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From the Crossing Numbers of $K_5 + P_n$ and $K_5 + C_n$ to the Crossing Numbers of $W_m + S_n$ and $W_m + W_n$

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Abstract: The crossing number of a graph is a significant measure that indicates the complexity of the graph and the difficulty of visualizing it. In this paper, we examine the crossing numbers of join products involving the complete graph K_5 with discrete graphs, paths, and cycles. We analyze optimal drawings of K_5 , identify all five non-isomorphic drawings, and address previously hypothesized crossing numbers for $K_5 + P_n$ and $K_5 + C_n$. Through a simplified approach, we first establish $cr(K_5 + D_n)$ and then extend our method to prove the crossing numbers $cr(K_5 + P_n)$ and $cr(K_5 + C_n)$. These results also lead to new hypotheses for $cr(W_m + S_n)$ and $cr(W_m + W_n)$ involving wheels and stars. Our findings correct previous inaccuracies in the literature and provide a foundation for future research.

Keywords: crossing number; complete graph K_5 ; discrete graphs; paths; cycles; good drawing; join product; separating cycles; wheels; stars

MSC: 05C10



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1. Introduction

Graph theory serves as a foundational framework for modeling and analyzing complex systems across various disciplines, ranging from computer science to social networks to biology. One fundamental aspect of graph theory is the concept of crossing numbers, which quantifies the minimum number of crossings required in a drawing of a graph [1]. The understanding and minimization of the crossing number of a graph have significant implications in numerous areas, encompassing both theoretical and practical domains. In computer science, minimizing the crossing number of a graph is essential for optimizing layout algorithms in circuit design, VLSI layout, and graph drawing applications. Moreover, in network design, reducing the crossing number enhances the efficiency and reliability of communication networks by minimizing signal interference and congestion. Furthermore, the crossing number has implications in spatial visualization and cartography, where minimizing crossings leads to clearer and more interpretable representations of geographic networks and transportation systems. In social network analysis, understanding the crossing number sheds light on the underlying structure and dynamics of social interactions, facilitating the identification of cohesive communities and influential nodes. Overall, reducing the number of crossings on graph edges can help in visualizing and understanding complex data, improving system performance, and optimizing graph algorithms [2,3].

In the paper, we will focus on the crossing number $cr(G)$ of a composite graph G obtained by combining two graphs G_i and G_j , known as their join product $G = G_i + G_j$. It is constructed by incorporating separate copies of G_i and G_j with no shared vertices and by adding all possible edges between the vertex sets $V(G_i)$ and $V(G_j)$. In the case where $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of G is composed of the disjoint sets of edges

$E(G_i)$, $E(G_j)$ and the complete bipartite graph $K_{m,n}$. Therefore, some parts of proofs will be based on Kleitman's result [4] on the crossing numbers for some $K_{m,n}$. He showed that

$$\text{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if } m \leq 6. \quad (1)$$

The graph terminology used is taken from the book [5].

One graph of our considered join product G will be a complete graph with five vertices K_5 , which is specific for its symmetries. As the crossing number $\text{cr}(K_5) = 1$, there is at least one crossing over all optimal drawings of K_5 with the vertex set $V(K_5) = \{v_1, \dots, v_5\}$. In the rest of the paper, let the edges v_1v_3 and v_2v_4 be crossed. The edges v_1v_2 , v_2v_3 , v_3v_4 , and v_4v_1 cannot cross each other because we are only interested in optimal subdrawings. Finally, all considered drawings of K_5 can be achieved from the nonplanar drawing of K_4 by adding one vertex v_5 with the four corresponding edges v_1v_5 , v_2v_5 , v_3v_5 , and v_4v_5 . As a result, there are only five mutually non-isomorphic drawings of K_5 , which are depicted in Figure 1.

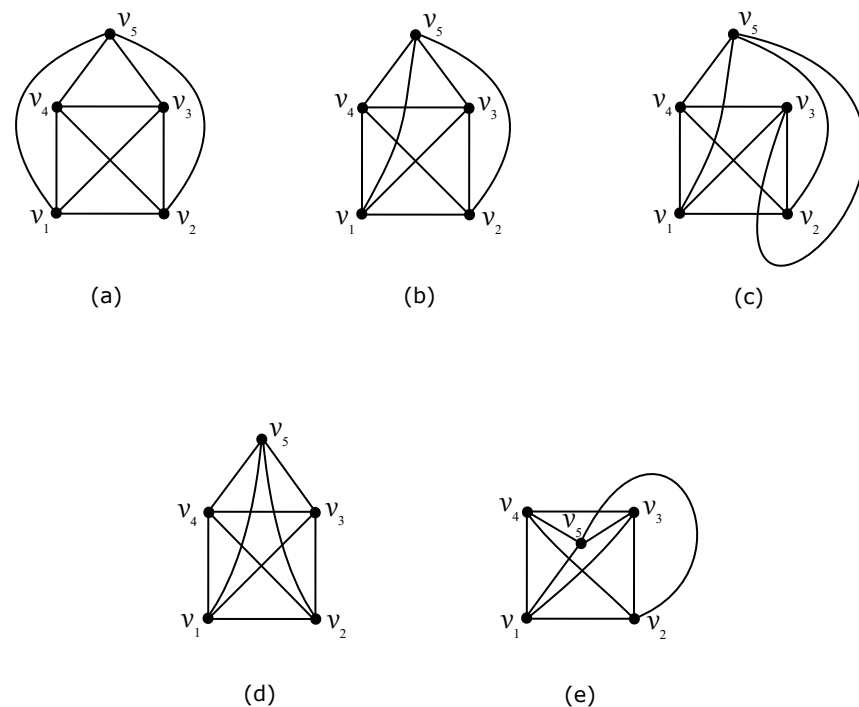


Figure 1. Five mutually non-isomorphic drawings of K_5 : (a) only one crossing; (b) additional crossing on the edge v_1v_5 ; (c) additional crossings on the edges v_1v_5 and v_3v_5 ; (d) additional crossings on the edges v_1v_5 and v_2v_5 ; (e) additional crossings on the edges v_1v_5 and v_2v_5 with the vertex v_5 located inside the inner region of K_4 .

