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Some New Estimations of Left and Right Interval Fractional Pachpatte's Type Integral Inequalities via Rectangle Plane

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Abstract: Inequalities involving fractional operators have been an active area of research, which is crucial in establishing bounds, estimates, and stability conditions for solutions to fractional integrals. In this paper, we initially presented a new class that is known as coordinated left and right $\hbar$-pre-invex interval-valued mappings ($\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-invex $I\cdot V$-M), as well classical convex and non-convex are also obtained. This newly defined class enabled us to derive novel inequalities, such as Hermite–Hadamard and Pachpatte's type inequalities. Furthermore, the obtained results allowed us to recapture several special cases of known results for different parameter choices, which can be applications of the main results. Finally, we discussed the validity of the main outcomes.

Keywords: interval-valued mappings over coordinates; coordinated left and right $\hbar$-convexity; double Riemann–Liouville fractional integral operator; Pachpatte's type inequalities

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1. Introduction

The theory of inequalities is fundamental in numerous scientific and engineering fields and is extensively taught due to its practical applications. Mathematical inequalities are used in system design and analysis, engineering, signal processing, and optimization problems [1,2]. They offer a robust framework for analyzing and comprehending the behavior of solutions to numerical and partial differential equations. Fractional analysis, an innovative extension of classical analysis to non-integer orders, plays a significant role in this context. In studying physical systems described by fractional differential equations, fractional integral inequalities are utilized to obtain energy estimates, see [3–6].

Convex functions have been a central topic in mathematics for a significant period. Their growing popularity is likely due to their essential role in optimization, machine learning, and various applications across scientific and engineering disciplines [7–11]. There is a profound and intricate relationship between the theory of inequalities and convex functions. Convex functions have been crucial in the discovery and development of numerous important and useful inequalities. The primary definition of a convex function on convex sets is as follows.

If $\mathcal{G} : K \to \mathfrak{N}$ is a convex function defined on the interval K of real numbers, and θ, $\lambda \in K$ with $\theta < \lambda$, then

$$\mathcal{G}(\kappa\theta + (1-\kappa)\lambda) \leq \kappa\mathcal{G}(\theta) + (1-\kappa)\mathcal{G}(\lambda). \tag{1}$$

Jensen's inequality is a foundational and widely recognized result for convex functions, with various related forms and extensions. Its extensive applications highlight its importance in numerous scientific, engineering, and computational fields, see [12,13]. Convex functions serve as the cornerstone of mathematical inequalities and are essential tools in contemporary research.

Fractional calculus offers a framework for describing and modeling systems with intricate dynamics, memory effects, and anomalous behaviors that classical calculus may not adequately capture. It is a specialized branch of applied mathematics that aids in the comprehension and analysis of complex real-world phenomena, see [14–16]. Fractional integral and derivative operators are vital in connecting mathematical models to real-world problems. These operators have substantially contributed to the development of new concepts in fractional analysis, broadening their application areas [17–19]. Fractional integrals and derivatives differ in terms of singularity, locality, and kernel. Fractional integrals, in particular, are powerful mathematical tools which are valuable for addressing a wide range of issues in science and engineering, see [20,21]. The development of fractional calculus has led to the introduction and analysis of various fractional integral operators, each with distinct properties and applications, see [22,23]. The Riemann–Liouville fractional integral is one of the most fundamental and widely utilized operators in this field. Additionally, operators introduced by Caputo, Fabrizio, and tempered operators add further depth and versatility to the theory. Due to the significance of fractional calculus, many mathematicians are interested in fractional operators and integral inequalities. Kashuri et al. [24,25] proved trapezoidal-type inequalities for differentiable convex functions with applications to matrices and the q-digamma function. Junjua et al. [26,27] obtained trapezoidal-type inequalities for exponential convex functions, applying them to certain real means. Furthermore, several articles [28–30] have demonstrated fractional trapezoidal-type inequalities for functions with convex second derivatives in the absolute value. Klasoom et al. [31,32] examined various modifications of trapezoidal-type inequalities for twice differentiable convex functions using (p,q)-integrals. For different fractional versions of various inequalities, see [33,34].

Due to the broad spectrum of fields where set-valued analysis is applicable, researchers have formulated integral inequalities using various operators and order relations within the realm of interval-valued mappings (I.V.F.). Taking inspiration from this approach, Zhao et al. [35,36] recently employed the conventional integral operator to establish inequality (1.2) within the framework of interval inclusion relations. The authors of [37] introduced pre-invex functions on the plane with interval values and derived multiple double inequalities. Wannalookkhee et al. [38] utilized quantum integrals to derive a double inequality on the plane and for other applications. Kalsoom et al. [39] developed a Hermite–Hadamard-type (H.H) inequality associated with coordinated higher-order generalized pre and quasi-invex mappings by utilizing the concept of quantum integrals. Utilizing two intriguing identities for functions of two variables, Akkurt et al. [40] innovatively formulated Hermite–Hadamard (H.H.)-type fractional integral inequalities for double fractional integrals. Shi et al. [41] established H.H. and its symmetric variation using interval-valued mappings ($I\cdot V$-M) and two forms of generalized convex functions. Saeed et al. [42] devised H.H. and Pachpatte-type integral inequalities employing coordinated up and down convex mappings with fuzzy-number values. Wu et al. [43] derived three fundamental integral identities based on the first- and second-order derivatives of a given function using fractional integrals with an exponential kernel. Ahmad et al. [44] introduced fractional operators with non-singular kernels to support H.H., Hermite–Hadamard–Fejer, Dragomir–Agarwal, and Pachpatte-type inequalities for convex functions. Khan et al. [45–49] formulated specific Hermite–Hadamard (H.H.) and its various variations of inequalities for exponential trigonometric convex mappings with fuzzy-number values, employing fuzzy fractional integral operators with exponential kernels.

In this paper, our focus lies on examining pseudo left–right interval order connections. Recent studies in this direction have yielded notable results: Saeed et al. [50] explored

coordinated h-convex mappings, leading to the derivation of three well-known inequalities within the framework of pseudo order relations. Stojiljkovic et al. [51,52] utilized Katugampola integrals to introduce a new class of p-convex mappings, establishing numerous generalized bounds for double inequalities through the utilization of left–right order relations. Khan et al. [53] employed convex interval-valued functions via log convex functions based on the pseudo order relation, revealing several novel properties and inequalities. For more information related to fractional integral operator and coordinated inequalities, see [54–59].

The rest of the paper is structured as follows: Section 2 reviews some basic notations that will be used throughout. In Section 3, we first introduce a new convex class on coordinates known as $\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex $I\cdot V$-M. By using fractional integral operators, various types of inequalities have been proved as well as Hermite–Hadamard and Pachpatte's type integral inequalities. Moreover, useful examples are also provided to check the validations of our main outcomes. For the applications of these results, our results will allow us to recover several special cases of known results for different parameter choices. In Section 4, we will draw some conclusions and provide future recommendations for interested readers.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ containing all bounded and closed intervals within $\mathbb{R}$. $\mathcal{CO} \in \mathbb{R}_I$ should be defined as follows:

$$\mathcal{CO} = [\mathcal{CO}_*,\ \mathcal{CO}^*] = \{y \in \mathbb{R} | \mathcal{CO}_* \leq y \leq \mathcal{CO}^*\},\ (\mathcal{CO}_*,\ \mathcal{CO}^* \in \mathbb{R}). \tag{2}$$

It is argued that $\mathcal{CO}$ is degenerate if $\mathcal{CO}_* = \mathcal{CO}^*$. The interval $[\mathcal{CO}_*,\ \mathcal{CO}^*]$ is referred to as positive if $\mathcal{CO}_* \geq 0$, $\mathbb{R}_I^+$ represents the set of all positive intervals and is defined as $\mathbb{R}_I^+ = \{[\mathcal{CO}_*,\ \mathcal{CO}^*] : [\mathcal{CO}_*,\ \mathcal{CO}^*] \in \mathbb{R}_I \text{ and } \mathcal{CO}_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$, $\mathcal{CO}, \mathcal{S} \in \mathbb{R}_I$ be defined by, with $\mathcal{S} = [\mathcal{S}_*,\ \mathcal{S}^*]$ and $\mathcal{CO} = [\mathcal{CO}_*,\ \mathcal{CO}^*]$, we may define the interval arithmetic as follows:

- Scaler multiplication:

$$\varrho.\mathcal{CO} = \begin{cases} [\varrho\mathcal{CO}_*,\ \varrho\mathcal{CO}^*] & \text{if } \varrho > 0, \\ \{0\} & \text{if } \varrho = 0, \\ [\varrho\mathcal{CO}^*, \varrho\mathcal{CO}_*] & \text{if } \varrho < 0. \end{cases} \tag{3}$$

- Minkowski difference:

$$[\mathcal{S}_*,\ \mathcal{S}^*] - [\mathcal{CO}_*,\ \mathcal{CO}^*] = [\mathcal{S}_* - \mathcal{CO}_*,\ \mathcal{S}^* - \mathcal{CO}^*], \tag{4}$$

- Addition:

$$[\mathcal{S}_*,\ \mathcal{S}^*] + [\mathcal{CO}_*,\ \mathcal{CO}^*] = [\mathcal{S}_* + \mathcal{CO}_*,\ \mathcal{S}^* + \mathcal{CO}^*], \tag{5}$$

- Multiplication:

$$[\mathcal{S}_*,\ \mathcal{S}^*] \times [\mathcal{CO}_*,\ \mathcal{CO}^*] = [min\{\mathcal{S}_*\mathcal{CO}_*,\ \mathcal{S}^*\mathcal{CO}_*,\ \mathcal{S}_*\mathcal{CO}^*,\ \mathcal{S}^*\mathcal{CO}^*\},\ max\{\mathcal{S}_*\mathcal{CO}_*,\ \mathcal{S}^*\mathcal{CO}_*,\ \mathcal{S}_*\mathcal{CO}^*,\ \mathcal{S}^*\mathcal{CO}^*\}]. \tag{6}$$

The inclusion " $\supseteq$ " means that
$\mathcal{CO} \supseteq \mathcal{S}$ if and only if, $[\mathcal{CO}_*,\ \mathcal{CO}^*] \supseteq [\mathcal{S}_*,\ \mathcal{S}^*]$, and if and only if

$$\mathcal{CO}_* \leq \mathcal{S}_*,\ \mathcal{S}^* \leq \mathcal{CO}^*. \tag{7}$$

Remark 1 ([58]). *(i) The relation " $\leq_p$" is defined on $\mathbb{R}_I$ by*

$$[\mathcal{S}_*,\ \mathcal{S}^*] \leq_p [\mathcal{CO}_*,\ \mathcal{CO}^*] \text{ if and only if } \mathcal{S}_* \leq \mathcal{CO}_*,\ \ \mathcal{S}^* \leq \mathcal{CO}^*, \tag{8}$$

for all $[\mathcal{S}_*, \mathcal{S}^*]$, $[\mathcal{C}_*, \mathcal{C}^*] \in \mathbb{R}_I$, *and it is a pseudo order relation. The relation* $[\mathcal{S}_*, \mathcal{S}^*] \leq_p [\mathcal{C}_*, \mathcal{C}^*]$ *coincident to* $[\mathcal{S}_*, \mathcal{S}^*] \leq [\mathcal{C}_*, \mathcal{C}^*]$ *on* $\mathbb{R}_I$ *when it is "* $\leq_p$*".*

