials. *Acta Sci. Math.* **2021**, *87*, 95–105. [\[CrossRef\]](#)
82. Jones, L.; Phillips, T. Infinite families of monogenic trinomials and their Galois groups. *Int. J. Math.* **2018**, *29*, 1850039. [\[CrossRef\]](#)
83. Jones, L.; White, D. Monogenic trinomials with non-squarefree discriminant. *Int. J. Math.* **2021**, *32*, 2150089. [\[CrossRef\]](#)
84. Gassert, T.A.; Smith, H.; Stange, K.E. A family of monogenic S_4 quartic fields arising from elliptic curves. *J. Number Theory* **2019**, *197*, 361–382. [\[CrossRef\]](#)
85. Yakkou, H.B. On indices and monogeneity of quartic number fields defined by quadrinomials. *arXiv* **2024**, arXiv:2401.12782.
86. Harrington, J.; Jones, L. Monogenic quartic polynomials and their Galois groups. *arXiv* **2024**, arXiv:2404.05487v2.
87. Jakhar, A.; Kalwaniya, R. On the index divisors of certain number fields. *arXiv* **2023**, arXiv:2303.00484.
88. Jones, L. Monogenic polynomials with non-squarefree discriminant. *Proc. Am. Math. Soc.* **2020**, *148*, 1527–1533. [\[CrossRef\]](#)
89. Jones, L. On necessary and sufficient conditions for the monogeneity of a certain class of polynomials. *Math. Slovaca* **2022**, *72*, 591–600. [\[CrossRef\]](#)

90. Jakhar, A.; Kaur, S.; Kumar, S. On power basis of a class of number fields. *Mediterr. J. Math.* **2023**, *20*, 315. [\[CrossRef\]](#)
91. Jakhar, A.; Kaur, S.; Kumar, S. On non-monogeneity of the number fields defined by certain quadrinomials. *Commun. Algebra* **2023**, *51*, 2448–2459. [\[CrossRef\]](#)
92. Jakhar, A. On primes dividing the index of a quadrinomial. *Rocky Mt. J. Math.* **2020**, *50*, 2117–2125. [\[CrossRef\]](#)
93. Jones, L. Infinite families of reciprocal monogenic polynomials and their Galois groups. *New York J. Math.* **2021**, *27*, 1465–1493.
94. Jones, L. Some new infinite families of monogenic polynomials with non-squarefree discriminant. *Acta Arith.* **2021**, *197*, 213–219. [\[CrossRef\]](#)
95. Jones, L. A brief note on some infinite families of monogenic polynomials. *Bull. Aust. Math. Soc.* **2019**, *100*, 239–244. [\[CrossRef\]](#)
96. Jones, L. Generating infinite families of monogenic polynomials using a new discriminant formula. *Albanian J. Math.* **2020**, *14*, 37–45. [\[CrossRef\]](#)
97. Jones, L. Reciprocal monogenic quintinomials of degree 2^n . *Bull. Aust. Math. Soc.* **2022**, *106*, 437–447. [\[CrossRef\]](#)
98. Jones, L. Infinite families of monogenic quadrinomials, quintinomials and sextinomials. *Colloq. Math.* **2022**, *169*, 1–10. [\[CrossRef\]](#)
99. Charkani, M.E.; Deajim, A. Generating a power basis over a Dedekind ring. *J. Number Theory* **2012**, *132*, 2267–2276. [\[CrossRef\]](#)
100. Deajim, A.; Fadil, L.E. A note on generating a power basis over a Dedekind ring. *Stud. Sci. Math. Hung.* **2021**, *58*, 367–370. [\[CrossRef\]](#)
101. Sahmoudi, M.; Charkani, M.E. On relative pure cyclic fields with power integral bases. *Math. Bohem.* **2023**, *148*, 117–128. [\[CrossRef\]](#)
102. Soullami, A.; Sahmoudi, M.; Boughaleb, O. On relative power integral basis of a family of numbers fields. *Rocky Mt. J. Math.* **2021**, *51*, 1443–1452. [\[CrossRef\]](#)
103. Boughaleb, O.; Soullami, A.; Sahmoudi, M. On relative monogeneity of a family of number fields defined by $X^{p^n} + aX^{p^s} - b$. *Bol. Soc. Mat. Mex., III. Ser.* **2023**, *29*, 5. [\[CrossRef\]](#)
104. Smith, H. The monogeneity of radical extensions. *Acta Arith.* **2021**, *198*, 313–327. [\[CrossRef\]](#)
105. Khanduja, S.K.; Jhorar, B. When is $R[\theta]$ integrally closed? *J. Algebra Appl.* **2016**, *15*, 1650091. [\[CrossRef\]](#)
106. Arpin, S.; Bozlee, S.; Herr, L.; Smith, H. The scheme of monogenic generators I: Representability. *Res. Number Theory* **2023**, *9*, 14. [\[CrossRef\]](#)
107. Arpin, S.; Bozlee, S.; Herr, L.; Smith, H. The scheme of monogenic generators. II: Local monogenicity and twists. *Res. Number Theory* **2023**, *9*, 43. [\[CrossRef\]](#)
108. Sekigawa, R. Rikuna's generic cyclic polynomial and the monogeneity. *J. Number Theory* **2022**, *231*, 239–250. [\[CrossRef\]](#)
109. Harrington, J.; Jones, L. Monogenic binomial compositions. *Taiwan. J. Math.* **2020**, *24*, 1073–1090. [\[CrossRef\]](#)
110. Jakhar, A.; Kalwaniya, R.; Yadav, P. A study of monogeneity of binomial composition. *arXiv* **2024**, arXiv:2402.10131.
