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Symmetric Identities Involving the Extended Degenerate Central Fubini Polynomials Arising from the Fermionic p-Adic Integral on $\mathbb{Z}_p$

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Abstract: Since the constructions of p-adic q-integrals, these integrals as well as particular cases have been used not only as integral representations of many special functions, polynomials, and numbers, but they also allow for deep examinations of many families of special numbers and polynomials, such as central Fubini, Bernoulli, central Bell, and Changhee numbers and polynomials. One of the key applications of these integrals is for obtaining the symmetric identities of certain special polynomials. In this study, we focus on a novel generalization of degenerate central Fubini polynomials. First, we introduce two variable degenerate w-torsion central Fubini polynomials by means of their exponential generating function. Then, we provide a fermionic p-adic integral representation of these polynomials. Through this representation, we investigate several symmetric identities for these polynomials using special p-adic integral techniques. Also, using series manipulation methods, we obtain an identity of symmetry for the two variable degenerate w-torsion central Fubini polynomials. Finally, we provide a representation of the degenerate differential operator on the two variable degenerate w-torsion central Fubini polynomials related to the degenerate central factorial polynomials of the second kind.

Keywords: central fubini polynomials; degenerate central fubini polynomials; degenerate central factorial polynomials of the second kind; fermionic p-adic integral; symmetric identities

MSC: 05A19; 05A40; 11B83

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Citation: Alatawi, M.S.; Khan, W.A.; Duran, U. Symmetric Identities Involving the Extended Degenerate Central Fubini Polynomials Arising from the Fermionic p-Adic Integral on $\mathbb{Z}_p$. *Axioms* **2024**, *13*, 421. https://doi.org/10.3390/axioms13070421

Academic Editors: Roman Dmytryshyn and Feliz Manuel Minhós

Received: 6 May 2024
Revised: 31 May 2024
Accepted: 20 June 2024
Published: 22 June 2024

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1. Introduction

Special functions have an importance role in many areas of engineering, physics, mathematics, and other linked disciplines involving areas like quantum mechanics, mathematical physics, functional analysis, numerical analysis, differential equations, and so on (cf. [1–26]). In the family of special functions, special polynomials possess are also an area of special interest. Recently, central-type special polynomials (and their corresponding numbers), including central Bell, central Fubini, and central factorial polynomials (and numbers), are some the most studied families of special polynomials (cf. [2,3,6,10,15–20,24,25]). For example, the central factorial numbers in [3], the degenerate central factorial numbers in [17,20], the central Fubini polynomials, and numbers in [2,6,15,19,24], the central degenerate Fubini polynomials and numbers in [25], the central Bell numbers and polynomials in [18], and the degenerate central Bell polynomials and numbers in [16] have been considered, and many of their relations and properties have been examined and investigated. Motivated by these types of studies, here, we consider the two variable degenerate w-torsion central Fubini polynomials and we research some of their symmetric relations and properties. To do this, we start by reviewing the relevant definitions and notations.

The two variable (abbreviated with "t.w.") central Fubini polynomials are defined (cf. [15,19,24]) as follows

$$\frac{1}{1-y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt}=\sum_{j=0}^{\infty}F_{j,c}(x;y)\frac{t^j}{j!}, \tag{1}$$

where $|t|<\left|\log|y^{-1}|\right|$ for $y\neq 1$ and $|t|<2\pi$ for $y=1$. Upon setting $x=0$, $F_{j,c}(0;y)=F_{j,c}(y)$ becomes central Fubini polynomials. Also, upon setting $y=1=x+1$, $F_{j,c}(0;1):=F_{j,c}$ becomes the usual central Fubini numbers (cf. [2,6,10,15,19,24]).