(ii) It can be easily seen that " $\leq_p$ *" looks like "left and right" on the real line* $\mathbb{R}$*, so we call "* $\leq_p$ *" is "left and right" (or "LR" order, in short).*

Definition 1 ([55,56]). *Let* $\mathcal{G} : [\mathfrak{n}, \mathfrak{k}] \to \mathbb{R}_I^+$ *be an interval-valued mapping (IVM) and* $\mathcal{G} \in \mathcal{IR}_{[\mathfrak{n}, \mathfrak{k}]}$*. Then, the interval Riemann–Liouville-type integrals of* $\mathcal{G}$ *are defined as*

$$\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}^+}\mathcal{G}(y) = \frac{1}{\Gamma(\mathfrak{i})}\int_{\mathfrak{n}}^{y}(y-\mathfrak{t})^{\mathfrak{i}-1}\mathcal{G}(\mathfrak{t})d\mathfrak{t} \quad (y > \mathfrak{n}), \tag{9}$$

$$\mathcal{I}^{\mathfrak{i}}_{\mathfrak{t}^-}\mathcal{G}(y) = \frac{1}{\Gamma(\mathfrak{i})}\int_{y}^{\mathfrak{t}}(\mathfrak{t}-y)^{\mathfrak{i}-1}\mathcal{G}(\mathfrak{t})d\mathfrak{t} \quad (y < \mathfrak{t}), \tag{10}$$

where $\mathfrak{i} > 0$ *and* Γ *is the gamma function.*

Theorem 1 ([59]). *Let* $\hbar : [0, 1] \to \mathbb{R}^+$ *and* $\mathcal{G} : [\mathfrak{z}, \mathfrak{z} + \mathcal{K}_2(\theta, \mathfrak{z})] \to \mathbb{R}_I^+$ *be a* $L{\cdot}R$*-*$\hbar$*-pre-invex* $I{\cdot}V$*-M on* $[\mathfrak{z}, \mathfrak{z} + \mathcal{K}_2(\theta, \mathfrak{z})]$*, given by* $\mathcal{G}(y) = [\mathcal{G}_*(y), \mathcal{G}^*(y)]$ *for all* $y \in [\mathfrak{z}, \mathfrak{z} + \mathcal{K}_2(\theta, \mathfrak{z})]$*. If* $\mathcal{G} \in L([\mathfrak{z}, \mathfrak{z} + \mathcal{K}_2(\theta, \mathfrak{z})], \mathbb{R}_I^+)$*, then*

$$\begin{aligned}\frac{1}{\mathfrak{i}\hbar\left(\frac{1}{2}\right)}\mathcal{G}\left(\frac{2\mathfrak{z}+\mathcal{K}_2(\theta,\mathfrak{z})}{2}\right) &\leq_p \frac{\Gamma(\mathfrak{i})}{(\mathcal{K}_2(\theta,\mathfrak{z}))^{\mathfrak{i}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{z}+\mathcal{K}_2(\theta,\mathfrak{z})) + \mathcal{I}^{\mathfrak{i}}_{\mathfrak{z}+\mathcal{K}_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{z})\right] \\ &\leq_p [\mathcal{G}(\mathfrak{z}) + \mathcal{G}(\theta)]\int_0^1 v^{\mathfrak{i}-1}[\hbar(v) + \hbar(1-v)]dv.\end{aligned} \tag{11}$$

Interval double Aumann's type integrals are defined as follows for coordinated IVM ($\mathcal{C}$-IVM) $\mathcal{G}(x, y)$.

Theorem 2 ([57]). *Let* $\mathcal{G} : \Omega = [\mathfrak{n}, \mathfrak{k}] \times [\mathfrak{z}, \theta] \subset \mathbb{R}^2 \to \mathbb{R}_I$ *be an* $I{\cdot}V$*-M on coordinates, given by* $\mathcal{G}(x, y) = [\mathcal{G}_*(x, y), \mathcal{G}^*(x, y)]$ *for all* $(x, y) \in \Omega = [\mathfrak{n}, \mathfrak{k}] \times [\mathfrak{z}, \theta]$*. Then,* $\mathcal{G}$ *is double integrable (FD-integrable) over* Ω *if and only if* $\mathcal{G}_*(x, y)$ *and* $\mathcal{G}^*(x, y)$ *both are D-integrable over* Ω*. Moreover, if* $\mathcal{G}$ *is FD-integrable over* Ω*, then*

$$(ID)\int_{\mathfrak{n}}^{\mathfrak{t}}\int_{\mathfrak{z}}^{\theta}\mathcal{G}(x,y)dydx = \left[(D)\int_{\mathfrak{n}}^{\mathfrak{t}}\int_{\mathfrak{z}}^{\theta}\mathcal{G}_*(x,y)dydx,\ (D)\int_{\mathfrak{n}}^{\mathfrak{t}}\int_{\mathfrak{z}}^{\theta}\mathcal{G}^*(x,y)dydx\right] \tag{12}$$

The family of all FD-integrables of $I{\cdot}V$-Ms over coordinates and D-integrable functions over coordinates are denoted by $\mathcal{FD}_\Omega$ and $\mathfrak{D}_\Omega$.

Here is the main definition of the fuzzy Riemann–Liouville fractional integral on the coordinates of the function $\mathcal{G}(x, y)$ by:

Definition 2 ([56]). *Let* $\mathcal{G} : \Omega \to \mathbb{R}_I$ *and* $\mathcal{G} \in \mathcal{FD}_\Omega$*. The double fuzzy interval Riemann–Liouville-type integrals* $\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\mathfrak{z}^+}$, $\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\theta^-}$, $\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{t}^-,\mathfrak{z}^+}$, $\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{t}^-,\theta^-}$ *of* $\mathcal{G}$ *order* $\mathfrak{i}, \mathfrak{j} > 0$ *are defined by*

$$\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\mathfrak{z}^+}\mathcal{G}(x,y) = \frac{1}{\Gamma(\mathfrak{i})\Gamma(\mathfrak{j})}\int_{\mathfrak{n}}^{x}\int_{\mathfrak{z}}^{y}(x-\mathfrak{t})^{\mathfrak{i}-1}(y-\mathfrak{s})^{\mathfrak{j}-1}\mathcal{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ (x > \mathfrak{n},\ y > \mathfrak{z}), \tag{13}$$

$$\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\theta^-}\mathcal{G}(x,y) = \frac{1}{\Gamma(\mathfrak{i})\Gamma(\mathfrak{j})}\int_{\mathfrak{n}}^{x}\int_{y}^{\theta}(x-\mathfrak{t})^{\mathfrak{i}-1}(\mathfrak{s}-y)^{\mathfrak{j}-1}\mathcal{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ (x > \mathfrak{n},\ y < \theta), \tag{14}$$

$$\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{t}^-,\mathfrak{z}^+}\mathcal{G}(x,y) = \frac{1}{\Gamma(\mathfrak{i})\Gamma(\mathfrak{j})}\int_{x}^{\mathfrak{t}}\int_{\mathfrak{z}}^{y}(\mathfrak{t}-x)^{\mathfrak{i}-1}(y-\mathfrak{s})^{\mathfrak{j}-1}\mathcal{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ (x < \mathfrak{t},\ y > \mathfrak{z}), \tag{15}$$

$$\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{t}^-,\theta^-}\mathcal{G}(x,y) = \frac{1}{\Gamma(\mathfrak{i})\Gamma(\mathfrak{j})}\int_{x}^{\mathfrak{t}}\int_{y}^{\theta}(\mathfrak{t}-x)^{\mathfrak{i}-1}(\mathfrak{s}-y)^{\mathfrak{j}-1}\mathcal{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ (x < \mathfrak{t},\ y < \theta). \tag{16}$$

Here is the newly defined concept of $\mathcal{C}{\cdot}L{\cdot}R$-$\hbar$-pre-invexity over fuzzy number space in the codomain via fuzzy relation given by

Definition 3. The $I\cdot V\text{-}M$ $\mathcal{G}:\Omega=[\mathfrak{n},\ \mathfrak{t}]\times[\mathfrak{z},\theta]\to\mathbb{R}_I^+$ is referred to be $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex $I\cdot V\text{-}M$ on Ω with respect to bifurcations $(\Lambda_1,\ \Lambda_2)$ if

$$\begin{gathered}\mathcal{G}(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \leq_p \hbar(\upsilon)\hbar(\kappa)\mathcal{G}(\mathfrak{n},\mathfrak{z})+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{n},\theta)+\hbar(1-\upsilon)\hbar(\kappa)\mathcal{G}(\mathfrak{t},\mathfrak{z})+\\ \hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{t},\theta),\end{gathered}\tag{17}$$

for all $(\mathfrak{n},\ \mathfrak{t})$, $(\mathfrak{z},\theta)\in\Omega$, and $\upsilon,\kappa\in[0,\ 1]$, where $\Lambda_1:\ [\mathfrak{n},\ \mathfrak{t}]\times[\mathfrak{n},\ \mathfrak{t}]\to[\mathfrak{n},\ \mathfrak{t}]$ and $\Lambda_2:\ [\mathfrak{z},\theta]\times[\mathfrak{z},\theta]\to[\mathfrak{z},\theta]$. If the inequality (17) is reversed, then $\mathcal{G}$ is referred to be $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-concave $I\cdot V\text{-}M$ on Ω.

Lemma 1. *Let $\mathcal{G}:\Omega\to\mathbb{R}_I^+$ be a coordinated $I\cdot V\text{-}M$ on Ω. Then, $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R\text{-}\hbar$- pre-invex $I\cdot V\text{-}M$ on Ω if and only if there exist two $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex $I\cdot V\text{-}M$s $\mathcal{G}_x:[\mathfrak{z},\theta]\to\mathbb{R}_I^+$, $\mathcal{G}_x(w)=\mathcal{G}(x,w)$ and $\mathcal{G}_y:[\mathfrak{n},\mathfrak{t}]\to\mathbb{R}_I^+$, $\mathcal{G}_y(z)=\mathcal{G}(z,y)$.*

Theorem 3. *Let $\mathcal{G}:\Omega\to\mathbb{R}_I^+$ be an $I\cdot V\text{-}M$ on Ω, given by*

$$\mathcal{G}(x,y)=[\mathcal{G}_*(x,y),\ \mathcal{G}^*(x,y)],\tag{18}$$

for all $(x,y)\in\Omega$. Then, $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex $I\cdot V\text{-}M$ on Ω, if and only if, $\mathcal{G}_*(x,y)$ and $\mathcal{G}^*(x,y)$ both are $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex.