111. Harrington, J.; Jones, L. Monogenic cyclotomic compositions. *Kodai Math. J.* **2021**, *44*, 115–125. [\[CrossRef\]](#)
112. Jones, L. Monogenically stable polynomials. *Albanian J. Math.* **2021**, *15*, 85–98. [\[CrossRef\]](#)
113. Jones, L. The monogeneity of power-compositional Eisenstein polynomials. *Ann. Math. Inform.* **2022**, *55*, 93–113. [\[CrossRef\]](#)
114. Jones, L. On the monogenicity of power-compositional Shanks polynomials. *Funct. Approx. Comment. Math.* **2023**, *69*, 93–103. [\[CrossRef\]](#)
115. Jones, L. The monogenicity of power-compositional characteristic polynomials. *arXiv* **2023**, arXiv:2311.08875.
116. Harrington, J.; Jones, L. The Irreducibility and Monogenicity of Power-Compositional Trinomials. *arXiv* **2022**, arXiv:2204.07784.
117. Kaur, S.; Kumar, S.; Remete, L. On the index of power compositional polynomials. *arXiv* **2024**, arXiv:2404.17351.
118. Jones, L. A connection between the monogenicity of certain power-compositional trinomials and k -Wall-Sun-Sun primes. *arXiv* **2022**, arXiv:2211.14834.
119. Jones, L. Generalized Wall-Sun-Sun primes and monogenic power compositional trinomials. *Albanian J. Math.* **2023**, *17*, 3–17. [\[CrossRef\]](#)
120. Jones, L. A new condition for k -Wall-Sun-Sun primes. *Taiwan. J. Math.* **2024**, *28*, 17–28. [\[CrossRef\]](#)
121. Harrington, J.; Jones, L. A note on generalised Wall-Sun-Sun primes. *Bull. Aust. Math. Soc.* **2023**, *108*, 373–378. [\[CrossRef\]](#)
122. Kang, M.; Kim, D. The proportion of monogenic orders of prime power indices of the pure cubic field. *arXiv* **2023**, arXiv:2306.13295.
123. Evertse, J.H. Orders with few rational monogenizations. *Acta Arith.* **2023**, *210*, 307–335. [\[CrossRef\]](#)
124. Akhtari, S. Counting monogenic cubic orders. In *Combinatorial and Additive Number Theory III*; Nathanson, M.B., Ed.; Papers Based on Talks Given at the CANT 2017 and 2018 Workshops, New York, NY, USA, May 2017 and May 2018; Springer: Cham, Switzerland, 2020.
125. Alpöge, L.; Bhargava, M.; Shnidman, A. A positive proportion of cubic fields are not monogenic yet have no local obstruction to being so. *arXiv* **2020**, arXiv:2011.01186.
126. Bhargava, M. On the number of monogenizations of a quartic order (with an appendix by Shabnam Akhtari). *Publ. Math. Debr.* **2022**, *100*, 513–531. [\[CrossRef\]](#)
127. Akhtari, S. Quartic index form equations and monogenizations of quartic orders. *Essent. Number Theory* **2022**, *1*, 57–72. [\[CrossRef\]](#)
128. Kim, H.H. Monogenic dihedral quartic extensions. *Ramanujan J.* **2019**, *50*, 459–464. [\[CrossRef\]](#)
129. Khan, N.; Katayama, S.; Nakahara, T.; Uehara, T. Monogeneity of totally real algebraic extension fields over a cyclotomic field. *J. Number Theory* **2016**, *158*, 348–355. [\[CrossRef\]](#)
130. Khan, N.; Nakahara, T.; Sekiguchi, H. An ideal theoretic proof on monogeneity of cyclic sextic fields of prime power conductor. *J. Number Theory* **2019**, *198*, 43–51. [\[CrossRef\]](#)

131. Gil-Munoz, D.; Tinková, M. Additive structure of non-monogenic simplest cubic fields. *arXiv* **2022**, arXiv:2212.00364.