The central factorial numbers of the second kind $T(j,k)$, for $j,k\geq 0$, are provided as follows (cf. [3])

$$x^j=\sum_{k=0}^{j}T(j,k)x^{[k]},(j\geq 0), \tag{2}$$

where $x^{[0]}=1$ and $x^{[k]}=x(x+\frac{k}{2}-1)(x+\frac{k}{2}-2)\cdots(x+\frac{k}{2}+1)$ for $k\geq 1$. The generating function of $T(j,k)$ is presented by:

$$\frac{1}{k!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k=\sum_{j=0}^{\infty}T(j,k)\frac{t^j}{j!}.$$

The degenerate exponential function is defined for $\lambda\in\mathbb{R}$, by (cf. [4,16,17,20,21,23,25])

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}\text{ with }e_\lambda(t):=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}}. \tag{3}$$

The series representations of the function $e_\lambda^x(t)$ is presented as follows:

$$e_\lambda^x(t)=\sum_{u=0}^{\infty}(x)_{u,\lambda}\frac{t^u}{u!}, \tag{4}$$

where $(x)_{0,\lambda}:=1$ and $(x)_{u,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(u-1)\lambda)$ for $u\geq 1$.

For $k\in\mathbb{N}$, the degenerate form of the central factorial numbers and polynomials of the second kind are provided as follows (cf. [17,20])

$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k=\sum_{j=k}^{\infty}T_\lambda(j,k)\frac{t^j}{j!} \tag{5}$$

and

$$\frac{e_\lambda^x(t)}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k=\sum_{j=k}^{\infty}T_\lambda(j,k:x)\frac{t^j}{j!}, \tag{6}$$

respectively. It is obvious that $\lim_{\lambda\to 0}T_\lambda(j,k)=T(j,k)$.

The type 2 degenerate central Fubini polynomials of t.w. are defined as follows (see [25])

$$\sum_{u=0}^{\infty}F_{u,\lambda}^{(C)}(x;y)\frac{t^u}{u!}=\frac{e_\lambda^x(t)}{1-y(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}, \tag{7}$$

where $|t|<\left|\log|y^{-1}|\right|$ for $y\neq 1$ and $|t|<2\pi$ for $y=1$. Upon setting $x=0$ in (7), $F_{u,\lambda}^{(C)}(0;y):=F_{u,\lambda}^{(C)}(y)$ are termed the type 2 degenerate central Fubini polynomials and upon letting $x+1=y=1$ in (7), $F_{u,\lambda}^{(C)}(0;1)=F_{u,\lambda}^{(C)}$ are termed the degenerate central Fubini numbers. So, we can write that

$$\sum_{u=0}^{\infty}F_{u,\lambda}^{(C)}(y)\frac{t^u}{u!}=\frac{1}{1-y(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}\text{ and }\sum_{u=0}^{\infty}F_{u,\lambda}^{(C)}\frac{t^u}{u!}=\frac{1}{1-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}.$$

Note that $\lim_{\lambda \longrightarrow 0} F_{u,\lambda}^{(C)}(x;y) = F_u^{(C)}(x;y)$.

Similarly, the following notations hold for p being a fixed odd prime number: $\mathbb{Z}_p$ denotes the ring of p-adic integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$. The normalized p-adic norm is provided by $|p|_p = p^{-1}$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the fermionic p-adic integral of f is provided (cf. [1,5,8,11–14,21,22,24,26]) as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)(-1)^x, \tag{8}$$

where $\mu_{-1}(x + p^n\mathbb{Z}_p) = (-1)^x$. It is apparent from (8) that

$$2f(0) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x). \tag{9}$$

With the construction and introduction of the fermionic p-adic integral (cf. [12,22]), it has been utilized for not only integral representations of many special numbers, polynomials, and functions, but also to provide a deep analysis of many families of special numbers and polynomials, such as Euler, tangent, Boole, Genocchi, Changhee, Frobenius–Euler, and Fubini polynomials and numbers (cf. [1,5,8,11–14,21,22,24,26]). One of the most useful aims of the fermionic p-adic integral (abbreviated with "f.p-a.i.") is to acquire more formulas and properties for special numbers and polynomials. In recent years, by utilizing the fermionic p-adic integrals on $\mathbb{Z}_p$, the symmetric identities of some special polynomials, such as q-Frobenius–Euler polynomials under S_5 in [1], Carlitz's-type twisted (h,q)-tangent polynomials in [5], power sums polynomials in [11], and degenerate q-Euler polynomials in [21], have been studied and investigated in detail. In this study, we focus on a novel extension of the degenerate central Fubini polynomials. First, we introduce the two variable degenerate (abbreviated as "t.w.d.") w-torsion central Fubini polynomials. Then, we provide a f.p-a.i. representation of the degenerate w-torsion central Fubini polynomials, through which we acquire diverse novel symmetric identities for the degenerate w-torsion central Fubini polynomials. Lastly, by using some series manipulation methods, we further obtain the identity of symmetry for the t.w.d. w-torsion central Fubini polynomials.