Currently, the crossing numbers of the graphs $K_5 + P_n$ and $K_5 + C_n$, where P_n and C_n are paths and cycles on n vertices, are given only as hypotheses. The crossing number of $K_5 + P_n$ graphs for $n \geq 2$ was previously investigated in Su's paper [6], but with several errors in the main proof caused by an inconsistent discussion of possible subcases (e.g., wrong drawings or not considering all the drawings with respect to a given subcase). To prove it correctly, we will first build the necessary mathematical apparatus for the case $\text{cr}(K_5 + D_n)$, where D_n denotes a discrete graph on n vertices. Lü and Huang [7] already proved it, albeit in a rather complicated way. We offer an alternative proof that is simpler, and its same main idea can be used later for determining $\text{cr}(K_5 + P_n)$. The intended idea lies in the use of the existence of different cycles of C_3 as subgraphs of K_5 that can perform the function of separation concerning the remaining two vertices of K_5 . By gradually adding edges, we will also determine the crossing number for the $K_5 + C_n$, which was an open problem until now. Additionally, the obtained values $\text{cr}(K_5 + P_n)$ and $\text{cr}(K_5 + C_n)$ will help

us establish completely new hypotheses for $\text{cr}(W_m + S_n)$ and $\text{cr}(W_m + W_n)$, where W_n and S_n are wheels and stars on $n + 1$ vertices, respectively [8].

Regarding the practical importance of the problem, it can be mentioned that some algorithms constructing the graph drawings with a minimal number of crossings have already been created [9–11]. For example, the algorithm located on the website [12] can find the crossing numbers of small sparse graphs using an integer linear programming (ILP) formulation based on Kuratowski subgraphs. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [11]. However, for our research, the capacity of the system is insufficient, because the join products considered in this paper belong to dense graphs. This confirms that examining the number of crossings of simple graphs is a classic but still challenging problem. Garey and Johnson [13] even proved that it belongs between NP-complete problems.

2. The Crossing Number of $K_5 + D_n$

The considered join product $K_5 + D_n$ is formed by combining a copy of the graph K_5 with n additional vertices labeled as $t_1, \dots, t_n$. Each of these newly added vertices t_i is connected to every vertex in the original graph K_5 . We define the subgraph T^i as the collection of five edges incident to the fixed vertex t_i . In the drawing D of $K_5 + D_n$, it is advantageous to categorize the n subgraphs T^i into three distinct subsets based on the number of times they intersect the edges of K_5 . Let R_D represent the set of subgraphs where $\text{cr}_D(K_5, T^i) = 0$ (indicating zero crossings), and let S_D represent the set of subgraphs where $\text{cr}_D(K_5, T^i) = 1$ (indicating one crossing). All remaining subgraphs T^i , which cross edges of K_5 at least twice in D , fall into the third subset. We note that the same categorization will also be used later in the drawing of $K_5 + P_n$ and $K_5 + C_n$ too.

It is evident that a drawing characterized by a minimal number of crossings, known as an optimal drawing, inherently possesses the desirable qualities of a *good* drawing. In such a drawing, the edges do not intersect themselves, no two edges cross multiple times, and there are no instances where two edges connected to the same vertex intersect [14]. In a good drawing D of some graph G , we say that a cycle C separates two different vertices of the subgraph $G \setminus C$ if they are contained in different components of $\mathbb{R}^2 \setminus C$, where $\mathbb{R}^2$ means a two-dimensional space. This considered cycle C is said to be a *separating cycle* of graph G in D .

To determine the value of the crossing number for the composite graph $K_5 + D_n$, we will divide five possible mutually non-isomorphic drawings of K_5 into two groups. First, we will look at those in which it is possible to find the separating cycle C_3 . In fact, there is such an option in all but one of the drawings in Figure 1. Their crossing numbers will be proved in Lemma 1 using the already known crossing number of the complete tripartite graph $K_{1,1,3}$ in combination with D_n , which is summarized in Theorem 1:

Theorem 1 ([15]). $\text{cr}(K_{1,1,3} + D_n) = 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ for $n \geq 1$.

Subsequently, in Lemma 2, we will deal with $\text{cr}(K_5 + D_n)$ for the two smallest possible n values. After that, we are ready to state the final Theorem 2, and in its proof, we take advantage of the previous two lemmas and discuss in more detail the last case when it is not possible to find a separating cycle C_3 in $D(K_5)$. As part of it, we will also use a concept of the *rotation*, $\text{rot}_D(t_i)$, of a vertex t_i in D of $K_5 + D_n$. It is defined as the cyclic permutation that documents the counterclockwise order, in which the edges departing from t_i appear. This rotation, introduced by Hernández-Vélez et al. [16] or Woodall [17], employs a notation (12345) to represent the counter-clockwise order of edges incident to the vertex t_i , such as $t_i v_1, t_i v_2, t_i v_3, t_i v_4$ and $t_i v_5$. It is important to note that the rotation signifies a cyclic permutation.

Lemma 1. For $n \geq 2$, let D be a good drawing of $K_5 + D_n$. If there is a separating cycle C_3 of K_5 in the subdrawing $D(K_5)$, then there are at least $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 1$ crossings in D .

Proof. By assumption, let us consider a separating cycle $C_3 = c_1c_2c_3c_1$ of K_5 in the subdrawing $D(K_5)$. Since the two remaining vertices of K_5 lie in different regions of $D(C_3)$, there is no subgraph T^i for which the edges of C_3 are not crossed. Hence, each subgraph T^i crosses the edges of C_3 at least once, which yields that $\text{cr}_D(C_3, \bigcup_{i=1}^n T^i) \geq n$. Let G be the graph difference of graphs K_5 and C_3 ; i.e., G is isomorphic to the complete tripartite graph $K_{1,1,3}$. The exact value for the crossing number of $G + D_n$ is given by Theorem 1; that is, $\text{cr}_D(G + D_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$. This enforces at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + n + 1$ crossings in D because some edge of the cycle C_3 is also crossed in $D(K_5)$. $\square$

Lemma 2. $\text{cr}(K_5 + D_1) = 3$ and $\text{cr}(K_5 + D_2) = 6$.