132. Jones, L. Monogenic Pisot and anti-Pisot polynomials. *Taiwan J. Math.* **2022**, *26*, 233–250. [[CrossRef](#)]
133. Jakhar, A.; Khanduja, S.K. On the index of an algebraic integer and beyond. *J. Pure Appl. Algebra* **2020**, *224*, 106281. [[CrossRef](#)]
134. Castillo, M. A dynamical characterization for monogeneity at every level of some infinite 2-towers. *Canad. Math. Bull.* **2022**, *65*, 806–814. [[CrossRef](#)]
135. Kashio, T.; Sekigawa, R. The characterization of cyclic cubic fields with power integral bases. *Kodai Math. J.* **2021**, *44*, 290–306. [[CrossRef](#)]
136. Tanoé, F.E. Chatelain’s integer bases for biquadratic fields. *Afr. Mat.* **2017**, *28*, 727–744. [[CrossRef](#)]
137. Kouakou, K.V.; Tanoé, F.E. Chatelain’s integral bases for triquadratic number fields. *Afr. Mat.* **2017**, *28*, 119–149. [[CrossRef](#)]
138. Tanoé, F.E.; Kouakou, K.V. Diophantine proof of non-monogeneity for triquadratic number fields with odd discriminant. *Fundam. J. Math. Math. Sci.* **2021**, *15*, 9–37.
139. Tanoé, F.E.; Kouassi, V. Generators of power integral bases of $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\sqrt{-3}, \sqrt{2}, \sqrt{-1})$. *Ann. Mathématiques Afr.* **2025**, *5*, 117–131.
140. Aruna C.; Vanchinathan, P. Exceptional Quartics are Ubiquitous. *arXiv* **2023**, arXiv:2306.17556.
141. Franušić, Z.; Jadrijević, B. Computing relative power integral bases in a family of quartic extensions of imaginary quadratic fields. *Publ. Math. Debr.* **2018**, *92*, 293–315. [[CrossRef](#)]
142. Gaál, I. Monogeneity in totally complex sextic fields, revisited. *JP J. Algebra Number Theory Appl.* **2020**, *47*, 87–98. [[CrossRef](#)]
143. Gaál, I. Monogeneity in totally real extensions of imaginary quadratic fields with an application to simplest quartic fields. *Acta Sci. Math.* **2023**, *89*, 3–12. [[CrossRef](#)]
144. Gaál, I. An experiment on the monogeneity of a family of trinomials. *JP J. Algebra Number Theory Appl.* **2021**, *51*, 97–111. [[CrossRef](#)]
145. Gaál, I. On the monogeneity of certain binomial compositions. *JP J. Algebra Number Theory Appl.* **2022**, *57*, 1–16. [[CrossRef](#)]
146. Gaál, I. Calculating generators of power integral bases in pure sextic fields. *Funct. Approx. Comment. Math.* **2024**, *70*, 85–100. [[CrossRef](#)]
147. Gaál, I.; Remete, L. On the monogeneity of pure quartic relative extensions of $\mathbb{Q}(i)$. *Acta Sci. Math.* **2023**, *89*, 357–371. [[CrossRef](#)]
148. Gaál, I. A note on the monogeneity of totally complex pure sextic fields. *JP J. Algebra Number Theory Appl.* **2023**, *60*, 85–96.
149. Gaál, I. On the monogeneity of totally complex pure octic fields. *arXiv* **2024**, arXiv:2402.09293.
150. Gaál, I. A note on the monogeneity of some trinomials of type $x^4 + ax^2 + b$. *JP J. Algebra Number Theory Appl.* **2024**, *63*, 265–279.
151. Gaál, I. Calculating power integral bases in some quartic fields corresponding to monogenic families of polynomials. *arXiv* **2024**, arXiv:2405.13429.
152. Gaál, I.; Pohst, M. Computing power integral bases in quartic relative extensions. *J. Number Theory* **2000**, *85*, 201–219. [[CrossRef](#)]

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