Proof. Assume that, $\mathcal{G}_*(x,y)$ and $\mathcal{G}^*(x,y)$ are $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex and $\hbar$-pre-invex on Ω, respectively. Then, from Equation (17), for all $(\mathfrak{n},\ \mathfrak{t})$, $(\mathfrak{z},\theta)\in\Omega$, υ, and $\kappa\in[0,\ 1]$, we have

$$\begin{gathered}\mathcal{G}_*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \leq \hbar(\upsilon)\hbar(\kappa)\mathcal{G}_*(\mathfrak{n},\mathfrak{z})+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{n},\theta)+\hbar(\kappa)\hbar(1-\upsilon)\mathcal{G}_*(\mathfrak{t},\mathfrak{z})\\ +\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{t},\theta),\end{gathered}\tag{19}$$

and

$$\begin{gathered}\mathcal{G}^*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \leq \hbar(\upsilon)\hbar(\kappa)\mathcal{G}^*(\mathfrak{n},\mathfrak{z})+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{G}^*(\mathfrak{n},\theta)+\hbar(\kappa)\hbar(1-\upsilon)\mathcal{G}^*(\mathfrak{t},\mathfrak{z})\\ +\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}^*(\mathfrak{t},\theta),\end{gathered}\tag{20}$$

Then, by Equations (18), (3), and (5), we obtain

$$\begin{gathered}\mathcal{G}(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ =[\mathcal{G}_*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})),\ \mathcal{G}^*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))]\\ \leq_p \hbar(\upsilon)\hbar(\kappa)[\mathcal{G}_*(\mathfrak{n},\mathfrak{z}),\ \mathcal{G}^*(\mathfrak{n},\mathfrak{z})]+\hbar(\upsilon)\hbar(1-\kappa)[\mathcal{G}_*(\mathfrak{n},\theta),\ \mathcal{G}^*(\mathfrak{n},\theta)]\\ +\hbar(\kappa)\hbar(1-\upsilon)[\mathcal{G}_*(\mathfrak{t},\mathfrak{z}),\ \mathcal{G}^*(\mathfrak{t},\mathfrak{z})]+\hbar(1-\upsilon)\hbar(1-\kappa)[\mathcal{G}_*(\mathfrak{t},\theta),\ \mathcal{G}^*(\mathfrak{t},\theta)]\end{gathered}$$

From (19) and (20), we have

$$\begin{gathered}\mathcal{G}(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \leq_p \hbar(\upsilon)\hbar(\kappa)\mathcal{G}(\mathfrak{n},\mathfrak{z})+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{n},\theta)+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{t},\mathfrak{z})+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{t},\theta),\end{gathered}$$

hence, $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R\text{-}\hbar$-pre-invex $I\cdot V\text{-}M$ on Ω.

Conversely, let $\mathcal{G}$ be $\mathcal{C}\cdot L\cdot R\text{-}\hbar$- pre-invex $I\cdot V\text{-}M$ on Ω. Then, for all $(\mathfrak{n},\ \mathfrak{t})$, $(\mathfrak{z},\theta)\in\Omega$, υ, and $\kappa\in[0,\ 1]$, we have

$$\begin{gathered}\mathcal{G}(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \leq_p \hbar(\upsilon)\hbar(\kappa)\mathcal{G}(\mathfrak{n},\mathfrak{z})+\hbar(\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{n},\theta)+\hbar(1-\upsilon)\hbar(\kappa)\mathcal{G}(\mathfrak{t},\mathfrak{z})+\hbar(1-\upsilon)\hbar(1-\kappa)\mathcal{G}(\mathfrak{t},\theta).\end{gathered}$$

Therefore, again from Equation (18), we have

$$\begin{gathered}\mathcal{G}((\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})))\\ =[\mathcal{G}_*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})),\ \mathcal{G}^*(\mathfrak{n}+(1-\upsilon)\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))].\end{gathered}\tag{21}$$

Again, by Equations (3) and (5), we obtain

$$\begin{aligned}&\hbar(v)\hbar(\kappa)\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\hbar(v)\hbar(1-\kappa)\mathcal{G}(\mathfrak{n}_\flat,\theta)+\hbar(1-v)\hbar(\kappa)\mathcal{G}(\mathfrak{t},\mathfrak{z})+\hbar(1-v)\hbar(1-\kappa)\mathcal{G}(\mathfrak{t},\theta)\\&=\hbar(v)\hbar(\kappa)[\mathcal{G}_*(\mathfrak{n}_\flat,\mathfrak{z}),\ \mathcal{G}^*(\mathfrak{n}_\flat,\mathfrak{z})]+\hbar(v)\hbar(1-\kappa)[\mathcal{G}_*(\mathfrak{n}_\flat,\theta),\ \mathcal{G}^*(\mathfrak{n}_\flat,\theta)]\\&+\hbar(\kappa)\hbar(1-v)[\mathcal{G}_*(\mathfrak{n}_\flat,\mathfrak{z}),\ \mathcal{G}^*(\mathfrak{n}_\flat,\mathfrak{z})]+\hbar(1-v)\hbar(1-\kappa)[\mathcal{G}_*(\mathfrak{n}_\flat,\theta),\ \mathcal{G}^*(\mathfrak{n}_\flat,\theta)],\end{aligned}\tag{22}$$

for all $\S,\omega\in\Omega$ and $v\in[0,\ 1]$. Then, from (21) and (22), we have for all $\S,\omega\in\Omega$ and $v\in[0,\ 1]$ such that

$$\begin{aligned}&\mathcal{G}_*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\&\leq\hbar(v)\hbar(\kappa)\mathcal{G}_*(\mathfrak{n}_\flat,\mathfrak{z})+\hbar(v)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{n}_\flat,\theta)+\hbar(1-v)\hbar(\kappa)\mathcal{G}_*(\mathfrak{t},\mathfrak{z})+\hbar(1-v)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{t},\theta),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{G}^*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\&\leq\hbar(v)\hbar(\kappa)\mathcal{G}^*(\mathfrak{n}_\flat,\mathfrak{z})+\hbar(v)\hbar(1-\kappa)\mathcal{G}^*(\mathfrak{n}_\flat,\theta)+\hbar(1-v)\hbar(\kappa)\mathcal{G}^*(\mathfrak{t},\mathfrak{z})+\hbar(1-v)\hbar(1-\kappa)\mathcal{G}^*(\mathfrak{t},\theta),\end{aligned}$$

Hence, the result follows. □

Example 1. *We consider the I·V-M $\mathcal{G}:[0,\ 1]\times[0,\ 1]\to\mathbb{R}_I^+$ defined by,*

$$\mathcal{G}(x)=[xy,(6+e^x)(6+e^y)].\tag{23}$$

Since endpoint functions $\mathcal{G}_*(x,y)$, $\mathcal{G}^*(x,y)$ are $\mathcal{C}$-$\hbar$-pre-invex functions. Hence, $\mathcal{G}(x,y)$ is $\mathcal{C}$·L·R-$\hbar$-pre-invex I·V-M.

From Lemma 1 and Example 1, we can easily note that each LR-$\hbar$-pre-invex I·V-M is $\mathcal{C}$·L·R-$\hbar$- pre-invex I·V-M. However, the converse is not true.

Remark 2. *Assuming that $\mathcal{G}_*(x,y)=\mathcal{G}^*(x,y)$, then $\mathcal{G}$ is referred to be as a classical $\mathcal{C}$·L·R-$\hbar$-pre-invex function if $\mathcal{G}$ meets the stated inequality here:*

$$\begin{aligned}&\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\&\leq\hbar(v)\hbar(\kappa)\mathcal{G}_*(\mathfrak{n}_\flat,\mathfrak{z})+\hbar(v)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{n}_\flat,\theta)+\hbar(\kappa)\hbar(1-v)\mathcal{G}_*(\mathfrak{n}_\flat,\mathfrak{z})+\hbar(1-v)\hbar(1-\kappa)\mathcal{G}_*(\mathfrak{n}_\flat,\theta).\end{aligned}\tag{24}$$

Assuming that $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and $\mathcal{G}_*(x,y)=\mathcal{G}^*(x,y)$, then $\mathcal{G}$ is referred to be as a classical $\mathcal{C}$-pre-invex function if $\mathcal{G}$ meets the stated inequality here:

$$\begin{aligned}&\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\&\leq v\kappa\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+v(1-\kappa)\mathcal{G}(\mathfrak{n}_\flat,\theta)+(1-v)\kappa\mathcal{G}(\mathfrak{t},\mathfrak{z})+(1-v)(1-\kappa)\mathcal{G}(\mathfrak{t},\theta).\end{aligned}\tag{25}$$

Let $\hbar(v)=v$, $\hbar(\kappa)=\kappa$. If $\mathcal{G}_*(x,y)$ and $\mathcal{G}^*(x,y)$ are affine and pre-concave functions with $\mathcal{G}_*(x,y)\neq\mathcal{G}^*(x,y)$, respectively, then the stated inequality we have, see [35],

$$\begin{aligned}&\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\supseteq v\kappa\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\\&+v(1-\kappa)\mathcal{G}(\mathfrak{n}_\flat,\theta)+(1-v)\kappa\mathcal{G}(\mathfrak{t},\mathfrak{z})+(1-v)(1-\kappa)\mathcal{G}(\mathfrak{t},\theta),\end{aligned}\tag{26}$$

is true.

Definition 4. *Let $\mathcal{G}:\Omega\to\mathbb{R}_I^+$ be a I·V-M on Ω. Then, we obtain*

$$\mathcal{G}(x,y)=[\mathcal{G}_*(x,y),\ \mathcal{G}^*(x,y)],$$

for all $(x,y)\in\Omega$. Then, $\mathcal{G}$ is left-$\mathcal{C}$·L·R-$\hbar$- pre-invex (pre-concave) I·V-M on Ω, if and only if, $\mathcal{G}_*(x,y)$ and $\mathcal{G}^*(x,y)$ are $\mathcal{C}$-$\hbar$- pre-invex (pre-concave) and affine functions on Ω, respectively.

Definition 5. *Let $\mathcal{G}:\Omega\to\mathbb{R}_I^+$ be a I·V-M on Ω. Then, we obtain*

$$\mathcal{G}(x,y)=[\mathcal{G}_*(x,y),\mathcal{G}^*(x,y)],$$

for all $(x, y) \in \Omega$. *Then,* $\mathcal{G}$ *is right-*$\mathcal{C}$·L·R-$\hbar$- *pre-invex (pre-concave)* I·V-M *on* Ω, *if and only if,* $\mathcal{G}_*(x, y)$ *and* $\mathcal{G}^*(x, y)$ *are* $\mathcal{C}$-$\hbar$-*affine and* $\hbar$-*(pre-concave) functions on* Ω, *respectively.*

Theorem 4. *Let* Ω *be a coordinated invex set, and let* $\mathcal{G} : \Omega \to \mathbb{R}_I^+$ *be a* I·V-M, *defined by*

$$\mathcal{G}(x, y) = [\mathcal{G}_*(x, y),\ \mathcal{G}^*(x, y)],$$

for all $(x, y) \in \Omega$. Then, $\mathcal{G}$ is $\mathcal{C}$·L·R-$\hbar$-pre-concave I·V-M on Ω, if and only if, $\mathcal{G}_*(x, y)$ and $\mathcal{G}^*(x, y)$ are $\mathcal{C}$-$\hbar$-pre-concave and $\mathcal{C}$-$\hbar$-pre-concave functions, respectively.