Proof. Notice that the graphs $K_5 + D_1$ and $K_5 + D_2$ are isomorphic to the join product of the complete graph K_4 with the paths P_2 and P_3 , respectively (Figure 2). Since it was proved in [18] that $\text{cr}(K_4 + P_2) = 3$ and $\text{cr}(K_4 + P_3) = 6$, due to isomorphism, we can directly state that $\text{cr}(K_5 + D_1) = 3$ and $\text{cr}(K_5 + D_2) = 6$. $\square$

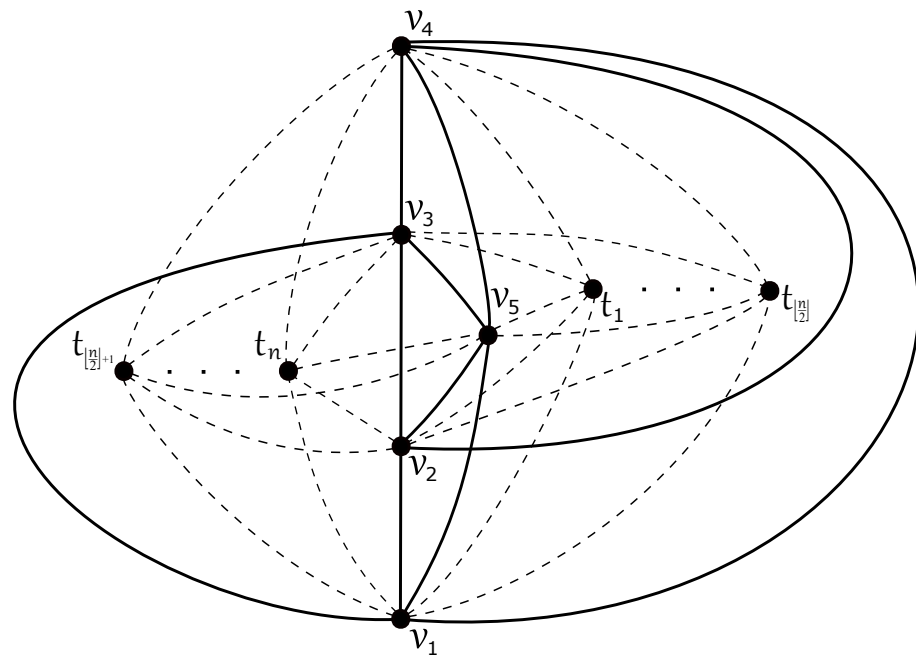


Figure 2. The good drawing of $K_5 + D_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 1$ crossings.

Theorem 2. $\text{cr}(K_5 + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 1$.

Proof. By Lemma 2, the result is true for $n = 1$ and $n = 2$. In Figure 2, the edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i , $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$ on the right side crosses edges of the complete graph K_5 three times, and each subgraph T^i , $i = \lfloor \frac{n}{2} \rfloor + 1, \dots, n$ on the left side crosses edges of K_5 exactly twice. As there is one crossing among edges of K_5 , we obtain $\text{cr}(K_5 + D_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 1$. To prove the reverse inequality by induction on n , suppose now that there is an optimal drawing D of $K_5 + D_n$ with

$$\text{cr}_D(K_5 + D_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor \text{ for some } n \geq 3, \quad (2)$$

and also let

$$\text{cr}(K_5 + D_m) = 4\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2m + \lfloor \frac{m}{2} \rfloor + 1 \text{ for any positive integer } m < n. \quad (3)$$

According to Lemma 1, we can suppose only the drawing of K_5 induced by D in such a way as shown in Figure 1d because there is a possibility of obtaining a separating cycle C_3 for all four remaining subdrawings $D(K_5)$. Note that such a single possible induced drawing of K_5 does not allow the addition of a subgraph T^k with $1 \leq \text{cr}_D(K_5, T^k) \leq 2$. Moreover, if $r = |R_D|$, the assumption (2), together with the well-known fact $\text{cr}_D(K_{5,n}) \geq 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, thanks to (1), implies that in D ,

$$5 + 0r + 3(n - r) \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor. \quad (4)$$

The obtained inequality (4) forces $3r \geq \left\lfloor \frac{n}{2} \right\rfloor + 5$, that is, $r \geq 2$ for n at least 3. For all subgraphs $T^i \in R_D$, there is only one subdrawing of $K_5 \cup T^i$ represented by the rotation (14532) (for our purposes, it does not matter which of the regions is unbounded). If there is a subgraph T^k such that $\text{cr}_D(T^i, T^k) = 0$, then the edges of K_5 are crossed at least five times by edges of $T^i \cup T^k$ based on the discussion of inserting the vertex t_k with corresponding edges over eleven considered regions of $D(K_5 \cup T^i)$. As a result, the number of crossings in D , we have

$$\begin{aligned} \text{cr}_D(K_5 + D_n) &= \text{cr}_D(K_5 + D_{n-2}) + \text{cr}_D(T^i \cup T^k) + \text{cr}_D(K_{5,n-2}, T^i \cup T^k) + \text{cr}_D(K_5, T^i \cup T^k) \\ &\geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2(n-2) + \left\lfloor \frac{n-2}{2} \right\rfloor + 1 + 4(n-2) + 5 \\ &= 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{aligned}$$

where each of $n - 2$ subgraphs T^l , $l \neq i, k$, crosses the edges of $T^i \cup T^k$ at least four times due to $\text{cr}_D(K_{5,3}) \geq 4$ again by (1). This contradiction with the assumption (2) confirms that the edges of all $T^i \in R_D$ must be crossed at least once by any other subgraph T^k . For two different subgraphs $T^i, T^j \in R_D$, $\text{cr}_D(T^i \cup T^j, T^k) \geq 4 + 4 = 8$ holds for any $T^k \in R_D$, $k \neq i, j$, provided by $\text{rot}_D(t_i) = \text{rot}_D(t_j) = \text{rot}_D(t_k)$. Moreover, the edges of $K_5 \cup T^i \cup T^j$ are crossed at least seven times by any subgraph $T^k \notin R_D$, because $\text{cr}_D(K_5 \cup T^i, T^k) \geq 6$ is fulfilled using the discussion over all possible regions of $D(K_5 \cup T^i)$. As $\text{cr}_D(K_5 \cup T^i \cup T^j) \geq 5 + 4 = 9$, by fixing the subgraph $K_5 \cup T^i \cup T^j$, we have

$$\begin{aligned} \text{cr}_D(K_5 + D_n) &\geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-2) + 7(n-r) + 9 = 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \\ &+ 7n + r - 7 \geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + 2 - 7 \geq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{aligned}$$

We have shown, in all cases, that there are at least $4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1$ crossings in each good drawing D of the graph $K_5 + D_n$. The proof of Theorem 2 is complete. $\square$