Let $t \in \mathbb{Z}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. The f.p-a.i. representations of the polynomials $F_u^{(C)}(y)$ and $F_u^{(C)}(x;y)$ are presented by (1) and (9), respectively, as follows (cf. [15,19]):

$$\frac{1}{2}\int_{\mathbb{Z}_p} (-1)^x (y(e^{\frac{t}{2}} - e^{-\frac{t}{2}}))^x d\mu_{-1}(x) = \frac{1}{1 - y((e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{u=0}^{\infty} F_u^{(C)}(y)\frac{t^u}{u!}, \tag{10}$$

and

$$\frac{e^{xt}}{2}\int_{\mathbb{Z}_p} (-1)^z (y(e^{\frac{t}{2}} - e^{-\frac{t}{2}}))^z d\mu_{-1}(z) = \frac{e^{xt}}{1 - y(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{u=0}^{\infty} F_u^{(C)}(x;y)\frac{t^u}{u!}. \tag{11}$$

Also, the f.p-a.i. representations of polynomials $F_{u,\lambda}^{(C)}(y)$ and $F_{u,\lambda}^{(C)}(x;y)$ are provided by (7) and (9), respectively, as follows (see [25]):

$$\frac{1}{2}\int_{\mathbb{Z}_p} (-1)^x (y(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^x d\mu_{-1}(x) = \frac{1}{1 - y(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} = \sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(y)\frac{t^u}{u!}, \tag{12}$$

and

$$\frac{e_\lambda^x(t)}{2}\int_{\mathbb{Z}_p} (-1)^z (y(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)))^z d\mu_{-1}(z) = \frac{e_\lambda^x(t)}{1 - y(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} = \sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(x;y)\frac{t^u}{u!}, \tag{13}$$

where $\lambda, t \in \mathbb{C}_p$ with $|t\lambda| < p^{(1-p)^{-1}}$.

From (3), (12), and (13), we see that

$$\begin{aligned} & \frac{e_\lambda^x(t)}{2} \int_{\mathbb{Z}_p} (-1)^z (y(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)))^z d\mu_{-1}(z) \\ = & \left(\sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!} \right) \left(\sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(y) \frac{t^u}{u!} \right) = \sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(x;y) \frac{t^u}{u!}, \end{aligned} \tag{14}$$

which yields $u \geq 0$, such that

$$F_{u,\lambda}^{(C)}(x;y) = \sum_{l=0}^{u} (x)_{l,\lambda} F_{u-l,\lambda}^{(C)}(y) \binom{u}{l}. \tag{15}$$

Also, it is observed that

$$\begin{aligned} \frac{1 - (e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^k y^k}{1 - (e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))y} &= \sum_{i=0}^{k-1} y^i (e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^i = \sum_{i=0}^{k-1} \sum_{l=0}^{i} \binom{i}{l} (-1)^{l-i} y^i e_\lambda^{l-\frac{i}{2}}(t) \\ &= \sum_{u=0}^{\infty} \left(\sum_{i=0}^{k-1} \sum_{l=0}^{i} y^i (l - \frac{i}{2})_{\lambda,u} \binom{i}{l} (-1)^{l-i} \right) \frac{t^u}{u!} \\ &= \sum_{u=0}^{\infty} \left(\sum_{i=0}^{k-1} y^i \delta^i(0)_{u,\lambda} \right) \frac{t^u}{u!} = \sum_{u=0}^{\infty} \left(\sum_{i=0}^{k-1} y^i i! T_2(u,i) \right) \frac{t^u}{u!}, \end{aligned} \tag{16}$$

where $\delta f(x) := f(x + \frac{1}{2}) - f(x - \frac{1}{2})$ and (cf. [19])

$$\delta^n(0)_{u,\lambda} = n! T_{2,\lambda}(u,n). \tag{17}$$

2. Main Results

In this section, we introduce the two variable degenerate w-torsion central Fubini polynomials in (18) by means of their exponential generating function. Thereafter, we provide a fermionic p-adic integral representation of these polynomials. Using this representation, we investigate two symmetric identities for these polynomials in Theorems 1 and 2, using some special p-adic integral techniques. Moreover, by utilizing some series manipulation methods, we obtain a more symmetric identity (Theorem 3) for the two variable degenerate w-torsion central Fubini polynomials.