Proof. The demonstration of proof of Theorem 4 is similar to the demonstration proof of Theorem 3. □

Example 2. *We consider the* I·V-M*s* $\mathcal{G} : [0,\ 1] \times [0,\ 1] \to \mathbb{R}_I^+$ *defined by,*

$$\mathcal{G}(x, y) = [(6 - e^x)(6 - e^y), 40xy]. \tag{27}$$

Then, we have since endpoint functions $\mathcal{G}_*(x, y), \mathcal{G}^*(x, y)$ which are both $\mathcal{C}$-$\hbar$- pre-concave functions. Hence $\mathcal{G}(x, y)$ is $\mathcal{C}$·L·R-$\hbar$- pre-concave I·V-M.

In the next results, to avoid confusion, we will not include the symbols (R), (IR), and (ID) before the integral sign.

3. Main Results

In this section, Hermite–Hadamard and Pachpatte-type inequalities for interval-value functions are given. We first present an inequality of Hermite–Hadamard via $\mathcal{C}$·L·R-$\hbar$-pre-convex I·V-Ms.

Theorem 5. *Let* $\mathcal{G} : \Omega \to \mathbb{R}_I^+$ *be a* $\mathcal{C}$·L·R-$\hbar$-*pre-invex* I·V-M *on* Ω, *where* $\mathcal{G}(x, y) = [\mathcal{G}_*(x, y),]$. $[.\mathcal{G}^*(x, y)]$ *for all* $(x, y) \in \Omega$ *and let* $\hbar : [0,\ 1] \to \mathbb{R}^+$. *If* $\mathcal{G} \in \mathcal{FD}_\Omega$, *then the following inequalities holds:*

$$\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
\leq_p\ &\tfrac{\Gamma(\mathfrak{i}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&+\tfrac{\Gamma(\mathfrak{j}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right]\\
&\leq_p\ \tfrac{\Gamma(\mathfrak{i}+1)\Gamma(\mathfrak{j}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&+\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\\
&\left.\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\
&\leq_p\ \tfrac{\mathfrak{j}\Gamma(\mathfrak{i}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}\begin{bmatrix}\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\end{bmatrix}\\
&\times\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\\
&+\tfrac{\mathfrak{i}\Gamma(\mathfrak{j}+1)}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\begin{bmatrix}\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\\+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\end{bmatrix}\\
&\times\int_0^1 v^{\mathfrak{i}-1}\hbar(v)+\hbar(1-v)dv\\
&\leq_p \mathfrak{i}\mathfrak{j}[\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)]\\
&\times\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\int_0^1 v^{\mathfrak{i}-1}[\hbar(v)+\hbar(1-v)]dv.
\end{aligned} \tag{28}$$

If $\mathcal{G}$ is $\mathcal{C}$·L·R-$\hbar$-pre-concave, then the inequality (28) is reversed such that

$$
\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\geq_p \tfrac{\Gamma(i+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&+\tfrac{\Gamma(j+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right]\\
&\geq_p \tfrac{\Gamma(i+1)\Gamma(j+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\\+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\end{bmatrix}\\
&\geq_p \tfrac{j\Gamma(i+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\begin{bmatrix}\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\end{bmatrix}\\
&\times\int_0^1\kappa^{j-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\\
&+\tfrac{i\Gamma(j+1)}{(\Lambda_2(\theta,\mathfrak{z}))^j}\begin{bmatrix}\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\\+\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\end{bmatrix}\\
&\times\int_0^1 v^{i-1}[\hbar(v)+\hbar(1-v)]dv\\
&\geq_p ij[\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)]\\
&\times\int_0^1\kappa^{j-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\int_0^1 v^{i-1}[\hbar(v)+\hbar(1-v)]dv.
\end{aligned}
\tag{29}
$$

Proof. Let $\mathcal{G}:[\mathfrak{n}_\flat,\ \mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)]\to\mathbb{R}_I^+$ be a $\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex $I\cdot V$-M. Then, by hypothesis, we have

$$
\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq_p \mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ (1-v)\mathfrak{z}+v\theta).
\end{aligned}
$$

By using Theorem 4, we have

$$
\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}_*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq \mathcal{G}_*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}_*(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})),\\
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}^*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq \mathcal{G}^*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}^*(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})).
\end{aligned}
$$

By using Lemma 1, we have

$$
\begin{aligned}
\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}_*\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)&\leq \mathcal{G}_*(x,\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}_*(x,\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})),\\
\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}^*\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)&\leq \mathcal{G}^*(x,\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}^*(x,\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})),
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}_*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)&\leq \mathcal{G}_*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y)+\mathcal{G}_*(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y),\\
\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}^*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)&\leq \mathcal{G}^*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y)+\mathcal{G}^*(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y).
\end{aligned}
\tag{31}
$$

From (30) and (31), we have

$$
\begin{aligned}
&\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\left[\mathcal{G}_*\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right),\mathcal{G}^*\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&\leq_p [\mathcal{G}_*(x,\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})),\mathcal{G}^*(x,\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))]\\
&+[\mathcal{G}_*(x,\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})),\mathcal{G}^*(x,\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))],
\end{aligned}
$$

and

$$\begin{aligned}&\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\left[\mathcal{G}_*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right),\mathcal{G}^*\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)\right]\\ \leq_p\ &[\mathcal{G}_*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y),\mathcal{G}^*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y)]\\ &+[\mathcal{G}_*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y),\mathcal{G}^*(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y)],\end{aligned}$$

It follows that

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}\left(x,\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\leq_p\mathcal{G}(x,\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}(x,\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})),\tag{32}$$

and

$$\begin{aligned}\tfrac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)\leq_p\ &\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y)\\ &+\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ y).\end{aligned}\tag{33}$$

Since $\mathcal{G}(x,.)$ and $\mathcal{G}(.,y)$, both are $\mathcal{C}\cdot L\cdot R$-$LR$-$\hbar$- pre-invex -$I\cdot V$-$M$s, then, we have

$$\begin{aligned}\tfrac{1}{\mathfrak{j}\hbar\left(\frac{1}{2}\right)}\mathcal{G}_x\left(\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\leq_p\ &\tfrac{\Gamma(\mathfrak{j})}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}_x(\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}_x(\mathfrak{z})\right]\\ \leq_p\ &[\mathcal{G}_x(\mathfrak{z})+\mathcal{G}_x(\theta)]\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\end{aligned}\tag{34}$$

and

$$\begin{aligned}\tfrac{1}{\mathfrak{i}\hbar\left(\frac{1}{2}\right)}\mathcal{G}_y\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}\right)\leq_p\ &\tfrac{\Gamma(\mathfrak{i})}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat^+}\mathcal{G}_y(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}_y(\mathfrak{n}_\flat)\right]\\ \leq_p\ &\left[\mathcal{G}_y(\mathfrak{n}_\flat)+\mathcal{G}_y(\mathfrak{t})\right]\int_0^1 v^{\mathfrak{i}-1}\hbar(v)+\hbar(1-v)dv\end{aligned}\tag{35}$$

Since $\mathcal{G}_x(w)=\mathcal{G}(x,w)$, then (34) can be written as

$$\begin{aligned}\tfrac{1}{\mathfrak{j}\hbar\left(\frac{1}{2}\right)}\mathcal{G}\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\leq_p\ &\tfrac{\Gamma(\mathfrak{j})}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{z}^+}\mathcal{G}(x,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(x,\mathfrak{z})\right]\\ \leq_p\ &[\mathcal{G}(x,\mathfrak{z})+\mathcal{G}(x,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))]\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned}\tag{36}$$

That is

$$\begin{aligned}&\tfrac{1}{\mathfrak{j}\hbar\left(\frac{1}{2}\right)}\mathcal{G}\left(x,\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\ \leq_p\ &\tfrac{1}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})-\kappa)^{\mathfrak{j}-1}\mathcal{G}(x,\kappa)d\kappa+\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(\kappa-\mathfrak{z})^{\mathfrak{j}-1}\mathcal{G}(x,\kappa)d\kappa\right]\\ \leq_p\ &[\mathcal{G}(x,\mathfrak{z})+\mathcal{G}(x,\theta)]\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned}$$

Multiplying the double inequality (36) by $\frac{(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}$ and integrating with respect to x over $[\mathfrak{n}_\flat,\ \mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)]$, we have

$$\begin{aligned}&\tfrac{1}{\mathfrak{j}(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}\hbar\left(\frac{1}{2}\right)}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}\left(x,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}dx\\ \leq_p\ &\tfrac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}(\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})-\kappa)^{\mathfrak{j}-1}\mathcal{G}(x,\kappa)d\kappa dx\\ &+\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}(\kappa-\mathfrak{z})^{\mathfrak{j}-1}\mathcal{G}(x,\kappa)d\kappa dx\\ \leq_p\ &\tfrac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}\left[\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}\mathcal{G}(x,\mathfrak{z})dx\right.\\ &\left.+\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)-x)^{\mathfrak{i}-1}\mathcal{G}(x,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))dx\right]\int_0^1\kappa^{\mathfrak{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.\end{aligned}\tag{37}$$

Again, multiplying the double inequality (36) by $\frac{(x-\mathfrak{n}_\flat)^{\mathfrak{i}-1}}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathfrak{i}}}$ and integrating with respect to x over $[\mathfrak{n}_\flat,\ \mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)]$, we have

$$\begin{aligned}
&\tfrac{1}{\mathrm{j}(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}\hbar\left(\frac{1}{2}\right)}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}\left(x,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)(x-\mathfrak{n}_\flat)^{\mathrm{i}-1}dx\\
&\leq_p \tfrac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\\
&\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(x-\mathfrak{n}_\flat)^{\mathrm{i}-1}(\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})-\kappa)^{\mathrm{j}-1}\mathcal{G}(x,\kappa)d\kappa dx\\
&+\tfrac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}(x-\mathfrak{n}_\flat)^{\mathrm{i}-1}(\kappa-\mathfrak{z})^{\mathrm{j}-1}\mathcal{G}(x,\kappa)d\kappa dx}\\
&\leq_p \tfrac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}(x-\mathfrak{n}_\flat)^{\mathrm{i}-1}\mathcal{G}(x,\mathfrak{z})dx+\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}(x-\mathfrak{n}_\flat)^{\mathrm{i}-1}\mathcal{G}(x,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))dx\right]\\
&\int_0^1\kappa^{\mathrm{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.
\end{aligned}\tag{38}$$

From (37), we have

$$\begin{aligned}
&\tfrac{\Gamma(\mathrm{i}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&\leq_p \tfrac{\Gamma(\mathrm{i}+1)\Gamma(\mathrm{j}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\right]\\
&\leq_p \tfrac{\mathrm{j}\Gamma(\mathrm{i}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\int_0^1\kappa^{\mathrm{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa.
\end{aligned}\tag{39}$$

From (38), we have

$$\begin{aligned}
&\tfrac{\Gamma(\mathrm{i}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&\leq_p \tfrac{\Gamma(\mathrm{i}+1)\Gamma(\mathrm{j}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\
&\leq_p \tfrac{\mathrm{j}\Gamma(\mathrm{i}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\int_0^1\kappa^{\mathrm{j}-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa
\end{aligned}\tag{40}$$

Similarly, since $\mathcal{G}_y(z)=\mathcal{G}(z,y)$ then, from (35), (41), and (42), we have

$$\begin{aligned}
&\tfrac{\Gamma(\mathrm{j}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)\right]\\
&\leq_p \tfrac{\Gamma(\mathrm{i}+1)\Gamma(\mathrm{j}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\\
&\leq_p \tfrac{\mathrm{i}\Gamma(\mathrm{j}+1)}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}^+}\ \mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right].
\end{aligned}\tag{41}$$

and

$$\begin{aligned}
&\tfrac{\Gamma(\mathrm{j}+1)}{2\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right]\\
&\leq_p \tfrac{\Gamma(\mathrm{i}+1)\Gamma(\mathrm{j}+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^{\mathrm{i},\mathrm{j}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\
&\leq_p \tfrac{\mathrm{i}\Gamma(\mathrm{j}+1)}{(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\left[\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\right].
\end{aligned}\tag{42}$$

The second, third, and fourth inequalities of (28) result from combining inequalities (41) and (42).