3. The Crossing Number of $K_5 + P_n$

In the drawing of $K_5 + D_n$ depicted in Figure 2, it is possible to add $n - 1$ new edges $t_i t_{i+1}$ for $i = 1, \dots, n - 1$ forming the path P_n^* on vertices of D_n with exactly three additional crossings. Thus, $\text{cr}(K_5 + P_n) \leq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 4$. Our goal is to show that the crossing number of $K_5 + P_n$ is equal to this upper bound. Note that this result was already predicted by Su [6], but we have found several mistakes in his proofs. Our Lemma 3 will confirm this result for $n = 2$ and $n = 3$, whereas its proof will be strongly based on isomorphisms. Then, we will focus in Lemma 4 on the case when a separating cycle C_3 of K_5 exists in the subdrawing $D(K_5)$. The crossing numbers of the join products of $K_{1,1,3}$ with paths P_n (summarized in Theorem 3), which have already been determined by Staš and Timková in [19], concludes the main idea of the proof.

Theorem 3 ([19], Theorem 3.5). $\text{cr}(K_{1,1,3} + P_n) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2$ for $n \geq 2$.

Subsequently, Lemma 4 will result in Corollary 1, which discusses the subdrawing of K_5 induced by D given in Figure 1a. This drawing of K_5 has only one crossing on the edges of C_3 in $D(K_5 \cup P_n^*)$, and therefore, the condition of Lemma 4 is not fulfilled. After that, the statement of the main Theorem 4 about $\text{cr}(K_5 + P_n)$, in the proof of which we take advantage of the pre-prepared Lemma 3, Lemma 4, and Corollary 1, can be established.

Lemma 3. $\text{cr}(K_5 + P_2) = 9$ and $\text{cr}(K_5 + P_3) = 15$.

Proof. Notice that the graphs $K_5 + P_2$ and $K_5 + P_3$ are isomorphic to the complete graph K_7 and $K_8 \setminus e$ obtained by removing one edge from K_8 , respectively. Guy [20] and Zheng et al. [21] proved that $\text{cr}(K_7) = 9$ and $\text{cr}(K_8 \setminus e) = 15$. Therefore, $\text{cr}(K_5 + P_2) = \text{cr}(K_7) = 9$ and $\text{cr}(K_5 + P_3) = \text{cr}(K_8 \setminus e) = 15$, and the proof of Lemma 3 is complete. $\square$

Lemma 4. For $n \geq 2$, let D be a good drawing of $K_5 + P_n$. If there is a separating cycle C_3 of K_5 in the subdrawing $D(K_5)$ with at least two crossings on the edges of C_3 in $D(K_5 \cup P_n^*)$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in D .

Proof. Using arguments similar to the proof of Lemma 1, the edges of any separating cycle C_3 in $D(K_5)$ are crossed at least n times by $\bigcup_{i=1}^n T^i$. If G is the graph difference of graphs K_5 and C_3 , then $\text{cr}_D(G + P_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ by Theorem 3 enforces at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + n + 4$ crossings in D . $\square$

Corollary 1. For $n \geq 4$, if D is a good drawing of $K_5 + P_n$ with the subdrawing of K_5 induced by D given in Figure 1a, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in D .

Proof. In Figure 1a, there are two separate cycles C_3 of K_5 , but both have only one crossing on their edges in the subdrawing $D(K_5)$. If some edge of the path P_n^* crosses any edge of these separating cycles, we obtain the desired result in D using Lemma 4. Of course, we also obtain at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 4$ crossings in D if the edges of some separating cycle C_3 are crossed more than once by edges of some subgraph T^i (see the proof of Lemma 1).

Now, we assume that there are exactly n crossings on the edges of each separating 3-cycle; that is, each subgraph T^i crosses the edges of any separating 3-cycle exactly once, and also, the edges of path P_n^* do not produce any crossing on the edges of both separating 3-cycles. In the rest of the proof, let all vertices t_i of P_n^* be placed in the outer region of $D(K_5)$ with three vertices v_1, v_2 , and v_5 or in the inner triangular region with two vertices v_1 and v_2 of K_5 on its boundary. Our assumption enforces no crossing on two edges v_3v_5 and v_4v_5 in D , which yields the fact that there are exactly $2n + 1$ crossings on four edges v_1v_3, v_1v_5, v_2v_4 , and v_2v_5 in D . Let us denote by H the subgraph of K_5 with the vertex set $V(K_5)$ and the edge set $\{v_1v_4, v_2v_3, v_3v_4, v_3v_5, v_4v_5\}$. By Klešč and Staš [22], it was proved that $\text{cr}(H + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, and thus, we receive the desired result if there are at least three crossings on the edge v_1v_2 in D .

Finally, let us turn to the possibility that there are at most two crossings on the edge v_1v_2 in D . For $n \geq 4$, we obtain at least one subgraph T^j (but there are definitely at least two) whose edges do not cross this edge v_1v_2 of the graph K_5 . This enforces the fact that no edge of K_5 is crossed by the edge t_jv_5 and $\text{cr}_D(K_5, T^j) = 2$ only for T^j with $\text{rot}_D(t_j) = (14532)$. It is easy to verify over six possible regions of $D(K_5 \cup T^j)$ that each of the $n - 1$ remaining subgraphs T^k crosses the edges of $K_5 \cup T^j$ at least five times. As $\text{cr}_D(K_5 \cup T^j) = 3$, by fixing the subgraph $K_5 \cup T^j$, we have

$$\text{cr}_D(K_5 + P_n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(n-1) + 3 + 1 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 4.$$

We could add +1 in the previous estimate because the number of crossings on edges of $K_5 \cup T^j$ can be increased up to 6 for each subgraph $T^k, k \neq j$ if both vertices t_j and t_k are placed in the same outer region of $D(K_5)$. The proof of Corollary 1 is done. $\square$

Theorem 4. $\text{cr}(K_5 + P_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 4$ for $n \geq 2$.