We first provide our main definition as follows.

Definition 1. *For $w \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we define the two variable degenerate w-torsion central Fubini polynomials as follows*

$$\sum_{u=0}^{\infty} F_{u,w}^{(C)}(x;y|\lambda) \frac{t^u}{u!} = \frac{e_\lambda^x(t)}{1 - y^w \left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t) \right)^w}, \tag{18}$$

where $|t| < \left|\log|y^{-1}|\right|$ for $y \neq 1$ and $|t| < 2\pi$ for $y = 1$.

We now analyze some special cases of polynomials (18), as follows:

Remark 1. *In some special cases, $F_{u,w}^{(C)}(y|\lambda) := F_{u,w}^{(C)}(0;y|\lambda)$ and $F_{u,w}^{(C)}(1|\lambda) := F_{u,w}^{(C)}(0;1|\lambda)$ are called the degenerate w-torsion central Fubini polynomials and the degenerate w-torsion central Fubini numbers, respectively.*

Remark 2. *Upon setting $w = 1$ in (18), the polynomials $F_{u,w}^{(C)}(x;y|\lambda)$ become the usual t.w.d. Fubini polynomials $F_u^{(C)}(x;y|\lambda)$ in (13).*

Remark 3. *Letting $w-1=\lambda \to 0$ in (18), the polynomials $F_{u,w}^{(C)}(x;y|\lambda)$ become the familiar t.w. central Fubini polynomials $F_u^{(C)}(x;y)$ in (11).*

Similar to (12) and (13), for $w \in \mathbb{N}$, the f.p-a.i. representations of polynomials $F_{u,w}^{(C)}(y|\lambda)$ and $F_{u,w}^{(C)}(x;y|\lambda)$ are provided by

$$\sum_{u=0}^{\infty} F_{u,w}^{(C)}(y|\lambda)\frac{t^u}{u!} = \frac{2}{1-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^w y^w} = \frac{1}{2}\int_{\mathbb{Z}_p}\left(-\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^w y^w\right)^x d\mu_{-1}(x) \tag{19}$$

and

$$\begin{aligned}\sum_{u=0}^{\infty} F_{u,w}^{(C)}(x;y|\lambda)\frac{t^u}{u!} &= \frac{e_\lambda^x(t)}{1-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^w y^w} \\ &= \frac{e_\lambda^x(t)}{2}\int_{\mathbb{Z}_p}\left(-\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^w y^w\right)^x d\mu_{-1}(x),\end{aligned} \tag{20}$$

where $\lambda, t \in \mathbb{C}_p$ with $|t\lambda| < p^{(1-p)^{-1}}$. We observe from (12) and (19) that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))y)^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}y^{w_1})^x d\mu_{-1}(x)} &= \frac{1-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}y^{w_1}}{1-(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))y} \\ &= \sum_{i=0}^{w_1-1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^i y^i \\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_1-1} y^i \delta^i(0)_{u,\lambda}\right)\frac{t^u}{u!}, (w_1 \in \mathbb{N}) \\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_1-1} T_{2,\lambda}(u,i)y^i i!\right)\frac{t^u}{u!}.\end{aligned} \tag{21}$$

Now, we provide our first symmetric identity for the two variable degenerate w-torsion central Fubini polynomials as follows.

Theorem 1. *The following symmetric identity*

$$\sum_{m=0}^{u}\binom{u}{m}\sum_{i=0}^{w_1-1} y^{w_2 i}\delta^{w_2 i}(0)_{m,\lambda}F_{u-m,w_1}^{(C)}(y|\lambda) = \sum_{m=0}^{u}\binom{u}{m}\sum_{i=0}^{w_2-1} y^{w_1 i}\delta^{w_1 i}(0)_{m,\lambda}F_{u-m,w_2}^{(C)}(y|\lambda) \tag{22}$$

holds for $0 \le u$, where w_1, w_2 are two odd numbers.