Now, we have the left portion of the inequality (11):

$$\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\leq_p \tfrac{\Gamma(\mathrm{j}+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^{\mathrm{j}}}\\
&\left[\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^{\mathrm{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right],
\end{aligned}\tag{43}$$

and

$$\begin{aligned}
&\tfrac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq_p \tfrac{\Gamma(\mathrm{i}+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^{\mathrm{i}}}\left[\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat{}^+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^{\mathrm{i}}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]
\end{aligned}\tag{44}$$

The following inequality results from combining inequalities (43) and (44):

$$\begin{aligned}&\frac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\ \leq_p\ &\frac{\Gamma(i+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\ &+\frac{\Gamma(j+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}\left(\frac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\frac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right].\end{aligned}$$

Similarly, given the set of I of $I\cdot V$-Ms $\mathcal{G}:\Omega\to\mathbb{R}^+_I$, the inequality can be expressed as follows:

$$\begin{aligned}&\frac{1}{\hbar^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\ \leq_p\ &\frac{\Gamma(i+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\ &+\frac{\Gamma(j+1)}{\hbar\left(\frac{1}{2}\right)(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}\left(\frac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)\ +\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\frac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right].\end{aligned}\tag{45}$$

This gives the first inequality in (28).
Again, we have the right-hand side of the inequality (11):

$$\begin{aligned}&\frac{\Gamma(j)}{(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\ \leq_p\ &[\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)]\times\int_0^1\kappa^{j-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\end{aligned}\tag{46}$$

$$\begin{aligned}&\frac{\Gamma(j)}{(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\right]\\ \leq_p\ &[\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{t},\theta)]\times\int_0^1\kappa^{j-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\end{aligned}\tag{47}$$

$$\begin{aligned}&\frac{\Gamma(i)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\ \leq_p\ &[\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})]\times\int_0^1 v^{i-1}\hbar(v)+\hbar(1-v)dv\end{aligned}\tag{48}$$

$$\begin{aligned}&\frac{\Gamma(i)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\\ \leq_p\ &[\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)]\times\int_0^1 v^{i-1}\hbar(v)+\hbar(1-v)dv\end{aligned}\tag{49}$$

By summing inequalities (46) to (49), and then multiplying the result by ij, we obtain

$$\begin{aligned}&\frac{j\Gamma(i+1)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right.\\ &\left.+\mathcal{I}^i_{\mathfrak{n}_\flat+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\\ &+\frac{i\Gamma(j+1)}{(\Lambda_2(\theta,\mathfrak{z}))^j}\begin{bmatrix}\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\\ +\mathcal{I}^j_{\mathfrak{z}+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\end{bmatrix}\\ &\leq_p[\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{t},\theta)]\\ &\times\int_0^1\kappa^{j-1}[\hbar(\kappa)+\hbar(1-\kappa)]d\kappa\int_0^1 v^{i-1}\hbar(v)+\hbar(1-v)dv.\end{aligned}\tag{50}$$

The final inequality of (28) marks the conclusion of the derivation.□

Example 3. *We assume the $I\cdot V$-Ms $\mathcal{G}:[0,\ 2]\times[0,\ 2]\to\mathbb{R}^+_I$ defined by,*

$$\mathcal{G}(x,y)=\left[\left(2-\sqrt{x}\right)\left(2-\sqrt{y}\right),2\left(2-\sqrt{x}\right)\left(2-\sqrt{y}\right)\right],\tag{51}$$

then, for each $\gamma\in[0,\ 1]$, we have. Since the endpoint functions $\mathcal{G}_*(x,y)$, $\mathcal{G}^*(x,y)$ are $\hbar$-pre-invex functions with respect to $\Lambda_1(\mathfrak{t},\mathfrak{n})=$ and $\Lambda_2(\theta,\mathfrak{z})$. Hence, $\widetilde{\mathcal{G}}(x,y)$ is $\mathcal{C}\cdot\mathcal{L}\cdot\mathcal{R}$-$\hbar$-pre-invex $I\cdot V$-M.

$$\mathcal{G}\left(\frac{2\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\ \frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)=[1,2]$$
$$\frac{\Gamma(\mathfrak{i}+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}))^{\mathfrak{i}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}^+}\mathcal{G}\left(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}\left(\mathfrak{n},\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]$$
$$+\frac{\Gamma(\mathfrak{j}+1)}{4(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}\left(\frac{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\frac{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\mathfrak{z}\right)\right]$$
$$=2-\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{8}\Lambda\cdot[1,2]$$
$$\frac{\Gamma(\mathfrak{i}+1)\Gamma(\mathfrak{j}+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}))^{\mathfrak{i}}(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.$$
$$+\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})+\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))$$
$$\left.+\mathcal{I}^{\mathfrak{i},\mathfrak{j}}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})\right]$$
$$=\frac{33}{8}-\sqrt{2}-\frac{\sqrt{2}}{2}\Lambda+\frac{\Lambda}{8}+\frac{\Lambda^2}{32}\cdot[1,2]$$
$$\frac{\Gamma(\mathfrak{i}+1)}{8(\Lambda_1(\mathfrak{t},\mathfrak{n}))^{\mathfrak{i}}}\left[\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.$$
$$\left.+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})+\mathcal{I}^{\mathfrak{i}}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]$$
$$+\frac{\Gamma(\mathfrak{j}+1)}{8(\Lambda_2(\theta,\mathfrak{z}))^{\mathfrak{j}}}\left[\begin{array}{c}\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})+\mathcal{I}^{\mathfrak{j}}_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})\end{array}\right]$$
$$=\frac{34\sqrt{2}+\left(\sqrt{2}-4\right)\Lambda-24}{8\sqrt{2}}\cdot[1,2]$$
$$\frac{\mathcal{G}(\mathfrak{z},\mathfrak{t})+\mathcal{G}(\sigma,\mathfrak{t})+\mathcal{G}(\mathfrak{z},\theta)+\mathcal{G}(\sigma,\theta)}{4}=\left(\frac{9}{2}-2\sqrt{2}\right)\cdot[1,2].$$

That is

$$[1,2]\leq_p 2-\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{8}\Lambda\cdot[1,2]\leq_p \frac{33}{8}-\sqrt{2}-\frac{\sqrt{2}}{2}\Lambda+\frac{\Lambda}{8}+\frac{\Lambda^2}{32}\cdot$$
$$[1,2]\leq_p \frac{34\sqrt{2}+\left(\sqrt{2}-4\right)\Lambda-24}{8\sqrt{2}}\cdot[1,2]\leq_p \left(\frac{9}{2}-2\sqrt{2}\right)\cdot[1,2].$$

Hence, Theorem 5 has been verified.

Remark 3. *Assuming that* $\mathfrak{i}=1$ *and* $\mathfrak{j}=1$, *and* $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, *then from (28), the following inequality emerges, see [53]:*

$$\begin{aligned}&\mathcal{G}\left(\frac{2\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\ \frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\&\leq_p \frac{1}{2}\left[\frac{1}{\Lambda_1(\mathfrak{t},\mathfrak{n})}\int_{\mathfrak{n}}^{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}\mathcal{G}\left(x,\frac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)dx\right.\\&\left.+\frac{1}{\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}\left(\frac{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},y\right)dy\right]\\&\leq_p \frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}))(\Lambda_2(\theta,\mathfrak{z}))}\int_{\mathfrak{n}}^{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)dydx\\&\leq_p \frac{1}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}))}\left[\int_{\mathfrak{n}}^{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}\mathcal{G}(x,\mathfrak{z})dx+\int_{\mathfrak{n}}^{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}\mathcal{G}(x,\theta)dx\right]\\&+\frac{1}{4(\Lambda_2(\theta,\mathfrak{z}))}\left[\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{n},y)dy+\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{t},y)dy\right]\\&\leq_p \frac{\mathcal{G}(\mathfrak{n},\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n},\theta)+\mathcal{G}(\mathfrak{t},\theta)}{4}.\end{aligned}\tag{52}$$

Assuming that $\mathfrak{i}=1$ and $\mathfrak{j}=1$, $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, and $\mathcal{G}$ is left-$\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex, then from (28), the following inequality emerges, see [35]:

$$\begin{aligned}
&\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\geq \frac{1}{2}\left[\frac{1}{\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}\left(x,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)dx\right.\\
&\left.+\frac{1}{\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)dy\right]\\
&\geq \frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))(\Lambda_2(\theta,\mathfrak{z}))}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)dydx\\
&\geq \frac{1}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))}\left[\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}(x,\mathfrak{z})dx+\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}(x,\theta)dx\right]\\
&+\frac{1}{4(\Lambda_2(\theta,\mathfrak{z}))}\left[\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{n}_\flat,y)dy+\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{t},y)dy\right]\\
&\geq \frac{\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)}{4}.
\end{aligned}\tag{53}$$

If $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, and $\mathcal{G}_*(x,y)\neq\mathcal{G}^*(x,y)$, then we can derive the following inequality from (28); for further information, see [53]:

$$\begin{aligned}
&\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq_p \frac{\Gamma(i+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}\left(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}\left(\mathfrak{n}_\flat,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&+\frac{\Gamma(j+1)}{4(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\mathfrak{z}\right)\right]\\
&\leq_p \frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\
&\leq_p \frac{\Gamma(i+1)}{8(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i}\left[\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\\
&+\frac{\Gamma(j+1)}{8(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}^+}\ \mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\right.\\
&\left.+\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\right]\\
&\leq_p \frac{\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)}{4}.
\end{aligned}\tag{54}$$

If $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, and $\mathcal{G}_*(x,y)\neq\mathcal{G}^*(x,y)$, then we can derive the following inequality from (28); for further information, see [58]:

$$\begin{aligned}
&\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq_p \frac{1}{2}\left[\frac{1}{\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}\left(x,\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)dx+\frac{1}{\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},y\right)dy\right]\\
&\leq_p \frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))(\Lambda_2(\theta,\mathfrak{z}))}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)dydx\\
&\leq_p \frac{1}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))}\left[\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}(x,\mathfrak{z})dx+\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\mathcal{G}(x,\theta)dx\right]\\
&+\frac{1}{4(\Lambda_2(\theta,\mathfrak{z}))}\left[\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{n}_\flat,y)dy+\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(\mathfrak{t},y)dy\right]\\
&\leq_p \frac{\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n}_\flat,\theta)+\mathcal{G}(\mathfrak{t},\theta)}{4}.
\end{aligned}\tag{55}$$