Proof. The proof proceeds similarly to the graph $K_5 + D_n$ in Theorem 2. Using Lemma 3, we suppose that there is an optimal drawing D of $K_5 + P_n$ with

$$\text{cr}_D(K_5 + P_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ for some } n \geq 4, \quad (5)$$

and that

$$\text{cr}(K_5 + P_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2m + \left\lfloor \frac{m}{2} \right\rfloor + 4 \text{ for any positive integer } m < n. \quad (6)$$

By Lemma 4 and Corollary 1, we consider only the subdrawing of the graph K_5 induced by D with $\text{cr}_D(K_5) = 5$ presented in Figure 1d. For $r = |R_D|$, the relation with respect to edge crossings of the subgraph K_5 in D produces

$$5 + 0r + 3(n - r) \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor + 3, \quad (7)$$

which again yields $r \geq 2$ for an n of at least four. Since $4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n - 5$ is at least as much as $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 4$, the proof of Theorem 4 is complete. $\square$

4. The Crossing Number of $K_5 + C_n$

To date, the crossing numbers of the join product $K_5 + C_n$ were not claimed in any hypothesis. The upper bound of $\text{cr}(K_5 + C_n)$ can be easily estimated from Figure 2 by the addition of n new edges $t_n t_1$ and $t_i t_{i+1}, i = 1, \dots, n - 1$ forming the cycle C_n^* on vertices of D_n with just six additional crossings. In such a way, we obtain

$$\text{cr}(K_5 + C_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7 \quad (8)$$

and we will show that the upper bound in (8) holds with equality for all n at least three in the following statements. Lemma 5 will confirm this result for $n = 3$, because $K_5 + C_3$ is isomorphic to the complete graph K_8 and it is shown by Guy [20] that $\text{cr}(K_8) = 18$. Similar to the previous two sections, we will focus first on Lemma 6 in the case when there is a separating cycle C_3 of K_5 in the subdrawing $D(K_5)$. As the crossing numbers of the join products of $K_{1,1,3}$ with the cycles C_n have been determined by Staš and Timková [19], they can be safely used in the last part of the proof of Lemma 6.

Theorem 5 ([19], Theorem 4.9). $\text{cr}(K_{1,1,3} + C_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 4$ for $n \geq 3$.

The condition of Lemma 6 for at least three crossings on edges of C_3 in $D(K_5 \cup C_n^*)$ is not fulfilled for the subdrawing of K_5 induced by D given in Figure 1a and Figure 1b, respectively. Therefore, these special cases will be discussed separately in Corollary 2 and Corollary 3. After that, we will turn our attention to the subdrawing of K_5 induced by D given in Figure 1d. It represents the only subdrawing of K_5 without the existence of a separating cycle C_3 and is addressed in Lemma 7 for $n = 4$; i.e., $\text{cr}(K_5 + C_4)$ is pre-prepared for the final statement. That will be introduced in Theorem 6 and will establish $\text{cr}(K_5 + C_n)$ for arbitrary $n \geq 3$.

Lemma 5. $\text{cr}(K_5 + C_3) = 18$.

Lemma 6. For $n \geq 3$, let D be a good drawing of $K_5 + C_n$. If there is a separating cycle C_3 of K_5 in the subdrawing $D(K_5)$ with at least three crossings on edges of C_3 in $D(K_5 \cup C_n^*)$, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 7$ crossings in D .

Proof. Using arguments similar to the proof of Lemma 1, the edges of any separating cycle C_3 in $D(K_5)$ are crossed at least n times by $\bigcup_{i=1}^n T^i$. If G is the graph difference of graphs K_5 and C_3 , then $\text{cr}_D(G + C_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ by Theorem 5 enforces at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + n + 7$ crossings in D . $\square$

Corollary 2. For $n \geq 4$, if D is a good drawing of $K_5 + C_n$ with the subdrawing of K_5 induced by D given in Figure 1a, then there are at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 7$ crossings in D .

Proof. In Figure 1a, there are two separate cycles C_3 of K_5 , namely 3-cycles $v_1v_3v_5v_1$ and $v_2v_5v_4v_2$. Both have only one crossing on their edges in the subdrawing $D(K_5)$. If some edge of the cycle C_n^* crosses any edge of one separating 3-cycle, we obtain the desired result in D using Lemma 6. Clearly, we also obtain at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor + 7$ crossings in D if there are at least $n + 2$ crossings between $\bigcup_{i=1}^n T^i$ and one separating cycle (see the proof of Lemma 1). In the following, suppose that a separating cycle C_3 can be crossed at most twice by only one subgraph T^i .

Now, we assume that the edges of cycle C_n^* do not produce any crossing on the edges of separating 3-cycles. In the rest of the paper, let all vertices t_i of C_n^* be placed in the outer region of $D(K_5)$ with three vertices v_1, v_2 , and v_5 or in the inner triangular region with two vertices v_1 and v_2 of K_5 on its boundary. We discuss two subcases:

1. $\text{cr}_D(K_5, T^i) \geq 3$ for each subgraph T^i , $i = 1, \dots, n$. If all vertices t_i of C_n^* are placed in the outer region of $D(K_5)$ with three vertices v_1, v_2 , and v_5 of K_5 on its boundary, then we obtain at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4 + 2n$ crossings in D , which is caused by at least $2n$ crossings on the edges of the 3-cycle $v_1v_2v_5v_1$ and also due to Theorem 5. Below, suppose that they are not placed only in the outer region of $D(K_5)$. Let us denote by H the subgraph of K_5 with the vertex set $V(K_5)$, and the edge set $\{v_1v_4, v_2v_3, v_3v_4, v_3v_5, v_4v_5\}$. Klešć and Staš [22] proved that $\text{cr}(H + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$. It is easy to verify that edges of the subgraph $K_5 - H$ are crossed at least twice by any subgraph T^i and just two crossings can be achieved only for at most two possible different subgraphs given our assumptions. This implies $\text{cr}_D(K_5 - H, \bigcup_{i=1}^n T^i) \geq 3n - \alpha$, where α is the number of such subgraphs forcing two crossings on edges of $K_5 - H$. Thus, we obtain at least

$$4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 + 3n - \alpha + \text{cr}_D(K_5 - H) + \text{cr}_D(v_1v_2, C_n^*) \quad (9)$$

crossings in D . As $\text{cr}_D(K_5 - H) = 1$ and $\text{cr}_D(v_1v_2, C_n^*) - \alpha \geq 0$, we are done.