Proof. For $w_1, w_2 \in \mathbb{N}$, we consider that

$$I = \frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^{x_1}d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^{x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(-y^{w_1 w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1 w_2})^{x}d\mu_{-1}(x)}, \tag{23}$$

which is invariant under the interchange of w_1 and w_2. Then, using (23), we find that

$$I = \int_{\mathbb{Z}_p}(-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x) \times \left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1 w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1 w_2})^x d\mu_{-1}(x)}\right). \tag{24}$$

First, we observe that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2}}{1-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2}}\\ &= \sum_{i=0}^{w_2-1} y^{w_2i}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2i}\\ &= \sum_{i=0}^{w_1-1} y^{w_2i}\sum_{l=0}^{w_2i}\binom{w_2i}{l}(-1)^{w_2i-l}e_\lambda^{l-\frac{i}{2}}(t)\\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_1-1} y^{w_2i}\delta^{w_2i}(0)_{u,\lambda}\right)\frac{t^u}{u!}\\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_1-1} y^{w_2i}(w_2i)!T_{2,\lambda}(u,w_2i)\right)\frac{t^u}{u!}. \end{aligned} \tag{25}$$

It can be discovered from (24) and (25) that

$$\begin{aligned} I &= \int_{\mathbb{Z}_p}(-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)}\right)\\ &= \left(\sum_{u=0}^{\infty}2F_{u,w_1}^{(C)}(y|\lambda)\frac{t^u}{u!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_1-1}y^{w_2i}\delta^{w_2i}(0)_{m,\lambda}\right)\frac{t^m}{m!}\right)\\ &= \sum_{u=0}^{\infty}\left(2\sum_{m=0}^{u}\binom{u}{m}\sum_{i=0}^{w_1-1}y^{w_2i}\Delta^{w_2i}(0)_{m,\lambda}F_{u-m,w_1}^{(C)}(y|\lambda)\right)\frac{t^u}{u!}. \end{aligned} \tag{26}$$

Interchanging the roles of w_1 and w_2, following (23), it can be written that

$$I=\int_{\mathbb{Z}_p}(-y^{w_2}(e_\lambda(t)-1)^{w_2})^x d\mu_{-1}(x)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)}\right). \tag{27}$$

Note that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2}}{1-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}}\\ &= \sum_{i=0}^{w_2-1} y^{w_1i}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1i}\\ &= \sum_{i=0}^{w_2-1} y^{w_1i}\sum_{l=0}^{w_1i}\binom{w_1i}{l}(-1)^{w_1i-l}e_\lambda^{l-\frac{i}{2}}(t)\\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_2-1} y^{w_1i}\delta^{w_1i}(0)_{u,\lambda}\right)\frac{t^u}{u!}\\ &= \sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_2-1} y^{w_1i}(w_1i)!T_{2,\lambda}(u,w_1i)\right)\frac{t^u}{u!}. \end{aligned} \tag{28}$$

It can be discovered from (27) and (28) that

$$
\begin{aligned}
I &= \int_{\mathbb{Z}_p} (-y^{w_2}(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_1}(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1 w_2}(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^{w_1 w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\sum_{u=0}^{\infty} 2F_{u,w_2}^{(C)}(y|\lambda) \frac{t^u}{u!} \right) \left(\sum_{m=0}^{\infty} \left(\sum_{i=0}^{w_2-1} y^{w_1 i} \delta^{w_1 i}(0)_{m,\lambda} \right) \frac{t^m}{m!} \right) \\
&= \sum_{u=0}^{\infty} \left(2 \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_2-1} y^{w_1 i} \delta^{w_1 i}(0)_{m,\lambda} F_{u-m,w_2}^{(C)}(y|\lambda) \right) \frac{t^u}{u!}. \qquad (29)
\end{aligned}
$$

So, the proof is completed as a result of the computations (26) and (29). □

In particular, using $w_1 = 1$ in (22), we obtain the following relationship.