If $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, and $\mathcal{G}_*(x,y)=\mathcal{G}^*(x,y)$, then we can derive the following classical inequality from (28):

$$\begin{aligned}
&\mathcal{G}\left(\tfrac{2\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\ \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq \tfrac{\Gamma(i+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}))^i}\left[\mathcal{I}^i_{\mathfrak{n}^+}\mathcal{G}\left(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)+\mathcal{I}^i_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}\left(\mathfrak{n},\tfrac{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\right]\\
&+\tfrac{\Gamma(j+1)}{4(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}\left(\tfrac{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})\right)+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}\left(\tfrac{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})}{2},\mathfrak{z}\right)\right]\\
&\leq \tfrac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&+\mathcal{I}^{i,j}_{\mathfrak{n}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{i,j}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})\right]\\
&\leq \tfrac{\Gamma(i+1)}{8(\Lambda_1(\mathfrak{t},\mathfrak{n}))^i}\left[\mathcal{I}^i_{\mathfrak{n}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})+\mathcal{G}\mathcal{I}^i_{\mathfrak{n}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^i_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})+\mathcal{I}^i_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right]\\
&+\tfrac{\Gamma(j+1)}{8(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^j_{\mathfrak{z}^+}\ \mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})\right.\\
&\left.+\mathcal{I}^j_{\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^j_{\theta^-}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})\right]\\
&\leq \tfrac{\mathcal{G}(\mathfrak{n},\mathfrak{z})+\mathcal{G}(\mathfrak{t},\mathfrak{z})+\mathcal{G}(\mathfrak{n},\theta)+\mathcal{G}(\mathfrak{t},\theta)}{4}.
\end{aligned}\tag{56}$$

In the upcoming results, we will discover intriguing findings derived from the product of two $\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-invex $I\cdot V$-Ms. These findings are commonly referred to as Pachpatte's inequalities.

Theorem 6. *Let $\mathcal{G}$, $\mathcal{J}:\Omega\to\mathbb{R}^+_I$ be two $\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex $I\cdot V$-Ms on Ω, given by $\mathcal{G}(x,y)=[\mathcal{G}_*(x,y),\ \mathcal{G}^*(x,y)]$ and $\mathcal{J}(x,y)=[\mathcal{J}_*(x,y),\ \mathcal{J}^*(x,y)]$ for all $(x,y)\in\Omega$ and let $\hbar_1,\hbar_2:[0,\ 1]\to\mathbb{R}^+$. If $\mathcal{G}\times\mathcal{J}\in\mathcal{F}\mathfrak{D}_\Omega$, then the following inequalities holds:*

$$\begin{aligned}
&\tfrac{\Gamma(i)\Gamma(j)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})\times\mathcal{J}(\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n}),\mathfrak{z})\right]\\
&+\tfrac{\Gamma(i)\Gamma(j)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n},\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}+\Lambda_1(\mathfrak{t},\mathfrak{n})^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n},\mathfrak{z})\times\mathcal{J}(\mathfrak{n},\mathfrak{z})\right]\\
&\leq_p \mathcal{M}(\mathfrak{n},\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)\\
&+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&+P(\mathfrak{n},\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)\\
&+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&+\mathcal{N}(\mathfrak{n},\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa\\
&+Q(\mathfrak{n},\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&+\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa.
\end{aligned}\tag{57}$$

If $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-concave, then the inequality (57) is reversed such that

$$\begin{aligned}
&\frac{\Gamma(i)\Gamma(j)}{(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\Big[\mathcal{I}^{i,j}_{n_\flat^+,z^+}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\times\mathcal{J}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\\
&\quad+\mathcal{I}^{i,j}_{n_\flat^+,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z)\times\mathcal{J}(n_\flat+\Lambda_1(t,n_\flat),z)\Big]\\
&+\frac{\Gamma(i)\Gamma(j)}{(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\Big[\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z^+}\mathcal{G}(n_\flat,z+\Lambda_2(\theta,z))\times\mathcal{J}(n_\flat,z+\Lambda_2(\theta,z))\\
&\quad+\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat,z)\times\mathcal{J}(n_\flat,z)\Big]\\
&\geq_p\mathcal{M}(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&\quad+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)\\
&\quad+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&\quad+P(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&\quad+\hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\
&\quad+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\
&\quad+\mathcal{N}(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&\quad+\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&\quad+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa\\
&\quad+Q(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\
&\quad+\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\
&\quad+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa.
\end{aligned}\tag{58}$$

where

$$\begin{aligned}
\mathcal{M}(n_\flat,t,z,\theta)&=\mathcal{G}(n_\flat,z)\times\mathcal{J}(n_\flat,z)+\mathcal{G}(t,z)\times\mathcal{J}(t,z)+\mathcal{G}(n_\flat,\theta)\times\mathcal{J}(n_\flat,\theta)+\mathcal{G}(t,\theta)\times\mathcal{J}(t,\theta),\\
P(n_\flat,t,z,\theta)&=\mathcal{G}(n_\flat,z)\times\mathcal{J}(t,z)+\mathcal{G}(t,z)\times\mathcal{J}(n_\flat,z)+\mathcal{G}(n_\flat,\theta)\times\mathcal{J}(t,\theta)+\mathcal{G}(t,\theta)\times\mathcal{J}(n_\flat,\theta),\\
\mathcal{N}(n_\flat,t,z,\theta)&=\mathcal{G}(n_\flat,z)\times\mathcal{J}(n_\flat,\theta)+\mathcal{G}(t,z)\times\mathcal{J}(t,\theta)+\mathcal{G}(n_\flat,\theta)\times\mathcal{J}(n_\flat,z)+\mathcal{G}(t,\theta)\times\mathcal{J}(t,z),\\
Q(n_\flat,t,z,\theta)&=\mathcal{G}(n_\flat,z)\times\mathcal{J}(t,\theta)+\mathcal{G}(t,z)\times\mathcal{J}(n_\flat,\theta)+\mathcal{G}(n_\flat,\theta)\times\mathcal{J}(t,z)+\mathcal{G}(t,\theta)\times\mathcal{J}(n_\flat,z),
\end{aligned}$$

and $\mathcal{M}(n_\flat,t,z,\theta)$, $P(n_\flat,t,z,\theta)$, $\mathcal{N}(n_\flat,t,z,\theta)$, and $Q(n_\flat,t,z,\theta)$ are defined as follows:

$$\begin{aligned}
\mathcal{M}(n_\flat,t,z,\theta)&=[\mathcal{M}_*(n_\flat,t,z,\theta),\ \mathcal{M}^*(n_\flat,t,z,\theta)],\\
P(n_\flat,t,z,\theta)&=[P_*(n_\flat,t,z,\theta),\ P^*(n_\flat,t,z,\theta)],\\
\mathcal{N}(n_\flat,t,z,\theta)&=[\mathcal{N}_*(n_\flat,t,z,\theta),\ \mathcal{N}^*(n_\flat,t,z,\theta)],\\
Q(n_\flat,t,z,\theta)&=[Q_*(n_\flat,t,z,\theta),\ Q^*(n_\flat,t,z,\theta)].
\end{aligned}$$

Proof. Let $\mathcal{G}$ and $\mathcal{J}$ be two $\mathcal{C}\cdot L\cdot R$-$\hbar_1$ and $\mathcal{C}\cdot L\cdot R$-$\hbar_2$- pre-invex $I\cdot V$-Ms on $[n_\flat,\ n_\flat+\Lambda_1(t,n_\flat)]\times[z,\ z+\Lambda_2(\theta,z)]$, respectively. Then,

$$\begin{aligned}
&\mathcal{G}(n_\flat+(1-v)\Lambda_1(t,n_\flat),\ z+(1-\kappa)\Lambda_2(\theta,z))\\
&\leq_p\hbar_1(v)\hbar_1(\kappa)\mathcal{G}(n_\flat,z)+\hbar_1(v)\hbar_1(1-\kappa)\mathcal{G}(n_\flat,\theta)\\
&+\hbar_1(1-v)\hbar_1(\kappa)\mathcal{G}(t,z)+\hbar_1(1-v)\hbar_1(1-\kappa)\mathcal{G}(t,\theta),\\
&\mathcal{G}(n_\flat+(1-v)\Lambda_1(t,n_\flat),\ z+\kappa\Lambda_2(\theta,z))\\
&\leq_p\hbar_1(v)\hbar_1(1-\kappa)\mathcal{G}(n_\flat,z)+\hbar_1(v)\hbar_1(\kappa)\mathcal{G}(n_\flat,\theta)\\
&+\hbar_1(1-v)\hbar_1(1-\kappa)\mathcal{G}(t,z)+\hbar_1(1-v)\hbar_1(\kappa)\mathcal{G}(t,\theta),\\
&\mathcal{G}(n_\flat+v\Lambda_1(t,n_\flat),\ z+(1-\kappa)\Lambda_2(\theta,z))\\
&\leq_p\hbar_1(1-v)\hbar_1(\kappa)\mathcal{G}(n_\flat,z)+\hbar_1(1-v)\hbar_1(1-\kappa)\mathcal{G}(n_\flat,\theta)\\
&+\hbar_1(v)\hbar_1(\kappa)\mathcal{G}(t,z)+\hbar_1(v)\hbar_1(1-\kappa)\mathcal{G}(t,\theta),\\
&\mathcal{G}(n_\flat+v\Lambda_1(t,n_\flat),\ z+\kappa\Lambda_2(\theta,z))\\
&\leq_p\hbar_1(1-v)\hbar_1(1-\kappa)\mathcal{G}(n_\flat,z)+\hbar_1(1-v)\hbar_1(\kappa)\mathcal{G}(n_\flat,\theta)\\
&+\hbar_1(v)\hbar_1(1-\kappa)\mathcal{G}(t,z)+\hbar_1(v)\hbar_1(\kappa)\mathcal{G}(t,\theta),
\end{aligned}$$

and

$$\begin{gathered}\mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ \leq_p \hbar_2(\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}) + \hbar_2(\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{n}_\flat,\theta)\\ +\hbar_2(1-\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{t},\mathfrak{z}) + \hbar_2(1-\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{t},\theta),\\ \mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))\\ \leq_p \hbar_2(\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}) + \hbar_2(\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{n}_\flat,\theta)\\ +\hbar_2(1-\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{t},\mathfrak{z}) + \hbar_2(1-\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{t},\theta),\\ \mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ \leq_p \hbar_2(1-\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}) + \hbar_2(1-\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{n}_\flat,\theta)\\ +\hbar_2(\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{t},\mathfrak{z}) + \hbar_2(\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{t},\theta),\\ \mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))\\ \leq_p \hbar_2(1-\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}) + \hbar_2(1-\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{n}_\flat,\theta)\\ +\hbar_2(\upsilon)\hbar_2(1-\kappa)\mathcal{J}(\mathfrak{t},\mathfrak{z}) + \hbar_2(\upsilon)\hbar_2(\kappa)\mathcal{J}(\mathfrak{t},\theta),\end{gathered}$$