2. There is a subgraph T^i such that $\text{cr}_D(K_5, T^i) < 3$. Without loss of generality, let T^n be a subgraph by which the edges of K_5 are crossed just twice, that is, $\text{rot}_D(t_n) = (14532)$. It is not difficult to verify over six possible regions of $D(K_5 \cup T^n)$ that the edges of $K_5 \cup T^n$ are crossed at least five times by each other subgraph T^j , $j \neq n$ and just five crossings can be achieved for several possible different subgraphs given our assumptions. This implies $\text{cr}_D(K_5 \cup T^n, \bigcup_{j=1}^{n-1} T^j) \geq 6(n-1) - \beta$, where β is the number of such subgraphs forcing five crossings on the edges of $K_5 \cup T^n$. For $\beta \leq 2$, by fixing the subgraph $K_5 \cup T^n$, we have at least

$$4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(n-1) - \beta + \text{cr}_D(K_5 \cup T^n) + \text{cr}_D(v_1v_2, C_n^*) \quad (10)$$

crossings in D . As $\text{cr}_D(K_5 \cup T^n) = 3$ and $\text{cr}_D(v_1v_2, C_n^*) - \beta \geq 0$, we are again done.

Finally, for $\beta \geq 3$, let H be the subgraph of K_5 defined in the same way as in the previous subcase. This implies $\text{cr}_D(K_5 - H, \bigcup_{j=1}^n T^j) \geq 2n + \beta$, which yields at least

$$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 + 2n + \beta + \text{cr}_D(v_1 v_2, C_n^*) \quad (11)$$

crossings in D . As $\beta + \text{cr}_D(v_1 v_2, C_n^*) \geq 5$, we achieve the desired result in D .

□

Corollary 3. For $n \geq 4$, if D is a good drawing of $K_5 + C_n$ with the subdrawing of K_5 induced by D given in Figure 1b, then there are at least $4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7$ crossings in D .

Proof. In Figure 1b, there are two separate cycles C_3 of K_5 but both have only two crossings on their edges in the subdrawing $D(K_5)$. Let us denote by H the subgraph of K_5 with the vertex set $V(K_5)$ and the edge set $\{v_1 v_3, v_1 v_4, v_2 v_3, v_3 v_4, v_3 v_5\}$. If some edge of the cycle C_n^* crosses any edge of these separating cycles, we obtain the desired result in D using Lemma 6. Now, we assume that no edge of C_n^* crosses any edge of these separating cycles, that is, all vertices t_i of C_n^* are placed in some common region of $D(K_5 - H)$. The crossing numbers of the join product of the graph H with cycles are already known by Staš [23], i.e., $\text{cr}(H + C_n) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$. Since the edges of the subgraph $K_5 - H$ (edges of both separating cycles together) are crossed at least $2n + 3$ times in D , we obtain at least $4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor + 2 + 2n + 3$ crossings, and this completes the proof of Corollary 3. □

Lemma 7. If D is a good drawing of $K_5 + C_4$ with the subdrawing of K_5 induced by D given in Figure 1d with the nonempty set R_D , then there are at least 25 crossings in D .

Proof. For easier reading, let $r = |R_D|$. By the assumption of Lemma 7, $\text{cr}_D(K_5) = 5$ and $r \geq 1$. We remember that there is only one subdrawing of $F^i = K_5 \cup T^i$ represented by the rotation (14532) for all subgraphs $T^i \in R_D$. If $n = r = 4$, we obtain at least $\binom{4}{2} 4 + 5 = 29$ crossings in D because every two different subgraphs from R_D with the same rotations produce at least four crossings on their edges. Below, let $T^1 \in R_D$ and let r be at most three. We discuss two possibilities:

1. Assume there exists a subgraph T^j , say T^4 , by which the edges of T^1 are not crossed. The edges of K_5 must be crossed at least nine times by the edges of $T^4 \cup C_n^*$ due to the location of the vertex t_4 in $D(K_5 \cup T^1)$. For $r = 3$, we obtain at least

$$\text{cr}_D \left(\bigcup_{i=1}^3 T^i \right) + \text{cr}_D(K_5) + \text{cr}_D(K_5, T^4 \cup C_n^*) \geq \binom{3}{2} 4 + 5 + 9 = 26$$

crossings in D . For $1 \leq r \leq 2$, we obtain at least

$$\begin{aligned} \text{cr}_D(T^1 \cup T^4, T^2 \cup T^3) + \text{cr}_D(K_5) + \text{cr}_D(K_5, T^4 \cup C_n^*) + \text{cr}_D(K_5, T^2 \cup T^3) \\ \geq 8 + 5 + 9 + 3 = 25 \end{aligned}$$

crossings in D because $\text{cr}_D(T^1 \cup T^4, T^2 \cup T^3) \geq 4 + 4$ thanks to (1), and at least one of the subgraphs T^2, T^3 crosses the edges of K_5 at least three times.

2. $\text{cr}_D(T^1, T^j) \neq 0$ for any $j = 2, 3, 4$. In the rest of the proof, let $T^4 \notin R_D$ in the case of $r = 3$. If $\text{cr}_D(T^i, T^4) = 0$ for some $i \in \{2, 3\}$, the proof can proceed in the same way as in the previous case. Taking into account the assumption that $\text{cr}_D(T^2 \cup T^3, T^4) \geq 2$, we obtain at least

$$\text{cr}_D \left(\bigcup_{i=1}^3 T^i \right) + \text{cr}_D(K_5) + \text{cr}_D(K_5 \cup T^1, T^4) + \text{cr}_D(T^2 \cup T^3, T^4) \geq \binom{3}{2} 4 + 5 + 6 + 2 = 25$$

crossings in D because the edges of $K_5 \cup T^1$ are crossed at least six times by any subgraph $T^i \notin R_D$. For $1 \leq r \leq 2$, we obtain at least

$$\text{cr}_D\left(K_5 \cup T^1, \bigcup_{i=2}^4 T^i\right) + \text{cr}_D(K_5) + \text{cr}_D\left(\bigcup_{i=2}^4 T^i\right) \geq (6 + 6 + 4) + 5 + 4 = 25$$

crossings in D , where $\text{cr}_D(T^2 \cup T^3 \cup T^4) \geq 4$, again thanks to (1).

□

Theorem 6. $\text{cr}(K_5 + C_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7$ for $n \geq 3$.