Remark 4. *The following relationship holds for* $0 \leq u$*:*

$$
F_u^{(C)}(y|\lambda) = \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_2-1} y^i \delta^i(0)_{m,\lambda} F_{u-m,w_2}^{(C)}(y|\lambda). \qquad (30)
$$

Corollary 1. *From Theorem 1 and Equation (17), the following relationship*

$$
\begin{aligned}
&\sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_1-1} y^{w_2 i} (w_2 i)! T_{2,\lambda}(u, w_2 i) F_{u-m,w_1}^{(C)}(y|\lambda) \qquad (31) \\
&= \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_2-1} y^{w_1 i} (w_1 i)! T_{2,\lambda}(u, w_1 i) F_{u-m,w_2}^{(C)}(y|\lambda)
\end{aligned}
$$

hold for $u \geq 0$*, with* w_1, w_2 *being two odd numbers.*

When selecting $w_1 = 1$ in (31), we obtain the following relationship.

Remark 5. *The following relationship holds for* $0 \leq u$*:*

$$
F_u^{(C)}(y|\lambda) = \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_2-1} y^i i! T_{2,\lambda}(u, i) F_{u-m,w_2}^{(C)}(y|\lambda). \qquad (32)
$$

Our second symmetric identity for the two variable degenerate w-torsion central Fubini polynomials is as follows.

Theorem 2. *The following symmetric identity*

$$
\begin{aligned}
&\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} y^{w_2 i} (-1)_{l,\lambda} F_{u,w_1}^{(C)}\left(\frac{w_2}{2} i - l, y|\lambda \right) \\
&= \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} \binom{w_1 i}{l} y^{w_1 i} (-1)_{l,\lambda} F_{u,w_2}^{(C)}\left(\frac{w_1}{2} i - l, y|\lambda \right) \qquad (33)
\end{aligned}
$$

holds for $0 \leq u$*, with* w_1, w_2 *being two odd numbers.*

Proof. It can be computed from (23) that

$$
\begin{aligned}
I &= \int_{\mathbb{Z}_p} (-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\int_{\mathbb{Z}_p} (-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x) \right) \times \left(\frac{1-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2}}{1-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2}} \right) \\
&= \left(\sum_{i=0}^{w_1-1} y^{w_2 i}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2 i} \right) \times 2 \left(\frac{1}{1-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}} \right) \\
&= \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} y^{w_2 i}(-1)_{l,\lambda} \frac{2e_\lambda^{(\frac{w_2}{2}i-l)}(t)}{1-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}} \\
&= 2\sum_{u=0}^{\infty} \left(\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} y^{w_2 i}(-1)_{l,\lambda} F_{u,w_1}^{(C)}(w_2 i-l, y|\lambda) \right) \frac{t^u}{u!}. \qquad (34)
\end{aligned}
$$

Interchanging the roles of w_1 and w_2, it is observed from (26) that

$$
\begin{aligned}
I &= \int_{\mathbb{Z}_p} (-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2})^x d\mu_{-1}(x)} \right) \\
&= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{1-y^{w_1w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1w_2}}{1-y^{w_1}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1}} \right) \\
&= \left(\sum_{i=0}^{w_2-1} y^{w_1 i}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_1 i} \right) \times 2 \left(\frac{1}{1-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2}} \right) \\
&= \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} \binom{w_1 i}{l} y^{w_1 i}(-1)_{l,\lambda} \frac{2e_\lambda^{((\frac{w_1}{2}i-l)}(t)}{1-y^{w_2}(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))^{w_2}} \\
&= 2\sum_{u=0}^{\infty} \left(\sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} \binom{w_1 i}{l} y^{w_1 i}(-1)_{l,\lambda} F_{u,w_2}^{(C)}(\frac{w_1}{2}i-l, y|\lambda) \right) \frac{t^u}{u!}. \qquad (35)
\end{aligned}
$$

So, the proof is completed as a result of the computations (34) and (35). □

In particular, choosing $w_1 = 1$ in (33), we obtain the following relationship.