Since $\mathcal{G}$ and $\mathcal{J}$ both are $\mathcal{C}\cdot L\cdot R$-$\hbar_1$ and $\mathcal{C}\cdot L\cdot R$-$\hbar_2$- pre-invex $I\cdot V$-Ms on $[\mathfrak{n}_\flat, \mathfrak{n}_\flat + \mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat)] \times [\mathfrak{z}, \mathfrak{z} + \mathcal{K}_2(\theta,\mathfrak{z})]$, we have

$$\begin{gathered}\mathcal{G}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ +\mathcal{G}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z})) + \mathcal{G}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\ +\mathcal{G}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z})) \times \mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))\\ \leq_p \mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)\\ +\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]\\ +P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]\\ +\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\ +\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]\\ +Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\ +\hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)].\end{gathered}$$

Taking the multiplication of the above fuzzy inclusion with $\upsilon^{i-1}\kappa^{j-1}$ and then taking the double integration of the resultant over $[0,\ 1] \times [0,\ 1]$ with respect to $(\upsilon,\ \kappa)$ such that

$$\begin{gathered}\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z})) \times \mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))d\upsilon d\kappa\\ +\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z})) \times \mathcal{J}(\mathfrak{n}_\flat + (1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))d\upsilon d\kappa\\ +\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z})) \times \mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + (1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))d\upsilon d\kappa\\ +\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z})) \times \mathcal{J}(\mathfrak{n}_\flat + \upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z} + \kappa\mathcal{K}_2(\theta,\mathfrak{z}))d\upsilon d\kappa\\ \leq_p \mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)\\ +\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\ +P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa) + \hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(\kappa) + \hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(\kappa)]d\upsilon d\kappa\\ +\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}[\hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(1-\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\ +\hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_1(\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa\\ +Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 \upsilon^{i-1}\kappa^{j-1}[\hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(1-\kappa)\hbar_2(\kappa)\\ +\hbar_1(1-\upsilon)\hbar_2(\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa) + \hbar_1(\upsilon)\hbar_2(1-\upsilon)\hbar_1(\kappa)\hbar_2(1-\kappa)]d\upsilon d\kappa\end{gathered} \tag{59}$$

From the right hand side of (59), we have

$$\begin{array}{c}\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))dvd\kappa\\ +\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))dvd\kappa\\ +\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))dvd\kappa\\ +\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))dvd\kappa\\ =\frac{\Gamma(i)\Gamma(j)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\ +\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\end{bmatrix}.\end{array}\tag{60}$$

Combining (59) and (60), we have

$$\begin{array}{c}\frac{\Gamma(i)\Gamma(j)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\Big[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\ \times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\Big]\\ \leq_p\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\ +P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(1-\kappa)+\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(\kappa)\\ +\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(\kappa)]dvd\kappa\\ +\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(1-v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(1-v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(v)\hbar_2(v)\hbar_1(1-\kappa)\hbar_2(\kappa)\\ +\hbar_1(v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa\\ +Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(v)\hbar_2(1-v)\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(1-v)\hbar_2(v)\hbar_1(\kappa)\hbar_2(1-\kappa)\\ +\hbar_1(v)\hbar_2(1-v)\hbar_1(\kappa)\hbar_2(1-\kappa)]dvd\kappa.\end{array}$$

Hence, the required result. □

Remark 4. *Assuming that $\mathcal{G}$is left-$\mathcal{C}$·L·R-$\hbar$-pre-invex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, $i=1$, and $j=1$, then from (59), the following inequality emerges, see [54]:*

$$\begin{array}{c}\frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)\times\mathcal{J}(x,y)dydx\\ \geq\frac{1}{9}\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\frac{1}{18}[P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)]+\frac{1}{36}Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).\end{array}\tag{61}$$

If $\mathcal{G}$ is $\mathcal{C}$·L·R-$\hbar$- pre-invex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and one assumes that $i=1$ and $j=1$, then from (59), the following inequality emerges, see [58]:

$$\begin{array}{c}\frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)\times\mathcal{J}(x,y)dydx\\ \leq_p\frac{1}{9}\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\frac{1}{18}[P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)]+\frac{1}{36}Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).\end{array}\tag{62}$$

If $\mathcal{G}_*(x,y) \neq \mathcal{G}^*(x,y)$ and $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, then we can derive the following inequality from (57); for further information, see [53]:

$$\begin{aligned}
&\frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{n_\flat^+,z^+}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\times J(n_\flat+\Lambda_1(t,n_\flat),\theta)\\ +\mathcal{I}^{i,j}_{n_\flat^+,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z)\times J(n_\flat+\Lambda_1(t,n_\flat),z)\end{bmatrix}\\
&+\frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z^+}\mathcal{G}(n_\flat,z+\Lambda_2(\theta,z))\times J(n_\flat,z+\Lambda_2(\theta,z))\\ +\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat,z)\times J(n_\flat,z)\end{bmatrix}\\
&\leq_p\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)\left(\frac{1}{2}-\frac{j}{(j+1)(j+2)}\right)\mathcal{M}(n_\flat,t,z,\theta)\\
&+\frac{i}{(i+1)(i+2)}\left(\frac{1}{2}-\frac{j}{(j+1)(j+2)}\right)P(n_\flat,t,z,\theta)\\
&+\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)\frac{j}{(j+1)(j+2)}\mathcal{N}(n_\flat,t,z,\theta)+\frac{j}{(j+1)(j+2)}\frac{i}{(i+1)(i+2)}Q(n_\flat,t,z,\theta).
\end{aligned}\tag{63}$$

If $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, and $\mathcal{G}_*(x,y) \neq \mathcal{G}^*(x,y)$, then we can derive the following inequality from (57); for further information, see [46]:

$$\begin{aligned}
&\frac{1}{(\Lambda_1(t,n_\flat))\Lambda_2(\theta,z)}\int_{n_\flat}^{n_\flat+\Lambda_1(t,n_\flat)}\int_z^{z+\Lambda_2(\theta,z)}\mathcal{G}(x,y)\times\mathcal{J}(x,y)dydx\\
&\leq_p\tfrac{1}{9}\mathcal{M}(n_\flat,t,z,\theta)+\tfrac{1}{18}[P(n_\flat,t,z,\theta)+\mathcal{N}(n_\flat,t,z,\theta)]+\tfrac{1}{36}Q(n_\flat,t,z,\theta).
\end{aligned}\tag{64}$$

If $\mathcal{G}_*(x,y) = \mathcal{G}^*(x,y)$ and $\mathcal{J}_*(x,y) = \mathcal{J}^*(x,y)$ and $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, then we can derive the following classical inequality from (66):

$$\begin{aligned}
&\frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{n_\flat^+,z^+}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\times J(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\\ +\mathcal{I}^{i,j}_{n_\flat^+,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z)\times J(n_\flat+\Lambda_1(t,n_\flat),z)\end{bmatrix}\\
&+\frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\begin{bmatrix}+\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z^+}\mathcal{G}(n_\flat,z+\Lambda_2(\theta,z))\times J(n_\flat,z+\Lambda_2(\theta,z))\\ +\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat,z)\times J(n_\flat,z)\end{bmatrix}\\
&\leq\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)\left(\frac{1}{2}-\frac{j}{(j+1)(j+2)}\right)\mathcal{M}(n_\flat,t,z,\theta)\\
&+\frac{i}{(i+1)(i+2)}\left(\frac{1}{2}-\frac{j}{(j+1)(j+2)}\right)P(n_\flat,t,z,\theta)\\
&+\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)\frac{j}{(j+1)(j+2)}\mathcal{N}(n_\flat,t,z,\theta)+\frac{j}{(j+1)(j+2)}\frac{i}{(i+1)(i+2)}Q(n_\flat,t,z,\theta).
\end{aligned}\tag{65}$$

Theorem 7. *Let $\mathcal{G}$, $\mathcal{J}:\Omega\to\mathbb{R}_I^+$ be two $\mathcal{C}\cdot\mathcal{L}\cdot\mathcal{R}$-$\hbar$-pre-invex I·V-Ms on Ω, given by $\mathcal{G}(x,y) = [\mathcal{G}_*(x,y),\ \mathcal{G}^*(x,y)]$ and $\mathcal{J}(x,y) = [\mathcal{J}_*(x,y),\ \mathcal{J}^*(x,y)]$ for all $(x,y)\in\Omega$ and let $\hbar:[0,1]\to\mathbb{R}^+$. If $\mathcal{G}\times\mathcal{J}\in\mathcal{FD}_\Omega$, then the following inequalities holds:*

$$\begin{aligned}
&\frac{1}{2ij\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\frac{2n_\flat+\Lambda_1(t,n_\flat)}{2},\ \frac{2z+\Lambda_2(\theta,z)}{2}\right)\times\mathcal{J}\left(\frac{2n_\flat+\Lambda_1(t,n_\flat)}{2},\ \frac{2z+\Lambda_2(\theta,z)}{2}\right)\\
&\leq_p\frac{\Gamma(i)\Gamma(j)}{2(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\Big[\mathcal{I}^{i,j}_{n_\flat^+,z^+}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\times\mathcal{J}(n_\flat+\Lambda_1(t,n_\flat),z+\Lambda_2(\theta,z))\\
&+\mathcal{I}^{i,j}_{n_\flat^+,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat+\Lambda_1(t,n_\flat),z)\times\mathcal{J}(n_\flat+\Lambda_1(t,n_\flat),z)\Big]\\
&+\frac{\Gamma(i)\Gamma(j)}{2(\Lambda_1(t,n_\flat))^i(\Lambda_2(\theta,z))^j}\Big[\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z^+}\mathcal{G}(n_\flat,z+\Lambda_2(\theta,z))\\
&\times\mathcal{J}(n_\flat,z+\Lambda_2(\theta,z))\quad+\mathcal{I}^{i,j}_{n_\flat+\Lambda_1(t,n_\flat)^-,z+\Lambda_2(\theta,z)^-}\mathcal{G}(n_\flat,z)\times\mathcal{J}(n_\flat,z)\Big]\\
&+\mathcal{M}(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]]dvd\kappa\\
&+P(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]]dvd\kappa\\
&+\mathcal{N}(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]]dvd\kappa\\
&+Q(n_\flat,t,z,\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dvd\kappa.
\end{aligned}\tag{66}$$

If $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-concave, then the inequality (66) is reversed such that

$$\begin{aligned}
&\frac{1}{2ij\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\ \frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\geq_p \frac{\Gamma(i)\Gamma(j)}{2(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\right]\\
&+\frac{\Gamma(i)\Gamma(j)}{2(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\left[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\right.\\
&\left.+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\times\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z})\right]\\
&+\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]]dvd\kappa\\
&+P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]]dvd\kappa\\
&+\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)\ \ +\hbar_2(1-v)\hbar_2(1-\kappa)]]dvd\kappa\\
&+Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dvd\kappa.
\end{aligned}\tag{67}$$

where $\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$, $P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$, $\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$, and $Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$ are given in Theorem 6.