Proof. Using Lemma 5, we suppose that there is an optimal drawing D of $K_5 + C_n$ with

$$\text{cr}_D(K_5 + C_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 6 \text{ for some } n \geq 4, \quad (12)$$

and that

$$\text{cr}(K_5 + C_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2m + \left\lfloor \frac{m}{2} \right\rfloor + 7 \text{ for any positive integer } m < n. \quad (13)$$

By Lemma 6 and Corollaries 2, 3, we only consider the drawing of K_5 induced by D presented in Figure 1d. For $r = |R_D|$, the relation with respect to edge crossings of the complete graph K_5 in D produces

$$5 + 0r + 3(n - r) \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor + 6, \quad (14)$$

which yields $r \geq 1$ for an n value of at least four. As the set R_D is nonempty, Lemma 7 enforces $n \geq 5$. Similarly as in the proof of Theorem 2 for a subgraph $T^i \in R_D$, if there is a T^k such that $\text{cr}_D(T^i, T^k) = 0$, then we can obtain a contradiction with the assumption (12) in D by fixing $T^i \cup T^k$. In the following, the edges of all $T^i \in R_D$ must be crossed at least once by each other subgraph T^k , and we will discuss two possibilities:

1. Assume $1 \leq r \leq \left\lfloor \frac{3n-6}{4} \right\rfloor$. By fixing the subgraph $K_5 \cup T^i$, we have

$$\begin{aligned} \text{cr}_D(K_5 + C_n) &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(r-1) + 6(n-r) + 5 \\ &= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6n - 2r + 1 \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6n - 2\left\lfloor \frac{3n-6}{4} \right\rfloor + 1 \\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7. \end{aligned}$$

2. Assume $\left\lfloor \frac{3n-6}{4} \right\rfloor < r \leq n$. By fixing the subgraph $K_5 \cup T^i \cup T^j$, we have

$$\begin{aligned} \text{cr}_D(K_5 + C_n) &\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-2) + 7(n-r) + 9 \\ &= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + r - 7 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + \left\lfloor \frac{3n-6}{4} \right\rfloor + 1 - 7 \\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7. \end{aligned}$$

The obtained contradictions in both subcases complete the proof of Theorem 6. □

5. The Crossing Number of $W_m + S_n$ and $W_m + W_n$

Let W_n and S_n denote the wheel and the star on $n + 1$ vertices, respectively. In general, the graphs $W_m + D_n$ and $W_m + C_n$ are isomorphic to $C_m + S_n$ and $C_m + W_n$ for all integers $m, n \geq 3$, respectively. Three conjectures about the crossing numbers of W_m in the join

product with D_n , P_n , and C_n have already been established, for more see [24–26]. Now, we can postulate that for all $m, n \geq 3$,

$$\text{cr}(W_m + S_n) = [Z(m) + m][Z(n) + n] - mn + n + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad (15)$$

where $Z(m) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ denotes Zarankiewicz's number [17,27].

In Figure 3, the edges of $K_{m+1,n}$ cross each other

$$\left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \binom{\left\lfloor \frac{n}{2} \right\rfloor}{2} + \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \binom{\left\lfloor \frac{n}{2} \right\rfloor}{2} = \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

times, each subgraph $T^i, i = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor$ on the right side crosses the edges of W_m exactly once, and each subgraph $T^i, i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \dots, n$ on the left side crosses the edges of W_m exactly $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ times. The other $\left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ crossings are contained on the edges of the subgraph T^0 . Thus,

$$\begin{aligned} & \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \\ & + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \\ & + n + 1 + \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \\ & + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) + n + 1 + \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left(m - \left\lfloor \frac{m}{2} \right\rfloor \right) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = \\ & = \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor n - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + n + 1 + \\ & + \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + m \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = \\ & = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left(\left\lfloor \frac{m+1}{2} \right\rfloor - 1 \right) + nZ(m) + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor \right) + \\ & + n + 1 + mZ(n) = Z(m)Z(n) + nZ(m) + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + n + 1 + mZ(n) = \\ & = [Z(m) + m][Z(n) + n] - mn + n + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + 1 \end{aligned}$$

crossings appear among the edges of the graph $W_m + S_n$ in this drawing.

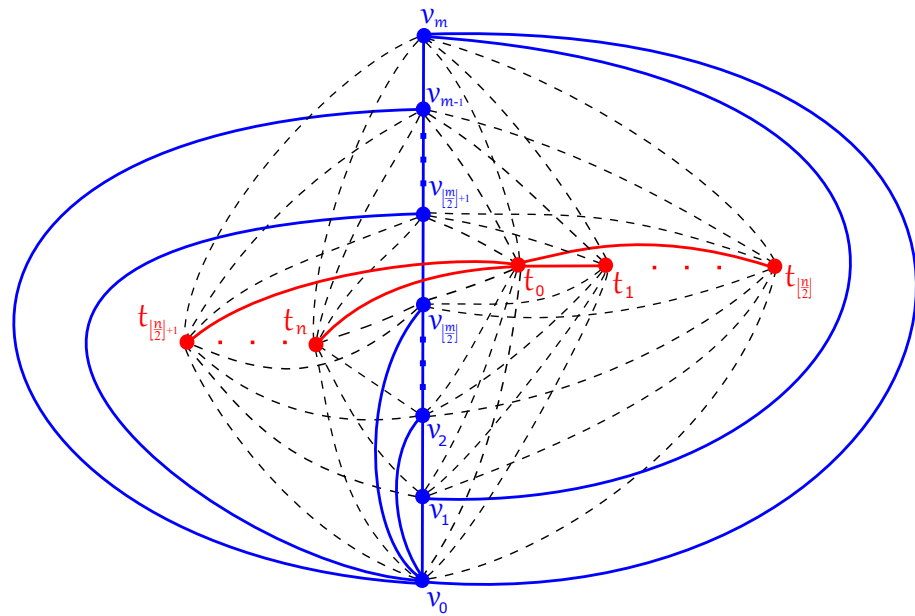


Figure 3. The good drawing of $W_m + S_n$ with $[Z(m) + m][Z(n) + n] - mn + n + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1$ crossings.

Usually, the star S_n is defined for an n of at least three, but the conjecture (15) can also be generalized for $n = 1$ and $n = 2$. In this case, the stars S_1 and S_2 are isomorphic to the paths P_2 and P_3 , respectively. The crossing numbers of the join products of W_m with P_2 and P_3 were determined by Staš and Valiska [25].

Theorem 7 ([25], Theorem 5.1). $\text{cr}(W_m + P_2) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2$ for $m \geq 3$.

Theorem 8 ([25], Theorem 5.2). $\text{cr}(W_m + P_3) = 2 \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 3$ for $m \geq 3$.