Remark 6. *The following relationship holds for* $0 \leq u$*:*

$$
F_u(y|\lambda) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{i} \binom{i}{l} y^i (-1)_{l,\lambda} F_{u,w_2}^{(C)}(\frac{i}{2}-l, y|\lambda). \qquad (36)
$$

A symmetric identity for t.w.d. w-torsion central Fubini polynomials utilizing the series manipulation methods is presented as follows.

Theorem 3. *The polynomials $F_{m,w}(\nu;\gamma|\lambda)$ fulfill the following identity for $m \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}$ and $m \geq 0$:*

$$\begin{aligned} & \sum_{k=0}^{u} \binom{u}{k} F_{u-k,w}^{(C)}(bx;y|b\lambda) a^{u-k} F_{k,w}^{(C)}(bx;y|b\lambda) b^{k} \\ = & \sum_{k=0}^{u} \binom{u}{k} F_{k,w}^{(C)}(bx;y|b\lambda) a^{k} F_{u-k,w}^{(C)}(bx;y|b\lambda) b^{u-k}. \end{aligned} \tag{37}$$

Proof. We choose that

$$\mathrm{Y} = \frac{e_{\lambda}^{2x}(abt)}{\left(1-\left(e_{b\lambda}^{\frac{1}{2}}(at)-e_{b\lambda}^{-\frac{1}{2}}(at)\right)^{w} y^{w}\right)\left(1-\left(e_{a\lambda}^{\frac{1}{2}}(bt)-e_{a\lambda}^{-\frac{1}{2}}(bt)\right)^{w} y^{w}\right)},$$

which is symmetric in a and b. We compute from (18) that

$$\begin{aligned} \mathrm{Y} &= \left[\sum_{u=0}^{\infty} F_{u,w}^{(C)}(bx;y|b\lambda)\frac{(at)^{u}}{u!}\right]\left[\sum_{u=0}^{\infty} F_{u,w}^{(C)}(ax;y|a\lambda)\frac{(bt)^{u}}{u!}\right] \\ &= \sum_{u=0}^{\infty}\left(\sum_{k=0}^{u}\binom{u}{k} F_{u-k,w}^{(C)}(bx;y|b\lambda) F_{k,w}^{(C)}(bx;y|b\lambda) a^{u-k} b^{k}\right)\frac{t^{u}}{u!} \end{aligned}$$

and, in the same way,

$$\mathrm{Y} = \sum_{u=0}^{\infty}\left(\sum_{k=0}^{u}\binom{u}{k} F_{k,w}^{(C)}(bx;y|b\lambda) F_{u-k,w}^{(C)}(bx;y|b\lambda) a^{k} b^{u-k}\right)\frac{t^{u}}{u!},$$

which implies the coveted relation (37). □

3. Further Remarks

In this part, we observe further remarks for the two variable degenerate w-torsion central Fubini polynomials related to the degenerate central factorial polynomials of the second kind. Also, eventually, we provide a representation of the degenerate differential operator on the two variable degenerate w-torsion central Fubini polynomials.

We observe from (6) and (18) that

$$\begin{aligned} \sum_{j=0}^{\infty} F_{j,w}^{(C)}(x;y|\lambda)\frac{t^{j}}{j!} &= \frac{e_{\lambda}^{x}(t)}{1-y^{w}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{w}} \\ &= \sum_{k=0}^{\infty} y^{kw}\left(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\right)^{kw} e_{\lambda}^{x}(t) \\ &= \sum_{k=0}^{\infty} y^{kw} k! \sum_{j=k}^{\infty} T_{\lambda}(j,k:x)\frac{t^{j}}{j!} \\ &= \sum_{j=k}^{\infty}\sum_{k=0}^{j} T_{\lambda}(j,k:x) y^{kw} k! \frac{t^{j}}{j!}, \end{aligned}$$

which yields $j \geq 0$, such that

$$F_{j,w}^{(C)}(x;y|\lambda) = \sum_{k=0}^{j} T_{\lambda}(j,k:x) y^{kw} k!, \tag{38}$$

which is the relation between the two variable degenerate w-torsion central Fubini polynomials and the degenerate central factorial polynomials of the second kind.