Proof. Since $\mathcal{G},\mathcal{J}:\Omega\to\mathbb{R}_I^+$ be two LR-$\hbar$- pre-invex $I\cdot V$-Ms, then from the inequality (17), we have

$$\begin{aligned}
&\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&=\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}+\tfrac{\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{\mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})}{2}+\tfrac{\mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\tfrac{\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}+\tfrac{\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{\mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z})}{2}+\tfrac{\mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
&\leq_p \hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\times\begin{bmatrix}\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))+\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\end{bmatrix}\\
&\times\begin{bmatrix}J(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))+J(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\+J(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))+J(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\end{bmatrix}\\
&\leq_p \hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\times\begin{bmatrix}\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+(1-v)\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+v\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\Lambda_2(\theta,\mathfrak{z}))\end{bmatrix}\\
&+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\times\begin{bmatrix}\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\\+\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\\+\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\end{bmatrix}\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\\
&\times\begin{bmatrix}\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\\+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]\\+\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\+\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\end{bmatrix}P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\\
&\times\begin{bmatrix}\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\\+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\+\hbar_1(1-v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]\\+\hbar_1(1-v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\end{bmatrix}\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)
\end{aligned}$$

$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)$$
$$\times\begin{bmatrix}\hbar_1(\upsilon)\hbar_1(\kappa)[\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\end{bmatrix}Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).$$

Taking the multiplication of the above fuzzy inclusion with $\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}$ and then taking the double integration of the resultant over $[0,\ 1]\times[0,\ 1]$ with respect to $(\upsilon,\ \kappa)$, we have

$$\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\mathcal{G}\left(\tfrac{\mathfrak{n}_\flat+\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{\mathfrak{z}+\mathcal{K}_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\tfrac{\mathfrak{n}_\flat+\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\tfrac{\mathfrak{z}+\mathcal{K}_2(\theta,\mathfrak{z})}{2}\right)d\upsilon d\kappa$$
$$\leq_p\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)$$
$$\times\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\begin{bmatrix}\mathcal{G}(\mathfrak{n}_\flat+(1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+(1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+\upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+(1-\kappa)\mathcal{K}_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+(1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\mathcal{K}_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+(1-\upsilon)\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\mathcal{K}_2(\theta,\mathfrak{z}))\\+\mathcal{G}(\mathfrak{n}_\flat+\upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\mathcal{K}_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\upsilon\mathcal{K}_1(\mathfrak{t},\mathfrak{n}_\flat),\ \mathfrak{z}+\kappa\mathcal{K}_2(\theta,\mathfrak{z}))\end{bmatrix}d\upsilon d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$$
$$\times\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\begin{bmatrix}\hbar_1(\upsilon)\hbar_1(\kappa)[\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(\kappa)[\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\end{bmatrix}d\upsilon d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$$
$$\times\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\begin{bmatrix}\hbar_1(\upsilon)\hbar_1(\kappa)[\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(\kappa)[\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\end{bmatrix}d\upsilon d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$$
$$\times\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\begin{bmatrix}\hbar_1(\upsilon)\hbar_1(\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(\kappa)[\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(1-\kappa)[\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\end{bmatrix}d\upsilon d\kappa$$
$$+\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)$$
$$\times\int_0^1\int_0^1\upsilon^{\mathfrak{i}-1}\kappa^{\mathfrak{j}-1}\begin{bmatrix}\hbar_1(\upsilon)\hbar_1(\kappa)[\hbar_2(1-\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(\upsilon)\hbar_2(1-\kappa)]\\+\hbar_1(1-\upsilon)\hbar_1(\kappa)[\hbar_2(\upsilon)\hbar_2(\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\\+\hbar_1(\upsilon)\hbar_1(1-\kappa)[\hbar_2(\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(1-\kappa)+\hbar_2(1-\upsilon)\hbar_2(\kappa)]\end{bmatrix}d\upsilon d\kappa,$$

which implies that

$$\begin{aligned}
&\frac{1}{ij}\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
\leq_p\ &\frac{\Gamma(i)\Gamma(j)\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\Big[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\
&+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times\mathcal{J}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\Big]\\
&+\frac{\Gamma(i)\Gamma(j)\hbar_1{}^2\left(\frac{1}{2}\right)\hbar_2{}^2\left(\frac{1}{2}\right)}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\Big[\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\
&+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\times\mathcal{J}(\mathfrak{n}_\flat,\mathfrak{z})\Big]\\
&+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(1-\kappa)\quad+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]]dvd\kappa\\
&+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)\quad+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]]dvd\kappa\\
&+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(v)\hbar_2(\kappa)\quad+\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(1-v)\hbar_2(1-\kappa)]]dvd\kappa\\
&+2\hbar_1{}^2\left(\tfrac{1}{2}\right)\hbar_2{}^2\left(\tfrac{1}{2}\right)Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\int_0^1 v^{i-1}\kappa^{j-1}[\hbar_1(v)\hbar_1(\kappa)[\hbar_2(1-v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)]\\
&+\hbar_1(v)\hbar_1(1-\kappa)[\hbar_2(1-v)\hbar_2(1-\kappa)+\hbar_2(v)\hbar_2(\kappa)+\hbar_2(v)\hbar_2(1-\kappa)]]dvd\kappa,
\end{aligned}$$

hence, the required result. □

Remark 5. *Assuming that $\mathcal{G}$ is left-$\mathcal{C}\cdot L\cdot R$-$\hbar$- pre-invex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and $i=1$ and $j=1$, then from (66), the following inequality emerges, see [35]:*

$$\begin{aligned}
&4\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
\supseteq\ &\frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)\times\mathcal{J}(x,y)dydx+\frac{5}{36}\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\frac{7}{36}[P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)]+\frac{2}{9}Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).
\end{aligned}\tag{68}$$

If $\mathcal{G}$ is $\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-invex with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$ and one assumes that $i=1$ and $j=1$, then from (66), the following inequality emerges, see [46]:

$$\begin{aligned}
&4\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
\leq_p\ &\frac{1}{(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))\Lambda_2(\theta,\mathfrak{z})}\int_{\mathfrak{n}_\flat}^{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}\int_{\mathfrak{z}}^{\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}\mathcal{G}(x,y)\times\mathcal{J}(x,y)dydx+\frac{5}{36}\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\frac{7}{36}[P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)+\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)]+\frac{2}{9}Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).
\end{aligned}\tag{69}$$

If $\mathcal{G}$ is left-$\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-invex and $\mathcal{G}_*(x,y)\neq\mathcal{G}^*(x,y)$ with $\hbar(v)=v$, $\hbar(\kappa)=\kappa$, then from (66), we derive the following inequality, see [53]:

$$\begin{aligned}
&4\mathcal{G}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\times\mathcal{J}\left(\frac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2},\frac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right)\\
\supseteq\ &\frac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j}\begin{bmatrix}\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat{}^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\\+\mathcal{I}^{i,j}_{\mathfrak{t}^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\\+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\times J(\mathfrak{n}_\flat,\mathfrak{z})\end{bmatrix}\\
&+\left[\frac{i}{2(i+1)(i+2)}+\frac{j}{(j+1)(j+2)}\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)\right]\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{i}{(i+1)(i+2)}\right)+\frac{i}{(i+1)(i+2)}\frac{j}{(j+1)(j+2)}\right]P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{j}{(j+1)(j+2)}\right)+\frac{i}{(i+1)(i+2)}\frac{j}{(j+1)(j+2)}\right]\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta)\\
&+\left[\frac{1}{4}-\frac{i}{(i+1)(i+2)}\frac{j}{(j+1)(j+2)}\right]Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).
\end{aligned}\tag{70}$$

If $\mathcal{G}_*(x,y) \neq \mathcal{G}^*(x,y)$ and $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, then we can derive the following inequality from (66); for further information, see [53]:

$$\begin{aligned}
&4\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}, \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right) \times \mathcal{J}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}, \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right) \\
\leq_p \tfrac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j} &\left[\begin{array}{c}
\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\times J(\mathfrak{n}_\flat,\mathfrak{z})
\end{array}\right] \\
&+\left[\tfrac{i}{2(i+1)(i+2)}+\tfrac{j}{(j+1)(j+2)}\left(\tfrac{1}{2}-\tfrac{i}{(i+1)(i+2)}\right)\right]\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{(i+1)(i+2)}\right)+\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{j}{(j+1)(j+2)}\right)+\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{4}-\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).
\end{aligned} \tag{71}$$

If $\mathcal{G}_*(x,y) = \mathcal{G}^*(x,y)$ and $\mathcal{J}_*(x,y) = \mathcal{J}^*(x,y)$ and $\hbar(v) = v$, $\hbar(\kappa) = \kappa$, then we can derive the following inequality from (66) such that

$$\begin{aligned}
&4\mathcal{G}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}, \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right) \times \mathcal{J}\left(\tfrac{2\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)}{2}, \tfrac{2\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})}{2}\right) \\
\leq \tfrac{\Gamma(i+1)\Gamma(j+1)}{4(\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat))^i(\Lambda_2(\theta,\mathfrak{z}))^j} &\left[\begin{array}{c}
\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat^+,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z})\times J(\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat),\mathfrak{z}) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}^+}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z}))\times J(\mathfrak{n}_\flat,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})) \\
+\mathcal{I}^{i,j}_{\mathfrak{n}_\flat+\Lambda_1(\mathfrak{t},\mathfrak{n}_\flat)^-,\mathfrak{z}+\Lambda_2(\theta,\mathfrak{z})^-}\mathcal{G}(\mathfrak{n}_\flat,\mathfrak{z})\times J(\mathfrak{n}_\flat,\mathfrak{z})
\end{array}\right] \\
&+\left[\tfrac{i}{2(i+1)(i+2)}+\tfrac{j}{(j+1)(j+2)}\left(\tfrac{1}{2}-\tfrac{i}{(i+1)(i+2)}\right)\right]\mathcal{M}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{(i+1)(i+2)}\right)+\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]P(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{j}{(j+1)(j+2)}\right)+\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]\mathcal{N}(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta) \\
&+\left[\tfrac{1}{4}-\tfrac{i}{(i+1)(i+2)}\tfrac{j}{(j+1)(j+2)}\right]Q(\mathfrak{n}_\flat,\mathfrak{t},\mathfrak{z},\theta).
\end{aligned} \tag{72}$$

4. Conclusions and Future Plans

This article aims to establish Hermite–Hadamard and Pachpatte's type integral inequalities for the newly defined nonconvex class which is known as $\mathcal{C}\cdot L\cdot R$-$\hbar$-pre-invex $I\cdot V$-Ms using the interval fractional operator. First, we present integral equalities that are crucial for deriving the main results of the paper by employing the fractional integral operators, where integrable mappings are $I\cdot V$-Ms. Additionally, the obtained results generalize many previous findings [53,54]. We also provide some exceptional cases as applications for the main results. In future work, we plan to expand and generalize our results using different classes of convex functions and other generalized fractional integral operators. We believe that our results and techniques will be useful to researchers working with various classes of convex functions and other generalized fractional integral operators.

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