Using the aforementioned isomorphisms, our conjecture holds for $n = 1$ and $n = 2$ thanks to Theorems 7 and 8. As the graph $W_m + S_3$ is isomorphic to $K_{1,1,3} + C_m$, Theorem 5 also confirms the validity of this conjecture for $n = 3$. Similarly, the graph $W_m + S_4$ is isomorphic to $K_{1,1,4} + C_m$, but only the crossing number of $K_{1,1,4} + D_m$ has been shown by Su and Klešč [28]. On the other hand, Theorem 2 again confirms the validity of our conjecture for $W_3 + S_n$ because the graph $K_5 + D_n$ is isomorphic to $W_3 + S_n$. We add that Wang and Huang [29] also dealt with establishing hypotheses for crossing numbers of connected graphs involving stars, specifically for $S_m + P_n$ and $S_m + C_n$. Our conjecture complements this knowledge for $S_m + W_n$.

Now, let us turn to postulate that for all $m, n \geq 3$

$$\text{cr}(W_m + W_n) = [Z(m) + m][Z(n) + n] - mn + m + n + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 4. \quad (16)$$

This makes it possible to add n new edges $t_i t_{i+1}$ and $t_n t_1$ for $i = 1, \dots, n-1$ into the drawing D of $W_m + S_n$ in Figure 3, which form the rim edge set of W_n with $m+3$ additional crossings. In Figure 4, the edges $t_{\lfloor \frac{n}{2} \rfloor} t_{\lfloor \frac{n}{2} \rfloor + 1}$ and $t_n t_1$ force $2 + \lfloor \frac{m}{2} \rfloor - 1$ and $1 + \lfloor \frac{m}{2} \rfloor + 1$ crossings on edges of $W_m + S_n$, respectively. In such a way, we obtain a good drawing of $W_m + W_n$ with exactly $\text{cr}_D(W_m + S_n) + m + 3 = [Z(m) + m][Z(n) + n] - mn + m + n + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 4$ crossings.

As the graph $W_m + W_3$ is isomorphic to $K_5 + C_m$, Theorem 6 confirms the validity of the conjecture (16) for $n = 3$. Due to the symmetry, the same holds for $m = 3$. The graph $W_m + W_4$ is isomorphic to $K_{1,1,2,2} + C_m$, but at the moment we do not even know the crossing number of the join product of $K_{1,1,2,2}$ with discrete graphs D_m , i.e., whether $\text{cr}(K_{1,1,2,2,m}) \leq 6 \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 3m + 2 \lfloor \frac{m}{2} \rfloor + 1$ holds with equality.

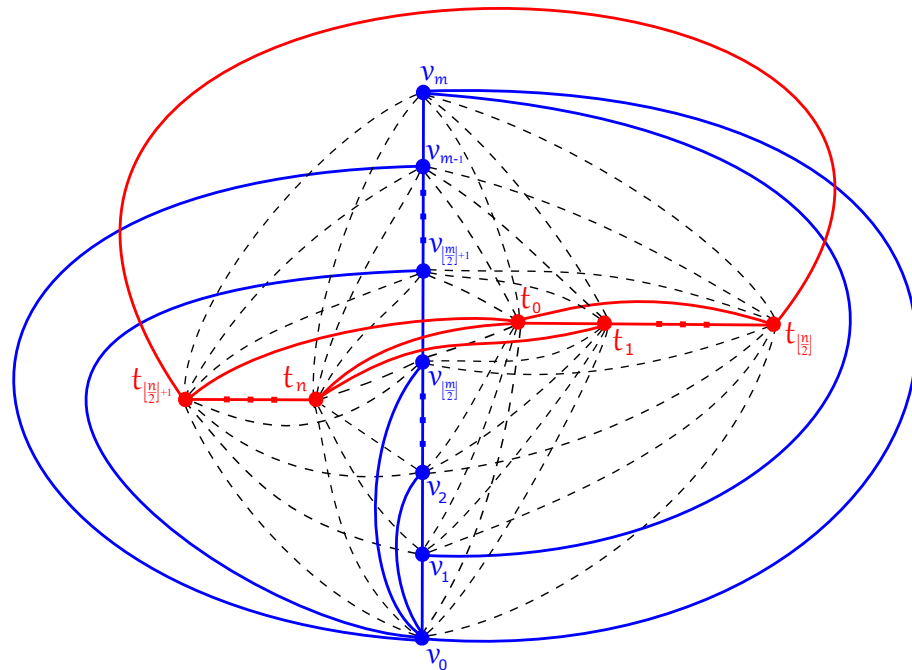


Figure 4. The good drawing of $W_m + W_n$ with $[Z(m) + m][Z(n) + n] - mn + m + n + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 4$ crossings.

6. Conclusions

In this article, we focused on proving the crossing numbers for three connected graphs that include a complete graph on five vertices K_5 . We began by explaining why K_5 has only five non-isomorphic drawings. Then, we used a similar procedure, involving the existence of different separating cycles C_3 , the concept of rotation, and the known crossing numbers of related graphs, to prove the crossing numbers $\text{cr}(K_5 + D_n)$, $\text{cr}(K_5 + P_n)$, and $\text{cr}(K_5 + C_n)$. With these results, we have resolved all open problems concerning connected graphs on five vertices because the last hypotheses regarding $\text{cr}(K_5 \setminus e + P_n)$ and $\text{cr}(K_5 \setminus e + C_n)$ were just recently proven in [30]. Regarding disconnected graphs on five vertices, there is only one graph left for which the crossing numbers in combination with D_n , P_n , and C_n are not exactly proven, namely the graph $2K_1 \cup K_3$. Since we obtain this graph by removing one edge from the graph $K_2 \cup K_3$, it is assumed that their crossing numbers in the respective join products will be the same. However, precise proof represents a challenge. If it is mastered, all graphs on five vertices in combination with discrete graphs, paths, and cycles will have exactly known crossing numbers, similar to the case of all graphs on four vertices [31,32].

Additionally, we proposed new hypotheses for $\text{cr}(W_m + S_n)$ and $\text{cr}(W_m + W_n)$, which we verified for the smallest possible values, $m = 3$ and $n = 3$, due to their isomorphisms with $K_5 + D_n$ and $K_5 + C_n$, respectively. Figures 3 and 4 provide upper bounds for these conjectures for all $m, n \geq 3$. However, the lower bounds still need to be proven, presenting an interesting topic for future research in this area.

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