The degenerate differential operator is considered by Kim et al. [23] as follows:

$$\left(x\frac{d}{dx}\right)_{k,\lambda} = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-\lambda\right)\left(x\frac{d}{dx}-2\lambda\right)\cdots\left(x\frac{d}{dx}-(k-1)\lambda\right). \tag{39}$$

From (39), we have

$$\left(x\frac{d}{dx}\right)_{k,\lambda} x^n = (n)_{k,\lambda}x^n.$$

Let f be a formal power series written as $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $k \geq 0$. Then, the degenerate differential operator of this series is given by

$$\left(x\frac{d}{dx}\right)_{k,\lambda} f(x) = \sum_{n=0}^{\infty} a_n (n)_{k,\lambda} x^n.$$

Kim et al. [23] found that the degenerate differential operator plays an important role in boson operators. Here, we focus on the representation of the degenerate differential operator on the t.w.d. w-torsion central Fubini polynomials (18). Here, we provide the following operator formula.

Theorem 4. *The following relation*

$$\left(y\frac{d}{dy}\right)_{k,\lambda} F_{j,w}^{(C)}(x;y|\lambda) = \sum_{s=0}^{j} s!T_\lambda(j,s:x)(sw)_{k,\lambda}y^{sw},$$

holds for $w \in \mathbb{N}$ *and* $k, j \geq 0$.

Proof. By (38) and (39), we observe that

$$\begin{aligned}\left(y\frac{d}{dy}\right)_{k,\lambda} F_{j,w}^{(C)}(x;y|\lambda) &= \left(y\frac{d}{dy}\right)_{k,\lambda}\left\{\sum_{s=0}^{j} s!T_\lambda(j,s:x)y^{sw}\right\}\\ &= \sum_{s=0}^{j} s!T_\lambda(j,s:x)\left\{\left(y\frac{d}{dy}\right)_{k,\lambda} y^{sw}\right\}\\ &= \sum_{s=0}^{j} s!T_\lambda(j,s:x)(sw)_{k,\lambda}y^{sw}.\end{aligned}$$

□

4. Conclusions

In recent years, after constructions of p-adic q-integrals by Teakyun Kim, a Korean mathematician, p-adic q-integrals as well as some of their special cases have been utilized not only as integral representations of many special polynomials and functions, but also to deeply analyze many families of special polynomials and numbers, such as central Bell, central Fubini, Bernoulli, and Euler polynomials and numbers.

Also, by means of p-adic integrals, several special techniques and methods have been utilized to obtain symmetric identities. These identities cover and generalize a lot of well-known special formulas and properties for polynomials, such as Raabe formulas, extended recurrence formulas, Miki identities, and Carlitz identities.

From these motivations, in the presented study, we focused on a novel generalization of degenerate central Fubini polynomials. We first considered two variable degenerate w-torsion central Fubini polynomials by means of their exponential generating function. Then, we provided a fermionic p-adic integral representation of the two variable degenerate w-torsion central Fubini polynomials. From this representation, we investigated

two symmetric identities (Theorems 1 and 2) for these polynomials, using special p-adic integral techniques.

Moreover, by utilizing series manipulation methods, we acquired a more symmetric identity (Theorem 3) for the two variable degenerate w-torsion central Fubini polynomials.

Furthermore, we provided a representation of the degenerate differential operator (Theorem 4) on the two variable degenerate w-torsion central Fubini polynomials related to the degenerate central factorial polynomials of the second kind.

To the best of our knowledge, the results obtained in this paper are novel and do not seem to be reported in the literature. The results presented here have the potential to be utilized in a lot of branches of statistics, probability, mathematics, engineering, and mathematical physics.

Author Contributions: Conceptualization, W.A.K. and U.D.; methodology, M.S.A., W.A.K. and U.D.; software, W.A.K.; validation, M.S.A., W.A.K. and U.D.; formal analysis, W.A.K. and U.D.; investigation, W.A.K. and U.D.; resources, M.S.A. and W.A.K.; data curation, W.A.K. and M.S.A.; writing—original draft preparation, W.A.K. and U.D.; writing—review and editing, M.S.A., W.A.K. and U.D.; visualization, W.A.K. and U.D.; supervision, W.A.K. and U.D.; project administration, W.A.K. and U.D.; funding acquisition, M.S.A. and W.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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